Existence and multiplicity of self-similar solutions for heat equations with nonlinear boundary conditions

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Abstract

This paper is concerned with self-similar solutions in the half-space for linear and semilinear heat equations with nonlinear boundary conditions. Existence, multiplicity and positivity of these solutions are analyzed. Self-similar profiles are obtained as solutions of a nonlinear elliptic PDE with drift term and a nonlinear Neumman boundary condition. For that, we employ a variational approach and derive some compact weighted embeddings for the trace operator.

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1 Introduction and main results

In the first part of this paper we consider the nonlinear problem

$$\begin{cases} v_t - \Delta v = 0, & \text{in } \mathbb{R}^N_+ \times (0,+\infty), \\ \dfrac{\partial v}{\partial \nu} = |v|^{p-2}v, & \text{on } \partial\mathbb{R}^N_+ \times (0,+\infty), \end{cases} \tag{1.1}$$

where $N \geq 3$, $2 < p < 2_* := 2(N-1)/(N-2)$ and $\frac{\partial u}{\partial \nu}$ denotes the partial outward normal derivative. Along all the paper the points $x \in \mathbb{R}^N$ will be written as $x = (x', x_N)$, with $x' \in \mathbb{R}^{N-1}$. The upper half-space is defined as $\mathbb{R}^N_+ = \{(x', x_N) : x_N > 0)\}$, and we identify $\partial\mathbb{R}^N_+ \simeq \mathbb{R}^{N-1}$.

In the last two decades, the problem (1.1) (and its variations) has been studied by many authors in bounded domains Ω or in the half-space $\mathbb{R}^N_+$; see, e.g., [5, 35, 33, 4, 23] and references therein. In these works, the reader can find several types of existence, uniqueness, blow-up or asymptotic behavior results in L^p-spaces for time-dependent solutions via parabolic approaches.

Here we are interested in another type of solutions and approach. We focus in self-similar solutions and investigate existence, multiplicity and positivity of solutions for (1.1) (and the second problem (1.6) below) via a variational approach for elliptic PDEs. In the forward case, self-similar solutions have the special form

$$v(x,t) = t^{-\mu} u(t^{-1/2}x), \tag{1.2}$$

for $\mu = 1/(2(p-2)) > 0$. A simple calculation shows that v is a solution for (1.1), if and only if, the profile u verifies

$$\begin{cases} -\Delta u - \dfrac{1}{2}(x \cdot \nabla u) = \mu u, & \text{in } \mathbb{R}^N_+, \\ \dfrac{\partial u}{\partial \nu} = |u|^{p-2}u, & \text{on } \mathbb{R}^{N-1}. \end{cases} \tag{P}$$

Besides providing qualitative properties like global existence, blow-up and asymptotic behavior (see e.g. [21, 25, 22]), self-similar solutions (or self-similar variables) are important because they preserve the PDE scaling and so carry simultaneously information about small and large scale behaviors.

Let us remark that, from a viewpoint of elliptic PDEs, the study of (P) has an interest of its own. In fact, the problem (P) is a counterpart in $\mathbb{R}^N_+$, with nonlinear boundary conditions, of the following PDE defined in the whole space

$$\mathcal{L}u := -\Delta u - \frac{1}{2}(x \cdot \nabla u) = \lambda u + f(u), \qquad x \in \mathbb{R}^N, \tag{1.3}$$

which is connected to self-similar solutions for

$$v_t - \Delta v = f(v) \text{ in } \mathbb{R}^N \times (0,\infty), \tag{1.4}$$

when $\lambda = 1/(p-2)$ and e.g. $f(v) = \pm|v|^{p-2}v$. In [21], Haraux and Weissler considered $f(v) = |v|^{p-2}v$ for $2 + 2/N < p < 2^* := 2N/(N-2)$. By using ODE techniques they obtained existence of radial solutions for (1.3) according to the value of λ (see also [6, 31, 38]). Still using ODE techniques, Brezis, Peletier and Terman [9] constructed a very singular solution for (1.4) with $f(v) = -|v|^{p-2}v$ and $1 < p < 2+2/N$ by obtaining radial solutions for (1.3) with the decay $|x|^{2/(p-2)}u(x) \to 0$ at infinity and $u'(0) = 0$ (see also [17]). An extension of this result for $f(x,v) = -|x|^{\beta}|v|^{p-2}v$ can be found in [37]. For $u(0) > 0$, $u'(0) = 0$ and $f(v) = |v|^{p-2}v$, Naito [28] obtained multiplicity of positive radial solutions for (1.3) depending on the size of $\lim_{r\to\infty} r^{2/(p-2)}u(r) > 0$.

In [10], Escobedo and Kavian presented a variational approach to deal with the operator $\mathcal{L}$. They considered a weighted Sobolev space and obtained some existence and multiplicity results for the same kind of nonlinearity f used in [21], and also for the critical case $p = 2^*$. The uniqueness of positive solutions given in [10] (in that weighted space) was proved by Naito and Suzuki [29] for $2 + 2/N < p < 2^*$. Thanks to the variational structure, it is possible to consider other variations of f (see [30, 16] and references therein). We also quote the paper of Escobedo and Zuazua [11], where a fixed point approach was used to treat a convection term on the nonlinearity. The problem (1.4) with $v(x,0) = |x|^{-2/(p-2)}$ and $f(v) = |v|^{p-2}v$ was studied by Galaktionov and Vazquez [18] who conjectured the existence of exactly two positive solutions for $2 + 2/N < p < 2^*$. After, Naito [27] considered (1.4) with the homogeneous initial data $v(x,0) = a(x/|x|)\,|x|^{-2/(p-2)}$, where $a \not\equiv 0$ and $0 \le a \in L^\infty(\mathbb{S}^{N-1})$, and gave a partially positive answer for that conjecture by studying solutions of (1.3) with the additional condition $|x|^{2/(p-2)}u(|x|\,\omega) \to a(\omega)$ as $|x| \to \infty$ for a.e. $\omega \in \mathbb{S}^{N-1}$ (see also [36, 26]). Another approach for looking for self-similar solutions, which goes back to Giga and Miyakawa [19], is to prove the well-posedness of the parabolic PDE (1.4) in scaling invariant spaces that contain the data $v(x,0) = a(x/|x|)\,|x|^{-2/(p-2)}$ where $a \in L^\infty(\mathbb{S}^{N-1})$ (see also [12]).

