Two solutions for an elliptic equation with fast increasing weight and concave-convex nonlinearities

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Abstract

We prove the existence of two nonnegative nontrivial solutions for the equation

$$-\operatorname{div}(K(x)\nabla u) = a(x)K(x)|u|^{q-2}u + b(x)K(x)|u|^{p-2}u, \;\; x \in \mathbb{R}^N,$$

where $N \geq 3$, $K(x) = \exp(|x|^\alpha/4)$, $\alpha \geq 2$ and the pontentials a and b have indefinite sign and satisfy some mild integrability conditions. The results hold when a has small norm in a suitable weighted Lebesgue space.

Key words: Elliptic equations; combined nonlinearities; critical problems; variational methods

1 Introduction

Let us consider the semilinear problem

$$-\Delta u = \lambda a(x)u^{q-1} + b(x)u^{p-1} \text{ in } \Omega, \ \ u \in H_0^1(\Omega), \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a domain, $N \geq 3$, $\lambda > 0$ is a parameter, $1 < q < 2 < p \leq 2^* := 2N/(N-2)$ and the potentials a and b satisfy some mild conditions. In the inspiring work of Ambrosetti, Brezis and Cerami [3] they supposed that Ω is boudned, $a(x) \equiv b(x) \equiv 1$ and prove that the problem has at least two positive solutions provided $\lambda \in (0, \Lambda)$. In [9], deFigueiredo, Gossez and Ubilla generalized this results by allowing that a and b were not constant sign changing functions. In this setting the Maximun Principle can fail and therefore the solutions are only nonnegative. There are many other results for concave-convex equations and indefinite problems on bounded domains. We refer the reader to [2,9,18,10,21,19] and references therein.

There are also some results for unbounded case $\Omega = \mathbb{R}^N$. In this setting we need to require some integrability conditions on a and b, in order to deal with the problem variationally. In [20], E. Tonkes obtained the existence of infinitely many solutions, but with no information about the sign of the solutions. We can also cite [1,4,15,17] where nonnegative solutions are obtained. In all of these papers some kind of sign restriction in one of the potentials is assumed.

In this paper we address the existence of multiple solutions for the following equation

$$(P) \qquad -\Delta u + \frac{\alpha}{2}|x|^{\alpha-2}(x \cdot \nabla u) = a(x)u^{q-1} + b(x)u^{p-1}, \ \ x \in \mathbb{R}^N,$$

where $N \geq 3$, $\alpha \geq 2$, $1 < q < 2 < p \leq 2^*$ and both the potentials a and b being indefinite in sign. This equation is in some sense related to the study of self-similar solutions for the heat equation as quoted in the works of Haraux and Weissler [13], Escobedo and Kavian [11] (see also [6,12]). In this direction, problem (P) arises naturally when one seek for solutions of the form

$$\omega(t, x) = t^{-1/(p-2)}u(t^{-1/2}x)$$

for the evolution equation

$$\omega_t - \Delta\omega = |\omega|^{p-2}\omega \quad \text{on} \quad (0, \infty) \times \mathbb{R}^N.$$

More precisely, $\omega(t, x)$ satisfies the previous equation if and only if $u : \mathbb{R}^N \to \mathbb{R}$ satisfies

$$-\Delta u - \frac{1}{2}(x \cdot \nabla u) = \lambda u + |u|^{p-2}u, \ \ x \in \mathbb{R}^N,$$

which is equivalent to the equation (P) with $\alpha = q = 2$, $a(x) \equiv \lambda$ and $b(x) \equiv 1$.

In order to present our main results we notice that our equation can be rewritten in a divergence form. Actually, if we set $K(x) := \exp(|x|^\alpha/4)$, a straightforward calculation proves that (P) is equivalent to

$$-\operatorname{div}(K(x)\nabla u) = a(x)K(x)|u|^{q-2}u + b(x)K(x)|u|^{p-2}u, \quad x \in \mathbb{R}^N.$$

Hence, it is natural to look for solutions on the space X given by the closure of $C_c^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\| := \left(\int_{\mathbb{R}^N} K(x)|\nabla u|^2 \mathrm{d}x\right)^{1/2}. \tag{1.2}$$

As quoted in [12, Proposition 2.1] X is a Banach space and the weighted Lebesgue spaces

$$L_K^s(\mathbb{R}^N) := \left\{u \text{ measurable in } \mathbb{R}^N \ : \ \|u\|_s^s := \int_{\mathbb{R}^N} K(x)|u|^s \mathrm{d}x < \infty\right\}$$

are such that the embeddings $X \hookrightarrow L_K^r(\mathbb{R}^N)$ are continuous for $2 \leq r \leq 2^*$ and compact for $2 \leq r < 2^*$.

