SUPERLINEAR ELLIPTIC PROBLEMS UNDER THE NONQUADRITICTY CONDITION AT INFINITY

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Abstract. We present some sufficient conditions to obtain compactness properties for the Euler-Lagrange functional of an elliptic equation. As an application we extend some existence and multiplicity results for superlinear problems.

1. Introduction

In this paper we consider the nonlinear elliptic equation

$$\begin{cases} -\Delta u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subseteq \mathbb{R}^N, N \geq 3$, is a bounded smooth domain and $f \in C(\Omega \times \mathbb{R}, \mathbb{R})$ satisfies the standard subcritical growth condition

(f_0) there exist $a_1 > 0$ and $p \in (2, 2^*)$ such that

$$|f(x,s)| \leq a_1(1 + |s|^{p-1}), \text{ for any } (x,s) \in \Omega \times \mathbb{R}.$$

Under this condition the weak solutions of the problem are precisely the critical points of the C^1-functional

$$I(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 \mathrm{d}x - \int_\Omega F(x,u)\mathrm{d}x, \ u \in H_0^1(\Omega),$$

where $F(x,s) := \int_0^s f(x,\tau)\mathrm{d}\tau$. Hence, we can use all the machinery of the Critical Point Theory to look for weak solutions. As it is well known, this theory is based on the existence of a linking structure and on deformation lemmas [1, 2, 25, 23]. In general, to be able to derive such deformation results, it is supposed that the functional satisfies some compactness condition. We use here the Cerami condition, which reads as: the functional I satisfies the Cerami condition at level $c \in \mathbb{R}$ ($(\mathrm{Ce})_c$ for short) if any sequence $(u_n) \subset H_0^1(\Omega)$ such that $I(u_n) \to c$ and $\|I'(u_n)\|_{H_0^1(\Omega)'}(1 + \|u_n\|) \to 0$ has a convergent subsequence.

Our main objective is presenting sufficient conditions to assure that the functional satisfies the Cerami condition. More specifically, we shall consider

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the *nonquadraticity condition at infinity* introduced by Costa and Magalhães [4], whose statement is

(NQ) setting $H(x,s) := f(x,s)s - 2F(x,s)$, we have that

$$\lim_{|s|\to\infty} H(x,s) = +\infty, \text{ uniformly for } x \in \Omega.$$

In the core result of this paper we show that the above condition and (f_0) are suffice to guarantee compactness for the functional I. More specifically, we prove the following result:

Theorem 1.1. *Suppose that f satisfies (f_0) and (NQ). Then the functional I satisfies the Cerami condition at any level $c \in \mathbb{R}$.*

As an application of this theorem we prove some new results for the problem (P) in the case that f is superlinear at infinity and at the origin. Furthermore, we give an unnified approach for any superlinear elliptic problem using the nonquadraticiy condition. In order to better explain our results we recall that, in their seminal work, Ambrosetti and Rabinowitz [1] introduced the condition

(AR) there exist $\theta > 2$ and $s_0 > 0$ such that

$$0 < \theta F(x,s) \leq sf(x,s), \text{ for any } x \in \Omega,\ |s| > s_0.$$

A straightforward calculations shows that it provides $c_1 > 0$ such that $F(x,s) \geq c_1|s|^\theta$ for $|s|$ large. Thus, the problem is called *superlinear* in the sense that the primitive of f lives above any parabola of the type $c_2 s^2$. Unfortunately, there are several nonlinearities which are superlinear but do not satisfy the above inequality. For example, if we take $f(s) = |s|\ln(1+|s|)$, we can easily check that $\lim_{s\to+\infty} F(s)/s^\theta = 0$ for any $\theta > 2$. So, it is natural to ask if we can replace (AR) condition for a more natural one, namely

(SL) the following limit holds

$$\lim_{|s|\to+\infty} \frac{2F(x,s)}{s^2} = +\infty, \text{ uniformly for } x \in \Omega.$$

One of the main feature of condition (AR) is that it provides the boundedness of Palais-Smale sequences. In the past 40 years many authors tried to obtain solution in situations where (AR) is no longer valid. Instead, they consider the condition (SL) with extra assumptions (see [4, 13, 20, 11, 21, 14, 12, 18, 10, 16, 15] and references therein). In the most of them, there are some kind of monotonicity assumption on the functions $F(x,s)$ or $f(x,s)/s$, or some convexity condition on the function $f(x,s)s - 2F(x,s)$.

