# QUASILINEAR ELLIPTIC PROBLEMS UNDER ASYMPTOTICALLY LINEAR CONDITIONS AT INFINITY AND AT THE ORIGIN

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Abstract. We obtain existence and multiplicity of solutions for the quasilinear Schrödinger equation

$$-\Delta u + V(x)u - \Delta(u^2)u = g(x,u),\ x \in \mathbb{R}^N,$$

where $V$ is a positive potential and the nonlinearity $g(x,t)$ behaves like $t$ at the origin, and like $t^3$ at infinity. In the proof we apply a changing of variables besides variational methods. The obtained solutions belong to $W^{1,2}(\mathbb{R}^N)$.

## 1. Introduction

In this paper we study the existence of solitary wave solutions for quasilinear Schrödinger equations of the form

$$i\partial_t z = -\Delta z + W(x)z - l(x,|z|^2)z - \kappa[\Delta\rho(|z|^2)]\rho'(|z|^2)z$$

where $z : \mathbb{R}^N \times \mathbb{R} \to \mathbb{C}$, $W : \mathbb{R}^N \to \mathbb{R}$ is a given potential, $\kappa$ is a real constant and $l, \rho$ are real functions. Equations of this type appear naturally in mathematical physics and have been accepted as models of several physical phenomena corresponding to various types of nonlinear terms $\rho$. They include equations in fluid mechanics, theory of Heisenberg ferromagnetism and magnons, dissipative quantum mechanics and matter theory (see [15, 16] and references therein).

We consider here the case of the superfluid film equation in plasma physics, namely $\rho(t) := t$ (see [11]). If we look for standing wave solutions $z(t,x) := \exp(-iEt)u(x)$ with $E > 0$, we are lead to consider the following elliptic equation

$$-\Delta u + V(x)u - \kappa\Delta(u^2)u = g(x,u),\ x \in \mathbb{R}^N,$$

with $V(x) := W(x) - E$ and $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ given by $g(x,t) := l(x,|t|^2)t$ is the new nonlinear term. Latter on, we shall consider precisely the hypotheses on $V$ and $g$.

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The semilinear case $\kappa = 0$ has been studied extensively in recent years with a huge variety of conditions on the potential $V$ and the nonlinearity $g$ (see e.g. [2, 8, 19, 1]). To the best of our knowledge the first results for the case $\kappa \neq 0$ are due to [18], where the existence of positive ground state solutions were constructed as minimizers of a constrained minimization problem. By using a change of variables the authors in [16] reduced the equation to a semilinear one and an Orlicz space framework was used to prove the existence of a positive solutions via Mountain Pass Theorem. The same method was also used in [3], but the usual Sobolev space $H^1(\mathbb{R}^N)$ framework was used as the working space. We refer the reader to [5, 20, 21, 6, 9] for more results. Usually the authors consider the case that the function $g(x,t)$ is sublinear at the origin and superlinear at infinity. Due to the change of variables introduced in [16] this behavior at infinity is related with the (modified) Ambrosetti-Rabinowitz condition $0 < \theta G(x,t) \leq g(x,t)t$ for some $\theta > 4$, any $x \in \mathbb{R}^N$, $t \neq 0$, where $G(x,t) := \int_0^t g(x,\tau)\mathrm{d}\tau$. As it is well know, this type of condition provides the boundedness of the Palais-Smale sequences of the associated functional. More generally, under suitable extra assumptions, it is possible to deal with the condition $\lim_{|t|\to+\infty} G(x,t)/t^4 = +\infty$ (see [20, 24]).

Differently from the aforementioned authors we do not suppose that our nonlinearity is superlinear. Instead, we are interested the case that $g(x,t) \sim t$ near the origin and $g(x,t) \sim t^3$ at infinity. As far we know there are few papers which deal with this type of nonlinearity at infinity. The first one is the work of Liu *et al* [16] which states, among other results, the existence of positive solution for the autonomous nonlinearity $g(x,t) = t^3$ under different kind of hypothesis on the potential $V$. We have recently learned that Silva and Vieira [22] have obtained some existence results under the condition

$$\lim_{|t|\to+\infty} \frac{G(x,t)}{t^4} > 0 \tag{1.1}$$

and other mild assumptions on $g$. We finally mention a recent paper of Fang and Szulkin [7] where they consider $g(x,t) = q(x)t^3$ and obtained the existence of infinite solutions under some symmetry conditions on the potential $V$. As far we know there are no other results concerning this "asymptotically linear" framework.

