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A Trudinger–Moser inequality in a weighted Sobolev space and applications

Marcelo F. Furtado[∗1], **Everaldo S. Medeiros**[∗∗2], and **Uberlandio B. Severo**[∗∗∗2]

[1] Departamento de Matemática, Universidade de Brasília, 70910-900, Brasília-DF, Brazil
[2] Departamento de Matemática, Universidade Federal da Paraíba, 58051-900, João Pessoa-PB, Brazil

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We establish a Trudinger–Moser type inequality in a weighted Sobolev space. The inequality is applied in the study of the elliptic equation

$$-\mathrm{div}(K(x)\nabla u) = K(x)f(u) + h \quad \text{in} \quad \mathbb{R}^2,$$

where $K(x) = \exp(|x|^2/4)$, f has exponential critical growth and h belongs to the dual of an appropriate function space. We prove that the problem has at least two weak solutions provided $h \neq 0$ is small.

1 Introduction

For $N \geq 2$, let $C_c^\infty(\mathbb{R}^N)$ be the space of infinitely differentiable functions with compact support and denote by X the closure of $C_c^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\| := \left(\int_{\mathbb{R}^N} K(x)|\nabla u|^2 \mathrm{d}x\right)^{1/2} \tag{1.1}$$

where $K(x) := \exp(|x|^2/4)$. For each $p \geq 1$ we also consider the weighted Lebesgue space $L_K^p(\mathbb{R}^N)$ of all the measurable functions $u : \mathbb{R}^N \to \mathbb{R}$ such that

$$\|u\|_p := \left(\int_{\mathbb{R}^N} K(x)|u|^p \mathrm{d}x\right)^{1/p} < \infty.$$

It is proved by Escobedo-Kavian in [8] that X is continuously embedded in $L_K^p(\mathbb{R}^N)$ for any $2 \leq p \leq 2^* := 2N/(N-2)$. The main purpose of this paper is to consider the limit case $N = 2$ also known as the Trudinger–Moser case.

We recall that if $\Omega \subset \mathbb{R}^2$ is a bounded domain then $W_0^{1,2}(\Omega) \hookrightarrow L^q(\Omega)$ for $1 \leq q < \infty$, but $W_0^{1,2}(\Omega) \not\hookrightarrow L^\infty(\Omega)$. The classical Trudinger–Moser inequality (see [15], [24]) gives an improvement to the limit case $N = 2$. More precisely, for all $u \in W_0^{1,2}(\Omega)$ and $\alpha > 0$ there holds $e^{\alpha u^2} \in L^1(\Omega)$ and there exists a constant $C = C(\Omega)$, which depends only on measure of Ω, such that

$$\sup_{\|\nabla u\|_2 \leq 1} \int_\Omega e^{\alpha u^2} \mathrm{d}x \leq C(\Omega), \quad \text{if} \quad \alpha \leq 4\pi. \tag{1.2}$$

Moreover, 4π is the best constant, in the sense that the above supremum is infinity if $\alpha > 4\pi$. The Trudinger–Moser result was extended for unbounded domains by D. M. Cao [5]. Explicitly, for any $u \in W^{1,2}(\mathbb{R}^2)$ and

∗ e-mail: mfurtado@unb.br
∗∗ Corresponding author: e-mail: everaldo@mat.ufpb.br, Phone: +55 083 3216 3474, Fax: +55 083 3216 7487
∗∗∗ e-mail: uberlandio@mat.ufpb.br

$\alpha > 0$ it holds $\left(e^{\alpha u^2} - 1\right) \in L^1(\mathbb{R}^2)$. Moreover, if $\|\nabla u\|_2 \leq 1$, $\|u\|_2 \leq M < \infty$ and $\alpha < 4\pi$, then there exists $C = C(M, \alpha)$ such that

$$\int_{\mathbb{R}^2} (e^{\alpha u^2} - 1)\,\mathrm{d}x \leq C(M, \alpha). \tag{1.3}$$

Recently, B. Ruf [19] proved that the result of J. Moser [15] can be fully extended to $\mathbb{R}^2$ if the Dirichlet norm $\|\nabla u\|_2$ is replaced by the full Sobolev usual norm $\|u\|_{1,2}$ (for a related result see also [3]). More precisely

$$\sup_{u \in H^1(\mathbb{R}^2)\,:\,\|u\|_{1,2} \leq 1} \int_{\mathbb{R}^2} (e^{\alpha u^2} - 1)\,\mathrm{d}x < \infty, \quad \text{if} \quad \alpha \leq 4\pi,$$

with the number 4π being again the best constant. Estimates of *Trudinger–Moser type* plays an important role in geometric analysis and partial differential equations.

As we will see in Lemma 2.1, the space X is embedded into the Lebesgue spaces $L^p_K(\mathbb{R}^2)$ for any $p \in [2, \infty)$. With the aid of inequality (1.3) we prove the following version of the Trudinger–Moser inequality in the space X.

Theorem 1.1 *For any $u \in X$ and $\beta > 0$ we have that $K(x)|u|^2\left(e^{\beta u^2} - 1\right) \in L^1(\mathbb{R}^2)$. Moreover, if $\|u\| \leq M$ and $\beta M^2 < 4\pi$, then there exists a constant $C = C(M, \beta) > 0$ such that*

$$\int_{\mathbb{R}^2} K(x)|u|^2\left(e^{\beta u^2} - 1\right)\mathrm{d}x \leq C(M, \beta).$$

As a byproduct of the proof of Theorem 1.1 we can prove the next corollary. It will be useful in the applications presented in the second part of our paper.

Corollary 1.2 *If $u \in X$, $\beta > 0$, $q > 0$ and $\|u\| \leq M$ with $\beta M^2 < 4\pi$, then there exists $C = C(\beta, M, q) > 0$ such that*

$$\int_{\mathbb{R}^2} K(x)|u|^{2+q}\left(e^{\beta u^2} - 1\right)\mathrm{d}x \leq C(\beta, M, q)\|u\|^{2+q}.$$

When dealing with PDE involving Trudinger–Moser critical growth one of the main difficulties is to handle the Palais-Smale sequences. For that matter, P.-L. Lions proved in [14] the following improvement of the Trudinger–Moser inequality: let (u_n) be a sequence of functions in $W_0^{1,2}(\Omega)$ with $\|\nabla u_n\|_2 = 1$ such that $u_n \rightharpoonup u \neq 0$ weakly in $W_0^{1,2}(\Omega)$. Then for any $0 < p < 4\pi\left(1 - \|\nabla u\|_2^2\right)^{-1}$ we have

$$\sup_{n \in \mathbb{N}} \int_{\Omega} e^{p u_n^2}\mathrm{d}x < \infty.$$

It is clear that this result gives more precise information than (1.2) when $u_n \rightharpoonup u$ weakly in $W_0^{1,2}(\Omega)$ with $u \neq 0$.

With the purpose to control the Palais-Smale sequences in our application we prove the following improvement of the Trudinger–Moser inequality considering our variational setting.

Theorem 1.3 *Let (v_n) in X with $\|v_n\| = 1$ and suppose that $v_n \rightharpoonup v$ weakly in X with $\|v\| < 1$. Then for each $0 < p < 4\pi\left(1 - \|v\|^2\right)^{-1}$, up to a subsequence, it holds*

$$\sup_{n \in \mathbb{N}} \int K(x)v_n^2\left(e^{p v_n^2} - 1\right)\mathrm{d}x < \infty.$$

As an application of the previous results we study the following semilinear elliptic equation

$$-\mathrm{div}(K(x)\nabla u) = K(x)f(u) + h \quad \text{in} \quad \mathbb{R}^2, \tag{P}$$

where f has critical exponential growth and the forcing term h belongs to the dual of X. Actually, we are able to show that this space is so big in order to contain any Lebesgue space $L^p(\mathbb{R}^2)$ for $p \geq 1$ (see Remark 2.3).

The above equation is closely related to the study of self-similar solutions for the heat equation as quoted in the works of Haraux-Weissler [12] and Escobedo-Kavian [8] (see also [6], [11]). In this direction, problem (P) arises naturally when one seek for solutions of the form

$$\omega(t, x) = t^{-1/(p-1)}u\left(t^{-1/2}x\right)$$

for the evolution equation

$$\omega_t - \Delta\omega = |\omega|^{p-1}\omega \quad \text{on} \quad (0, \infty) \times \mathbb{R}^2.$$

More precisely, $\omega(t, x)$ satisfies the previous equation if and only if $u : \mathbb{R}^2 \to \mathbb{R}$ satisfies

$$-\Delta u - \frac{1}{2}(x \cdot \nabla u) = \lambda u + |u|^{p-1}u, \quad \text{on} \quad \mathbb{R}^2,$$

which is equivalent to the equation (P) with $\lambda = 1/(p-1)$, $f(u) = \lambda u + |u|^{p-1}u$ and $h = 0$.