For any $s > 1$, we denote by s' its conjugated exponent, namely the unique $s' > 1$ such that $1/s + 1/s' = 1$. The basic assumptions on the potentials a and b are the following:

(a_1) $a \in L_K^{\sigma_q}(\mathbb{R}^N)$ for some

$$\left(\frac{p}{q}\right)' < \sigma_q \leq \left(\frac{2}{q}\right)';$$

(a_2) the set $\Omega_a^+ := \{x \in \mathbb{R}^N : a(x) > 0\}$ has an interior point;
(b_1) $b \in L^\infty(\mathbb{R}^N)$;
(b_2) the set $\Omega_b^+ := \{x \in \mathbb{R}^N : b(x) > 0\}$ has an interior point.

The main results of this paper are stated below.

Theorem 1.1 (Subcritical case) *If $1 < q < 2 < p < 2^*$ and a, b satisfy $(a_1), (a_2), (b_1), (b_2)$, then (P) has at least two nonnegative nontrivial solutions if $\|a\|_{\sigma_q}$ is small.*

Theorem 1.2 (Critical case) *If $1 < q < 2 < p = 2^*$ and a, b satisfy (a_1), $(a_2), (b_1), (b_2)$ and*

(b_3) *there exist $x_0 \in \mathbb{R}^N$ and $\delta > 0$ such that $B_\delta(x_0) \subset (\Omega_a^+ \cap \Omega_b^+)$ and*

$$\|b\|_\infty - b(x) \leq M|x - x_0|^\gamma,$$

for a.e. $x \in B_\delta(x_0)$, with $M > 0$ and $\gamma > N/2$.

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Then (P) has at least two nonnegative nontrivial solutions if $\|a\|_{\sigma_q}$ is small and $\alpha > (N-2)/2$.

As it is well known, critical problems become more delicate in the presence of variable coefficients ([5], p. 454). In order to be able to handle this case we impose the technical condition (b_3). It is crucial for proving precise estimates of the minimax level of the associated functional. It is worthwhile to mention that similar conditions have already appeared in other papers (see ([9,10,14,16] for instance). Moreover, it is clearly satisfied by constant potentials.

In the proofs we shall apply variational techniques. The first solution will be obtained by a minimzation argument and the second one by applying the Mountain Pass Theorem. In the first theorem the subcriticality of the power p enable us to prove the required compactness condition for the associated functional. This is not the case in the setting of Theorem 1.2, since the embedding $X \hookrightarrow L_K^{2^*}(\mathbb{R}^N)$ is no longer compact. Actually, the same kind of difficult appears in the bounded domain case. In order to overcome this problem we apply the ideias introduced by Brezis and Nirenberg [5] to get a local compactness result. Thanks to some delicated estimates we can prove that the minimax level of the associated funcional belongs to the correct range. At this point we need to require the technical condition $\alpha > (N-2)/2$. We do not know if the result is true without this hypothesys. However, we emphasize that a related condition has already appeared in [17, Theorem 1.7], where a concave-convex problem with the same kind of operator was considered under more restrictive conditions on the potentials. We also refer to the paper of the Catrina, Furtado and Montenegro [6] where the author obtained some existence results of the Brezis-Nirenberg type and showed that the number α affects the critical dimension of the equation.

The main results of this paper can be view as versions of the results presented in [9] for bounded domains. They also complement the results in [4,1,15]. We finally mention that, with some slight modifications of the arguments presented in [20], we could obtain the existence of infinitely many solutions for the equation (P) with a high variety of conditions on the potentials a and b. However, as in [20], we can not give information on the sign of these solution.

The paper contains two more sections, each one dedicated to the proof of one the theorems.

2 Proof of Theorem 1.1

Throughout the paper we write $\int u$ instead of $\int_{\mathbb{R}^N} u(x)\mathrm{d}x$. Moreover, for any $u \in X$ we set $u^+(x) := \max\{u(x), 0\}$. Given $u \in X$, it follows from Hölder's inequality that

$$\left| \int K(x)a(x)(u^+)^q \right| \leq \|a\|_{\sigma_q} \left(\int K(x)|u|^{q\sigma_q'} \right)^{1/\sigma_q'}. \tag{2.1}$$

Since $2 \leq q\sigma_q' < p \leq 2^*$ the right hand side above is finite. Thus, by using some standard calculations we can show that the functional $I : X \to \mathbb{R}$ given by

$$I(u) := \frac{1}{2} \int K(x)|\nabla u|^2 - \frac{1}{q} \int K(x)a(x)(u^+)^q - \frac{1}{p} \int K(x)b(x)(u^+)^p$$

is well defined. Moreover it belongs to $C^1(X, \mathbb{R})$ and its critical points are exactly the nonnegative solutions of the equation (P).

