NONQUADRATICITY CONDITION ON SUPERLINEAR PROBLEMS

MARCELO F. FURTADO AND EDCARLOS D. SILVA

Abstract. It is established existence of weak solution for a semilinear superlinear elliptic problems on bounded domains. The main feature of the paper is to prove that, for superlinear problems, the nonquadraticity condition introduced by Costa and Magalhães in [4] is sufficient to get the compactness required by minimax procedures.

Dedicated to Prof. Djairo de Figueiredo on the occasion of his 80th birthday

1. Introduction

In this paper we consider the nonlinear elliptic equation

$$(P) \qquad \begin{cases} -\Delta u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subseteq \mathbb{R}^N, N \geq 3$, is a bounded smooth domain and $f \in C(\Omega \times \mathbb{R}, \mathbb{R})$ is subcritical and superlinear in the following sense:

(f_0) there exist $a_1 > 0$ and $p \in (2, 2^*)$ such that

$$|f(x,s)| \leq a_1(1 + |s|^{p-1}), \text{ for any } (x,s) \in \Omega \times \mathbb{R}.$$

(f_1) for $F(x,s) := \int_0^s f(x,\tau)d\tau$, uniformly in $x \in \Omega$ there holds

$$\lim_{|s|\to\infty} \frac{F(x,s)}{s^2} = +\infty.$$

The weak solutions of the problem are precisely the critical points of the C^1-functional

$$I(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 \mathrm{d}x - \int_\Omega F(x,u)\mathrm{d}x, \; u \in H_0^1(\Omega).$$

and therefore we can use all the machinery of the Critical Point Theory. This theory is based on the existence of a linking structure and deformation lemmas [1, 2, 3, 21, 20]. In general, to be able to derive such deformation results, it is supposed that the functional satisfies some compactness condition. We use here the well known Cerami condition, which reads as: the functional I satisfies the Cerami condition at level $c \in \mathbb{R}$ ($(\mathrm{Ce})_c$ for short) if any sequence $(u_n) \subset H_0^1(\Omega)$ such that $I(u_n) \to c$ and $\|I'(u_n)\|_{H_0^1(\Omega)'}(1 + \|u_n\|) \to 0$ has a convergent subsequence.

In order to get compactness we shall assume the following condition (see [4])

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(NQ) setting $H(x,s) := f(x,s)s - 2F(x,s)$, we have that

$$\lim_{|s|\to\infty} H(x,s) = +\infty, \text{ uniformly for } x \in \Omega.$$

The behaviour of the nonlinearity at the origin will be done by the condition

(f_2) there exists $K_0 \in L^t(\Omega)$, $t > N/2$, with nontrivial positive part such that

$$\lim_{s\to 0} \frac{2F(x,s)}{s^2} = K_0(x), \text{ uniformly for } x \in \Omega.$$

As it is well known (see deFigueiredo [5]), under this condition the weighted eigenvalue problem

$$(LP) \qquad -\Delta u = \lambda K_0(x) u, \ u \in H_0^1(\Omega)$$

has an increasing sequence of eigenvalues $(\lambda_j(K_0))_{j\in\mathbb{N}}$ with $\lambda_1(K_0) > 0$.

We establish the existence of one weak solution by assuming a crossing condition at the origin. Related conditions on weighted eigenvalue problems have already appeared in the paper of deFigueiredo and Massabó [7] (see also [8]). Our main result can be stated as follows

Theorem 1.1. *Suppose that f satisfies $(f_0), (f_1)$ and (NQ). If f also satisfies (f_2) with*

$$\lambda_m(K_0) < 1 < \lambda_{m+1}(K_0),$$

then the problem (P) has at least one nonzero solutions.