Our results concerning the problem (P) are stated below.

Theorem 1.2. *Suppose that f satisfies (f_0), (NQ) and (SL). Then the problem* (1.1) *has at least one nonzero weak solution provided we have that*

(f_1) *there holds*

$$\limsup_{s\to 0} \frac{F(x,s)}{s^2} = 0, \text{ uniformly for } x \in \Omega.$$

If $f(x,s)$ is odd in s then we can drop the condition (f_1) and obtain infinitely many weak solutions.

We notice that, for the existence result, we can suppose that the limit in (SL) holds only for $x \in \Omega_0$, where $\Omega_0 \subset \Omega$ is a subset with positive measure (see the proof of Theorem 1.2). So, we can deal with nonlinearities which are locally superlinear at infinity.

In order to compare our existence result with the literature, we start by citing again the paper of Costa and Magalhães [4], where the authors supposed, among other conditions, that

(F_μ) there exist $a_2 > 0$ and $\mu > \frac{N}{2}(p-2)$ such that

$$\liminf_{|s|\to\infty} \frac{H(x,s)}{|s|^\mu} \geq a_2, \text{ uniformly for } x \in \Omega,$$

where the number $p \in (2, 2^*)$ comes from (f_0). Since $\mu > 0$, we see that (NQ) is weaker than (F_μ), and therefore our existence result extend [4, Theorem 1]. It also extend the main theorem of a recent paper by Miyagaki and Souto [18], where the conditions (f_1) and (NQ) are replaced by

$(\widehat{f_1})$ $f(x,s) = o(s)$ as $s \to 0$, uniformly for $x \in \Omega$;

(M_1) the function $f(x,s)/|s|$ is increasing in $|s|$ for $|s| > s_1$.

Beyond their condition at the origin be stronger than ours, the main point is that (M_1) and (SL) together imply (NQ). Indeed, it can be proved that (M_1) implies that $H(x,s)$ is increasing in $|s|$ for $|s| > s_2$. Hence, if $s > s_2$, we have that

$$\begin{aligned} \frac{F(x,s)}{s^2} - \frac{F(x,s_2)}{s_2^2} &= \int_{s_2}^{s} \frac{d}{d\tau}\left\{\frac{F(x,\tau)}{\tau^2}\right\} \mathrm{d}\tau = \int_{s_2}^{s} \frac{H(x,\tau)}{\tau^3}\mathrm{d}\tau \\ &\leq H(x,s)\left(-\frac{1}{2s^2} + \frac{1}{2s_2^2}\right), \end{aligned} \tag{1.2}$$

and therefore

$$\frac{F(x,s)}{s^2} \leq c_3 + c_4 H(x,s),$$

for some c_3, $c_4 > 0$. It follows from (SL) that $\lim_{s\to+\infty} H(x,s) = +\infty$. An analogous argument shows that the same occurs as $s \to -\infty$. In [7], Fang and Liu have obtained one nonzero solution by assuming (f_0), (SL), $(\widehat{f_1})$ and

(J) there exists $\theta \geq 1$ such that $H(x,ts) \leq \theta H(x,s)$ for any $(x,s) \in \Omega \times \mathbb{R}$ and $t \in [0,1]$.

This quasi-monotonicity condition was introduced by Jeanjean in [11]. The same argument used in (1.2) shows that (J) together with (SL) imply (NQ), and therefore Theorem 1.2 extends [7, Theorem 1.1].

Our existence result also complements many other works of the updated literature. For example, in [17], Liu and Wang obtained a nonzero solution under $(\widehat{f_1})$, (SL) and the following version of (M_1)

($\widehat{M_1}$) the function $H(x,s)$ is nondecreasing in $|s|$ and increasing for $|s|$ small.