Given $2 \leq r \leq 2^*$, the existence of the embedding $X \hookrightarrow L_K^r(\mathbb{R}^N)$ enables us to define

$$S_r := \inf \left\{ \int K(x)|\nabla u|^2 : \int K(x)|u|^r = 1 \right\} < +\infty. \tag{2.2}$$

When $r = 2^*$ we shall write only $S := S_{2^*}$. Despite this is not important here we would like to quote that this constant is equals to the best constant of the Sobolev embedding $\mathcal{D}^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$ (see [6]).

Lemma 2.1 *There exist $\rho, \alpha > 0$ such that $I(u) \geq \alpha > 0$, for any $u \in X$ such that $\|u\| = \rho$, provided $\|a\|_{\sigma_q}$ is small enough.*

Proof. It follows from (2.1) and (2.2) that

$$I(u) \geq \frac{1}{2} \|u\|^2 - \frac{1}{q} \|a\|_{\sigma_q} \|u\|_{q\sigma_q'}^q - \frac{1}{p} \|b\|_\infty \|u\|_p^p$$

$$\geq \frac{\|u\|^q}{2} \left\{ \|u\|^{2-q} - \frac{2}{p} \|b\|_\infty S_p^{-p/2} \|u\|^{p-q} - \frac{2}{q} \|a\|_{\sigma_q} S_{q\sigma_q'}^{-q/2} \right\}.$$

For $B := (2/p) \|b\|_\infty S_p^{-p/2}$, the function $f : (0, +\infty) \to \mathbb{R}$ given by

$$f(t) := t^{2-q} - Bt^{p-q}$$

achieves its maximum value at

$$\rho := \left[\frac{(2-q)}{B(p-q)} \right]^{1/(p-2)}.$$

Let $M := f(\rho)$ and notice that, for any $\|u\| = \rho$, there holds

$$I(u) \geq \frac{\rho^q}{2}\left\{M - \frac{2}{q}\|a\|_{\sigma_q} S_{q\sigma_q'}^{-q/2}\right\} \geq \frac{\rho^q}{2}\frac{M}{2} = \alpha > 0,$$

whenever

$$\|a\|_{\sigma_q} \leq \frac{Mq}{4}S_{q\sigma_q'}^{q/2}. \tag{2.3}$$

The lemma is proved. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proposition 2.2 *Suppose the function a satisfies (a_2) and (2.3), and let $\rho > 0$ be given by Lemma 2.1. Then the infimum*

$$-\infty < c_0 := \inf_{u \in B_\rho(0)} I(u) < 0$$

is achieved at some $u_0 \in B_\rho(0)$.

Proof. Since I maps bounded sets in bounded sets we have that $c_0 > -\infty$. By condition (a_2) there exists $x_0 \in \mathbb{R}^N$ and $\delta > 0$ such that $B_\delta(x_0) \subset \Omega_a^+$. If we take $\varphi \in C_0^\infty(B_\delta(x_0))$ such that $\int K(x)a(x)\varphi^q > 0$ we obtain

$$\frac{I(t\varphi)}{t^q} \leq \frac{t^{2-q}}{2}\|\varphi\|^2 - \frac{1}{q}\int K(x)a(x)\varphi^q - \frac{t^{p-q}}{p}\int K(x)b(x)\varphi^p.$$

Hence

$$\limsup_{t \to 0^+} \frac{I(t\varphi)}{t^q} \leq -\frac{1}{q}\int K(x)a(x)\varphi^q < 0$$

and therefore we have that $I(t\varphi) < 0$ for any $t > 0$ small. This proves that $c_0 < 0$.

Let $(u_n) \subset \overline{B}_\rho(0)$ be a minimizing sequence for c_0. By Ekeland's Variational Principle we may assume that

$$I(u_n) \to c_0 \text{ and } I'(u_n) \to 0.$$

Since (u_n) is bounded and $2 \leq q\sigma_q' < p \leq 2^*$, up to a subsequence we have that

$$\begin{cases} u_n \rightharpoonup u_0 \text{ weakly in } X, \\\\ u_n \to u_0 \text{ strongly in } L_K^{q\sigma_q'}(\mathbb{R}^N), \\\\ u_n^+(x) \to u_0^+(x), |u_n(x)| \leq \psi(x) \text{ for a.e. } x \in \mathbb{R}^N, \end{cases} \tag{2.4}$$

for some $\psi \in L_K^{q\sigma_q'}(\mathbb{R}^N)$. Hence we have that, for a.e. $x \in \mathbb{R}^N$, there holds

$$|K(x)a(x)(u_n^+)^q| \leq K(x)|a(x)|\psi(x)|^q \leq \frac{1}{\sigma_q}K(x)a(x)^{\sigma_q} + \frac{1}{\sigma_q'}K(x)\psi(x)^{q\sigma_q'}.$$