The role played by the nonhomogeneous term h in producing multiple solutions was investigated in many works (see [1], [18]). Recently, many authors have been interested in the perturbed problem

$$-\text{div}(K(x)\nabla u) + cu = |u|^{p-1}u + \mu f \quad \text{in} \quad \Omega,$$

involving both critical and subcritical Sobolev exponents in bounded and unbounded domains of $\mathbb{R}^N$, $N \geq 3$. The case Ω bounded has been widely studied and the exponent p was crucial in the arguments (see [18], [21] and references therein). If $\Omega = \mathbb{R}^N$, this problem has been studied recently by many authors and the nonhomogeneous term h plays an important role in their analysis (see [1], [7], [13]).

We are interested here in the case that the function f has the maximal growth which allows to deal with (P) variationally. According to our abstract results, we can use here the same notion of criticality introduced in [4], [9], namely

(f_0) there exists $\alpha_0 > 0$ such that

$$\lim_{|s|\to+\infty} \frac{f(s)}{e^{\alpha s^2}} = \begin{cases} 0, & \text{if} \quad \alpha > \alpha_0, \\ +\infty, & \text{if} \quad \alpha < \alpha_0. \end{cases}$$

Perturbed problems involving critical exponential growth in bounded domains of $\mathbb{R}^2$ have been studied by many authors (see [16], [17], [22], [23]). When dealing with problems on the entire space the authors usually use the inequality (1.3) (see [2], [5], [19] and references therein). Due to the presence of the weight $K(x)$ in Equation (P) we are not able to use the usual Sobolev spaces. As quoted in [8], the natural space to look for rapid decay solutions is the space X and this was the main motivation for the establishment of Theorems 1.1 and 1.3.

In order to perform the minimax approach to problem (P) we also need to make some suitable assumptions on the behavior of f. More precisely, we shall assume the following conditions:

(f_1) $\lim_{s\to 0} f(s)/s = 0$;

(f_2) there exists $\theta_0 > 2$ such that

$$0 \leq \theta_0 F(s) := \theta_0 \int_0^s f(t)\,\mathrm{d}t \leq sf(s), \quad \text{for all} \quad s \in \mathbb{R}.$$

Now, we are ready to state our first existence result.

Theorem 1.4 *Suppose f satisfies (f_0)–(f_2). Then there exists $\delta_1 > 0$ such that, if $0 < \|h\|_{X^{-1}} < \delta_1$, the problem (P) has a weak solution $u_h \in X$. Moreover, we have that $\|u_h\| \to 0$ as $\|h\|_{X^{-1}} \to 0$.*

In our next result we study the effect of the smallness of h on the existence of multiple solutions for the problem (P). In this case we need to do some fine estimates which are related with a version of the Strauss Lemma for radial functions (see Lemma 4.3) and therefore we work in the subspace of the radial functions of X. Actually, we shall look for solution in X_{rad}, which is defined as the closure of $C^\infty_{c,rad}(\mathbb{R}^2)$ with respect to the norm (1.1). Here $C^\infty_{c,rad}(\mathbb{R}^2)$ stands for the subspace of radial functions of $C^\infty_c(\mathbb{R}^2)$. We will denote by X^{-1}_{rad} the topological dual space of X_{rad}.

Concerning the nonlinearity f we make the following additional assumptions:

(f_3) for each $\theta > 2$, there exists $R_\theta > 0$ such that

$$0 \leq \theta F(s) \leq sf(s), \quad \text{for all} \quad |s| \geq R_\theta.$$

(f_4) there exists $\beta_0 > 0$ such that

$$\liminf_{|s|\to\infty} \frac{f(s)s}{e^{\alpha_0 s^2}} \geq \beta_0 > \frac{4}{\alpha_0} \min_{r>0} \frac{1}{r^2} \exp\left(\frac{r^2}{4} + \frac{r^4}{256}\right).$$

Our multiplicity result can be stated as follows.

Theorem 1.5 *Suppose f satisfies (f_0)–(f_4). Then there exists $\delta_2 > 0$ such that, if $0 < \|h\|_{X_{rad}^{-1}} < \delta_2$, then the problem (P) has at least two weak solutions.*

We remark that our first theorem ensures the existence of one weak solution for the model nonlinearity

$$f(s) = e^{\alpha s^2} - 1.$$

Concerning the second result, it is straightforward to check that

$$f(s) = \frac{\left(e^{s^2} - 1\right)s\beta_0}{1 + s^2},$$

satisfies (f_0)–(f_4) with $\alpha_0 = 1$ and $\beta_0 > 3$.

Theorem 1.4 will be proved by a minimization argument. Actually, in this first result we can replace the condition (f_2) by the natural superlinear assumption that $F(s)/s^2 \to +\infty$ as $s \to +\infty$ (see Remark 3.2). For the proof of Theorem 1.5 we shall use the Mountain Pass Theorem centered at the local minimum u_h. The main difficult here is the handling of the Palais-Smale sequence. Since the embedding of X in the Orlicz space $L_A(\mathbb{R}^2)$ (with the $N-$function $A(t) = e^{\alpha t^2} - 1$) is not compact, beside the abstracts results of the first of the paper, we shall perform some careful estimations of the critical level of the functional associated with problem (P).

Hypothesis (f_3), which has already appeared in [16], [25], is essential in order to get some convergence results. It says that the nonlinearity f satisfies the Ambrosetti-Rabinowitz condition for any $\theta > 2$. Although this appears to be very restrictive, the models functions with critical exponential growth satisfies (f_3). Moreover, this condition is implied by

$(\widehat{f_3})$ there exist constants $R_0, M_0 > 0$ such that

$$0 < F(s) \leq M_0 f(s), \quad \text{for all} \quad |s| \geq R_0,$$

which has been used for instance in the papers [9], [10].

Concerning the correct localization of the minimax level we use the technical condition (f_4) and adapt some calculations performed in [16] by using Green's functions considered by Moser in [15]. It is worthwhile to mention that our condition (f_4) is more general from the analogous one considered in [16], since here the number β_0 may be finite. Actually, a numerical computation shows that the relation between α_0 and β_0 in the condition (f_4) is satisfied if, for instance, $\alpha_0\beta_0 > 9$.

The paper is organized as follows. In Section 2 we present the proof of our abstract results for the space X. In Section 3 we prove our existence result Theorem 1.4 and in the final Section 4 we prove Theorem 1.5.

Throughout the paper we write $\int u$ instead of $\int_{\mathbb{R}^2} u(x)\, dx$.

2 Proof of the abstract results

In this section we present the proof of Theorems 1.1 and 1.3. To this end, we first recall that X was defined as the closure of $C_c^\infty(\mathbb{R}^2)$ with respect to the norm

$$\|u\| := \left(\int K(x)|\nabla u|^2\right)^{1/2}$$

where $K(x) = \exp\left(|x|^2/4\right)$. The following result establishes the embedding of X in the weighted Lebesgue spaces.

Lemma 2.1 *The space X is compactly embedded in $L_K^p(\mathbb{R}^2)$ for any $p \in [2, +\infty)$.*

P r o o f . For any given $u \in C_c^\infty(\mathbb{R}^2)$ we have that

$$\int \left|\nabla(K(x)^{1/2}u)\right|^2 = \int K(x)|\nabla u|^2 + \int \nabla\big(K(x)^{1/2}u^2\big)\nabla\big(K(x)^{1/2}\big).$$

Integrating by parts we get

$$\int \left|\nabla(K(x)^{1/2}u)\right|^2 = \int K(x)|\nabla u|^2 - \frac{1}{2}\int K(x)u^2\left(\Delta\theta + \frac{1}{2}|\nabla\theta|^2\right) \tag{2.1}$$

where $\theta(x) := |x|^2/4$. Since

$$\Delta\theta(x) + \frac{1}{2}|\nabla\theta(x)|^2 = 1 + \frac{1}{8}|x|^2 \geq 1$$

it follows that

$$\int K(x)u^2 \leq 2\int K(x)|\nabla u|^2. \tag{2.2}$$

By density we have the same inequality for any $u \in X$. This establishes the continuous embedding $X \hookrightarrow L_K^2(\mathbb{R}^2)$.