This hypothesis plays an important role in their proof, since they apply the Nehari method. Finally, Schechter and Zou in [22] have assumed (f_0), (SL) and $(\widehat{f_1})$. Moreover, they additionally assume that $H(x,s)$ was convex on s or

(SZ) there exist $\theta > 2$, $a_3 \geq 0$ and $s_3 \geq 0$ such that

$$\theta F(x,s) - sf(x,s) \leq a_3(1+s^2), \text{ for any } x \in \Omega, \ |s| \geq s_3.$$

Since we do not require any kind of monotonicity nor convexity, our existence result extended or complement the aforementioned works. It also complement other results on superlinear problems (see [24, 20, 21, 26, 17] and references therein). As a matter of fact, we can consider here the nonlinearity f such that $H(x,s) = a(x)s^2(1+\cos(s)) + \ln(1+|s|)$, with $a \in C^\infty(\Omega)$ being positive. Hence, the arguments presented in the cited papers do not work in our setting.

Concerning the multiplicy statement of Theorem 1.2 we quote that it complements many results on multiplicity of solutions for superlinear problems (see, for instance, [1, 27, 9] and references therein). The main novelty here is to consider the nonquadraticity condition on the superlinear setting. We emphasize that, in some of the aforementioned works, the proof of existence is given by showing that the (bounded) Palais-Smale sequence weakly converges to a nonzero critical point of I. Hence, the authors can not obtain multiple solutions, even if the function f is odd. Since here we prove compactness for I, we are able to use the Symetric Mountain Pass Theorem to obtain infinitely many solutions in this context.

In the next section we prove our main result, namely Theorem 1.1. The result is applied in the Section 3 where we present the proof of Theorem 1.2. It is worthwhile to mention that our ideas could be used in many different settings of linking type. So, we add a final section with some words concerning possible extensions of the study of problem (P).

2. Proof of the main result

Throughout the paper we suppose that the function f satisfies (f_0). For save notation, we write only $\int_\Omega g$ and instead of $\int_\Omega g(x)\mathrm{d}x$. For any $1 \leq t < \infty$, $|g|_t$ denotes the norm in $L^t(\Omega)$.

We denote by H the Hilbert space $H_0^1(\Omega)$ endowed with the norm

$$\|u\|^2 = \left(\int |\nabla u|^2\right)^{1/2}, \text{ for any } u \in H.$$

As stated in the Introduction the weak solutions of (P) are precisely the critical points of the C^1-functional

$$I(u) := \frac{1}{2}\|u\|^2 - \int F(x,u), \text{ for any } u \in H.$$

By using some carefull estimates we can prove our compactness result as follows:

Proof of Theorem 1.1. Let $(u_n) \subset H$ be such that

$$I(u_n) \to c, \ \|I'(u_n)\|_{H'}(1+\|u_n\|) \to 0,$$

where $c \in \mathbb{R}$. Since f has subcritical growth it suffices to prove that (u_n) is bounded.

Arguing by contradiction we suppose that, along a subsequence, $\|u_n\| \to +\infty$ as $n \to +\infty$. For each $n \in \mathbb{N}$, let $t_n \in [0,1]$ be such that

$$I(t_n u_n) = \max_{t \in [0,1]} I(t u_n). \tag{2.1}$$

Setting $v_n := u_n/\|u_n\|$ we obtain $v \in H$ such that, along a subsequence,

$$\begin{cases} v_n \rightharpoonup v \text{ weakly in } H, \\ v_n \to v \text{ strongly in } L^q(\Omega), \text{ for any } 1 \le q < 2^*, \\ v_n(x) \to v(x). \end{cases} \tag{2.2}$$

In what follows we prove that $v \neq 0$. Indeed, suppose by contradiction that $v = 0$. Then it follows from (f_0) and the strong convergence in (2.2) that $\int F(x, \sqrt{4m} v_n) \to 0$, as $n \to +\infty$, for any fixed $m > 0$. Since we may suppose that $\sqrt{4m} < \|u_n\|$, it follows from the definition of t_n in (2.1) that

$$I(t_n u_n) \ge I\left(\frac{\sqrt{4m}}{\|u_n\|} u_n\right) = 2m - \int F(x, \sqrt{4m} v_n) \ge m > 0, \tag{2.3}$$

for any $n \ge n_0$, where $n_0 \in \mathbb{N}$ depends only on m.