We emphasize that our existence result works without the well known Ambrosetti-Rabinowitz condition [1]. It reads as: there exist $\theta > 2$, $R > 0$ such that

$$(AR) \qquad \theta F(x,s) \le s f(x,s), \quad x \in \Omega, \, |s| \ge R.$$

The main role of (AR) condition is to ensure the boundedness of the Palais-Smale sequences for I. It is not hard to verify that it implies that $F(x,s) \ge c_1|s|^\theta - c_2$ for any $x \in \Omega$, $t \in \mathbb{R}$, in such way that F goes to infinity at least like $|s|^\theta$. We observe that the condition (f_1) ia a more natural superlinear condition. Indeed, there are many superlinear functions which do not behave like $|s|^\theta$, $\theta > 2$, at infinity. For instance, we can take $f(s) = \lambda s + s\log(1+|s|)$, with $\lambda > 0$, and easily conclude that it does not verify (AR), but (NQ) holds. Actually, we can verify that hypotheses $(f_0) - (f_2)$ are also satisfied with $K_0 \equiv 1$ and $\lambda_j(K_0) = \lambda_j \in \sigma(-\Delta, H_0^1(\Omega)), \lambda \in (\lambda_m, \lambda_{m+1})$ for some fixed $m \in \mathbb{N}$. Thus our results extends and complements earlier results on superlinear problems.

As it will be clear from the proofs the ideas presented here can be used to handle with other settings of conditions on f. We could consider the case $\lambda_1(K_0) > 1$ as an application of the classical Mountain Pass theorem. If $K_0 \equiv 0$, the same ideas of the proof provides a weak solution. Also we can deal with the existence of multiple solutions under some symmetry assumptions (see [9] for instance). The main point here is to guarantee compactness. We show that the condition (NQ), introduced by Costa-Magalhães [4], is a powerful tool. More precisely, we prove that (f_0) and (NQ) are sufficient to prove that the functional I satisfies the Cerami condition, this being the main novelty of this work.

Semilinear superlinear elliptic problems have been considered during the past forty years, see [4, 6, 11, 13, 12, 21, 14]. In all these works some condition on the nonlinearity ensure some kind of compactness condition. For example, in [6] de

Figueiredo *et al.* considered superlinear elliptic problems such as (P) which satisfies the following conditions

(FLN_1) there exist $\theta \in (0, 2^*)$ such that

$$\limsup_{|s|\to\infty} \frac{f(x,s)s - \theta F(x,s)}{s^2 f(s)^{2/N}} \leq 0, \text{ uniformly for } x \in \Omega.$$

(FLN_2) for each $x \in \Omega$, the function $s \mapsto f(x,s)/s^{2^*-1}$ is nonincreasing.

They proved that problem (P) admits at least one positive solution using topological methods. Posteriorly, Jeanjean [10] considered the problem (P) requiring convexity in s for the function $H(x,s)$ defined in (NQ). We also mention the papers [18, 19] and references therein for similar results. Superlinear elliptic problems have been also studied under monotonicity conditions for the function $s \mapsto f(x,s)/s$, for $|s| \geq R$ (see [15]). In other works [13, 11] monotonicity was imposed on $H(x,\cdot)$ (see also [16, 17, 18]).

Here we do not assume any kind of monotonicity or convexity on the nonlinear therm f nor in the function H defined in (NQ). Hence, our result complement and/or extended the aforementioned works.

The paper hast just one more section, where we present the variational setting of the problem and prove Theorem 1.1. Throughout the paper we suppose that the function f satisfies (f_0). For save notation, we write only $\int_\Omega g$ and instead of $\int_\Omega g(x)\mathrm{d}x$. For any $1 \leq t < \infty$, $|g|_t$ denotes the norm in $L^t(\Omega)$.