If $p > 2$ and $u \in X$, we can use (2.1) and (2.2) to get

$$\int \left(\left|\nabla(K(x)^{1/2}u)\right|^2 + K(x)u^2\right) \leq \int K(x)|\nabla u|^2 + \frac{1}{2}\int K(x)u^2 \leq 2\|u\|^2.$$

Thus we conclude that $K^{1/2}u \in H^1(\mathbb{R}^2)$. Hence, we can use (2.1) again to infer that

$$\begin{aligned}\int K(x)|\nabla u|^2 &\geq \int \left|\nabla(K(x)^{1/2}u)\right|^2 + \frac{1}{2}\int K(x)|u|^2\\ &= \left\|K^{1/2}u\right\|^2_{H^1(\mathbb{R}^2)} - \frac{1}{2}\int K(x)|u|^2\\ &\geq C_p\left(\int K(x)^{p/2}|u|^p\right)^{2/p} - \frac{1}{2}\int K(x)|u|^2,\end{aligned}$$

where $C_p > 0$ is related with the embedding $H^1(\mathbb{R}^2) \hookrightarrow L^p(\mathbb{R}^2)$. Since $K(x) \geq 1$ and $p \geq 2$ we have that $K(x)^{p/2} \geq K(x)$. It follows from (2.2) that

$$C_p\left(\int K(x)|u|^p\right)^{2/p} \leq C_p\left(\int K(x)^{p/2}|u|^p\right)^{2/p} \leq 2\int K(x)|\nabla u|^2, \tag{2.3}$$

and therefore $X \hookrightarrow L_K^p(\mathbb{R}^2)$, for $p \geq 2$.

It is proved in [8], Proposition 11] that the embedding $X \hookrightarrow L_K^2(\mathbb{R}^2)$ is compact. For the case $p > 2$, we take a sequence $(u_n) \subset X$ such that $u_n \rightharpoonup 0$ weakly in X. Fix $\tilde{p} > p$ and consider $\tau \in (0,1)$ such that $p = (1-\tau)2 + \tau\tilde{p}$. Hölder's inequality with exponents $1/(1-\tau)$ and $1/\tau$ provides

$$\begin{aligned}\int K(x)|u_n|^p &= \int K(x)^{(1-\tau)}|u_n|^{(1-\tau)2}K(x)^\tau|u_n|^{\tau\tilde{p}}\\ &\leq \left(\int K(x)|u_n|^2\right)^{1-\tau}\left(\int K(x)|u_n|^{\tilde{p}}\right)^\tau \leq c\|u_n\|_2^{2(1-\tau)}.\end{aligned}$$

Up to a subsequence, we have that $u_n \to 0$ in $L_K^2(\mathbb{R}^2)$. The above expression implies that $u_n \to 0$ in $L_K^p(\mathbb{R}^2)$. □

Remark 2.2 As a byproduct of the above calculations we see that $X \hookrightarrow H^1(\mathbb{R}^2)$. Indeed, for any $u \in X$ we infer from (2.2) that

$$\int \left(|\nabla u|^2 + |u|^2\right) \leq \int K(x)\big(|\nabla u|^2 + |u|^2\big) \leq 3\|u\|^2.$$

We also quote for future references that, in view of the second inequality (2.3), for any $r \geq 1$ there exists $C = C(r)$ such that

$$\left(\int K(x)^r |u|^{2r}\right)^{1/r} \leq C \int K(x)|\nabla u|^2, \quad \text{for all} \quad u \in X. \tag{2.4}$$

Remark 2.3 Another interesting remark is that $X \hookrightarrow L^p(\mathbb{R}^2)$ for any $p \geq 1$. Hence, the element h of the dual of X can belongs to any Lebesgue space. In order to verify the last embedding we notice that, since $K \geq 1$, we have $L^p_K(\mathbb{R}^2) \hookrightarrow L^p(\mathbb{R}^2)$ for $p \geq 2$. On the other hand, if $u \in X$ then

$$\int |u| \leq \left(\int e^{-|x|^2/4}\right)^{1/2} \left(\int K(x)|u|^2\right)^{1/2} < \infty,$$

which shows that $X \hookrightarrow L^1(\mathbb{R}^2)$. By using interpolation we conclude that $X \hookrightarrow L^p(\mathbb{R}^2)$ also for $p > 1$.

We are now ready to present the proof of our abstract results.

Proof of Theorem 1.1. Let $r_i > 1$, $i = 1, 2$, be such that $1/r_1 + 1/r_2 = 1$. Hölder's inequality implies that

$$\int K(x)|u|^2 \left(e^{\beta u^2} - 1\right) \leq \left(\int K(x)^{r_1} |u|^{2r_1}\right)^{1/r_1} \left(\int (e^{\beta u^2} - 1)^{r_2}\right)^{1/r_2}. \tag{2.5}$$

By applying the inequality $(1+t)^r \geq 1 + t^r$ with $t = e^s - 1 \geq 0$, we obtain

$$(e^s - 1)^r \leq (e^{rs} - 1), \quad \text{for all} \quad r \geq 1,\ s \geq 0. \tag{2.6}$$

Using this inequality in (2.5) and recalling (2.4) we get

$$\int K(x)|u|^2 \left(e^{\beta u^2} - 1\right) \leq C\|u\|^2 \left(\int \left(e^{\beta r_2 u^2} - 1\right)\right)^{1/r_2}. \tag{2.7}$$

By choosing r_2 close to 1, in such way that $\alpha := \beta r_2 M^2 < 4\pi$, we obtain

$$\int \left(e^{\beta r_2 u^2} - 1\right) \leq \int \left(e^{\beta r_2 M^2 (u/\|u\|)^2} - 1\right) = \int \left(e^{\alpha v^2} - 1\right),$$

with $v := u/\|u\|$. Since $\int |\nabla v|^2 \leq \|v\|^2 = 1$ we have that $\|\nabla v\|_{L^2(\mathbb{R}^2)} \leq 1$. Moreover, it follows from (2.2) that $\|v\|_{L^2(\mathbb{R}^2)} \leq 2$. Hence, we can invoke inequality (1.3) to obtain a positive constant $C(M, \beta)$ such that

$$\int \left(e^{\beta r_2 u^2} - 1\right) \leq C(M, \beta).$$

The results follows from the above estimate and (2.7). □

Proof of Corollary 1.2. Let $r_i > 1$, $i = 1, 2, 3$, be such that $1/r_1 + 1/r_2 + 1/r_3 = 1$ and $qr_2 \geq 1$. Hölder's inequality implies that

$$\int K(x)|u|^{2+q} \left(e^{\beta u^2} - 1\right) \leq \left(\int K(x)^{r_1} |u|^{2r_1}\right)^{1/r_1} \|u\|^q_{L^{qr_2}(\mathbb{R}^2)} \left(\int \left(e^{\beta u^2} - 1\right)^{r_3}\right)^{1/r_3}.$$

Using the embedding $X \hookrightarrow L^{qr_2}(\mathbb{R}^2)$ and arguing as in the proof of Theorem 1.2 we obtain the desired result. □

Proof of Theorem 1.3. Given $r > 1$ it follows from Hölder's inequality, (2.4) and (2.6) that

$$\begin{aligned} \int K(x) v_n^2 \left(e^{p v_n^2} - 1\right) &\leq \left(\int K(x)^{r'} |v_n|^{2r'}\right)^{1/r'} \left(\int \left(e^{r p v_n^2} - 1\right)\right)^{1/r} \\ &\leq C \left(\int \left(e^{r p v_n^2} - 1\right)\right)^{1/r}, \end{aligned}$$

where $1/r + 1/r' = 1$.

In order to obtain an uniform estimate of the right-hand side above we recall that, if $a,\ b,\ \varepsilon > 0$, Young's inequality implies that

$$a^2 = (a-b)^2 + b^2 + 2\varepsilon(a-b)\frac{b}{\varepsilon}$$
$$\leq (1+\varepsilon^2)(a-b)^2 + \left(1+\frac{1}{\varepsilon^2}\right)b^2.$$

Hence, we can use Young's inequality again to get

$$\int \left(e^{rpv_n^2}-1\right) \leq \int \left(e^{rp(1+\varepsilon^2)(v_n-v)^2} e^{rp(1+1/\varepsilon^2)v^2} - \frac{1}{\gamma} - \frac{1}{\gamma'}\right)$$
$$\leq \frac{1}{\gamma}\int \left(e^{r\gamma p(1+\varepsilon^2)(v_n-v)^2}-1\right) + \frac{1}{\gamma'}\int \left(e^{r\gamma' p(1+1/\varepsilon^2)v^2}-1\right)$$

where $\gamma > 1$ and $1/\gamma + 1/\gamma' = 1$. The last integral above is finite and therefore it suffices to prove that

$$\sup_{n\in\mathbb{N}} \int \left(e^{r\gamma p(1+\varepsilon^2)(v_n-v)^2}-1\right) < \infty.$$