We look for a contradiction by considering two cases:

Case 1: along a subsequence, $t_n < (2/\|u_n\|)$

In this case we first use the condition (f_0) and the Sobolev embeddings to obtain c_1, $c_2 > 0$ such that

$$\left| \int H(x, t_n u_n) \right| \le c_1 t_n \|u_n\| + c_2 t_n^p \|u_n\|^p \le 2c_1 + c_2 2^p = c_3.$$

If $t_n > 0$, it follows from $I'(t_n u_n)(t_n u_n) = 0$ that

$$0 = t_n^2 \|u_n\|^2 - \int f(x, t_n u_n)(t_n u_n) = 2I(t_n u_n) - \int H(x, t_n u_n),$$

and therefore

$$I(t_n u_n) = \frac{1}{2} \int H(x, t_n u_n) \le \frac{c_3}{2}.$$

The above inequality also holds if $t_n = 0$, and therefore we obtain a contradiction with (2.3), since the number $m > 0$ in that expression is arbitrary. Hence, the case 1 cannot occurs.

It remains to discard the

Case 2: along a subsequence, $t_n \ge (2/\|u_n\|)$

In this setting we fix $\gamma > 0$ in such way that

$$3\gamma|\Omega| > 4, \tag{2.4}$$

where $|\Omega|$ stands for the Lebesgue measure of Ω. In view of (NQ) we can obtain $s_0 > 0$ such that $H(x,s) \geq \gamma$ for any $x \in \Omega$, $|s| \geq s_0$. On the other hand, since H has a subcritical growth, we have that $H(x,s) \geq -C|s|$ for any $x \in \Omega$, $|s| \leq s_1$, where $s_1 > 0$ is small.

We consider the nonnegative cut off function $\psi_\varepsilon : \mathbb{R} \to \mathbb{R}$ given by

$$\psi_\varepsilon(s) = \begin{cases} e^{-\varepsilon/s^2}, & \text{if } s \neq 0, \\ 0, & \text{if } s = 0, \end{cases}$$

with $\varepsilon > 0$ free for now. We mention that ψ_ε is smooth and

$$\lim_{s\to 0} \psi_\varepsilon(s) = \lim_{s \to 0} \psi_\varepsilon'(s) = 0.$$

These limits, (f_0) and the continuity of H provide $C_{\gamma,\varepsilon} > 0$ such that

$$H(x,s) \geq \gamma\psi_\varepsilon(s) - C_{\gamma,\varepsilon}|s|, \text{ for any } (x,s) \in \Omega \times \mathbb{R}.$$

Given $0 < s < t$, we can use the above inequality and the definition of H to get

$$\begin{aligned} \frac{I(tu_n)}{t^2\|u_n\|^2} - \frac{I(su_n)}{s^2\|u_n\|^2} &= -\int_\Omega \int_s^t \frac{d}{d\tau}\left(\frac{F(x,\tau u_n)}{\tau^2\|u_n\|^2}\right) \mathrm{d}\tau\, \mathrm{d}x \\ &= -\int_\Omega \int_s^t \frac{H(x,\tau u_n)}{\tau^3\|u_n\|^2} \mathrm{d}\tau\, \mathrm{d}x \\ &\leq \int_\Omega \int_s^t \left(\frac{C_{\gamma,\varepsilon}}{\|u_n\|}\frac{|u_n|}{\|u_n\|}\tau^{-2} - \frac{\gamma\psi_\varepsilon(\tau u_n)}{\|u_n\|^2}\tau^{-3}\right) \mathrm{d}\tau\, \mathrm{d}x \end{aligned}$$

from which it follows that

$$\frac{I(tu_n)}{t^2\|u_n\|^2} \leq \frac{I(su_n)}{s^2\|u_n\|^2} + C_{\gamma,\varepsilon}\frac{|v_n|_1}{s\|u_n\|} - \gamma\int_\Omega\int_s^t \frac{\psi_\varepsilon(|\tau u_n|)}{\|u_n\|^2}\tau^{-3}\mathrm{d}\tau\, \mathrm{d}x.$$