The main goal in this paper is to consider the problem

$$(P) \qquad \begin{cases} -\Delta u + V(x)u - \Delta(u^2)u = g(x,u), \ x \in \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases}$$

where $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ and $V : \mathbb{R}^N \to \mathbb{R}$ are continuous functions. Throughout the paper we shall assume the following basic hypothesis on the potential $V$:

$(V_1)$ $\inf_{x\in\mathbb{R}^N} V(x) \geq V_0 > 0$ ;

$(V_2)$ for any $M > 0$ there holds

$$\text{measure}\big(\{x \in \mathbb{R}^N : V(x) \le M\}\big) < +\infty.$$

We are looking for solution on a suitable subspace of $H^1(\mathbb{R}^N)$. Hence, we impose some growth conditions on the nonlinearity $g$. More specifically we shall assume that

$(g_1)$ there exist $\alpha_0 > N/2$ and $a$, $b \in L^{\alpha_0}(\mathbb{R}^N)$ such that

$$|g(x,t)| \le a(x)|t| + b(x)|t|^3, \text{ for all } (x,t) \in \mathbb{R}^N \times \mathbb{R};$$

$(g_2)$ there exist $0 < q_1 < 2$ and $(2 \cdot 2^*/q_1)' \le \tau \le (4/q_1)'$ such that

$$g(x,t)t - 4G(x,t) \ge -\Gamma_1(x) - \Gamma_2(x)|t|^{q_1}, \text{ for all } (x,t) \in \mathbb{R}^N \times \mathbb{R},$$

where $\Gamma_1 \in L^1(\mathbb{R}^N)$ and $\Gamma_2 \in L^\tau(\mathbb{R}^N)$.

In the condition $(g_2)$ above we are denoting by $s'$ the conjugated exponent of $s > 1$, namely the unique $s' > 1$ satisfying $1/s + 1/s' = 1$.

For any $w \in L^q(\mathbb{R}^N)$ we set $w^+(x) := \max\{w(x), 0\}$, define

$$\mathcal{F} := \big\{w : \mathbb{R}^N \to \mathbb{R} : w^+ \not\equiv 0,\, w \in L^\alpha(\mathbb{R}^N) \text{ for some } \alpha > N/2\big\}$$

and consider the asymptotic assumptions near the origin and at infinity:

$(G_0)$ there exists $K_0 \in \mathcal{F}$ such that

$$\limsup_{t\to 0} \frac{2G(x,t)}{t^2} = K_0(x), \text{ uniformly for a.e. } x \in \mathbb{R}^N;$$

$(G_\infty)$ there exists $K_\infty \in \mathcal{F}$ such that

$$\liminf_{|t|\to+\infty} \frac{4G(x,t)}{t^4} = K_\infty(x), \text{ uniformly for a.e. } x \in \mathbb{R}^N.$$

In order to state our main theorem we need to introduce the space

$$X := \left\{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)u^2 \mathrm{d}x < \infty\right\}. \tag{1.2}$$

It is well known (see [10]) that, under conditions $(V_0) - (V_1)$, the space $X$ is a closed subspace of $H^1(\mathbb{R}^N)$. Moreover, the embedding $X \hookrightarrow L^q(\mathbb{R}^N)$ is compact for any $2 \le q < 2^*$. Hence, for any given $K \in \mathcal{F}$, we can prove that the eigenvalue problem

$$-\Delta u + V(x)u = \lambda K(x)u \text{ in } \mathbb{R}^N,\ u \in X, \tag{1.3}$$

has a first positive eigenvalue $\lambda_1(K) > 0$.