In order to deal with the operator $\mathcal{L}$ in the half-space, we first introduce the correct framework. Following [10], we take the exponential-type weight

$$K(x) = \exp\left(\frac{|x|^2}{4}\right)$$

and, using $\nabla K = \frac{1}{2}xK$, notice that the first equation in (P) becomes

$$-\mathrm{div}(K(x)\nabla u) = \mu K(x)u \text{ in } \mathbb{R}^N_+.$$

Thus, as in the whole space case, the problem has a variational structure and we can define the space $\mathcal{D}^{1,2}_K(\mathbb{R}^N_+)$ as being the closure of $C^\infty_c(\overline{\mathbb{R}^N_+})$ with respect to the norm

$$\|u\|_{\mathcal{D}^{1,2}_K(\mathbb{R}^N_+)} = \left(\int_{\mathbb{R}^N_+} K(x)|\nabla u|^2 \mathrm{d}x\right)^{1/2}.$$

We are going to show (see Section 2) that this space has good properties of embeddings in weighted Lebesgue spaces, including compact trace ones. It will allow to control the nonlinear boundary term in (P). Hence, using Critical Point Theory we are able to prove the following result.

Theorem 1.1. *If $2 < p < 2_*$ and $\mu < (N/2)$, then the problem (P) has a positive solution.*

Recalling that self-similar solutions of (1.1) are obtained when $\mu = 1/(2(p-2))$, we can apply the above theorem to obtain the following existence result.

Corollary 1.2. *The problem (1.1) has a positive self-similar solutions provided*

$$2 + \frac{1}{N} < p < 2\frac{(N-1)}{(N-2)}. \tag{1.5}$$

In the proof of Theorem 1.1 we use the classical Mountain Pass Theorem. We do not know if the number $N/2$ is optimal to the existence of positive solution for (P). It is the first eigenvalue of the associated linear problem (see Section 2). In some sense, this first eigenvalue plays the same role of that in the problem settled in the whole $\mathbb{R}^N$ (see [10, Theorem 0.14]).

It is worthwhile to mention that the embedding results proved in [10] seem not to be suffice to deal with the problem (P). Even the variational arguments performed there do not work here. Actually, in order to correctly establish our framework, we need to obtain some weighted embeddings which are proved by adapting some classical techniques of Sobolev spaces and using an interpolation argument (see Lemmas 2.2 and 2.4 for details). These embeddings, specially the trace one, could be applied to other kinds of problems in half-spaces. They complement previous weighted embedding results which can be found, for instance, in [2, 14, 15, 24, 32, 3, 20] and references therein. For example, the growth of our weight $K(x)$ is not of log or polynomial type. Indeed, it does not belong to the Muckenhoupt class A_r.

In the second part of the paper we consider (P) with an additional power-type reaction term inside the domain

$$\begin{cases} -\Delta u - \dfrac{1}{2}(x \cdot \nabla u) = \mu u + |u|^{q-2}u, & \text{in } \mathbb{R}^N_+, \\ \dfrac{\partial u}{\partial \nu} = |u|^{p-2}u, & \text{on } \mathbb{R}^{N-1}, \end{cases} \tag{$\widetilde{P}$}$$

for $2 < p < 2_*$ and $2 < q < 2^*$. For $q = (2p-2)$, considering $v(x,t) = t^{-\mu}u(t^{-1/2}x)$ with $\mu = \frac{1}{2(p-2)} = \frac{1}{q-2}$, the problem $(\widetilde{P})$ provides self-similar solutions for

$$\begin{cases} v_t - \Delta v = |v|^{q-2}v, & \text{in } \mathbb{R}^N_+ \times (0,+\infty), \\ \dfrac{\partial v}{\partial \nu} = |v|^{p-2}v, & \text{on } \mathbb{R}^{N-1} \times (0,+\infty), \end{cases} \tag{1.6}$$

which is doubly nonlinear with a reaction term inside the domain and another one acting on the boundary. As well as (1.1), the problem (1.6) has been object of interest in the PDEs literature (see e.g. [5, 35, 4]). Thanks to the extra term $|u|^{q-2}u$ and the symmetry of the problem, we are able to obtain multiple self-similar solutions. Actually, we complement the multiplicity results in [10, 38] by proving the following theorem.

Theorem 1.3. *Suppose that $2 < p < 2_*$ and $2 < q < 2^*$. Then there exists a countable set $\Gamma \subset \mathbb{R}$ such that the problem $(\widetilde{P})$ has infinitely many solutions provided $\mu \notin \Gamma$. Consequently, for $2 < p < 2_*$ and $q = (2p-2)$ such that $\frac{1}{2(p-2)} \notin \Gamma$, we obtain infinitely many self-similar solutions for (1.6).*

In the proof of Theorem 1.3 we apply a variant of the Symmetric Mountain Pass Theorem. We notice that, as in Theorem 1.1, the range of applicability for the parameter μ is related with the eigenvalues of the linear problem. Even the nonlinearity being superlinear, it seems that there exists some resonant effect when we take the number μ in the set Γ. The restriction appears when we try to prove that almost critical point sequences are bounded. It could be interesting to study how we can recover compactness in the case $\mu \in \Gamma$. Also, we point out that the condition $q = (2p-2)$ (necessary to self-similarity) maps injectively the boundary subcritical range $(2, 2_*)$ onto the inside one $(2, 2^*)$. This provides a link between self-similarity, variational methods and compactness in problems with multiple nonlinearities.

We notice that we do not have information about the sign of the solutions given by Theorem 1.3. It is natural to ask if you can argue as in the first part of the paper and obtain positive solutions. In our last result, we shall present a positive answer to this question.

Theorem 1.4. *If $2 < p < 2_*$, $2 < q < 2^*$ and $\mu < (N/2)$, then the problem $(\widetilde{P})$ has a positive solution. In particular, the problem (1.6) has a positive self-similar solution when $2 + \frac{1}{N} < p < 2_*$ and $q = (2p-2)$.*

As commented before, the existence of self-similar solutions provides some qualitative properties for PDEs. For instance, concerning non-uniqueness, we have the following: the critical L^r-space for the problems (1.1) and (1.6) is that with index $r_0 = \frac{N}{2\mu} = N(p-2) = \frac{N(q-2)}{2} < 2^*$. The solutions u of (P) and $(\widetilde{P})$ belong to $H^1_K(\mathbb{R}^N_+) \subset L^{\widetilde{r}}_K(\mathbb{R}^N_+)$ with $2 \leq \widetilde{r} \leq 2^*$, and then $u \in L^{\widetilde{r}}_K(\mathbb{R}^N_+) \subset L^r(\mathbb{R}^N_+)$ for all $1 \leq r \leq 2^*$. In view of Corollary 1.2 and Theorem 1.4, for $1 \leq r < r_0$, we obtain that $v(x,t) = t^{-\mu}u(t^{-1/2}x) \in BC([0,T); L^r(\mathbb{R}^N_+))$ is positive with

$$\|v(\cdot,t)\|_{L^r} = t^{-\frac{1}{2(p-2)}+\frac{N}{2r}} \|u\|_{L^r} \to 0, \text{ as } t \to 0^+,$$

where $BC(I;X)$ stands for bounded continuous functions from I to X. Since $v_2 \equiv 0$ also is solution, we obtain non-uniqueness of non-negative solutions in $BC([0,T); L^r(\mathbb{R}^N_+))$ for the problems (1.1) and (1.6). In fact, allowing solutions change their signs, for $2 < p < 2_*$ with $\frac{1}{2(p-2)} \notin \Gamma$, one obtains infinitely many solutions in $BC([0,T); L^r(\mathbb{R}^N_+))$ for (1.6).