We now set

$$s = s_n = \frac{1}{\|u_n\|} < \frac{2}{\|u_n\|} \leq t_n.$$

Since $\int_{s_n}^{t_n} \tau^{-3}\mathrm{d}\tau = (1/2)(\|u_n\|^2 - t_n^2)$ we have that

$$\begin{aligned} \frac{I(t_nu_n)}{t_n^2\|u_n\|^2} &\leq I(v_n) + C_{\gamma,\varepsilon}|v_n|_1 - \frac{\gamma|\Omega|}{2}\left(1 - \frac{1}{t_n^2\|u_n\|^2}\right) + \gamma A_n \\ &\leq B_\gamma + C_{\gamma,\varepsilon}|v_n|_1 - \int F(x,v_n) + \gamma A_n, \end{aligned} \tag{2.5}$$

with

$$A_n = \int_{s_n}^{t_n}\int_\Omega \frac{1-\psi_\varepsilon(|\tau u_n|)}{\|u_n\|^2}\tau^{-3}\mathrm{d}x\, \mathrm{d}\tau \geq 0$$

and

$$B_\gamma = \frac{1}{2}\left(1 - \frac{3}{4}\gamma|\Omega|\right) < 0,$$

where we have used (2.4) in the last inequality.

We shall verify in a few moments that, uniformly in $n \in \mathbb{N}$, the following limit holds

$$\lim_{\varepsilon\to 0}\int_{s_n}^{t_n}\int_\Omega \frac{1-\psi_\varepsilon(|\tau u_n|)}{\|u_n\|^2}\tau^{-3}\mathrm{d}x\,\mathrm{d}\tau = 0. \tag{2.6}$$

If this is true, we can choose $\varepsilon > 0$ in such way that $\gamma A_n < -B_\gamma/2$, for all $n \in \mathbb{N}$. Since we are supposing that $v = 0$, it follows from (2.2) and (f_0) that $|v_n|_1 = o_n(1)$ and $\int F(x, v_n) = o_n(1)$, as $n \to +\infty$. Hence, we can take the limit in (2.5) to obtain

$$\limsup_{n\to+\infty}\frac{I(t_n u_n)}{t_n^2\|u_n\|^2} \le B_\gamma - \frac{B_\gamma}{2} = \frac{B_\gamma}{2} < 0,$$

and therefore $I(t_n u_n) < 0$, for n large, contradicting (2.3) again.

We proceed now with the proof that the limit in (2.6) is uniform. We start by considering $\delta > 0$ and splitting the term A_n in two integrals

$$\int_{s_n}^{t_n}\int_\Omega \frac{1-\psi_\varepsilon(|\tau u_n|)}{\|u_n\|^2}\tau^{-3}\mathrm{d}x\,\mathrm{d}\tau = \int_{s_n}^{t_n}\int_{|\tau u_n|\ge\delta}(\cdots) + \int_{s_n}^{t_n}\int_{|\tau u_n|<\delta}(\cdots).$$

In order to save notation we call $A^+_{n,\delta}$ the first integral on the right-hand side above and $A^-_{n,\delta}$ the second one. It suffices to show that these quantities go to 0, uniformly in n, as $\varepsilon \to 0$.

Since ψ_ε is nondecreasing we have that

$$\begin{aligned} A^+_{n,\delta} &\le \frac{1-e^{-\varepsilon/\delta^2}}{\delta\|u_n\|^2}\int_{s_n}^{t_n}\int_{|\tau u_n|\ge\delta}|\tau u_n|\tau^{-3}\mathrm{d}x\,\mathrm{d}\tau \\ &\le \frac{1-e^{-\varepsilon/\delta^2}}{\delta\|u_n\|}\left(\frac{1}{s_n}-\frac{1}{t_n}\right)\int_\Omega\frac{|u_n|}{\|u_n\|} \\ &\le \left(\frac{1-e^{-\varepsilon/\delta^2}}{\delta}\right)|v_n|_1, \end{aligned}$$

since $s_n\|u_n\| = 1$. Recalling that $(|v_n|_1)$ is uniformly bounded, we conclude that the limit $\lim_{\varepsilon\to 0} A^+_{n,\delta} = 0$ is uniform.