2. Proof of the main theorem

We denote by H the Hilbert space $H_0^1(\Omega)$ endowed with the norm

$$\|u\|^2 = \left(\int |\nabla u|^2\right)^{1/2}, \text{ for any } u \in H.$$

By the Sobolev theorem we know that, for any $2 \leq \sigma \leq 2^*$ fixed, the embedding $H \hookrightarrow L^\sigma(\Omega)$ is continuous and therefore we can find $S_\sigma > 0$ such that

$$\int |u|^\sigma \leq S_\sigma \|u\|^\sigma. \tag{2.1}$$

If $\sigma < 2^*$, the Rellich-Kondrachov theorem implies that the above embedding is also compact.

As quoted in the introduction, the linear problem (LP) has a sequence of eigenvalues $(\lambda_j(K_0))_{j\in\mathbb{N}}$ with $\lambda_1(K_0) > 0$. If we denote by φ_j the eigenfunction associated with $\lambda_j(K_0)$, we set

$$V := \text{span}\{\varphi_1, \ldots, \varphi_m\}, \quad W := V^\perp.$$

and write H as being $H = V \oplus W$. The following variational inequalities hold

$$\|u\|^2 \leq \lambda_m(K_0) \int K_0(x)u^2, \ \forall\ u \in V, \tag{2.2}$$

and

$$\|u\|^2 \geq \lambda_{m+1}(K_0) \int K_0(x)u^2, \ \forall\ u \in W. \tag{2.3}$$

As a consequence of our assumption at the origin we have the following

Lemma 2.1. *Suppose that f satifies (f_0) and (f_2) with $\lambda_m(A_0) < 1 < \lambda_{m+1}(A_0)$. Then I has a local link at the origin, i.e.,*

(i) *there exists $\rho_1 > 0$ such that $I(z) \le 0$, for all $z \in V \cap B_{\rho_1}(0)$,*
(ii) *there exists $\rho_2 > 0$ such that $I(z) > 0$, for all nonzero $z \in W \cap B_{\rho_2}(0)$.*

Proof. Given $\varepsilon > 0$, we can use (f_0) and (f_2) to obtain $A_\varepsilon > 0$ such that

$$\frac{1}{2}K_0(x)s^2 - \frac{\varepsilon}{2}|s|^2 - A_\varepsilon|s|^p \le F(x,s) \le \frac{1}{2}K_0(x)s^2 + \frac{\varepsilon}{2}|s|^2 + A_\varepsilon|s|^p, \tag{2.4}$$

for any $(x,s) \in \Omega \times \mathbb{R}$. By taking $\varepsilon > 0$ sufficiently small we can use (2.4), (2.2), (2.1) and $\lambda_m(A_0) < 1$ to obtain

$$\begin{aligned} I(u) &\le \frac{1}{2}\|u\|^2 - \frac{1}{2}\int K_(x)u^2 + \frac{\varepsilon}{2}\int |u|^2 + A_\varepsilon \int |u|^p \\ &\le \frac{1}{2}\left(1 - \frac{1}{\lambda_m(A_0)} + \varepsilon S_2\right)\|u\|^2 + A_\varepsilon S_p\|u\|^p \\ &\le \left(\frac{\kappa}{2} + A_\varepsilon S_p\|u\|^{p-2}\right)\|u\|^2 \end{aligned}$$

for some $\kappa < 0$ and for all $u \in V$. Hence the condition (i) holds for $\rho_1 := (-\kappa/2A_\varepsilon S_p)^{1/(p-2)} > 0$.

In order to verify (ii), we choose $\varepsilon > 0$ small and use (2.4), (2.3), (2.1) and $\lambda_{m+1}(A_0) > 1$, to get

$$\begin{aligned} I(u) &\ge \frac{1}{2}\left(1 - \frac{1}{\lambda_{m+1}(A_0)} - \varepsilon S_2\right)\|u\|^2 - A_\varepsilon S_p\|u\|^p \\ &\ge \left(\frac{\mu}{2} + A_\varepsilon S_p\|z\|^{p-2}\right)\|u\|^2, \end{aligned}$$

for some $\mu > 0$ and for all $u \in W$. As before, we can check that (ii) holds for $\rho_2 := (\mu/2A_\varepsilon S_p)^{1/(p-2)} > 0$. The lemma is proved. $\square$

We are now ready to prove our main theorem.