We finally mention that, fixed $T > 0$, one can replace t by $(T - t)$ in (1.2) and consider the backward self-similar solution

$$v(x,t) = (T-t)^{-\mu} u((T-t)^{-1/2}x), \text{ for all } 0 \le t < T.$$

Then v is a solution of the first equation in (1.1) if and only if u satisfies

$$-\Delta u + \frac{1}{2}(x \cdot \nabla u) - \mu u = 0, \qquad \text{in } \mathbb{R}^N_+. \tag{1.7}$$

If this is the case for $u \in L^r(\mathbb{R}^N_+)$ with $r_0 < r \le 2^*$, it follows that $v(x,t) \in C([0,T); L^r(\mathbb{R}^N_+))$ and

$$\|v(\cdot,t)\|_{L^r} = (T-t)^{-\frac{1}{2(p-2)} + \frac{N}{2r}} \|u\|_{L^r} \to \infty, \text{ as } t \to T^-,$$

which shows existence of L^r-solutions that blow-up. The equation in (1.7) can be written as

$$-\text{div}(\widetilde{K}(x)\nabla u) = -\mu \widetilde{K}(x) u, \qquad \text{in } \mathbb{R}^N_+,$$

where $\widetilde{K}(x) = \exp(-|x|^2/4)$. Unfortunately, the calculations performed here are not enough to consider this case, since this weight strongly decay to zero at infinity. For the one dimensional case of (1.7), we refer to [13, pg.202].

The paper is organized as follows. Section 2 is devoted to establish the variational framework and prove the trace type embeddings involving weighted Sobolev spaces. The analysis of the problem (P) is performed in Section 3. The last section is devoted to the problem $(\widetilde{P})$.

2 The variational setting

For any $\Omega \subset \mathbb{R}^N$ we denote by $\mathcal{D}^{1,2}_K(\Omega)$ the closure of $C^\infty_c(\overline{\Omega})$ with respect to the norm

$$\|u\|_{\mathcal{D}^{1,2}_K(\Omega)} = \left(\int_\Omega K(x) |\nabla u|^2 \right)^{1/2}.$$

For $s \ge 2$, we also denote by $L^s_K(\Omega)$ the weighted Lebesgue space

$$L^s_K(\Omega) = \left\{ u \in L^s(\Omega) : \int_\Omega K(x)|u|^s dx < \infty \right\}.$$

Analogously, $H^s_K(\Omega)$ stands for the Sobolev space $H^1(\Omega)$ with weight $K(x)$.

When $\Omega = \mathbb{R}^N$, we have the following embedding result proved by Escobedo and Kavian [10].

Lemma 2.1. *There holds $\mathcal{D}^{1,2}_K(\mathbb{R}^N) \hookrightarrow L^s_K(\mathbb{R}^N)$ for any $2 \le s \le 2^*$. Moreover, the embedding is compact if $2 \le s < 2^*$.*

For save notation we denote $\mathcal{D}^{1,2}_K(\mathbb{R}^N_+)$ by X and $\|\cdot\|_{\mathcal{D}^{1,2}_K(\Omega)}$ by $\|\cdot\|$. In order to prove our embedding results, we define an extension of the function $u \in X$ to the whole $\mathbb{R}^N$ by setting

$$\bar{u}(x', x_N) = \begin{cases} u(x', x_N), & \text{if } x_N > 0, \\ -2u(x', -2x_N) + 3u(x', -x_N), & \text{if } x_N \leq 0, \end{cases} \tag{2.1}$$

for each $x = (x_1, x_2, \ldots, x_N) = (x', x_N) \in \mathbb{R}^N$. Notice that, if $x_N > 0$, then $\nabla\bar{u}(x) = \nabla u(x)$. On the other hand, if $x_N < 0$, we have that

$$\bar{u}_{x_i}(x', x_N) = \begin{cases} -2u_{x_i}(x', -2x_N) + 3u_{x_i}(x', -x_N), & \text{if } 1 \leq i \leq N-1, \\ 4u_{x_i}(x', -2x_N) - 3u_{x_i}(x', -x_N), & \text{if } i = N, \end{cases} \tag{2.2}$$

which shows that the extension (2.1) preserves C^1-regularity.

Lemma 2.2. *The embedding $X \hookrightarrow L^s_K(\mathbb{R}^N_+)$ is continuous for any $s \in [2, 2^*]$. Moreover, the embedding is compact if $s \in [2, 2^*)$.*

Proof. Given $u \in X$, we first prove that $\bar{u} \in \mathcal{D}^{1,2}_K(\mathbb{R}^N)$. Indeed, we have that

$$\int_{\mathbb{R}^N} K(x)|\nabla\bar{u}|^2 dx = \int_{\mathbb{R}^N_+} K(x)|\nabla u|^2 + \int_{\mathbb{R}^N_-} K(x)|\nabla\bar{u}|^2 dx,$$

where $\mathbb{R}^N_- = \{(x', x_N) : x_N < 0\}$. According to (2.2), we can estimate the last integral in the following way

$$\int_{\mathbb{R}^N_-} K(x)|\nabla\bar{u}|^2 dx \leq c_1 \left\{ \sum_{i=1}^N \int_{\mathbb{R}^N_-} u^2_{x_i}(x', -2x_N) e^{|x|^2/4} dx' dx_N \right.$$
$$\left. + \sum_{i=1}^N \int_{\mathbb{R}^N_-} u^2_{x_i}(x', -x_N) e^{|x|^2/4} dx' dx_N \right\}.$$

Since the weight $\exp(|x|^2/4)$ is radial, the change of variables $(x', x_N) \mapsto (x', -x_N)$ in the last integral provides

$$\int_{\mathbb{R}^N_-} K(x)|\nabla\bar{u}|^2 dx \leq c_1 \left\{ \sum_{i=1}^N \int_{\mathbb{R}^N_-} u^2_{x_i}(x', -2x_N) e^{|x|^2/4} dx' dx_N \right.$$
$$\left. + \int_{\mathbb{R}^N_+} K(x)|\nabla u|^2 dx \right\}$$

The weight is also increasing in the radial direction, and therefore

$$\int_{\mathbb{R}^N_-} u^2_{x_i}(x', -2x_N) e^{|x|^2/4} dx' dx_N \leq \int_{\mathbb{R}^N_-} u^2_{x_i}(x', -2x_N) e^{|(x', -2x_N)|^2/4} dx' dx_N.$$

The changing $(x', x_N) \mapsto (x', -2x_N)$ and the above inequalities imply that

$$\int_{\mathbb{R}^N} K(x)|\nabla \bar{u}|^2 dx \le c_2 \int_{\mathbb{R}^N_+} K(x)|\nabla u|^2 dx, \tag{2.3}$$

and therefore $\bar{u} \in \mathcal{D}^{1,2}_K(\mathbb{R}^N)$.