Since $\psi \in L_K^{q\sigma_q'}(\mathbb{R}^N)$, the right-hand side above belongs to $L^1(\mathbb{R}^N)$, and therefore it follows from (2.4) and the Lebesgue Theorem that

$$\lim_{n\to+\infty} \int K(x)a(x)(u_n^+)^q = \int K(x)a(x)(u_0^+)^q.$$

We now claim that $I'(u_0) = 0$. Assuming the claim, the above equality and the weak convergence of (u_n) imply that

$$
\begin{aligned}
c_0 &= \liminf_{n\to+\infty} \left(I(u_n) - \frac{1}{p}I'(u_n)u_n \right)\\
&= \liminf_{n\to+\infty} \left\{ \left(\frac{1}{2} - \frac{1}{p} \right) \|u_n\|^2 + \left(\frac{1}{p} - \frac{1}{q} \right) \int K(x)a(x)(u_n^+)^q \right\}\\
&\geq \left(\frac{1}{2} - \frac{1}{p} \right) \|u_0\|^2 + \left(\frac{1}{p} - \frac{1}{q} \right) \int K(x)a(x)(u_0^+)^q\\
&= I(u_0) - \frac{1}{p}I'(u_0)u_0 = I(u_0),
\end{aligned}
$$

and therefore $I(u_0) = c_0 < 0$. By Lemma (2.1) that conclude that $u_0 \in \partial B_\rho(0)$ cannot happen.

It suffices to prove that $I'(u_0) = 0$. So, let us consider $\varphi \in C_0^\infty(\mathbb{R}^N)$ and $A := \operatorname{supp}(\varphi)$. Since $\sigma_q > (p/q)' = p/(p-q)$, we can choose $2 < p_0 < p$ sufficiently close to p and such that

$$\sigma_q > \frac{p_0}{p_0 - q} > \frac{p_0}{(p_0 + 1) - q}.$$

Hence, there exists $\tau > 1$ satisfying

$$\frac{1}{\sigma_q} + \frac{1}{p_0/(q-1)} + \frac{1}{\tau} = 1.$$

The strong convergence $u_n \to u_0$ in $L^{p_0}(A)$ provides $\psi_{p_0} \in L^{p_0}(A)$ such that $|u_n(x)| \leq \psi_{p_0}(x)$ a.e. in A. Thus, by Young's inequality we get

$$|K(x)a(x)(u_n^+)^{q-1}\varphi| \leq C\left(|a(x)|^{\sigma_q} + |u_n(x)|^{p_0} + |\varphi(x)|^\tau\right)$$

$$\leq C\left(|a(x)|^{\sigma_q} + |\psi_{p_0}(x)|^{p_0} + |\varphi(x)|^\tau\right),$$

a.e. in A. Since φ is smooth, the right-hand side above belongs to $L^1(A)$. It follows from the Lebesgue Theorem that

$$\lim_{n\to+\infty} \int K(x)a(x)(u_n^+)^{q-1}\varphi = \int K(x)a(x)(u_0^+)^{q-1}\varphi.$$

Since $b \in L^\infty(\mathbb{R}^N)$, a simpler argument shows that

$$\lim_{n \to +\infty} \int K(x)b(x)(u_n^+)^{2^*-1}\varphi = \int K(x)b(x)(u_0^+)^{2^*-1}\varphi.$$

The two above equations and the weak convergence of (u_n) imply that

$$0 = \lim_{n \to +\infty} I'(u_n)\varphi = I'(u_0)\varphi,$$

for any $\varphi \in C_0^\infty(\mathbb{R}^N)$. The conclusion follows by density. $\quad\square$