Proof of Theorem 1.1. According to the last lemma the functional I has a local linking ar the origin. For any given $k \in \mathbb{N}$, let $H_k \subset H$ be a k-dimensional subspace. Since all the norms in H_k are equivalent, there exists $c_1 > 0$ such that $\|u\|^2 \le c_1 \int u^2$ for any $u \in H_k$. Given $M > (2/c_1)$, it follows from (f_1) that $F(x,s) \ge Ms^2 - c_2$ for any $x \in \Omega$ and $s \in \mathbb{R}$. Hence,

$$I(u) \le \frac{1}{2}\left(1 - \frac{2M}{c_2}\right)\|u\|^2 + c_1|\Omega|,$$

and we conclude that $I(u) \to -\infty$ as $\|u\| \to +\infty$, $u \in H_k$. Moreover, by (f_0), we can easily see that I maps bounded sets into bounded sets.

The above consideration shows that the functional I satisfy all the geometric condition of the Local Linking Theorem proved by Li and Willem in [12, Theorem 2]. Hence, if we can prove that I satisfies the Cerami condition, this last theorem provides a nonzero critical point for I. Here we mention that Theorem 2 in [12] is stated for a Palais-Smale type condition. However, as it is well know (see [3]), the deformation lemma used in [12] also holds for the Cerami condition.

It remains to check that I satisfies the Cerami condition. Let $(u_n) \subset H$ be such that

$$I(u_n) \to c, \ \|I'(u_n)\|_{H'}(1+\|u_n\|) \to 0,$$

where $c \in \mathbb{R}$. Since f has subcritical growth it suffices to prove that (u_n) is bounded.

Arguing by contradiction we suppose that, along a subsequence, $\|u_n\| \to +\infty$ as $n \to +\infty$. For each $n \in \mathbb{N}$, let $t_n \in [0,1]$ be such that

$$I(t_n u_n) = \max_{t \in [0,1]} I(t u_n). \tag{2.5}$$

Setting $v_n := u_n/\|u_n\|$ we obtain $v \in H$ such that, along a subsequence,

$$\begin{cases} v_n \rightharpoonup v \text{ weakly in } H, \\ v_n \to v \text{ strongly in } L^q(\Omega), \text{ for any } 1 \le q < 2^*, \\ v_n(x) \to v(x). \end{cases} \tag{2.6}$$

In what follows we prove that $v \neq 0$. Indeed, suppose by contradiction that $v = 0$. Then it follows from (f_0) and the strong convergence in (2.6) that $\int F(x, \sqrt{4m} v_n) \to 0$, as $n \to +\infty$, for any fixed $m > 0$. Since we may suppose that $\sqrt{4m} < \|u_n\|$, the definition of t_n in (2.5) provides

$$I(t_n u_n) \ge I\left(\frac{\sqrt{4m}}{\|u_n\|} u_n\right) = 2m - \int F(x, \sqrt{4m} v_n) \ge m > 0, \tag{2.7}$$

for any $n \ge n_0$, where $n_0 \in \mathbb{N}$ depends only on m.

We look for a contradiction by considering two cases:

Case 1: along a subsequence, $t_n < (2/\|u_n\|)$

In this case we use condition (f_0) and the Sobolev embeddings to obtain c_1, $c_2 > 0$ such that

$$\left| \int H(x, t_n u_n) \right| \le c_1 t_n \|u_n\| + c_2 t_n^p \|u_n\|^p \le 2c_1 + c_2 2^p = c_3.$$

If $t_n > 0$, it follows from $I'(t_n u_n)(t_n u_n) = 0$ that

$$0 = t_n^2 \|u_n\|^2 - \int f(x, t_n u_n)(t_n u_n) = 2I(t_n u_n) - \int H(x, t_n u_n),$$

and therefore

$$I(t_n u_n) = \frac{1}{2} \int H(x, t_n u_n) \le \frac{c_3}{2}.$$

The above inequality also holds if $t_n = 0$, and therefore we obtain a contradiction with (2.7), since the number $m > 0$ in that expression is arbitrary. Hence, the case 1 cannot occurs.