We are able to state the main result of this paper as follows.

**Theorem 1.1.** *Suppose $V$ satisfies $(V_1)$–$(V_2)$, and $g$ satisfies $(g_1)$, $(g_2)$, $(G_0)$ and $(G_\infty)$. Then there exists $\eta > 0$ such that problem $(P)$ has at least one nontrivial solution provided*

$$\lambda_1(K_\infty) < (1+\eta) \quad \textit{and} \quad \lambda_1(K_0) > 1.$$

A very interesting point in the above theorem is that we can deal with cases where there is no crossing of eigenvalues. Actually, our hypothesis on the eigenvalues are weaker than the usual one $\lambda_1(K_\infty) < 1 < \lambda_1(K_0)$, since the number $\eta$ in Theorem 1.1 is positive. Hence, whenever $\lambda_1(K_\infty) < 1+\eta$, we can consider all the following situations:

$$1 < \lambda_1(K_\infty) = \lambda_1(K_0), \quad 1 < \lambda_1(K_0) < \lambda_1(K_\infty) \quad \text{or} \quad \lambda_1(K_\infty) \le 1 < \lambda_1(K_0).$$

This occurs due to the nature of the changing of variables performed in the equation. We also have a good expression for the number $\eta$, which is essentially related with some kind of interaction between the potential $V$ and the limit function $K_\infty$ (see Lemma 3.3 for details).

Although we apply standard variational techniques, the novelty in this paper is to consider existence of solution for $(P)$ under asymptotically conditions at infinity and at the origin. There are some related results for the semilinear case $\kappa = 0$ (see [13, 14, 23] and references therein), but we do not know any work which deal with this type of problem with $\kappa \neq 0$. As it is well known, the main difficult is to find sufficient conditions to prove the compactness required by the classical minimax theorems. The technicality involved by the presence of the term $\Delta(u^2)$ is not trivial and we perform some fine estimates to avoid this difficult. We believe that our compactness result (see Proposition 4.1) can be used to consider many other kinds of linking situations for the equation $(P)$.

For $j \in \mathbb{N}$, if we denote by $\lambda_j(K_\infty)$ the $j$-th positive eigenvalue of the problem (1.3) with weight $K = K_\infty$, problems like $(P)$ are usually called resonant at infinity if $\lambda_j(K_\infty) = 1$. We notice that we can deal here with this case without any kind of extra assumptions like Landesman-Lazer or nonquadraticity conditions [12, 4]. Actually, as you can see in Proposition 4.1, the conditions $(g_1)$ and $(g_2)$ are suffices to get the required compactness properties. We also emphasize that the function $K_\infty$ can change sign, and therefore the assumption (1.1) may be not true here.

It is worthwhile to mention that the our existence result can be improved by an usual truncation argument. More specifically, as a by product of the calculations performed in the proof of Theorem 1.1, we get the following result.

**Theorem 1.2.** *Under the same hypothesis of Theorem 1.1 the problem $(P)$ has at least two nontrivial solutions. One of them is positive and the another one is negative.*

Before finishing the introduction we present some typical examples of functions $g$ which satisfy our assumptions. We first fix $K_0$, $K_\infty \in \mathcal{F}$ and define

$$g(x,t) = a_0 K_0(x) h_0(t) + a_\infty K_\infty(x) h_\infty(t),$$

where $a_0$, $a_\infty$ are positive numbers picked in such way that $a_0 < \lambda_1(K_0)$ and $a_\infty > \lambda_1(K_\infty)$. The function $h_0$ is continuous, odd, has compact support and verifies $\lim_{t\to 0} h_0(t)/t = 1$. If we take $h_\infty(t) = t^3$ and suppose that

$K_0 \in L^1(\mathbb{R}^N)$ is nonnegative, then the condition $(g_2)$ holds with $\Gamma_1 = K_0$ and $\Gamma_2 \equiv 0$, and therefore our main results apply in this situation. There are many other nonlinearities which can be considered when $K_\infty$ is nonnegative, for instance we can take $h_\infty(t) = t^5/(1+t^2)$, $h_\infty(t) = t^3 - \mu t^2\sqrt{|t|}$ or $h_\infty(t) = t^3 - \mu t \ln(t^2)$ for $t \neq 0$, $h_\infty(0) = 0$, where $\mu > 0$ is arbitrary.