Since $v_n \rightharpoonup v$ and $\|v_n\| = 1$, we conclude that

$$\lim_{n\to\infty} \|v_n - v\|^2 = 1 - \|v\|^2 < \frac{4\pi}{p}.$$

Thus, we can take $0 < \alpha < 4\pi$ and choose $r,\ \gamma$ close to 1 and $\varepsilon > 0$ small in such way that

$$r\gamma p\big(1+\varepsilon^2\big)\|v_n - v\|^2 < \alpha < 4\pi. \tag{2.8}$$

We now set $u_n := \frac{(v_n-v)}{\|v_n-v\|}$ and notice that, since $\int |\nabla u_n|^2 \leq \|u_n\|^2 = 1$ we have that $\|\nabla u_n\|_{L^2(\mathbb{R}^2)} \leq 1$. Moreover, it follows from (2.2) that $\|u_n\|_{L^2(\mathbb{R}^2)} \leq 2$. Hence, we can invoke inequalities (1.3) and (2.8) to obtain $C_1 > 0$, independent of n, such that

$$\int \left(e^{r\gamma p(1+\varepsilon^2)(v_n-v)^2}-1\right) = \int \left(e^{r\gamma p(1+\varepsilon^2)\|v_n-v\|^2 u_n^2}-1\right)$$
$$\leq \int \left(e^{\alpha u_n^2}-1\right) \leq C_1,$$

and the theorem is proved. □

3 Proof of Theorem 1.4

Let $\alpha > \alpha_0$ be given by (f_0) and $q \geq 1$. By using the critical growth of f we obtain

$$\lim_{|s|\to+\infty} \frac{f(s)}{|s|^{q-1}(e^{\alpha s^2}-1)} = 0.$$

This and (f_1) imply that, for any given $\varepsilon > 0$, there hold

$$|f(s)| \leq \varepsilon|s| + c_1|s|^{q-1}\big(e^{\alpha s^2}-1\big), \quad \text{for all} \quad s\in\mathbb{R}, \tag{3.1}$$

and

$$|F(s)| \leq \frac{\varepsilon}{2}s^2 + c_2|s|^q\big(e^{\alpha s^2}-1\big), \quad \text{for all} \quad s\in\mathbb{R}. \tag{3.2}$$

Given $u \in X$ we can use the above inequality with $q = 2$ to obtain

$$\int K(x)F(u) \leq c_3 \int K(x)|u|^2 + c_4 \int K(x)|u|^2\big(e^{\alpha u^2}-1\big) < +\infty,$$

where we have used Lemma 2.1 and Theorem 1.1. Hence, we can use standard calculations to conclude that the functional

$$I(u) := \frac{1}{2}\|u\|^2 - \int K(x)F(u) - \langle h, u\rangle, \quad u \in X,$$

is well defined. Moreover, $I \in C^1(X, \mathbb{R})$ with derivative given by

$$I'(u)\varphi = \int K(x)\nabla u \cdot \nabla\varphi - \int K(x) f(u)\varphi - \langle h, \varphi\rangle,$$

and therefore the critical points of I are precisely the weak solutions of the problem (P).

In order to obtain the link structure for the functional I we use the following technical result.

Lemma 3.1 *Suppose f satisfies (f_0)–(f_2). Then there exists $\delta_1 > 0$ such that, for each $h \in X^{-1}$ with $0 < \|h\|_{X^{-1}} \le \delta_1$, there hold*

(i) *there exist $\gamma_h,\ \rho_h > 0$ such that*

$$I(u) \ge \gamma_h > 0, \quad \text{for all} \quad u \in \partial B_{\rho_h}(0).$$

Furthermore, ρ_h can be chosen such that $\rho_h \to 0$ as $\|h\|_{X^{-1}} \to 0$.

(ii) *there exists $e_h \in X$ such that*

$$I(e_h) < \inf_{B_{\rho_h}(0)} I < 0.$$

P r o o f . By using (3.2) with $q > 2$, Corollary 1.2 and the continuous embedding $X \hookrightarrow L^2_K(\mathbb{R}^2)$ we obtain

$$\begin{aligned} I(u) &\ge \frac{1}{2}\|u\|^2 - \frac{\varepsilon}{2}\int K(x)|u|^2 - C_\varepsilon \int K(x)|u|^{2+(q-2)}\left(e^{\alpha u^2} - 1\right) - \langle h, u\rangle \\ &\ge \frac{1}{2}\left(1 - c_1\varepsilon\right)\|u\|^2 - c_2\|u\|^q - \|h\|_{X^{-1}}\|u\|, \end{aligned}$$

whenever $\|u\| \le M < (4\pi/\alpha)^{1/2}$. Choosing $\varepsilon = 1/(2c_1)$ we get

$$I(u) \ge \|u\|\left(\frac{1}{4}\|u\| - c_3\|u\|^{q-1} - \|h\|_{X^{-1}}\right).$$

If we define $\phi(t) := t/4 - c_3 t^{q-1}$, a straightforward calculation shows that, for any $0 < \|h\|_{X^{-1}} < \max_{t\ge 0}\phi(t)$, there exists $0 < \rho_h < (4\pi/\alpha)^{1/2}$ such that

$$\frac{1}{4}\rho_h - c_3\rho_h^{q-1} = \frac{\|h\|_{X^{-1}}}{2},$$

and therefore

$$I(u) \ge \frac{\rho_h}{2}\|h\|_{X^{-1}} > 0, \quad \text{if} \quad \|u\| = \rho_h.$$

Moreover, the number ρ_h can be chosen in such way that $\rho_h \to 0$ as $\|h\|_{X^{-1}} \to 0$.

In order to verify (ii) we note that, from (f_2), we get

$$F(s) \ge c_4|s|^{\theta_0} - c_5, \quad \text{for all} \quad s \in \mathbb{R}. \tag{3.3}$$

If we take a nonzero function $\varphi \in C_c^\infty(\mathbb{R}^2)$ and denote by A the support of φ, we have for $t \ge 0$

$$I(t\varphi) \le \frac{t^2}{2}\|\varphi\|^2 - c_4 t^{\theta_0}\int K(x)|\varphi|^{\theta_0} + c_5\|K\|_{L^1(A)} + t\|h\|_{X^{-1}}\|\varphi\|.$$

Hence $I(t\varphi) \to -\infty$ as $t \to \infty$. Thus, if we set $e_h := t\varphi$ for $t > 0$ large enough, we conclude that $I(e_h) < \inf_{u\in B_{\rho_h}(0)} I(u)$.

It remains to prove that the infimun of I on $B_{\rho_h}(0)$ is negative. For this purpose we denote by v_h the unique function of X satisfying (in the weak sense)

$$-\mathrm{div}(K(x)\nabla v_h) = h \quad \text{in} \quad \mathbb{R}^2.$$

This function can be obtained by the Riesz Theorem. We have that $\langle h, v_h\rangle = \|v_h\|^2 > 0$, whenever $h \neq 0$. Since $f(0) = 0$, it follows by continuity that there exists $\eta_h > 0$ such that

$$\frac{d}{dt} I(tv_h) = t\|v_h\|^2 - \int K(x) f(tv_h)v_h - \langle h, v_h\rangle < 0,$$

for all $0 < t < \eta_h$. Hence $t \mapsto I(tv_h)$ is decreasing in $(0, \eta_h)$. Since $I(0) = 0$, we must have $I(tv_h) < 0$ for all $0 < t < \eta_h$, and the result follows. □

Remark 3.2 It is worthwhile to mention that the above lemma remains true if we replace the Ambrosetti-Rabinowitz condition (f_2) by the following weaker condition

$(\widehat{f_2})$ $\displaystyle\lim_{s\to\infty} \frac{F(s)}{s^2} = +\infty.$

Indeed, notice that (f_2) was used only to show that $I(t\varphi) \to -\infty$ for some $\varphi \in X$. So, we need only check that the above condition suffices to get the same result. Given $M > 0$, it follows from $(\widehat{f_2})$ that $F(s) \geq Ms^2 - C_M$, for any $s \geq 0$ and some $C_M > 0$. If we take a nonzero nonnegative function $\varphi \in C_c^\infty(\mathbb{R}^2)$ and denote by A the support of φ, we have

$$I(t\varphi) \leq \frac{t^2}{2}\left(\|\varphi\|^2 - 2M\int K(x)\varphi^2\right) + c_1\|K\|_{L^1(A)} + t\|h\|_{X^{-1}}\|\varphi\|.$$

Since $M > 0$ is arbitrary, the result follows.

We are ready to prove our existence result.