The calculations for $A^-_{n,\delta}$ are more involved. We first notice that, for each $|s| \le \delta$ fixed, the function $\varepsilon \mapsto \psi_\varepsilon(s)$ is smooth. Hence, it follows from Taylor's Theorem that, for $h(s) = s^{-2}e^{-\varepsilon,s^2}$, there holds

$$1-\psi_\varepsilon(s) = \varepsilon s^{-2}e^{-\varepsilon/s^2} + r(\varepsilon, s) = \varepsilon\left(h(s) + \frac{r(\varepsilon,s)}{\varepsilon}\right) \le \varepsilon(h(s)+1),$$

since the continuous remainder term r is such that $\lim_{\varepsilon\to 0} r(\varepsilon, s)/\varepsilon = 0$ uniformly in the compact set $|s| \le \delta$. By applying Taylor's Theorem again we get, for $|s| \le \delta$,

$$h(s) = h(0) + h'(0)s + r_1(\varepsilon, s) = r_1(\varepsilon, s),$$

with $r_1(\varepsilon, s) = o(|s|)$ as $s \to 0$. Thus, we conclude that, if $\delta > 0$ is small,

$$1 - \psi_\varepsilon(s) \le \varepsilon(1 + |s|), \quad \text{for any } |s| \le \delta.$$

The above inequality and the definition of $A^-_{n,\delta}$ provide

$$\begin{aligned} A^-_{n,\delta} &= \int_{s_n}^{t_n} \int_{|\tau u_n| < \delta} \frac{1 - \psi_\varepsilon(|\tau u_n|)}{\|u_n\|^2} \tau^{-3} \mathrm{d}x \, \mathrm{d}\tau \\ &\le \varepsilon \int_{s_n}^{t_n} \int_\Omega \frac{\tau^{-3}}{\|u_n\|^2} \mathrm{d}x \, \mathrm{d}\tau + \varepsilon \int_{s_n}^{t_n} \int_\Omega \frac{|u_n|}{\|u_n\|^2} \tau^{-2} \mathrm{d}x \, \mathrm{d}\tau \\ &= \varepsilon \frac{|\Omega|}{2} \left(1 - \frac{1}{t_n^2 \|u_n\|^2} \right) + \frac{\varepsilon}{\|u_n\|} \left(1 - \frac{1}{t_n \|u_n\|} \right) \int_\Omega |v_n| \mathrm{d}x \\ &\le \varepsilon \left(\frac{|\Omega|}{2} + |v_n|_1 \right), \end{aligned}$$

since we may assume that $\|u_n\| > 1$. This implies that, uniformly in n, there holds $\lim_{\varepsilon \to 0} A^-_{n,\delta} = 0$. This finishes the proof that the weak limit v is nonzero.

After proving that $v \neq 0$ we can prove the theorem in the following way: the set $\widetilde{\Omega} := \{x \in \Omega : v(x) \neq 0\}$ has positive measure. Moreover, since $\|u_n\| \to +\infty$, we have that $|u_n(x)| \to +\infty$ a.e. in $\widetilde{\Omega}$. Thus, the continuity of H, Fatou's Lemma and (NQ) provide

$$\begin{aligned} 2c &= \lim_{n \to +\infty} (2I(u_n) - I'(u_n)u_n) \\ &\ge \operatorname{meas}(\Omega \setminus \widetilde{\Omega}) \cdot \min_{\overline{\Omega} \times \mathbb{R}} H + \int_{\widetilde{\Omega}} \liminf_{n \to +\infty} H(x, u_n) = +\infty, \end{aligned}$$

which is a contradiction. Hence, we have that (u_n) is bounded and the theorem is proved. $\square$

3. Proof of Theorem 1.2

In this section we prove our results concerning problem (P). For the multiplicty part we need the following version of the Symmetric Moutain Pass Theorem [19, Theorem 9.12] (see [2, Theorem 1.3] for the proof that the deformation lemma used in [19] also holds with the Cerami condition).

Theorem 3.1. *Let X be an infinite dimensional Banach space and let $\mathcal{I} \in C^1(X, \mathbb{R})$ be even, satisfy $(Ce)_c$ for any $c \in \mathbb{R}$, and $\mathcal{I}(0) = 0$. If $X = V \oplus W$, where V is finite dimensional, and $\mathcal{I}$ satisfies*

$(\mathcal{I}_1)$ *there exist α, $\rho > 0$ such that*

$$\mathcal{I}(u) \ge \alpha, \ \textit{for any } u \in \partial B_\rho(0) \cap W;$$

$(\mathcal{I}_2)$ *for any finite dimensional subspace $\widehat{X} \subset X$ there exists $R = R(\widehat{X})$ such that*

$$\mathcal{I}(u) \le 0, \ \textit{for any } u \in \widehat{X} \setminus B_R(0),$$

then $\mathcal{I}$ possesses an unbounded sequence of critical values.