It remains to discard the

Case 2: along a subsequence, $t_n \ge (2/\|u_n\|)$

We fix $\gamma > 0$ in such way that

$$3\gamma|\Omega| > 4, \tag{2.8}$$

where $|\Omega|$ stands for the Lebesgue measure of Ω. In view of (NQ) we can obtain $s_0 > 0$ such that $H(x,s) \ge \gamma$ for any $x \in \Omega$, $|s| \ge s_0$. On the other hand, since H has a subcritical growth, we have that $H(x,s) \ge -C|s|$ for any $x \in \Omega$, $|s| \le s_1$, where $s_1 > 0$ is small.

We consider the nonnegative cut off function $\psi_\varepsilon : \mathbb{R} \to \mathbb{R}$ given by

$$\psi_\varepsilon(s) = \begin{cases} e^{-\varepsilon/s^2}, & \text{if } s \neq 0, \\ 0, & \text{if } s = 0, \end{cases}$$

with $\varepsilon > 0$ free for now. We mention that ψ_ε is smooth and

$$\lim_{s\to 0} \psi_\varepsilon(s) = \lim_{s\to 0} \psi_\varepsilon^{'}(s) = 0.$$

These limits, (f_0) and the continuity of H provide $C_{\gamma,\varepsilon} > 0$ such that

$$H(x,s) \geq \gamma\psi_\varepsilon(s) - C_{\gamma,\varepsilon}|s|, \text{ for any } (x,s) \in \Omega \times \mathbb{R}.$$

Given $0 < s < t$, we can use the above inequality and the definition of H to get

$$\begin{aligned} \frac{I(tu_n)}{t^2\|u_n\|^2} - \frac{I(su_n)}{s^2\|u_n\|^2} &= -\int_\Omega \int_s^t \frac{d}{d\tau}\left(\frac{F(x,\tau u_n)}{\tau^2\|u_n\|^2}\right) \mathrm{d}\tau\,\mathrm{d}x \\ &= -\int_\Omega \int_s^t \frac{H(x,\tau u_n)}{\tau^3\|u_n\|^2}\,\mathrm{d}\tau\,\mathrm{d}x \\ &\leq \int_\Omega \int_s^t \left(\frac{C_{\gamma,\varepsilon}}{\|u_n\|}\frac{|u_n|}{\|u_n\|}\tau^{-2} - \frac{\gamma\psi_\varepsilon(\tau u_n)}{\|u_n\|^2}\tau^{-3}\right)\mathrm{d}\tau\,\mathrm{d}x \end{aligned}$$

from which it follows that

$$\frac{I(tu_n)}{t^2\|u_n\|^2} \leq \frac{I(su_n)}{s^2\|u_n\|^2} + C_{\gamma,\varepsilon}\frac{|v_n|_1}{s\|u_n\|} - \gamma\int_\Omega\int_s^t \frac{\psi_\varepsilon(|\tau u_n|)}{\|u_n\|^2}\tau^{-3}\mathrm{d}\tau\,\mathrm{d}x.$$