Lemma 2.3 *Suppose b satisfies (b_2) and let $B_\delta(x_1) \subset \Omega_b^+$. If $\varphi \in C_0^\infty(B_\delta(x_1)) \setminus \{0\}$ is nonnegative then*

$$\lim_{t \to +\infty} I(u_0 + t\varphi) = -\infty.$$

Proof. Since $\varphi = 0$ outside $B_\delta(x_1) \subset \Omega_b^+$ a straightforward computation provides

$$I(u_0 + t\varphi) \le O(t^2) + O(t^q) - O(1) - \frac{1}{p}\int_{\Omega_b^+} K(x)b(x)(u_0 + t\varphi)^p \, \mathrm{d}x$$

$$\le O(t^2) - C\int_{B_\delta(x_1)} b(x)(u_0 + t\varphi)^p \, \mathrm{d}x$$

$$\le O(t^2) + O(1) - Ct^p \int_{B_\delta(x_1)} b(x)\varphi^p \, \mathrm{d}x,$$

as $t \to +\infty$. By the choice of φ the last integral above is positive and therefore the result follows from $p > 2$. $\quad\square$

Let us recall that, if E is a real Banach space, we say that $J \in C^1(E, \mathbb{R})$ satisfies the Palais-Smale condition at level $c \in \mathbb{R}$, in short $(PS)_c$, if any sequence $(u_n) \subset E$ such that

$$\lim_{n \to +\infty} J(u_n) = c, \quad \lim_{n \to \infty} \|J'(u_n)\|_{E^*} = 0$$

has a convergent subsequence.

Lemma 2.4 *If $2 < p < 2^*$ then the functional I satisfies the $(PS)_c$ condition for any $c \in \mathbb{R}$.*

Proof. Let $(u_n) \subset X$ be such that $I(u_n) \to c$ and $I'(u_n) \to 0$. Hölder's

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inequality provides, as $n \to +\infty$,

$$c + o(1)\|u_n\| = I(u_n) - \frac{1}{p}I'(u_n)u_n$$

$$\geq \left(\frac{1}{2} - \frac{1}{p}\right)\|u_n\|^2 - \left(\frac{1}{q} - \frac{1}{p}\right)S_{q\sigma_q'}^{-q/2}\|a\|_{\sigma_q}\|u_n\|^q.$$

Since $q < 2 < p$ we conclude that (u_n) is bounded in X.

Up to a subsequence, we have that $u_n \rightharpoonup u$ weakly in X and $u_n \to u$ strongly in $L_K^s(\mathbb{R}^N)$ for any $2 \leq s \leq p < 2^*$. The choice of σ_q enable us to obtain $2 \leq p_0 < p$ such that $\sigma_q = (p_0/q)'$. Hence, Hölder's inequality with exponents σ_q, $p_0/(q-1)$ and p_0 imply that

$$\left|\int K(x)a(x)(u_n^+)^{q-1}(u_n - u)\right| \leq \|a\|_{\sigma_q}\|u_n\|_{p_0}^{q-1}\|u_n - u\|_{p_0} \to 0,$$

as $n \to +\infty$. On the other hand, by using Hölder's inequality again, we get

$$\left|\int K(x)b(x)(u_n^+)^{p-1}(u_n - u)\right| \leq \|b\|_\infty\|u_n\|_p^{p-1}\|u_n - u\|_p \to 0.$$

It follows from the two above convergences that

$$o(1) = I'(u_n)(u_n - u) = \|u_n\|^2 - \|u\|^2 + o(1),$$

as $n \to +\infty$. Hence, $\|u_n\| \to \|u\|$ and the result follows from the weak convergence of (u_n) in X. □

We are now ready to prove our first result.

Proof of Theorem 1.1. If we suppose that a satisfies (2.3) we can use Proposition 2.2 to obtain a first solution $u_0 \in X$ with negative energy. For the second one we take $\rho > 0$ given by Lemma 2.1 and consider $\varphi \in X$ as in the statement of Lemma 2.3. We can obtain $t_0 > 0$ large in such way that $e := u_0 + t\varphi$ satisfies $I(e) \leq I(u_0)$. If we define

$$c_M := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I(\gamma(t)) \tag{2.5}$$

where $\Gamma := \{\gamma \in C([0,1], X) : \gamma(0) = u_0, \gamma(1) = e\}$. Lemma 2.1 and an usual intersection argument show that $c_M \geq \alpha > 0$. The Mountain Pass Theorem provides a sequence $(u_n) \subset X$ such that $I(u_n) \to c$ and $I'(u_n) \to 0$. It follows from Lemma 2.4 that, along a subsequence, $u_n \to u$ strongly in X with $I(u) = c_M > 0$ and $I'(u) = 0$. So, we have obtained a second nontrivial solution. □

3   Proof of Theorem 1.2

In this section we prove our existence result for the critical case. We first notice that the same argument of the last section provides a first nontrivial critical point u_0 (see Proposition 2.2) whenever a is small. However, since we do not have compactness of the embedding $X \hookrightarrow L_K^{2^*}(\mathbb{R}^N)$, the $(PS)_c$ condition can fail at some sublevels.