We are now ready to obtaining the solutions for (P).

Proof of Theorem 1.2. The conditions (f_0), (f_1) and standard arguments imply that $\int F(x,u) = o(\|u\|^2)$ as $\|u\| \to 0$. Hence, there exists α, $\rho > 0$ such that $I(u) \geq \alpha$ whenever $u \in \partial B_\rho(0) \subset H$. Suppose that the limit in (SL) holds for $x \in \Omega_0 \subset \Omega$ of positive measure. If we take a positive function $\phi \in H_0^1(\Omega_0)$ we can use (SL) to conclude that $I(t\phi) \to -\infty$ as $t \to +\infty$. Since I satisfies the Cerami condition it follows from the Moutain Pass Theorem that I has a nonzero critical point.

For proving the multiplicity part we shall apply Theorem 3.1 with $X = H$ and $\mathcal{I} = I$. Since f is odd in the second variable, I is even. Recalling that $I(0) = 0$ and I satifies the Cerami condition it remains to check the geometric conditions $(\mathcal{I}_1)$ and $(\mathcal{I}_2)$.

Let $\widehat{X} \subset H$ be a finite dimensional subspace. Since all the norms in $\widehat{X}$ are equivalent there exists $c_1 > 0$ such that $\|u\|^2 \leq c_1 \int u^2$ for any $u \in \widehat{X}$. Given $M > (2/c_1)$, it follows from (SL) that $F(x,s) \geq Ms^2 - c_2$ for any $x \in \Omega$ and $s \in \mathbb{R}$. Hence,

$$I(u) \leq \frac{1}{2}\left(1 - \frac{2M}{c_2}\right)\|u\|^2 + c_1|\Omega|,$$

and we conclude that $I(u) \to -\infty$ as $\|u\| \to +\infty$, $u \in \widehat{X}$. This establishes $(\mathcal{I}_2)$.

In order to verify $(\mathcal{I}_1)$ we set, for each $k \in \mathbb{N}$,

$$V_k := \operatorname{span}\{\varphi_1, \ldots, \varphi_k\},\ W_k = V_k^\perp,$$

where $(\varphi_k)_{k\in\mathbb{N}}$ are the eigenfunctions of $(-\Delta, H_0^1(\Omega))$. Integrating the inequality in (f_0) we get

$$I(u) \geq \frac{1}{2}\|u\|^2 - c_3|u|_p^p - c_4,$$

for some c_3, $c_4 > 0$. Since $2 < p < 2^*$, the interpolation inequality $|u|_p \leq |u|_2^\theta |u|_{2^*}^{1-\theta}$, for some $\theta \in (0,1)$, provides

$$I(u) \geq \frac{1}{2}\|u\|^2 - c_3|u|_2^{p\theta}|u|_{2^*}^{p(1-\theta)} - c_4 \geq \frac{1}{2}\|u\|^2 - c_5|u|_2^{p\theta}\|u\|^{p(1-\theta)} - c_4,$$

where $c_5 > 0$ and we have used the embedding $H \hookrightarrow L^{2^*}(\Omega)$.

The above inequality holds for any $u \in H$. If we take $u \in W_k$, we can use the variational inequality $\|u\|^2 \geq \lambda_{k+1}|u|_2^2$ to obtain

$$I(u) \geq \frac{1}{2}\|u\|^2 - \frac{c_5}{\lambda_{k+1}^{p\theta/2}}\|u\|^{p\theta}\|u\|^{p(1-\theta)} - c_4 = \left(\frac{1}{2} - \frac{c_5}{\lambda_{k+1}^{p\theta/2}}\|u\|^{p-2}\right)\|u\|^2 - c_4.$$

We now set $\rho = 2\sqrt{c_4+1}$ and choose $k \in \mathbb{N}$ in such way that

$$\frac{c_5}{\lambda_{k+1}^{p\theta/2}}\rho^{p-2} \leq \frac{1}{4}. \tag{3.1}$$

This is always possible, since $\lambda_k \to +\infty$. It follows that, for any $u \in \partial B_\rho(0) \cap W_k$, there holds

$$I(u) \geq \left(\frac{1}{2} - \frac{1}{4}\right)\rho^2 - c_4 = \frac{1}{4}(2\sqrt{c_4+1})^2 - c_4 = 1.$$