Using (2.3) and Lemma 2.1 we get, for any $s \in [2, 2^*]$,

$$\int_{\mathbb{R}^N_+} K(x)|u|^s dx \le \int_{\mathbb{R}^N} K(x)|\bar{u}|^s dx \le c_3 \int_{\mathbb{R}^N} K(x)|\nabla \bar{u}|^2 dx \le c_4 \int_{\mathbb{R}^N_+} K(x)|\nabla u|^2 dx,$$

and therefore the embedding $X \hookrightarrow L^s_K(\mathbb{R}^N_+)$ is continuous.

Let $2 < s < 2^*$ and take $\theta \in (0,1)$ such that $s = 2\theta + 2^*(1-\theta)$. Hölder's inequality provides

$$\begin{aligned} \int_{\mathbb{R}^N_+} K(x)|u|^s dx &= \int_{\mathbb{R}^N_+} K(x)^\theta |u|^{2\theta} K(x)^{1-\theta} |u|^{2^*(1-\theta)} dx \\ &\le \left(\int_{\mathbb{R}^N_+} K(x)|u|^2 dx \right)^\theta \left(\int_{\mathbb{R}^N_+} K(x)|u|^{2^*} dx \right)^{1-\theta}. \end{aligned}$$

Hence, we have that

$$\|u\|_{L^s_K(\mathbb{R}^N_+)} \le \|u\|^{2\theta/s}_{L^2_K(\mathbb{R}^N_+)} \|u\|^{2^*(1-\theta)/s}.$$

The above inequality shows that, in order to prove the compactness part of the lemma, it is sufficient to consider the case $s = 2$.

If $(u_n) \subset X$ is bounded, then $(\bar{u}_n)$ is bounded in $\mathcal{D}^{1,2}_K(\mathbb{R}^N)$. It follows from Lemma 2.1 that, along a subsequence, $\bar{u}_n \to \bar{u}$ strongly in $L^2_K(\mathbb{R}^N)$. If we denote by u the restriction of $\bar{u}$ to $\mathbb{R}^N_+$ we get

$$\int_{\mathbb{R}^N_+} K(x)|u_n - u|^2 dx \le \int_{\mathbb{R}^N} K(x)|\bar{u}_n - \bar{u}|^2 dx \to 0,$$

as $n \to +\infty$. Hence, the embedding of X into $L^2_K(\mathbb{R}^N_+)$ is compact.

■

Remark 2.3. *As a byproduct of the above proof, we obtain the following: there exists* $c_1, c_2 > 0$ *such that,*

$$\|\bar{u}\|_{H^1_K(\mathbb{R}^N)} \le c_1 \|u\|_{H^1_K(\mathbb{R}^N_+)} \le c_2 \|u\|,$$

for any $u \in X$.

In our next result we prove a trace embedding result for the space $X = \mathcal{D}^{1,2}_K(\mathbb{R}^N_+)$.

Lemma 2.4. *We have the compact embedding* $X \hookrightarrow L^q_K(\mathbb{R}^{N-1})$ *for any* $2 < q < 2(N-1)/(N-2)$.

Proof. Given $u \in X$, we have that $u \in H^1(\mathbb{R}^N_+)$, since $K(x) \geq 1$. Arguing as in the proof of the previous lemma, we can conclude that $\bar{u} \in H^1(\mathbb{R}^N)$, and therefore $\bar{u}$ belongs to the fractional Sobolev space $H^s(\mathbb{R}^N)$ for any $0 < s < 1$ (see [1, Theorem 7.63]). According to [1, Theorem 7.58], the space $H^s(\mathbb{R}^N)$ embeds in $L^t(\mathbb{R}^{N-1})$ whenever $2 \leq t \leq q^*(s) = 2(N-1)/(N-2s)$. If we choose s sufficiently close to 1, we have that $2 < q < q^*(s)$, and therefore we obtain $H^s(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^{N-1})$.

We now set $\bar{v} := \bar{u}\exp(|x|^2/(4q))$. Since $u \equiv \bar{u}$ in $\{x_N = 0\}$, we have that

$$\|u\|_{L^q_K(\mathbb{R}^{N-1})} = \|\bar{u}\|_{L^q_K(\mathbb{R}^{N-1})} = \|\bar{v}\|_{L^q(\mathbb{R}^{N-1})} \leq \|\bar{v}\|_{H^s(\mathbb{R}^N)}$$

The interpolation inequality presented in [8, Theorem 6.4.5] provides $c_1 > 0$ such that

$$\|u\|_{L^q_K(\mathbb{R}^{N-1})} \leq c_1 \|\bar{v}\|^s_{H^1(\mathbb{R}^N)} \|\bar{v}\|^{1-s}_{L^2(\mathbb{R}^N)}. \tag{2.4}$$

In what follows we shall estimate the two norms on the right-hand side above. A direct computation provides

$$\nabla\bar{v} = \exp(|x|^2/(4q))\left(\nabla\bar{u} + \bar{u}\frac{x}{2q}\right)$$

for a.e. $x \in \mathbb{R}^N$. Hence, recalling that $2q \geq 4$, we get

$$\begin{aligned} |\nabla\bar{v}|^2 &\leq c_2 \exp(|x|^2/(2q))\left(|\nabla\bar{u}|^2 + \frac{|\bar{u}|^2|x|^2}{4q^2}\right) \\ &\leq c_2\left(|\nabla\bar{u}|^2\exp(|x|^2/4) + \frac{|\bar{u}|^2|x|^2}{4q^2}\exp(|x|^2/(2q))\right). \end{aligned}$$

Since we also have $|\bar{v}(x)|^2 \leq |\bar{u}(x)|^2\exp(|x|^2/4)$, almost everywhere in $\mathbb{R}^N_+$, we conclude that

$$\|\bar{v}\|_{H^1(\mathbb{R}^N)} \leq c_3\left\{\|\bar{u}\|_{H^1_K(\mathbb{R}^N)} + \left\|\bar{u}\frac{|x|}{2q}\exp(|x|^2/(4q))\right\|_{L^2(\mathbb{R}^N)}\right\}. \tag{2.5}$$