The paper is organized as follows: In Section 1 we consider some preliminaries results related to the dual principle. In Section 2 we prove the mountain pass geometry. In Section 3 we prove Theorem 1.1 by showing that the associated functional satisfies the Palais-Smale condition. In the final Section 4 we briefly describe how the arguments of the previous sections can be used to prove Theorem 1.2.

## 2. Variational Framework

Throughout the paper we write $\int u$ instead of $\int_{\mathbb{R}^N} u(x)\mathrm{d}x$. Moreover, for any $p \geq 1$, we denote by $\|u\|_p$ the $L^p(\mathbb{R}^N)$-norm of a function.

From the standard variational point of view, the problem $(P)$ is formally the Euler-Lagrange equation associated to the functional

$$u \mapsto \frac{1}{2}\int (1+2u^2)|\nabla u|^2 + \int V(x)u^2 - \int G(x,u). \tag{2.1}$$

Nevertheless, as quoted in [3], the above functional is not well defined in $H^1(\mathbb{R}^N)$, $N \geq 3$. In order to overcome this difficult we follow the idea introduced in [16] and the variational approach used in [3]. So we reformulate the problem $(P)$ using the change of variable $f : \mathbb{R} \to \mathbb{R}$ given by

$$\begin{cases} f'(t) = \dfrac{1}{\sqrt{1+2f(t)^2}}, & t \geq 0, \\ f(t) = -f(-t), & t \leq 0. \end{cases} \tag{2.2}$$

For an easy reference we list below the main properties of the function $f$. They will be extensively used in the rest of the paper.

**Lemma 2.1.** *The function $f$ satisfies the following properties:*

$(f_1)$ *$f$ is uniquely determined, $C^\infty$ and invertible;*
$(f_2)$ $0 < f'(t) \leq 1$ *for all* $t \in \mathbb{R}$*;*
$(f_3)$ $|f(t)| \leq |t|$ *for all* $t \in \mathbb{R}$*;*
$(f_4)$ $\lim_{t\to 0} \dfrac{f(t)}{t} = 1$*;*
$(f_5)$ $\lim_{t\to +\infty} \dfrac{f(t)}{\sqrt{t}} = 2^{1/4}$*;*
$(f_6)$ $\dfrac{f(t)}{2} \leq tf'(t) \leq f(t)$ *for all* $t \geq 0$*;*
$(f_7)$ $|f(t)| \leq 2^{1/4}\sqrt{|t|}$ *for all* $t \in \mathbb{R}$*;*
$(f_8)$ *there exists $\kappa > 0$ such that*

$$|f(t)| \geq \begin{cases} \kappa\,|t|\,, & |t| \leq 1, \\ \kappa\,|t|^{1/2}\,, & |t| \geq 1\,; \end{cases}$$

($f_9$) $|f(t)f'(t)| \leq 2^{-1/2}$ *for all* $t \in \mathbb{R}$;

($f_{10}$) *the function* $f^2$ *is strictly convex. In particular,* $f^2(st) \leq sf^2(t)$ *for all* $t \in \mathbb{R}$, $s \in [0,1]$;

($f_{11}$) $f^2(st) \leq s^2 f^2(t)$ *for all* $t \in \mathbb{R}$, $s \geq 1$;

($f_{12}$) $f^2(s-t) \leq 4(f^2(s) + f^2(t))$ *for all* $s, t \in \mathbb{R}$.