Following the ideas of Brezis and Nirenberg [5] we can prove the following local compactness result.

Lemma 3.1 *If zero and u_0 are the unique critical points of I, then I satisfies $(PS)_c$ condition for every*

$$c < \bar{c} := I(u_0) + \frac{1}{N} \frac{1}{\|b\|_{\infty}^{(N-2)/2}} S^{N/2}.$$

Proof. Let $(u_n) \subset X$ be such that $I(u_n) \to c$ and $I'(u_n) \to 0$. The same argument of Lemma 2.4 shows that (u_n) is bounded in X. Hence, along a subsequence, we have that $u_n \rightharpoonup u$ weakly in X and $u_n \to u$ strongly in $L_K^{q\sigma_q'}(\mathbb{R}^N)$. This strong convergence implies that $\int K(x)a(x)(u_n^+)^q \to \int K(x)a(x)(u^+)^q$. So, if we set $v_n := u_n - u$, we can use the Brezies-Lieb lemma to get

$$\begin{aligned}
0 = I'(u_n)u_n &= \|u_n\|^2 - \int K(x)a(x)(u_n^+)^q - \int Kb(x)(u_n^+)^{2^*} \\
&= \|u\|^2 + \|v_n\|^2 - \int K(x)a(x)(u^+)^q + o(1) \\
&\quad - \int K(x)b(x)(u^+)^{2^*} - \int K(x)b(x)(v_n^+)^{2^*} \\
&= I'(u)u + \|v_n\|^2 - \int K(x)b(x)(v_n^+)^{2^*}.
\end{aligned}$$

As in the proof of Proposition 2.2 we have $I'(u) = 0$, and therefore there exists $l \geq 0$ such that

$$\lim_{n \to +\infty} \|v_n\|^2 = l = \lim_{n \to +\infty} \int K(x)b(x)(v_n^+)^{2^*}.$$

We claim that $l = 0$. If this is true, it follows that $\|u_n - u\| \to 0$ and we have done.

In order to prove that $l = 0$ we first notice that

$$\int K(x)b(x)(v_n^+)^{2^*} \leq \|b\|_{\infty} S^{-2^*/2} \left(\int K(x)|\nabla v_n|^2 \right)^{2^*/2}.$$

Taking the limite we obtain $l \leq \|b\|_{\infty} S^{-2^*/2} l^{2^*/2}$. If $l > 0$ we infer from this

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last inequality that

$$l \geq \frac{1}{\|b\|_\infty^{(N-2)/2}} S^{N/2}. \tag{3.1}$$

On the other hand, the same argument of the beginning of the proof provides

$$c + o(1) = I(u_n) = I(u) + \frac{1}{2}\|v_n\|^2 - \frac{1}{2^*} \int K(x)b(x)(v_n^+)^{2^*} + o(1).$$

Taking the limit and using (3.1) we get

$$c = I(u) + \left(\frac{1}{2} - \frac{1}{2^*}\right) l = I(u) + \frac{1}{N}l \geq I(u) + \frac{1}{N}\frac{1}{\|b\|_\infty^{(N-2)/2}} S^{N/2}.$$

But $I'(u) = 0$, and therefore by hypotheses either $u = 0$ or $u = u_0$. Since $\max\{I(0), I(u_0)\} \leq 0$, the above inequality contradicts $c < \bar{c}$. $\qquad \square$

Given $x_0 \in \text{int}(\Omega_a^+ \cap \Omega_b^+)$ and $\delta > 0$ from (b_3), we consider $\eta > 0$ small in such way that $B_{2\eta}(x_0) \subset B_\delta(x_0)$. We take a smooth function satisfying $\varphi \equiv 1$ in $B_\delta(x_0)$ and $\varphi \equiv 0$ outside $B_{2\delta}(x_0)$. We consider the function

$$u_\varepsilon(x) := K(x)^{-1/2}\varphi(x)\left(\frac{1}{\varepsilon + |x - x_0|^2}\right)^{(N-2)/2},$$

and set

$$v_\varepsilon(x) := \frac{u_\varepsilon(x)}{\|u_\varepsilon\|_{2^*}}. \tag{3.2}$$

Without loss of generality we can suppose that $x_0 = 0$. This will be assumed from now on.