For the last norm in (2.5) we have that

$$\left\|\bar{u}\frac{|x|}{2q}\exp(|x|^2/(4q))\right\|^2_{L^2(\mathbb{R}^N)} = \frac{1}{4q^2}\int_{\mathbb{R}^N} a(x)K(x)|\bar{u}|^2 dx, \tag{2.6}$$

where

$$a(x) := |x|^2\exp\left(\left(\frac{1}{2q} - \frac{1}{4}\right)|x|^2\right).$$

Since $q > 2$, the function a belongs to $L^\infty(\mathbb{R}^N)$. Thus, we infer from (2.5) and (2.6) that

$$\|\bar{v}\|_{H^1(\mathbb{R}^N)} \leq c_3\left\{\|\bar{u}\|_{H^1_K(\mathbb{R}^N)} + \|\bar{u}\|_{L^2_K(\mathbb{R}^N)}\right\} \leq c_4\|\bar{u}\|_{H^1_K(\mathbb{R}^N)}. \tag{2.7}$$

By arguing as above we also conclude that

$$\|\bar{v}\|_{L^2(\mathbb{R}^N)} \leq c_5\|\bar{u}\|_{L^2_K(\mathbb{R}^N)}. \tag{2.8}$$

These two last inequalities, (2.4) and Remark 2.3 imply that

$$\|u\|_{L^q_K(\mathbb{R}^{N-1})} \le c_6 \|\bar{u}\|_{H^1_K(\mathbb{R}^N)} \le c_7 \|u\|.$$

Hence, we have the continuous embedding $X \hookrightarrow L^q_K(\mathbb{R}^{N-1})$.

In order to prove the compactness of the embedding we take a bounded sequence $(u_n) \subset X$. Then $(\bar{u}_n)$ is bounded in $\mathcal{D}^{1,2}_K(\mathbb{R}^N)$ and therefore, by Lemma 2.1, along a subsequence $\bar{u}_n \to \bar{u}$ strongly in $L^2_K(\mathbb{R}^N)$. If we call u the restriction of $\bar{u}$ to the set $\mathbb{R}^N_+$, we can use (2.7), (2.8) and (2.4) to get

$$\|u_n - u\|_{L^q_K(\mathbb{R}^{N-1})} \le c_8 \|\bar{u}_n - \bar{u}\|^s_{H^1_K(\mathbb{R}^N)} \|\bar{u}_n - \bar{u}\|^{1-s}_{L^2_K(\mathbb{R}^N)}.$$

But Lemma 2.1 implies that $\|\cdot\|_{H^1_K(\mathbb{R}^N)}$ and $\|\cdot\|_{\mathcal{D}^{1,2}_K(\mathbb{R}^N)}$ are equivalent. Hence, it follows from the above inequality and the strong convergence in $L^2_K(\mathbb{R}^N)$ that $u_n \to u$ strongly in $L^q_K(\mathbb{R}^{N-1})$, as required.

■

We finish this section with some important remarks concerning the linear problem associated to (P), namely the eigenvalue problem

$$\begin{cases} -\mathrm{div}(K(x)\nabla u) = \lambda K(x) u, & \text{in } \mathbb{R}^N_+, \\ \dfrac{\partial u}{\partial \nu} = 0, & \text{on } \mathbb{R}^{N-1}. \end{cases} \qquad (LP)$$

Since the constant functions do not belong to X, any eigenvalue is positive. The weak formulation of (P) reads as

$$\int_{\mathbb{R}^N_+} K(x)(\nabla u \cdot \nabla v) dx = \lambda \int_{\mathbb{R}^N_+} K(x) uv \, dx.$$

Hence, $\lambda > 0$ is an eigenvalue if, and only if, λ^{-1} is an eigenvalue of the bounded linear operator $T : X \to X$ given by

$$\langle Tu, v \rangle := \int_{\mathbb{R}^N_+} K(x) uv \, dx, \quad \forall\, u, \, v \in X.$$

Invoking Lemma 2.2 we have that T is an compact operator, and therefore it follows from the spectral theory of compact operators that this problem has a sequence of eigenvalues $(\lambda_j)_{j \in \mathbb{N}}$ such that

$$0 < \lambda_1 < \lambda_2 \le \cdots \le \lambda_j \le \cdots$$

with $\lim_{j \to +\infty} \lambda_j = +\infty$. Moreover, we have the following characterization

$$\frac{N}{2} = \lambda_1 = \inf_{u \in X \setminus \{0\}} \frac{\int_{\mathbb{R}^N_+} K(x) |\nabla u|^2 dx}{\int_{\mathbb{R}^N_+} K(x) u^2 dx}. \qquad (2.9)$$

Actually, the second equality in (2.9) follows from the spectral theory of compact operator in Hilbert spaces, while the first one will be proved in the sequel.

Let φ_1 be an eigenfunction of (LP) associated to λ_1, and denote by $\psi \in H^1_K(\mathbb{R}^N)$ its extension given by $\psi \equiv \varphi$ in $\mathbb{R}^N_+$, and $\psi(x', x_N) = \varphi_1(x', -x_N)$ if $x_N < 0$. Since the weight K is radial, changing variables we get

$$\begin{aligned}\int_{\mathbb{R}^N} K(x)|\nabla\psi|^2 dx &= \int_{\mathbb{R}^N_+} K(x)|\nabla\varphi_1|^2 dx + \int_{\mathbb{R}^N_-} K(x)|\nabla\varphi_1(x', -x_N)|^2 dx \\ &= 2\int_{\mathbb{R}^N_+} K(x)|\nabla\varphi_1|^2 dx,\end{aligned}$$

with an analogous equality holding for the integral $\int_{\mathbb{R}^N} K(x)\psi^2 dx$. It is proved in [10, Proposition 2.3] that the first eigenvalue of the linear PDE in (LP) settled in $\mathbb{R}^N$ is equal to $N/2$. Hence, using the characterization of first eigenvalue, we obtain

$$\frac{N}{2} \le \frac{\int_{\mathbb{R}^N} K(x)|\nabla\psi|^2 dx}{\int_{\mathbb{R}^N} K(x)\psi^2 dx} = \frac{\int_{\mathbb{R}^N_+} K(x)|\nabla\varphi_1|^2 dx}{\int_{\mathbb{R}^N_+} K(x)\varphi_1^2 dx} = \lambda_1,$$

and therefore $N/2 \le \lambda_1$. On the other hand, the first eigenfunction of the problem in the whole space $\mathbb{R}^N$ is $e^{-|x|^2/4}$. Since it is radial, we can use a similar argument to obtain the reverse inequality $N/2 \ge \lambda_1$.