*Proof.* We only prove properties $(f_{10}), (f_{11})$ and $(f_{12})$. The other ones can be proved by using the ODE in (2.2) and arguing as in the papers [16, 3, 17]. A straightforward calculation shows that $(f^2)'' > 0$ in $(0+\infty)$, and therefore item $(f_{10})$ follows. In order to prove $(f_{11})$ we notice that, since $f'' \leq 0$ in $[0, +\infty)$, we have that $f'$ is non-increasing in this interval. For any $t \geq 0$ fixed we consider the function $h(s) := f(st) - sf(t)$ defined for $s \geq 1$. We have that $h'(s) = tf'(st) - f(t) \leq tf'(t) - f(t) \leq 0$, by $(f_6)$. Since $h(1) = 0$ we conclude that $h(s) \leq 0$ for any $s \geq 1$, that is $f(st) \leq sf(t)$ for any $t \geq 0$ and $s \geq 1$. Thus

$$f^2(st) \leq s^2 f^2(t)$$

for any $t \geq 0$ and $s \geq 1$. Since $f^2$ is even the proof of item $(f_{11})$ follows. To establish the item $(f_{12})$, we use the fact that $f^2$ is even and increasing in $(0, +\infty)$ together with $(f_{10})$ and $(f_{11})$ to get, for all $s, t \in \mathbb{R}$,

$$\begin{aligned} f^2(s-t) &= f^2(|s-t|) \leq f^2(|s| + |t|) \\ &\leq f^2(2\max\{|s|, |t|\}) \leq 4(f^2(s) + f^2(t)), \end{aligned}$$

and we have done. □

We now consider the following Orlicz-Sobolev space

$$E := \left\{ v \in H^1(\mathbb{R}^N) : \int V(x) f^2(v) < \infty \right\}. \tag{2.3}$$

It can be proved (see [17, 16]) that it is a Banach space when endowed with the norm

$$\|v\| := \|\nabla v\|_2 + |v|_f, \text{ for any } v \in E, \tag{2.4}$$

where

$$|v|_f := \inf_{\xi > 0} \frac{1}{\xi} \left\{ 1 + \int V(x) f^2(\xi v) \right\}.$$

By a weak solution of $(P)$ we mean a function $u \in H^1(\mathbb{R}^N) \cap L^\infty_{loc}(\mathbb{R}^N)$ such that

$$\int [(1 + 2u^2)\nabla u \nabla \varphi + 2u|\nabla u|^2 \varphi + V(x) u \varphi] = \int g(x, u)\varphi,$$

for all $\varphi \in C_0^\infty(\mathbb{R}^N)$. After the change of variables $u = f(v)$ in the map given in (2.1), we obtain the following functional

$$J(v) := \frac{1}{2} \int \left( |\nabla v|^2 + \int V(x) f^2(v) \right) - \int G(x, f(v)), \tag{2.5}$$

for any $v \in E$. Under the growth condition $(g_1)$ the functional $J$ belongs to $C^1(E, \mathbb{R})$ and its critical points are weak solutions of the problem

$$-\Delta v + V(x) f'(v) f(v) = g(x, f(v)) f'(v), \ v \in E. \tag{2.6}$$

Moreover, if $v \in E \cap C^2(\mathbb{R}^N)$ is a critical point of $J$ then the function $u = f(v)$ is a classical solution of $(P)$ (see [3] for details). Thus, we deal in the sequel with the modified problem described above.

The next two propositions summarizes the main properties of the space $E$.

**Proposition 2.2.** *Suppose $V$ satisfies $(V_1)$ and $(V_2)$. Then the Orlicz space $E$ has the following properties:*

(1) *If $v_n(x) \to v(x)$ a. e. in $\mathbb{R}^N$ and*

$$\lim_{n\to+\infty} \int V(x) f^2(v_n) = \int V(x) f^2(v),$$

*then*

$$\lim_{n\to+\infty} |v_n - v|_f = 0$$

(2) *The embeddings $E \hookrightarrow \mathcal{D}^{1,2}(\mathbb{R}^N)$, $E \hookrightarrow H^1(\mathbb{R}^N)$ and $X \hookrightarrow E$ are continuous. Here, $X$ denotes the space defined in (1.2).*

(3) *The map $v \to f(v)$ from $E$ to $L^q(\mathbb{R}^N)$ is continuous for each $q \in [2, 2\cdot 2^*]$, and it is compact for each $q \in [2, 2\cdot 2^*)$.*