Proof of Theorem 1.4. Let ρ_h be given by Lemma 3.1. We can choose $\|h\|_{X^{-1}}$ small in such way that $\rho_h < (4\pi/\alpha_0)^{1/2}$. Let

$$c_0 := \inf_{\|u\|\leq\rho_h} I(u) < 0.$$

By using the Ekeland Variational Principle we obtain a minimizing sequence $(u_n) \subset \overline{B}_{\rho_h}(0)$ such that $I(u_n) \to c_0$ and $I'(u_n) \to 0$. Notice that

$$\liminf_{n\to\infty} \|u_n\|^2 \leq \rho_h^2 < \frac{4\pi}{\alpha_0}.$$

We claim that, along a subsequence, $u_n \to u$ strongly in X. If this is true it follows that $I(u) = c_0 < 0$ and therefore u is a (nonzero) weak solution of I.

It remains to prove the claim. Since $(u_n) \subset X$ is bounded we may suppose that $u_n \rightharpoonup u$ weakly in X. We set $w_n := u_n - u$ and notice that, since $w_n \rightharpoonup 0$ weakly in X, we have that

$$\begin{aligned} o_n(1) = I'(u_n)w_n &= \langle u_n, u_n - u\rangle - \int K(x) f(u_n)w_n - \langle h, w_n\rangle \\ &= \|u_n\|^2 - \|u\|^2 - \int K(x) f(u_n)w_n + o_n(1). \end{aligned} \tag{3.4}$$

It suffices to prove that

$$\lim_{n\to\infty} \int K(x) f(u_n)w_n = 0. \tag{3.5}$$

If this is true, it follows from (3.4) that $\|u_n\| \to \|u\|$. Hence, the weak convergence of (u_n) implies that $u_n \to u$ strongly in X.

We now argue along the same lines of the proof of Theorem 1.1 in order to prove (3.5). We first use (3.1) with $q = 3$ to get

$$\left|\int K(x) f(u_n) w_n\right| \le \varepsilon \int K(x)|u_n||w_n| + c_1 \int K(x)|u_n|^2|w_n|\big(e^{\alpha u_n^2} - 1\big)$$
$$\le \varepsilon \|u_n\|_2\|w_n\|_2 + c_1 D_n, \tag{3.6}$$

where

$$D_n := \int K(x)|u_n|^2|w_n|\big(e^{\alpha u_n^2} - 1\big).$$

Since the embedding $X \hookrightarrow L^2_K\big(\mathbb{R}^2\big)$ is compact, we have that $\|w_n\|_2 \to 0$. Hence, it is enough to verify that $D_n \to 0$. By taking $r_i > 1, i = 1, 2, 3$, such that $1/r_1 + 1/r_2 + 1/r_3 = 1$ and $r_2 > 2$, we can use Hölder inequality again to get

$$D_n \le c_2 \left(\int K(x)^{r_1}|u_n|^{2r_1}\right)^{1/r_1} \|w_n\|_{L^{r_2}(\mathbb{R}^2)} \left(\int \left(e^{\alpha r_3\|u_n\|^2(u_n/\|u_n\|)^2}\right)\right)^{1/r_3}$$
$$\le c_3\|u_n\|^2 o_n(1) \left(\int \left(e^{\alpha r_3\|u_n\|^2(u_n/\|u_n\|)^2}\right)\right)^{1/r_3},$$

where we have used (2.4), (2.6) and the compactness of $X \hookrightarrow L^{r_2}\big(\mathbb{R}^2\big)$. Since $\|u_n\|^2 \to \gamma < 4\pi/\alpha_0$, we can choose r_3 close to 1 and $\alpha > \alpha_0$ close to α_0 in such way that $\alpha r_3\|u_n\|^2 \le \tilde{\gamma} < 4\pi$. It follows from inequality (1.3) that the last term into the parenthesis above is bounded. Hence, $D_n \to 0$ and the theorem is proved. □

4 Proof of Theorem 1.5

In this section we prove our multiplicity result. As quoted in the Introduction, in order to get the correct estimates, we need now to work with radial functions. We then denote by X_{rad} the closure of $C^\infty_{c,rad}\big(\mathbb{R}^2\big)$ with respect to the norm (1.1). A simple inspection of the proofs present in the two last sections show that all the results also hold if we replace X by X_{rad}. Thus, we can use the same variational setting earlier presented with the functional I defined only on the space X_{rad}. We point out that, in order to simplify the explanation, in the sequel we will write only X to denote the subspace X_{rad}.

From now on we will suppose that $0 < \|h\|_{X^{-1}} < \delta_1$, with $\delta_1 > 0$ given by Theorem 1.4. We shall denote by u_h the weak solution provided by that theorem.

The proof we are going to present is based on an indirect application of the Mountain Pass Theorem. There are two main points: to obtain a local compactness result and making a careful estimate of the minimax level of the functional I. We state below these two core results.

Proposition 4.1 *Suppose f satisfies (f_2) and (f_3). The functional I satisfies the (PS)$_d$ condition for any*

$$d < I(u_h) + \frac{2\pi}{\alpha_0},$$

provided 0 *and u_h are the only critical points of I.*

Proposition 4.2 *Suppose f satisfies (f_0)–(f_2) and (f_4) and let $\delta_1 > 0$ and $u_h \in X$ be given by Theorem 1.4. Then there exists $0 < \delta_2 \le \delta_1$ such that, for all $h \in X^{-1}$ such that $0 < \|h\|_{X^{-1}} < \delta_2$, there exists $v \in X$ with compact support such that*

$$\max_{t\ge 0} I(tv) < I(u_h) + \frac{2\pi}{\alpha_0}. \tag{4.1}$$

The above propositions will be proved in the next two subsections. In what follows, we show how they can be applied to prove our multiplicity result.

Proof of Theorem 1.5. We shall prove the theorem for δ_2 given by the last proposition. Arguing by contradiction, we suppose that $0 < \|h\|_{X^{-1}} < \delta_2$ but the functional I has no critical points other than 0 and u_h.

Let v be given by Proposition 4.2 and denote by $A \subset \mathbb{R}^2$ its support. It follows from (3.3) that, for any $t > 0$,

$$I(tv) \le \frac{t^2}{2}\|v\|^2 - c_4 t^{\theta_0} \int_A K(x)|v|^{\theta_0} \mathrm{d}x - c_5|A| + t\|h\|_{X^{-1}}\|v\|.$$

Since $\theta_0 > 2$ we conclude that $I(tv) \to -\infty$ as $t \to \infty$. Hence, $I(t_0 v) < 0$ for some $t_0 > 0$ large enough. This and item (i) of Lemma 3.1 show that I has the Mountain Pass Geometry, and therefore we can define the minimax level

$$c_M := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)),$$

where $\Gamma := \{\gamma \in C([0, 1], X) : \gamma(0) = 0,\ \gamma(1) = t_0 v\}$. The definition of c_M and (4.1) imply that

$$c_M \le \max_{t \ge 0} I(tv) < I(u_h) + \frac{2\pi}{\alpha_0}.$$

By Proposition 4.1 the functional I satisfies the Palais-Smale condition at the level c_M. It follows from the Mountain Pass Theorem that I possesses a critical point $u_M \in X$ with $I(u_M) > 0$. Since $I(0) = 0$ and $I(u_h) < 0$ we have that $u_M \notin \{0,\ u_h\}$, which is a contradiction, since we are supposing that the only critical points of I are 0 and u_h. The theorem is proved. □

4.1 The local compactness result

We devote this subsection to the proof of Proposition 4.1. It will be done in several steps and it was inspired by similar arguments developed in [16]. We start by establishing a variant of a well-known radial lemma of Strauss [20].

Lemma 4.3 *There exists $c_0 > 0$ such that, for all $v \in X$, there holds*

$$|v(x)| \le c_0 |x|^{-1/2} e^{-\frac{|x|^2}{8}} \|v\|, \quad \text{for all} \quad x \in \mathbb{R}^2 \setminus \{0\}.$$

Proof. It suffices to prove the lemma for $v \in C^\infty_{c,rad}(\mathbb{R}^2)$. Let $r = |x|$ and $\varphi : [0, +\infty) \to \mathbb{R}$ be such that $\varphi(r) = v(|x|)$. We have that

$$\begin{aligned}
\varphi(r)^2 &= -2 \int_r^\infty \varphi(s)\varphi'(s)\, \mathrm{d}s \\
&\le 2 \int_r^\infty e^{-s^2/4} s^{-1} |\varphi(s)||\varphi'(s)| e^{s^2/4} s\, \mathrm{d}s \\
&\le r^{-1} e^{-r^2/4} \int_r^\infty \big(\varphi(s)^2 + \varphi'(s)^2\big) e^{s^2/4} s\, \mathrm{d}s \\
&\le c_1 r^{-1} e^{-r^2/4} \int \big(K(x)|\nabla v|^2 + K(x) v^2\big).
\end{aligned}$$

Since $X \hookrightarrow L^2_K(\mathbb{R}^2)$, we get

$$\varphi(r)^2 \le c r^{-1} e^{-r^2/4} \|v\|^2,$$

and the lemma follows. □

In order to make the proof of our results more direct and effective, we state a technical lemma.