Proposition 3.2 *Suppose a, b verifies (a_2), (b_2), (b_3) and that $\alpha > (N-2)/2$. Then, for any $\varepsilon > 0$ small, the function v_ε defined above satisfies*

$$\max_{t>0} I(u_0 + tv_\varepsilon) < \bar{c} := I(u_0) + \frac{1}{N}\frac{1}{\|b\|_\infty^{(N-2)/2}} S^{N/2}.$$

Proof. For any $\varepsilon > 0$, it follows from Lemma 2.3 that the function $t \mapsto I(u_0 + tv_\varepsilon)$ achieves its maximum at a point $t_\varepsilon > 0$. Moreover, arguing as in [8, Lemma 4.1], we can prove that (t_ε) is bounded for $\varepsilon \in (0, 1]$.

If we set $\Omega := \text{supp}(v_\varepsilon) \subset \left(\Omega_a^+ \cap \Omega_b^+\right)$ we can use $I'(u_0)v_\varepsilon = 0$ and a straightforward calculation to get

$$m_\varepsilon := I(u_0 + t_\varepsilon v_\varepsilon) = I(u_0) + \frac{t_\varepsilon^2}{2}\|v_\varepsilon\|^2 - \frac{1}{q}A_\varepsilon - \frac{1}{2^*}D_\varepsilon, \tag{3.3}$$

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where
$$A_\varepsilon := \int_\Omega K(x)a(x)\left[(u_0 + t_\varepsilon v_\varepsilon)^q - u_0^q - qt_\varepsilon u_0^{q-1}v_\varepsilon\right]\mathrm{d}x,$$

and
$$D_\varepsilon := \int_\Omega K(x)b(x)\left[(u_0 + t_\varepsilon v_\varepsilon)^{2^*} - u_0^{2^*} - 2^* t_\varepsilon u_0^{2^*-1}v_\varepsilon\right]\mathrm{d}x.$$

By using the Mean Value Theorem we obtain $\theta(x) \in [0,1]$ such that

$$(u_0(x) + t_\varepsilon v_\varepsilon(x))^q - u_0(x)^q = q(u_0(x) + \theta(x)t_\varepsilon v_\varepsilon(x))^{q-1}t_\varepsilon v_\varepsilon$$

$$\geq qt_\varepsilon u_0(x)^{q-1}v_\varepsilon(x),$$

for any $x \in \Omega$. Since $a \geq 0$ in Ω, it follows that $A_\varepsilon \geq 0$. In order to estimate D_ε we recall that, for a, $b \geq 0$, $r > 2$ and $1 < \mu < r - 1$ there holds (see [7])

$$(a+b)^r \geq a^r + b^r + ra^{r-1}b + rab^{r-1} - C_\mu b^\mu a^{r-\mu},$$

for some $C_\mu > 0$. If we choose $a = u_0$, $b = t_\varepsilon v_\varepsilon$ and $r = 2^*$ we obtain

$$D_\varepsilon \geq \int_\Omega K(x)b(x)\left[t_\varepsilon^{2^*}v_\varepsilon^{2^*} + 2^* t_\varepsilon^{2^*-1}u_0 v_\varepsilon^{2^*} - C_\mu t_\varepsilon^\mu u_0^{2^*-\mu}v_\varepsilon^\mu\right]\mathrm{d}x.$$

Replacing this inequality and $A_\varepsilon \geq 0$ in (3.3), we obtain

$$
\begin{aligned}
m_\varepsilon \leq I(u_0) &+ \left(\frac{t_\varepsilon^2}{2}\|v_\varepsilon\|^2 - \frac{t_\varepsilon^{2^*}}{2^*}\|b\|_\infty\right)\\
&+ \frac{t_\varepsilon^{2^*}}{2^*}\int K(x)(\|b\|_\infty - b(x))v_\varepsilon^{2^*}\\
&- t_\varepsilon^{2^*-1}\int K(x)b(x)u_0 v_\varepsilon^{2^*-1} + C_\mu \frac{t_\varepsilon^\mu}{2^*}\int K(x)b(x)u_0^{2^*-\mu}v_\varepsilon^\mu,
\end{aligned}
\tag{3.4}
$$

where we have used $\int K(x)v_\varepsilon^{2^*} = 1$ (see (3.2)) and also that $v_\varepsilon \equiv 0$ outside Ω.