We now set

$$s = s_n = \frac{1}{\|u_n\|} < \frac{2}{\|u_n\|} \leq t_n.$$

Since $\int_{s_n}^{t_n}\tau^{-3}\mathrm{d}\tau = (1/2)(\|u_n\|^2 - t_n^{-2})$ we have that

$$\begin{aligned} \frac{I(t_nu_n)}{t_n^2\|u_n\|^2} &\leq I(v_n) + C_{\gamma,\varepsilon}|v_n|_1 - \frac{\gamma|\Omega|}{2}\left(1 - \frac{1}{t_n^2\|u_n\|^2}\right) + \gamma A_n \\ &\leq B_\gamma + C_{\gamma,\varepsilon}|v_n|_1 - \int F(x,v_n) + \gamma A_n, \end{aligned} \tag{2.9}$$

with

$$A_n = \int_{s_n}^{t_n}\int_\Omega \frac{1-\psi_\varepsilon(|\tau u_n|)}{\|u_n\|^2}\tau^{-3}\mathrm{d}x\,\mathrm{d}\tau \geq 0$$

and

$$B_\gamma = \frac{1}{2}\left(1 - \frac{3}{4}\gamma|\Omega|\right) < 0,$$

where we have used (2.8) in the last inequality.

We shall verify in a few moments that, uniformly in $n \in \mathbb{N}$, the following limit holds

$$\lim_{\varepsilon\to 0}\int_{s_n}^{t_n}\int_\Omega \frac{1-\psi_\varepsilon(|\tau u_n|)}{\|u_n\|^2}\tau^{-3}\mathrm{d}x\,\mathrm{d}\tau = 0. \tag{2.10}$$

If this is true, we can choose $\varepsilon > 0$ in such way that $\gamma A_n < -B_\gamma/2$, for all $n \in \mathbb{N}$. Since we are supposing that $v = 0$, it follows from (2.6) and (f_0) that $|v_n|_1 = o_n(1)$

and $\int F(x, v_n) = o_n(1)$, as $n \to +\infty$. Hence, we can take the limit in (2.9) to obtain

$$\limsup_{n\to+\infty} \frac{I(t_n u_n)}{t_n^2\|u_n\|^2} \le B_\gamma - \frac{B_\gamma}{2} = \frac{B_\gamma}{2} < 0,$$

and therefore $I(t_n u_n) < 0$, for n large, contradicting (2.7) again.

We proceed now with the proof that the limit in (2.10) is uniform. We start by considering $\delta > 0$ and splitting the term A_n in two integrals

$$\int_{s_n}^{t_n} \int_\Omega \frac{1-\psi_\varepsilon(|\tau u_n|)}{\|u_n\|^2}\tau^{-3}\mathrm{d}x\,\mathrm{d}\tau = \int_{s_n}^{t_n}\int_{|\tau u_n|\ge\delta}(\cdots) + \int_{s_n}^{t_n}\int_{|\tau u_n|<\delta}(\cdots).$$

In order to save notation we call $A^+_{n,\delta}$ the first integral on the right-hand side above and $A^-_{n,\delta}$ the second one. It suffices to show that these quantities go to 0, uniformly in n, as $\varepsilon \to 0$.

Since ψ_ε is nondecreasing we have that

$$\begin{aligned} A^+_{n,\delta} &\le \frac{1-e^{-\varepsilon/\delta^2}}{\delta\|u_n\|^2}\int_{s_n}^{t_n}\int_{|\tau u_n|\ge\delta}|\tau u_n|\tau^{-3}\mathrm{d}x\,\mathrm{d}\tau \\ &\le \frac{1-e^{-\varepsilon/\delta^2}}{\delta\|u_n\|}\left(\frac{1}{s_n}-\frac{1}{t_n}\right)\int_\Omega\frac{|u_n|}{\|u_n\|} \\ &\le \left(\frac{1-e^{-\varepsilon/\delta^2}}{\delta}\right)|v_n|_1, \end{aligned}$$

since $s_n\|u_n\| = 1$. Recalling that $(|v_n|_1)$ is uniformly bounded, we conclude that the limit $\lim_{\varepsilon\to0} A^+_{n,\delta} = 0$ is uniform.