Therefore $(\mathcal{I}_1)$ is satisfied with $\alpha = 1$, $\rho = 2\sqrt{c_4+1}$ and the decomposition of H being $H = V_k \oplus W_k$. The multiplicity result follows from Theorem 3.1. □

4. Further remarks

In this final section we present many variants which could be considered. For example, concerning the condition at the origin, we could suppose that

$$\lim_{s\to 0} \frac{2F(x,s)}{s^2} = K_0(x), \text{ uniformly for } x \in \Omega,$$

where $K_0 \in L^t(\Omega)$ for some $t > N/2$ and the positive part of K_0 is nontrivial. In this case the linear problem

$$-\Delta u = \lambda K_0(x)u, \ u \in H_0^1(\Omega).$$

has a sequence of eigenvalues $(\lambda_j(K_0))_{j\in\mathbb{N}}$ with $\lambda_1(K_0) > 0$. A simple inspection of the proof of Theorem 1.2 shows that it remains true if we suppose that $\lambda_1(K_0) > 1$ instead of condition (f_1). Indeed, we can deal with *nonresonance at the origin* in the following sense: suppose that $\lambda_m(K_0) < 1 < \lambda_{m+1}(K_0)$ for some $m \geq 1$. In this case we can apply the Local Linking Theorem given by Li and Willem [13], together with our compactness result, to obtain a nonzero solution. So, it is possible to generalize the main theorems contained in [12, 6, 14]

We could also treat the asymptotically linear case, by replacing (SL) by the following condition:

$$\lim_{|s|\to +\infty} \frac{2F(x,s)}{s^2} = K_\infty(x), \text{ uniformly for } x \in \Omega,$$

where $K_\infty \in L^t(\Omega)$ for some $t > N/2$ and the positive part of K_∞ is nontrivial. If $\lambda_m(K_\infty) = 1$ for some $m \geq 1$, we could use the Saddle Point Theorem to extend the existence result of [4, Theorem 2] (see also [8] for related results). This means that, under the nonquadraticity condition, we give here an unified approach for nonlinear elliptic problems that are superlinear or asymptotically linear at infinity. Actually, our Theorem 1.1 presents another proof of [4, Lemma 1.2] but with weaker conditions. Hence, we could also consider the *double resonant* case:

(DR) there exists $j \geq 1$ such that

$$\lambda_j \leq \liminf_{|t|\to\infty} \frac{f(x,t)}{t} \leq \limsup_{|t|\to\infty} \frac{f(x,t)}{t} \leq \lambda_{j+1}$$

where λ_j is the sequence of eigenvalues on $(-\Delta, H_0^1(\Omega))$. In this case it is allowed the resonant phenomena in two consecutive eigenvalues. The main point here is to obtain compactness for the associated functional. But now, this is a consequence of condition (NQ). Thus, we can obtain a nontrivial solution under the assumption

$$\lambda_m < \lim_{s\to 0} \frac{f(x,s)}{s} < \lambda_{m+1}$$

for some $m \geq 1$, as a consequence of the Local Linking Theorem (see [3, 5, 8] for more details on double resonant problems).

Finally, under the hypothesis of Theorem 1.2, it is possible to argue as in [1] to obtain two solutions, one positive and other negative. Indeed, to obtain the first one we define

$$f^+(x,s) := \begin{cases} f(x,s), & \text{if } s \geq 0, \\ 0, & \text{if } s < 0, \end{cases}$$

and consider the functional

$$I^+(u) := \frac{1}{2}\|u\|^2 - \int F^+(x,u), \ \ u \in H_0^1(\Omega),$$

where $F^+(x,s) := \int_0^s f^+(x,\tau)\mathrm{d}\tau$. We have that F^+ is superlinear at infinity and nonquadratic at infinity in one direction. More precisely,

$$\lim_{s\to\infty} (sf^+(x,s) - 2F^+(x,s)) = +\infty, \text{ uniformly in } x \in \Omega,$$

and we can argue as in the proof of Theorem 1.2 to obtain a positive solution. The negative solution can be obtained with the analogous truncation f^-.

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