A simple computation provides

$$\max_{t \geq 0}\left(\frac{t^2}{2}\|v_\varepsilon\|^2 - \frac{t^{2^*}}{2^*}\|b\|_\infty\right) = \frac{1}{N}\frac{1}{\|b\|_\infty^{(N-2)/2}}\|v_\varepsilon\|^N. \tag{3.5}$$

Moreover, for some constant $c_\mu > 0$, there holds (see [5,8])

$$\int v_\varepsilon^\mu = c_\mu \varepsilon^{(N(2-\mu)+2\mu)/4},$$

whenever $N/(N-2) < \mu < 2^*$. Thus, if we choose $\mu := (N+1)/(N-2) < 2^*-1$ and recall that v_ε has compact support and (t_ε) is bounded, we obtain

$$C_\mu \frac{t_\varepsilon^\mu}{2^*}\int K(x)b(x)u_0^{2^*-\mu}v_\varepsilon^\mu = O(\varepsilon^{(N-1)/4}), \tag{3.6}$$

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as $\varepsilon \to 0^+$. By the same reason we obtain $A_0 > 0$ such that

$$t_\varepsilon^{2^*-1} \int K(x)b(x)u_0 v_\varepsilon^{2^*-1} = A_0 O(\varepsilon^{(N-2)/4}). \tag{3.7}$$

On the other hand, according to [6, p. 1165], we have that

$$\|u_\varepsilon\|_{2^*}^{2^*} = \int K(x)|u_\varepsilon|^{2^*} = \varepsilon^{-N/2} A_1 + O(1),$$

where $A_1 := \int (1 + |x|^2)^{-N}$. Hence, we can use condition (b_3) to compute

$$\int K(x)(\|b\|_\infty - b(x))v_\varepsilon^{2^*} = \frac{1}{\|u_\varepsilon\|_{2^*}^{2^*}} \int K(x)(\|b\|_\infty - b(x))u_\varepsilon^{2^*}$$

$$\leq C \frac{\varepsilon^{N/2}}{A_1 + O(\varepsilon^{N/2})} \int_{B_{2\eta}(0)} |x|^\gamma (\varepsilon + |x|^2)^{-N} dx$$

$$\leq C\varepsilon^{N/2} \int_{B_{2\eta}(0)} |x|^{\gamma-2N} dx = O(\varepsilon^{N/2}).$$

By replacing this estimate and (3.5)-(3.7) in (3.4), and recalling that $\varepsilon > 0$ is small, we obtain

$$m_\varepsilon \leq I(u_0) + \frac{1}{N} \frac{1}{\|b\|_\infty^{(N-2)/2}} (\|v_\varepsilon\|^2)^{N/2}$$

$$+ O(\varepsilon^{(N-1)/4}) + O(\varepsilon^{N/2}) - A_0 O(\varepsilon^{(N-2)/4}). \tag{3.8}$$

We now refer to [12, p. 1043-1046] for the following estimates

$$\|v_\varepsilon\|^2 = \begin{cases} S + O(\varepsilon^{\alpha/2}), & \text{if } N > \alpha + 2, \\ S + O(\varepsilon^{\alpha/2-\upsilon}) \text{ for any } 0 < \upsilon < \alpha/2, & \text{if } N = \alpha + 2, \\ S + O(\varepsilon^{(N-2)/2}), & \text{if } 2 < N < \alpha + 2. \end{cases}$$

We first consider the case $N > \alpha + 2$. From the first equality above and (3.8) we get

$$m_\varepsilon \leq I(u_0) + \frac{1}{N} \frac{1}{\|b\|_\infty^{(N-2)/2}} S^{N/2} + O(\varepsilon^{\alpha/2}) - O(\varepsilon^{(N-2)/4})$$

$$= \bar{c} + \varepsilon^{(N-2)/4} \left(O(\varepsilon^{(2\alpha-N+2)/4}) - A_0 \right),$$

as $\varepsilon \to 0^+$. The above expression implies that $m_\varepsilon < \bar{c}$ provided $\alpha > (N-2)/2$. If $N = \alpha + 2$ we take the number υ sufficiently close to 0 and obtain the same

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result. For the last case $2 < N < \alpha + 2$ it suffices to notice that

$$m_\varepsilon \le I(u_0) + \frac{1}{N} \frac{1}{\|b\|_\infty^{(N-2)/2}} S^{N/2} + O(\varepsilon^{(N-1)/4}) - O(\varepsilon^{(N-2)/4}),$$

and therefore the result follows as before. $\qquad\square$

Proof of Theorem 1.2. The proof is the same of Theorem 1.1. We just notice that, by Lemma 2.3, we have $\lim_{t\to+\infty} I(u_0 + tv_\varepsilon) = -\infty$ and therefore we can take $e := u_0 + tv_\varepsilon$ with $t > 0$ large and define the minimax level of the Moutain Pass Theorem as in (2.5). According to Proposition 3.2, for $\varepsilon > 0$ small enough we have that $c_M < \bar{c}$, where $\bar{c}$ comes from Lemma 3.1. Hence, we have compactness on the level c_M and we can argue as before to obtain a second critical point with positive energy. $\qquad\square$

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