Lemma 4.4 *Suppose $G \in C(\mathbb{R}, \mathbb{R})$ satisfies*

$$G(s) \le c_1 s^4 \big(e^{\alpha s^2} - 1\big), \quad \text{for all} \quad s \in \mathbb{R},$$

with $c_1,\ \alpha > 0$. *Then there exists* $c > 0$ *such that, for any* $R > 1$ *and* $u \in X$, *there holds*

$$\int_{B_R(0)^c} K(x)G(u)\,\mathrm{d}x \le \frac{c}{R}\|u\|^4\Big(e^{\alpha c_0^2\|u\|^2} - 1\Big),$$

where $c_0 > 0$ *comes from Lemma 4.3.*

Proof. It follows from the Monotone Convergence Theorem that

$$\begin{aligned}\int_{B_R(0)^c} K(x)G(u)\,\mathrm{d}x &\le c_1\int_{B_R(0)^c} K(x)|u|^4(e^{\alpha u^2} - 1)\,\mathrm{d}x\\ &= c_1\sum_{j=1}^{+\infty}\frac{\alpha^j}{j!}\int_{B_R(0)^c} K(x)|u|^{2j+4}\,\mathrm{d}x.\end{aligned} \tag{4.2}$$

By using Lemma 4.3 we can estimate the last integral above as follows

$$\begin{aligned}\int_{B_R(0)^c} K(x)|u|^{2j+4}\,\mathrm{d}x &\le (c_0\|u\|)^{2j+4}\int_{B_R(0)^c} e^{\frac{|x|^2}{4}(1-j-2)}|x|^{-j-2}\,\mathrm{d}x\\ &\le 2\pi(c_0\|u\|)^{2j+4}\int_R^\infty s^{-j-2}s\,\mathrm{d}s\\ &= 2\pi(c_0\|u\|)^{2j+4}\frac{1}{jR^j}\\ &\le \frac{2\pi}{R}(c_0\|u\|)^{2j+4},\end{aligned}$$

where we have used that $j \ge 1$ and $R > 1$. The above expression and (4.2) provide

$$\begin{aligned}\int_{B_R(0)^c} K(x)G(u)\,\mathrm{d}x &\le \frac{2\pi}{R}c_1(c_0\|u\|)^4\sum_{j=1}^{\infty}\frac{(\alpha c_0^2\|u\|^2)^j}{j!}\\ &= \frac{c}{R}\|u\|^4\Big(e^{\alpha c_0^2\|u\|^2} - 1\Big),\end{aligned}$$

with $c := 2\pi c_1 c_0^4 > 0$. The proof is complete. □

Lemma 4.5 *Suppose* f *satisfies* (f_2) *and* (f_3). *If* $(u_n) \subset X$ *is a (PS)*$_c$ *sequence of* I *then, up to a subsequence, we have that*

(i) $u_n \rightharpoonup u$ *weakly in* X *with* $I'(u) = 0$;
(ii) $\lim_{n\to\infty}\int K(x)F(u_n) = \int K(x)F(u)$;
(iii) $\limsup_{R\to\infty}\int_{B_R(0)^c} K(x)f(u_n)u_n\,\mathrm{d}x = 0$.

Proof. Let $(u_n) \subset X$ be such that $I'(u_n) \to 0$ and $I(u_n) \to c$. Notice that

$$\begin{aligned}c + o_n(1)\|u_n\| + o_n(1) &= I(u_n) - \frac{1}{\theta_0}I'(u_n)u_n\\ &= \left(\frac{1}{2} - \frac{1}{\theta_0}\right)\|u_n\|^2 - \left(1 - \frac{1}{\theta_0}\right)\langle h, u_n\rangle\\ &\quad - \int K(x)\left(F(u_n) - \frac{1}{\theta_0}f(u_n)u_n\right),\end{aligned}$$

and therefore it follows from (f_2) that (u_n) is bounded in X. Hence, up to a subsequence, $u_n \rightharpoonup u$ weakly in X and $u_n(x) \to u(x)$ a.e. in $\mathbb{R}^2$. Moreover, $I'(u_n)u_n = o_n(1)$, the above expression, (f_2) and the boundedness of (u_n) provide $c_1 > 0$ such that

$$\int K(x) f(u_n)u_n \le c_1, \quad \int K(x)\left(\frac{1}{\theta_0} f(u_n)u_n - F(u_n)\right) \le c_1. \tag{4.3}$$

The first estimate above, $K \ge 1$, and condition (f_2) again imply that

$$\int |f(u_n)u_n| = \int f(u_n)u_n \le \int K(x) f(u_n)u_n \le c_1.$$

Thus, it follows from [9, Lemma 2.1] that $f(u_n) \to f(u)$ in $L^1_{loc}(\mathbb{R}^2)$. Hence, given $\varphi \in C^\infty_{c,rad}(\mathbb{R}^2)$, we have that

$$I'(u_n)\varphi = \langle u_n, \varphi\rangle - \int K(x) f(u_n)\varphi - \langle h, \varphi\rangle.$$

Since φ has compact support $\int K(x)f(u_n)\varphi \to \int K(x) f(u)\varphi$. Taking the limit in the above expression we conclude that $I'(u)\varphi = 0$. By density $I'(u) = 0$.

Let $M > 0$ be such that $\|u_n\|,\ \|u\| \le M$. Given $R,\ \varepsilon > 0$, we can use (3.2) with $q = 4$, Lemma 4.4 and the embedding $X \hookrightarrow L^2_K(\mathbb{R}^2)$ to get

$$\int_{B_R(0)^c} K(x)F(u_n)\,\mathrm{d}x \le \frac{\varepsilon}{2}\int K(x)|u_n|^2 + \frac{c}{R}\|u_n\|^4\left(e^{\alpha c_0^2\|u_n\|^2} - 1\right) \le c_2\varepsilon + \frac{c_3}{R},$$

with $c_2,\ c_3 > 0$ depending only on M. It follows that

$$\limsup_{R\to\infty} \int_{B_R(0)^c} K(x)F(u_n)\,\mathrm{d}x \le c_2\varepsilon. \tag{4.4}$$

Moreover, since $K(x)F(u) \in L^1(\mathbb{R}^2)$, it holds

$$\limsup_{R\to\infty} \int_{B_R(0)^c} K(x)F(u)\,\mathrm{d}x = 0. \tag{4.5}$$

Fixing $R > 0$, we claim that

$$\lim_{n\to\infty} \int_{B_R(0)} K(x)F(u_n)\,\mathrm{d}x = \int_{B_R(0)} K(x)F(u)\,\mathrm{d}x. \tag{4.6}$$

If this is true we can use the above convergence, (4.4) and (4.5) to get (ii).

We now use (f_3) to prove (4.6) in the following way. Consider $\theta > 0$ such that $(\theta_0 c_1)/(\theta - \theta_0) < \varepsilon$. If $R_\theta > 0$ is given by (f_3) and $A_n := \{|u_n| \ge R_\theta\}$, we case use the second inequality in (4.3) and (f_2) to obtain

$$\begin{aligned}\theta_0 c_1 &\ge \int_{A_n} K(x)(f(u_n)u_n - \theta_0 F(u_n))\,\mathrm{d}x \\ &= (\theta - \theta_0)\int_{A_n} K(x)F(u_n)\,\mathrm{d}x + \int_{A_n} K(x)(f(u_n)u_n - \theta F(u_n))\,\mathrm{d}x,\end{aligned}$$

and therefore it follows from (f_3) and the choice of θ that

$$\int_{A_n} K(x)F(u_n)\,\mathrm{d}x \le \frac{\theta_0 c_1}{\theta - \theta_0} < \varepsilon. \tag{4.7}$$

Applying Egoroff's Theorem we found a measurable set $A \subset B_R(0)$ such that $|A| < \varepsilon$ and $u_n(x) \to u(x)$ uniformly on $(B_R(0) \setminus A)$. Hence

$$\begin{aligned}\left|\int_{B_R(0)} K(x)(F(u_n) - F(u))\,\mathrm{d}x\right| &\le \int_A K(x)F(u_n)\,\mathrm{d}x \\ &\quad + \int_A K(x)F(u)\,\mathrm{d}x + o_n(1).\end{aligned} \tag{4.8}$$

Given $\alpha > \alpha_0$ it follows from (f_0) that $F(s) \le c_4 e^{\alpha s^2}$ for all $s \in \mathbb{R}$ and some $c_4 > 0$. Thus, given $\gamma > 1$, we can use Hölder's inequality to get

$$\int_A K(x)F(u)\,\mathrm{d}x \le M_1 c_4 |A|^{1/\gamma} \left(\int e^{\alpha\gamma' u^2} \right)^{1/\gamma'} \le c_5 \varepsilon^{1/\gamma}, \tag{4.9}$$

with $1/\gamma + 1/\gamma' = 1$. On the other hand, using (4.7), Lebesgue Theorem and the above expression we obtain

$$\begin{aligned} \int_A K(x)F(u_n)\,\mathrm{d}x &\le \int_{A\cap A_n} K(x)F(u_n)\,\mathrm{d}x + \int_{A\cap\{|u_n|<R_\theta\}} K(x)F(u_n)\,\mathrm{d}x \\ &\le \varepsilon + \int_{A\cap\{|u_n|<R_\theta\}} K(x)F(u)\,\mathrm{d}x + o_n(1) \\ &\le \varepsilon + c_5\varepsilon^{1/\gamma} + o_n(1). \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, the convergence in (4.6) follows from (4.8), (4.9) and the above inequality.