3 The problem (P)

In this section we present the proof of the existence results for our first problem. Recall the notations $X = \mathcal{D}^{1,2}_K(\mathbb{R}^N_+)$ and $\|\cdot\| = \|\cdot\|_{\mathcal{D}^{1,2}_K(\mathbb{R}^N_+)}$. We call $u \in X$ a weak solution of the problem (P) if

$$\int_{\mathbb{R}^N_+} K(x)(\nabla u \cdot \nabla v) - \mu \int_{\mathbb{R}^N_+} K(x)uv = \int_{\mathbb{R}^{N-1}} K(x', 0)|u|^{p-2}uv, \quad \forall\, v \in X.$$

With this definition, the critical points of the energy functional $I : X \to \mathbb{R}$ given by

$$I(u) := \frac{1}{2}\int_{\mathbb{R}^N_+} K(x)|\nabla u|^2 dx - \frac{\mu}{2}\int_{\mathbb{R}^N_+} K(x)u^2 dx - \frac{1}{p}\int_{\mathbb{R}^{N-1}} K(x', 0)(u^+)^p dx',$$

with $u^+(x) := \max\{u(x), 0\}$, are weak solutions of (P).

Since we are intending to use Critical Point Theory, we need to prove some compactness properties for the functional I. We recall that $(u_n) \subset X$ is a Cerami sequence at level $c \in \mathbb{R}$, if

$$I(u_n) \to c, \qquad \|I'(u_n)\|_{X'}(1 + \|u_n\|) \to 0. \tag{3.1}$$

We say that I satisfies the Cerami condition at level $c \in \mathbb{R}$ if any such sequence has a convergent subsequence.

Lemma 3.1. *If $\mu < (N/2)$, then the functional I satisfies the Cerami condition at any level.*

Proof. Suppose that $(u_n) \subset X$ verifies (3.1). A simple computation provides

$$c + o(1) = I(u_n) - \frac{1}{2}I'(u_n)u_n = \left(\frac{1}{2} - \frac{1}{p}\right)\|u_n^+\|^p_{L^p_K(\mathbb{R}^{N-1})},$$

and therefore $\|u_n^+\|_{L^p_K(\mathbb{R}^{N-1})} \le M_1$, for some $M_1 > 0$.

Suppose, by contradiction, that $\|u_n\| \to +\infty$. Setting $v_n := u_n/\|u_n\|$ we may assume that, for some function $v \in X$, there hold $v_n \rightharpoonup v$ weakly in X and $v_n \to v$ strongly in $L^2_K(\mathbb{R}^N_+)$.

For any $\varphi \in X$, we have that

$$\frac{1}{\|u_n\|}I'(u_n)\varphi = \int_{\mathbb{R}^N_+} K(x)(\nabla v_n \cdot \nabla\varphi)dx - \mu\int_{\mathbb{R}^N_+} K(x)v_n\varphi dx - \frac{1}{\|u_n\|}C_n, \tag{3.2}$$

where

$$C_n := \int_{\mathbb{R}^{N-1}} K(x',0)(u_n^+)^{p-1}\varphi dx'.$$

The boundedness of $(\|u_n\|_{L^p_K(\mathbb{R}^{N-1})})$ and Holder's inequality imply that C_n is bounded. Hence, taking the limit in (3.2) we conclude that

$$\int_{\mathbb{R}^N_+} K(x)(\nabla v \cdot \nabla\varphi)dx = \mu\int_{\mathbb{R}^N_+} K(x)v\varphi dx,$$

for any $\varphi \in X$. Since $\mu < (N/2) = \lambda_1$, the number μ is not an eigenvalue of (LP), and therefore $v = 0$.

On the other hand,

$$o(1) = \frac{1}{\|u_n\|^2}I'(u_n)u_n = 1 - \mu\|v_n\|_{L^2_K(\mathbb{R}^N_+)} - \frac{1}{\|u_n\|^2}\|u_n^+\|^p_{L^p_K(\mathbb{R}^{N-1})}.$$

Since $v_n \to 0$ in $L^2_K(\mathbb{R}^N_+)$ and the last term above also goes to zero, we obtain a contradiction taking $n \to +\infty$. Hence, the sequence (u_n) is bounded in X.

Since (u_n) is bounded in X, up to a subsequence, $u_n \rightharpoonup u$ weakly in X. Thus,

$$o(1) = I'(u_n)(u_n - u) = \|u_n\|^2 - \|u\|^2 - \mu A_n - B_n + o(1), \tag{3.3}$$

with

$$A_n := \int_{\mathbb{R}^N_+} K(x)u_n(u_n - u)dx, \quad B_n := \int_{\mathbb{R}^{N-1}} K(x',0)(u_n^+)^{p-1}(u_n - u)dx'.$$

By Lemma 2.4 we have that, up to a subsequence, $u_n \to u$ strongly in $L^p_K(\mathbb{R}^{N-1})$. Thus, we can use Holder's inequality to get

$$\begin{aligned} |B_n| &\le \int_{\mathbb{R}^{N-1}} K(x',0)^{(p-1)/p}|u_n|^{p-1}K(x',0)^{1/p}|u_n - u|dx' \\ &\le \|u_n\|_{L^p_K(\mathbb{R}^{N-1})}\|u_n - u\|_{L^p_K(\mathbb{R}^{N-1})}, \end{aligned}$$

and therefore $B_n \to 0$. Lemma 2.2 and an easier argument provide $A_n \to 0$. Hence, we infer from (3.3) that $\|u_n\|^2 \to \|u\|^2$. This is equivalent to $u_n \to u$ strongly in X and the lemma is proved.

■

We finish this section with the proof of our first main result.

Proof of Theorem 1.1. We first check that I satisfies the Mountain Pass Geometry, namely

(i) there exist α, $\rho > 0$ such that $I(u) \geq \alpha$ for any $\|u\| = \rho$, and

(ii) there exist $e \in X$ such that $\|e\| > \rho$ and $I(e) < 0$.