The calculations for $A^-_{n,\delta}$ are more involved. We first notice that, for each $|s| \le \delta$ fixed, the function $\varepsilon \mapsto \psi_\varepsilon(s)$ is smooth. Hence, it follows from Taylor's Theorem that, for $h(s) = s^{-2}e^{-\varepsilon/s^2}$, there holds

$$1-\psi_\varepsilon(s) = \varepsilon s^{-2}e^{-\varepsilon/s^2} + r(\varepsilon,s) = \varepsilon\left(h(s)+\frac{r(\varepsilon,s)}{\varepsilon}\right) \le \varepsilon(h(s)+1),$$

since the continuous remainder term r is such that $\lim_{\varepsilon\to0} r(\varepsilon,s)/\varepsilon = 0$ uniformly in the compact set $|s| \le \delta$. By applying Taylor's Theorem again we get, for $|s| \le \delta$,

$$h(s) = h(0) + h'(0)s + r_1(\varepsilon,s) = r_1(\varepsilon,s),$$

with $r_1(\varepsilon,s) = o(|s|)$ as $s \to 0$ uniformly in $\varepsilon \in (0,1]$. Thus, we conclude that, if $\delta > 0$ is small,

$$1-\psi_\varepsilon(s) \le \varepsilon(1+|s|), \quad \text{for any } |s| \le \delta.$$

The above inequality and the definition of $A^-_{n,\delta}$ provide

$$\begin{aligned} A^-_{n,\delta} &= \int_{s_n}^{t_n}\int_{|\tau u_n|<\delta}\frac{1-\psi_\varepsilon(|\tau u_n|)}{\|u_n\|^2}\tau^{-3}\mathrm{d}x\,\mathrm{d}\tau \\ &\le \varepsilon\int_{s_n}^{t_n}\int_\Omega\frac{\tau^{-3}}{\|u_n\|^2}\mathrm{d}x\,\mathrm{d}\tau + \varepsilon\int_{s_n}^{t_n}\int_\Omega\frac{|u_n|}{\|u_n\|^2}\tau^{-2}\mathrm{d}x\,\mathrm{d}\tau \\ &= \varepsilon\frac{|\Omega|}{2}\left(1-\frac{1}{t_n^2\|u_n\|^2}\right) + \frac{\varepsilon}{\|u_n\|}\left(1-\frac{1}{t_n\|u_n\|}\right)\int_\Omega|v_n|\mathrm{d}x \\ &\le \varepsilon\left(\frac{|\Omega|}{2}+|v_n|_1\right), \end{aligned}$$

since we may assume that $\|u_n\| > 1$. This implies that, uniformly in n, there holds $\lim_{\varepsilon\to 0} A^-_{n,\delta} = 0$. This finishes the proof that the weak limit v is nonzero.

After proving that $v \neq 0$ we can prove the theorem in the following way: the set $\widetilde{\Omega} := \{x \in \Omega : v(x) \neq 0\}$ has positive measure. Moreover, since $\|u_n\| \to +\infty$, we have that $|u_n(x)| \to +\infty$ a.e. in $\widetilde{\Omega}$. Thus, the continuity of H, Fatou's Lemma and (NQ) provide

$$\begin{aligned} 2c &= \lim_{n\to+\infty}(2I(u_n) - I'(u_n)u_n) \\ &\geq \operatorname{meas}(\Omega \setminus \widetilde{\Omega}) \cdot \min_{\overline{\Omega}\times\mathbb{R}} H + \int_{\widetilde{\Omega}} \liminf_{n\to+\infty} H(x,u_n) = +\infty, \end{aligned}$$

which is a contradiction. Hence, we have that (u_n) is bounded and the theorem is proved. □

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Universidade de Brasília, Departamento de Matemática, 70910-900 Brasília-DF, Brazil
E-mail address: mfurtado@unb.br

Universidade Federal de Goiás, Instituto de Matemática e Estatística, 74001-970 Goiânia-GO, Brasil
E-mail address: edcarlos@mat.ufg.br