In order to prove (iii) it suffices to use (3.1) with $q = 4$ and proceed as in the proof of (4.4). □

Now we are ready to prove our compactness result.

Proof of Proposition 4.1. Let $(u_n) \subset X$ be such that $I'(u_n) \to 0$ and $I(u_n) \to d < I(u_h) + 2\pi/\alpha_0$. According to the previous lemma we have that $u_n \rightharpoonup u$ weakly in X with $I'(u) = 0$ and $\int K(x)F(u_n) \to \int K(x)F(u)$. Moreover, the weak convergence of (u_n) also implies that $\langle h, u_n\rangle \to \langle h, u\rangle$. We shall consider the two possible cases.

Case 1. $u = 0$.

In this case, the aforementioned convergence imply that

$$\frac{1}{2}\|u_n\|^2 = d + \int K(x)F(u) + \langle h, u\rangle + o_n(1).$$

Thus, recalling that $u = 0$ and $I(u_h) < 0$, we get

$$\lim_{n\to\infty} \|u_n\|^2 = 2d < 2I(u_h) + \frac{4\pi}{\alpha_0} < \frac{4\pi}{\alpha_0}.$$

Hence, we can argue as in the proof of Theorem 1.4 to conclude that, along a subsequence, $u_n \to u = 0$ strongly in X.

Case 2. $u = u_h$.

In this setting we shall verify that

$$\lim_{n\to\infty} \int K(x)f(u_n)u_n = \int K(x)f(u)u. \tag{4.10}$$

If this is true we obtain

$$\begin{aligned} o_n(1) = I'(u_n)u_n &= \|u_n\|^2 - \int K(x)f(u_n)u_n - \langle h, u_n\rangle \\ &= \|u_n\|^2 - \int K(x)f(u)u - \langle h, u\rangle + o_n(1) \\ &= \|u_n\|^2 - \|u\|^2 + I'(u)u + o_n(1). \end{aligned}$$

Since $I'(u) = 0$ we conclude that $\|u_n\| \to \|u\|$ and the proposition follows from the weak convergence of u_n.

It remains to check (4.10). In view of item (iii) of the previous lemma and since $\limsup_{R\to\infty} \int_{B_R(0)^c} K(x)f(u)u = 0$, it suffices to prove that, for any $R > 0$, the following holds:

$$\lim_{n\to\infty} \int_{B_R(0)} K(x)f(u_n)u_n\,\mathrm{d}x = \int_{B_R(0)} K(x)f(u)u\,\mathrm{d}x. \tag{4.11}$$

In order to check the above convergence we first notice that, as in the first case, we have

$$\lim_{n\to\infty} \|u_n\|^2 = 2\left(d + \int K(x)F(u) + \langle h, u\rangle \right) = 2(d + d_0) > 0, \tag{4.12}$$

where $d_0 := \int K(x)F(u) + \langle h, u\rangle$. We may suppose that $\|u_n\| \neq 0$ for all $n \geq n_0$, and therefore is well defined $v_n := u_n/\|u_n\|$. The weak convergence of u_n and the above expression imply that

$$v_n \rightharpoonup v := \frac{u_h}{\sqrt{2(d+d_0)}}, \quad \text{weakly in} \quad X.$$

If $\alpha > \alpha_0$ is such that $d < I(u_h) + 2\pi/\alpha$, then

$$1 - \|v\|^2 < \frac{4\pi/\alpha}{2(d+d_0)}.$$

Indeed, since

$$2d < 2I(u_h) + \frac{4\pi}{\alpha} \quad \text{and} \quad 2d_0 = 2\int K(x)F(u_h) + 2\langle h, u_h\rangle$$

we get

$$2d + 2d_0 < \|u_h\|^2 + \frac{4\pi}{\alpha}.$$

Thus,

$$1 - \|v\|^2 = \frac{2d + 2d_0 - \|u_h\|^2}{2(d+d_0)} < \frac{4\pi/\alpha}{2(d+d_0)}.$$

Using (4.12) we obtain $p_0 > 0$ such that $\alpha\|u_n\|^2 < p_0 < (4\pi)/\big(1 - \|v\|^2\big)$. We now choose $q > 1$ sufficiently close to 1 in such way that

$$\alpha q\|u_n\|^2 < p_0 q < \frac{4\pi}{1 - \|v\|^2}.$$

From Theorem 1.3 with $p = p_0 q$ we conclude that

$$\sup_{n\in\mathbb{N}} \int K(x)v_n^2\Big(e^{\alpha q\|u_n\|^2 v_n^2} - 1\Big) < \sup_{n\in\mathbb{N}} \int K(x)v_n^2\big(e^{p_0 q v_n^2} - 1\big) < \infty. \tag{4.13}$$

Up to a subsequence, we have that $u_n \to u$ strongly in $L^2(B_R(0))$, and therefore there exists $\Psi \in L^1(B_R(0))$ such that $|u_n(x)|^2 \leq \Psi(x)$ a.e. on $B_R(0)$. Since $K \in L^\infty(B_R(0))$, (3.1) implies that

$$\int_A K(x)f(u_n)u_n\,\mathrm{d}x \leq c_2\int_A \Psi(x)\,\mathrm{d}x + c_1\int_A K(x)|u_n|^{2/q}\big(e^{\alpha u_n^2} - 1\big)\mathrm{d}x, \tag{4.14}$$

for any measurable subset $A \subset B_R(0)$. We can use Hölder's inequality and (2.6) in the last integral above to get

$$\begin{aligned}
&\int K(x)|u_n|^{2/q}\big(e^{\alpha u_n^2} - 1\big) \\
&\leq \left(\int_A K(x)\,\mathrm{d}x\right)^{1/q'} \left(\int_A K(x)u_n^2\big(e^{\alpha q u_n^2} - 1\big)\mathrm{d}x\right)^{1/q} \\
&\leq \|u_n\|^{2/q}\|K\|_{L^1(A)}^{1/q'} \left(\int K(x)v_n^2\Big(e^{\alpha q\|u_n\|^2 v_n^2} - 1\Big)\right)^{1/q}.
\end{aligned}$$

By replacing this inequality in (4.14), using (4.13) and the boundedness of (u_n) we conclude that

$$\int_A K(x)f(u_n)u_n\,\mathrm{d}x \leq c_2\|\psi\|_{L^1(A)} + c_3\|K\|_{L^1(A)}^{1/q'}.$$

Since $\Psi,\ K \in L^1(B_R(0))$ and the set $A \subset B_R(0)$ is arbitrary, we conclude that the first integral above is uniformly small provided the measure of A is small. Hence, the set $\{K(x)f(u_n)u_n\}$ is uniformly integrable. This fact and a standard application of Egoroff's Theorem imply that $K(x)f(u_n)u_n \to K(x)f(u)u$ in $L^1(B_R(0))$. The convergence in (4.11) is proved. □

4.2 Minimax estimate

In this subsection we prove Proposition 4.2. As in the last subsection we divide the proof in several steps. Firstly we consider a little modification in the sequence of scaled truncated Green's functions considered by Moser (see [15]). More specifically, we define

$$\widetilde{M}_n(x) := (2\pi)^{-1/2} \begin{cases} K(r/n)^{-1/2}(\log n)^{1/2}, & \text{if} \quad |x| \le r/n, \\ K(x)^{-1/2} \dfrac{\log\left(\frac{r}{|x|}\right)}{(\log n)^{1/2}}, & \text{if} \quad r/n \le |x| < r, \\ 0, & \text{if} \quad |x| \ge r, \end{cases}$$

with $r > 0$ fixed. Notice that $\widetilde{M}_n \in H^1(\mathbb{R}^2)$ and $\operatorname{supp}(\widetilde{M}_n) = \overline{B}_r(0)$. Moreover, the following holds:

Lemma 4.6 *There exists $D = D(r) > 0$ and a sequence $(d_n) \subset \mathbb{R}$, which also depends on r, such that*

$$\|\widetilde{M}_n\|^2 = 1 + \frac{D}{\log n} - d_n,$$

with $\lim_{n\to\infty} d_n \log n = 0$. In particular,

$$\lim_{n\to\infty} \|\widetilde{M}_n\|^2 = 1. \tag{4.15}$$

Proof. We set

$$A_n := B_r(0) \setminus B_{r/n}(0)$$

and notice that $\nabla \widetilde{M}_n$ is zero outside the set A_n and

$$\nabla \widetilde{M}_n(x) = -e^{-|x|^2/8}(2\pi \log n)^{-1/2}\left(\frac{x}{|x|^2} + \frac{x}{4}\log(r/|x|)\right), \quad x \in A_n.$$

Hence, we can compute

$$\begin{aligned} \int K(x)\left|\nabla \widetilde{M}_n\right|^2 &= \frac{1}{2\pi \log n}\int_{A_n}\left(\frac{1}{|x|^2} + \frac{|x|^2}{16}\log^2(r/|x|) + \frac{1}{2}\log(r/|x|)\right)\mathrm{d}x \\ &= \frac{1}{\log n}\int_{r/n}^{r}\left(\frac{1}{s} + \frac{s^3}{16}\log^2(r/s) + \frac{1}{2}s\log(r/s)\right)\mathrm{d}s \\ &= \frac{1}{\log n}\left(\log n + \frac{r^2}{8} + \frac{r^4}{512} - \Gamma_{r,n,1} - \Gamma_{r,n,2}\right) \end{aligned}$$

with

$$\Gamma_{r,n,1} := \frac{r^2}{8}\left(\frac{2\log n}{n^2} + \frac{1}{n^2}\right), \quad \Gamma_{r,n,2} := \frac{r^4}{512}\left(\frac{8\log^2 n}{n^4} + \frac{4\log n}{n^4} + \frac{1}{n^4}\right).$$

It suffices now to set

$$D := \frac{r^2}{8} + \frac{r^4}{512}, \quad d_n := (\log n)^{-1}(\Gamma_{r,n,1} + \Gamma_{r,n,2}) \tag{4.16}$$

to get the conclusions of the lemma. □

For the next result we normalize the Green function and consider the function M_n defined by

$$M_n := \frac{\widetilde{M}_n}{\|\widetilde{M}_n\|}.$$

Notice that $\|M_n\| = 1$. Moreover, we have the following:

Lemma 4.7 *Suppose that (f_2), (f_3) and (f_4) hold. Then there exists $n \in \mathbb{N}$ such that*

$$\max_{t\geq 0}\left\{\frac{t^2}{2}-\int K(x)F(tM_n)\right\}<\frac{2\pi}{\alpha_0}. \tag{4.17}$$

P r o o f . Since M_n has compact support we can argue as in the proof of Lemma 3.1 to conclude that the function

$$g_n(t):=\frac{t^2}{2}-\int K(x)F(tM_n),\quad t\geq 0,$$

goes to $-\infty$ as $t\to+\infty$. Hence, it attains its global maximum at a point $t_n>0$ such that $g_n'(t_n)=0$, that is,

$$t_n^2=\int_{B_r(0)}K(x)t_nM_nf(t_nM_n)\,\mathrm{d}x. \tag{4.18}$$

Suppose, by contradiction, that the lemma is false. Then, for any $n\in\mathbb{N}$ there holds

$$\frac{t_n^2}{2}-\int K(x)F(t_nM_n)\geq\frac{2\pi}{\alpha_0}$$

and therefore

$$t_n^2\geq\frac{4\pi}{\alpha_0},\quad \text{for all}\quad n\in\mathbb{N}. \tag{4.19}$$

We claim that $(t_n)\subset\mathbb{R}$ is bounded. Indeed, let $\beta_0>0$ be given by (f_4) and $0<\varepsilon<\beta_0$. The condition (f_4) provides $R=R(\varepsilon)>0$ such that

$$sf(s)\geq(\beta_0-\varepsilon)\exp\left(\alpha_0 s^2\right),\quad \text{for all}\quad |s|\geq R. \tag{4.20}$$

The definition of M_n, (4.19) and $\|\widetilde{M}_n\|\to 1$ imply that, for any large values of n, there holds

$$t_nM_n(x)\geq e^{-r^2/8}\sqrt{\frac{2\log n}{\alpha_0}}\geq R,\quad \text{for all}\quad x\in B_{r/n}(0).$$

It follows from (4.18), (4.20), the above expression, $K\geq 1$ and the definition of M_n that

$$\begin{aligned} t_n^2 &\geq \int_{B_{r/n}(0)}K(x)t_nM_nf(t_nM_n)\,\mathrm{d}x\\ &\geq(\beta_0-\varepsilon)\int_{B_{r/n}(0)}\exp\left(\alpha_0(t_nM_n)^2\right)\mathrm{d}x\\ &=(\beta_0-\varepsilon)\int_{B_{r/n}(0)}\exp\left(\alpha_0 t_n^2\frac{\log n}{2\pi}\frac{e^{-r^2/(4n^2)}}{\|\widetilde{M}_n\|^2}\right)\mathrm{d}x. \end{aligned}$$

This, (4.15), the equation $1/n^2=\exp(-2\log n)$ and a straightforward calculation provide

$$t_n^2\geq(\beta_0-\varepsilon)\pi r^2\exp\left(2\left(\frac{e^{-r^2/(4n^2)}}{\|\widetilde{M}_n\|^2}\frac{\alpha_0}{4\pi}t_n^2-1\right)\log n\right). \tag{4.21}$$

Since $\exp(x)\geq x$ we can invoke Lemma 4.6 and the above expression to conclude that (t_n) is bounded.

By going to a subsequence, we may use (4.19) to conclude that $t_n^2\to\gamma\geq 4\pi/\alpha_0$. Since $e^{-r^2/(4n^2)}\|\widetilde{M}_n\|^{-2}\to 1$, we can take the limit in (4.21) to conclude that $\gamma>4\pi/\alpha_0$ cannot occur. Hence

$$\lim_{n\to\infty}t_n^2=\frac{4\pi}{\alpha_0}. \tag{4.22}$$

By using (4.19) and (4.21) again we get

$$t_n^2\geq(\beta_0-\varepsilon)\pi r^2\exp\left(\frac{-2}{\|\widetilde{M}_n\|^2}\left(\|\widetilde{M}_n\|^2-e^{-r^2/(4n^2)}\right)\log n\right). \tag{4.23}$$

It follows from Lemma 4.6 and L'Hopital rule that

$$\left(\|\widetilde{M}_n\|^2 - e^{-r^2/(4n^2)}\right)\log n = \left(1 - e^{-r^2/(4n^2)}\right)\log n + D - d_n \log n = D + o_n(1).$$

Hence, recalling that $\left\|\widetilde{M}_n\right\|^2 \to 1$, we can take the limit in (4.23) and use (4.22) to obtain

$$\frac{4\pi}{\alpha_0} \geq (\beta_0 - \varepsilon)\pi r^2 e^{-2D}.$$

Letting $\varepsilon \to 0$ and using the expression of $D = D(r)$ given in (4.16) we conclude that

$$\alpha_0\beta_0 \leq \frac{4}{r^2}\exp\left(\frac{r^2}{4} + \frac{r^4}{256}\right).$$

Since $r > 0$ is arbitrary, the above expression contradicts (f_4) and the lemma is proved. □

We are ready to finish the paper by presenting the

P r o o f o f P r o p o s i t i o n 4.2. Let $n \in \mathbb{N}$ be given by the above lemma and set $v := M_n$. Since $\langle h, v\rangle \leq \|h\|_{X^{-1}}$ we can use (4.17) to obtain $0 < \delta_2 < \delta_1$ such that $\max_{t\geq 0} I(tv) < 2\pi/(\alpha_0)$, whenever $0 < \|h\|_{X^{-1}} < \delta_2$. In view of the first item of Lemma 3.1 we have that $u_h \to 0$ as $\rho_0 \to 0$ and $\rho \to 0$ as $\|h\|_{X^{-1}} \to 0$. Thus, taking δ_2 smaller if necessary, we may suppose that $I(u_h)$ is so close to zero in such way that

$$\max_{t\geq 0} I(tv) < I(u_h) + \frac{2\pi}{\alpha_0}.$$

The proposition is proved. □

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