Indeed, it follows from Lemma 2.4 that $\|u\|_{L^p_K(\mathbb{R}^{N-1})} \leq c_1\|u\|$ for some $c_1 > 0$ and any $u \in X$. Hence, using (2.9) and arguing as in the proof of Lemma 3.1, we get

$$I(u) \geq \frac{1}{2}\left(1 - \frac{\mu^+}{\lambda_1}\right)\|u\|^2 - c_1\|u\|^p = \|u\|^p\left(\beta_0\|u\|^{2-p} - c_1\right),$$

for $\beta_0 := (\lambda_1 - \mu^+)/(2\lambda_1) > 0$. Thus, item (i) holds for $\rho := [(1 + c_1)/\beta_0]^{1/(2-p)}$ and $\alpha := \rho^p$. For any $u \in X$ such that $\|u^+\|_{L^p_K(\mathbb{R}^{N-1})} \neq 0$ we have that

$$I(tu) = \frac{t^2}{2}\|u\|^2 - \mu\frac{t^2}{2}\|u\|^2_{L^2(\mathbb{R}^N_+)} - \frac{t^p}{p}\|u^+\|_{L^p_K(\mathbb{R}^{N-1})}.$$

Since $p > 2$ we conclude that $I(tu) \to -\infty$ as $t \to +\infty$. Hence, we can set $e := tu$, with t large, to get (ii).

Since $\mu < (N/2)$ the functional I satisfies the Cerami condition at any level. Hence, it follows from the Mountain Pass Theorem that I possesses a critical point $u \in X\setminus\{0\}$. As quoted before, u is a weak solution of (P), and therefore it remains to show that it is positive.

If we set $u^-(x) := \max\{-u(x), 0\}$, we have that

$$0 = I'(u)(u^-) = -\|u^-\|^2 + \mu\int_{\mathbb{R}^N_+} K(x)(u^-)^2 \leq -\left(1 - \frac{\mu^+}{\lambda_1}\right)\|u^-\|^2.$$

Hence $u \geq 0$ a.e. in $\mathbb{R}^N_+$. It follows from the Maximum Principle that $u > 0$ in $\mathbb{R}^N_+$.

■

4 The problem $(\widetilde{P})$

In order to obtain multiple solutions for $(\widetilde{P})$ we change the definition of the functional I, by setting

$$I(u) := \frac{1}{2}\|u\|^2 - \frac{\mu}{2}\|u\|^2_{L^2_K(\mathbb{R}^N_+)} - \frac{1}{p}\|u\|^p_{L^p_K(\mathbb{R}^{N-1})} - \frac{1}{q}\|u\|^q_{L^q_K(\mathbb{R}^N_+)}.$$

We shall verify that it possesses infinitely many critical points.

If $\mu > \lambda_1$, we take $j \in \mathbb{N}$ in such way that $\lambda_j < \mu < \lambda_{j+1}$, and set

$$V := \text{span}\{\varphi_1, \ldots, \varphi_j\}, \qquad W := V^{\perp},$$

where φ_k is a normalized λ_k-eigenfunction of the linear problem (LP). If $\mu < \lambda_1$ we just define $V := \{0\}$ and $W := X$.

Lemma 4.1. *There hold*

(i) there are constants ρ, $\alpha > 0$ such that $I|_{\partial B_\rho(0) \cap W} \geq \alpha$, and

(ii) for each finite dimensional subspace $\widetilde{X} \subset X$, there is an $R = R(\widetilde{X})$ such that $I \leq 0$ on $\widetilde{X} \setminus B_R(0)$.

Proof. We first recall that, in view of the variational characterization of the eigenvalue λ_{j+1}, we have that

$$\|u\|^2 \geq \lambda_{j+1} \int_{\mathbb{R}^N_+} K(x) u^2 dx, \qquad \forall\, u \in W.$$

Hence, we can proceed as in the proof of Theorem 1.1 to get

$$I(u) \geq \frac{1}{2}\left(1 - \frac{\mu^+}{\lambda_{j+1}}\right)\|u\|^2 - c_1\|u\|^p - c_2\|u\|^q, \qquad \forall\, u \in W.$$

If we set $\beta_0 := (\lambda_{j+1} - \mu^+)/(2\lambda_{j+1}) > 0$, $\theta := \max\{p, q\} > 2$ and $c_3 := c_1 + c_2 > 0$, we obtain

$$I(u) \geq \|u\|^\theta(\beta_0\|u\|^{2-\theta} - c_3), \qquad \forall\, u \in W \cap B_1(0),$$

and therefore the item (i) holds for $\rho := \min\{[(1 + c_3)/\beta_0]^{1/(2-\theta)}, 1\}$ and $\alpha := \rho^\theta$.

The verification of (ii) is easy. Indeed, suppose that $\widetilde{X}$ is a finite dimensional subspace of X. Since all norms in $\widetilde{X}$ are equivalent, we can argue as above to get

$$I(u) \leq \frac{1}{2}\|u\|^2 - c_4\|u\|^q, \qquad \forall\, u \in \widetilde{X}.$$

Thus, $I(u) \to -\infty$ as $\|u\| \to \infty$, $u \in \widetilde{X}$, and (ii) follows. ∎

In what follows we prove our multiplicity result.

Proof of Theorem 1.3. Arguing along the same lines of the proof of Lemma 3.1, we can check that I satisfies the Cerami condition if μ is not an eigenvalue of (LP). Indeed, with the same notation of that proof, we see that this fact enable us to conclude that the weak limit of the normalization of the Cerami sequence is zero. Thus, it suffices to repeat the argument recalling that, by Lemma 2.2, the embedding $X \hookrightarrow L^q_K(\mathbb{R}^N_+)$ is compact.

We have that I satisfies the Cerami condition, is even and $I(0) = 0$. Hence, in view of the geometric conditions given by Lemma 4.1, we may invoke the Symmetric Mountain Pass Theorem [34, Theorem 9.12] (see [7] for the deformation lemma with Cerami instead of Palais-Smale condition) to obtain infinitely many critical points for I. This concludes the proof.

■

Proof of Theorem 1.4. Arguing as in the above proof, we can check that the functional

$$u \mapsto \frac{1}{2}\|u\|^2 - \frac{\mu}{2}\|u\|^2_{L^2_K(\mathbb{R}^N_+)} - \frac{1}{p}\|u^+\|^p_{L^p_K(\mathbb{R}^{N-1})} - \frac{1}{q}\|u^+\|^q_{L^q_K(\mathbb{R}^N_+)},$$

satisfies all the hypotheses of the Mountain Pass Theorem. The positivity of the obtained critical point can be proved as in Theorem 1.1.

■

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References

[1] R. Adams, *Sobolev spaces*, Pure and Applied Mathematics, Vol. 65. Academic Press, New York-London, 1975.

[2] R. Adams, *Compact imbeddings of weighted Sobolev spaces on unbounded domains*, J. Differential Equations **9** (1971), 325–334.

[3] C. Amrouche, F. Bonzom, *Exterior problems in the half-space for the Laplace operator in weighted Sobolev spaces*, J. Differential Equations **246** (2009), 1894–1920.

[4] J.M. Arrieta, *On boundedness of solutions of reaction-diffusion equations with nonlinear boundary conditions*. Proc. Amer. Math. Soc. **136** (2008), 151–160.

[5] J.M. Arrieta, A.N. Carvalho, A. Rodríguez-Bernal, *Parabolic problems with nonlinear boundary conditions and critical nonlinearities*. J. Differential Equations **156** (1999), 376–406.

[6] F.V. Atkinson, L.A. Peletier, *Sur les solutions radiales de l'équation* $\Delta u + \frac{1}{2}x \cdot \nabla u + \frac{1}{2}\lambda u + |u|^{p-1}u = 0$. (French) C. R. Acad. Sci. Paris Sér. I Math. **302** (1986), 99–101.

[7] P. Bartolo, V. Benci, D. Fortunato, *Abstract critical point theorems and applications to some nonlinear problems with strong resonance at infinity*, Nonlinear Anal. **7** (1983), 981–1012.

[8] J. Bergh, J. Löfström, *Interpolation spaces, An introduction.* Grundlehren der Mathematischen Wissenschaften, no. 223. Springer-Verlag, Berlin-New York, 1976.

[9] H. Brezis, L.A. Peletier, D. Terman, *A very singular solution of the heat equation with absorption*, Arch. Ration. Mech. Anal. **95** (1986), 185–206.

[10] M. Escobedo and O. Kavian, *Variational problems related to self-similar solutions of the heat equation*, Nonlinear Anal. **11** (1987), 1103–1133.

[11] M. Escobedo and E. Zuazua, *Self-similar solution for a convection-diffusion equation with absorption in* $\mathbb{R}^N$, Israel J. Math. **74** (1991), 47–64.

[12] L.C.F. Ferreira, E.J. Villamizar-Roa, *Self-similar solutions, uniqueness and long-time asymptotic behavior for semilinear heat equations*, Differential and Integral Equations **19** (2006), 1349–1370.

[13] M. Fila, P. Quittner, *The blow-up rate for the heat equation with a nonlinear boundary condition*, Math. Methods Appl. Sci. **14** (3) (1991), 197–205.

[14] B. Franchi, *Trace theorems for anisotropic weighted Sobolev spaces in a corner*, Math. Nachr. **127** (1986), 25–50.

[15] B. Franchi, *Pointwise estimates for a class of strongly degenerate elliptic operators: a geometrical approach*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **14** (1987), 527–568.

[16] M.F. Furtado, O.H. Miyagaki and J.P.P. da Silva, *On a class of nonlinear elliptic equations with fast increasing weight and critical growth*, J. Differential Equations **249** (2010), 1035–1055.

[17] V.A. Galaktionov, S. Kurdyumov, A. Samarskii, *On asymptotic "eigenfunctions" of the Cauchy problem for a nonlinear parabolic equation*, Math. USSR Sbornik **54** (1986), 421–455.

[18] V.A. Galaktionov, J.L. Vazquez, *Continuation of blowup solutions of nonlinear heat equations in several space dimensions*, Comm. Pure Appl. Math. **50** (1997), 1–67.

[19] Y. Giga, T. Miyakawa, *Navier-Stokes flow in* $\mathbb{R}^3$ *with measures as initial vorticity and Morrey spaces*, Comm. Partial Differential Equations **14** (1989), 577–618.

[20] V. Goldshtein, A. Ukhlov, *Weighted Sobolev spaces and embedding theorems*, Trans. Amer. Math. Soc. **361** (2009), 3829–385.

[21] A. Haraux, F.B. Weissler, *Nonuniqueness for a semilinear initial value problem.* Indiana Univ. Math. J. **31** (1982), 167–189.

[22] L. Herraiz, *Asymptotic behaviour of solutions of some semilinear parabolic problems*, Ann. Inst. H. Poincaré Anal. Non Linéaire **16** (1999), 49–105.

[23] K. Ishige, T. Kawakami, *Global solutions of the heat equation with a nonlinear boundary condition*, Calc. Var. Partial Differential Equations **39** (2010), 429–457.

[24] A. Kufner and B. Opic, *Remark on compactness of imbeddings in weighted spaces*, Math. Nachr. **133** (1987), 63–69.

[25] N. Mizoguchi, E. Yanagida, *Critical exponents for the blow-up of solutions with sign changes in a semilinear parabolic equation*, Math. Ann. **307** (1997), 663–675.

[26] Y. Naito, *Self-similar solutions for a semilinear heat equation with critical Sobolev exponent*, Indiana Univ. Math. J. **57** (2008), 1283–1315.

[27] Y. Naito, *Non-uniqueness of solutions to the Cauchy problem for semilinear heat equations with singular initial data*, Math. Ann. **329** (2004), 161–196.

[28] Y. Naito, *An ODE approach to the multiplicity of self-similar solutions for semilinear heat equations*, Proc. Roy. Soc. Edinburgh Sect. A **136** (2006), 807–835.

[29] Y. Naito, T. Suzuki, *Radial symmetry of self-similar solutions for semilinear heat equations*, J. Differential Equations **163** (2000), 407–428.

[30] H. Ohya, *Existence results for some quasilinear elliptic equations involving critical Sobolev exponents*, Adv. Differential Equations **9** (2004), 1339–1368.

[31] L. A. Peletier, D. Terman, F.B. Weissler, *On the equation* $\Delta u + \frac{1}{2}x \cdot \nabla u + f(u) = 0$. Arch. Rational Mech. Anal. **94** (1986), 83–99.

[32] K. Pflüger, *Nonlinear boundary value problems in weighted Sobolev spaces.* Nonlinear Anal. **30** (1997), 1263–1270.

[33] P. Quittner, P. Souplet, *Bounds of global solutions of parabolic problems with nonlinear boundary conditions.* Indiana Univ. Math. J. **52** (2003), 875–900.

[34] P.H. Rabinowitz, *Minimax methods in critical point theory with applications to differential equations*, CBMS Regional Conference Series in Mathematics, 65. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1986.

[35] A. Rodríguez-Bernal, *Attractors for parabolic equations with nonlinear boundary conditions, critical exponents, and singular initial data.* J. Differential Equations **181** (2002), 165–196.

[36] P. Souplet, F.B. Weissler, *Regular self-similar solutions of the nonlinear heat equation with initial data above the singular steady state*. Ann. Inst. H. Poincaré Anal. Non Linéaire **20** (2003), 213–235.

[37] A. Shishkov, L. Véron, *Singular solutions of some nonlinear parabolic equations with spatially inhomogeneous absorption*, Calc. Var. Partial Differential Equations **33** (2008), 343–375.

[38] F.B. Weissler, *Rapidly decaying solutions of an ordinary differential equation with applications to semilinear elliptic and parabolic partial differential equations*. Arch. Rat. Mechanics Analysis **91** (1985), 247-266.