(4) *For any $v \in E$ there holds*

$$\left\| \frac{f(v)}{f'(v)} \right\| \leq 4\|v\|.$$

(5) *If $v_n \rightharpoonup 0$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ and $\left(\int V(x) f^2(v_n)\right)$ is bounded then, up to a subsequence, $f(v_n) \to 0$ strongly in $L^q(\mathbb{R}^N)$ for any $2 \leq q < 2\cdot 2^*$.*

(6) *For any $v \in E$ there holds*

$$|v|_f \leq 2 \max\left\{ \int V(x) f^2(v), \left( \int V(x) f^2(v) \right)^{1/2} \right\}.$$

*Proof.* The proof of items $1, 2$ and $3$ can be found in [17]. In order to prove item 4 we take $v \in E$ and notice that, by using the ODE satisfied by $f$ and a straightforward calculation, we get

$$\nabla \left( \frac{f(v)}{f'(v)} \right) = \left( 1 + \frac{2f^2(v)}{1 + 2f^2(v)} \right) \nabla v \tag{2.7}$$

and therefore

$$\left\| \nabla \left( \frac{f(v)}{f'(v)} \right) \right\|_2 \leq 2\|\nabla v\|_2. \tag{2.8}$$

By $(f_6)$, the following holds $1 \leq f(t)/(tf'(t)) \leq 2$, for any $t \neq 0$. Hence, we can use $(f_{11})$ to get

$$f^2\left(\xi \frac{f(t)}{f'(t)}\right) = f^2\left(\frac{f(t)}{tf'(t)}\xi t\right) \leq \left(\frac{f(t)}{tf'(t)}\right)^2 f^2(\xi t) \leq 4f^2(\xi t),$$

for any $t \in \mathbb{R}$, $\xi > 0$. Thus, using the last estimates, we see that

$$\left|\frac{f(v)}{f'(v)}\right|_f = \inf_{\xi>0}\left\{\frac{1}{\xi}\left(1+\int V(x)f^2\left(\xi\frac{f(v)}{f'(v)}\right)\right)\right\} \leq 4|v|_f.$$

Statement 4 follows from the above inequality and (2.8).

We now prove item 5. We may suppose that $v_n(x) \to 0$ a.e. in $\mathbb{R}^N$. Since

$$\|f(v_n)\|_X^2 = \int \left(\frac{|\nabla v_n|^2}{1+2f^2(v_n)} + V(x)f^2(v_n)\right) \leq \int \left(|\nabla v_n|^2 + V(x)f^2(v_n)\right),$$

the sequence $(f(v_n))$ is bounded in $X$. Hence, up to a subsequence, it weakly converges in $X$. The compactness of embedding $X \hookrightarrow L^q(\mathbb{R}^N)$, for $2 \leq q < 2^*$, and the pointwise convergence $f(v_n(x)) \to f(0) = 0$ a.e. in $\mathbb{R}^N$ imply that the weak limit is zero. So, $f(v_n) \to 0$ strongly in $L^q(\mathbb{R}^N)$, whenever $2 \leq q < 2^*$.

It follows from $(f_9)$ that

$$|\nabla(f^2(v_n))|^2 = 4(f(v_n)f'(v_n))^2|\nabla v_n|^2 \leq 2|\nabla v_n|^2.$$

Hence, we can use Sobolev inequality to get

$$\begin{aligned} \|f(v_n)\|_{2\cdot 2^*} &= \|f^2(v_n)\|_{2^*}^{1/2} \leq c_1\|\nabla(f^2(v_n))\|_2^{1/2} \\ &\leq 2^{1/4}c_1\left(\int |\nabla v_n|^2\right)^{1/4} \leq c_2 < \infty. \end{aligned} \tag{2.9}$$

It follows from the interpolation inequality that $f(v_n) \to 0$ in $L^q(\mathbb{R}^N)$ for any $2 \leq q < 2\cdot 2^*$.

For the proof of item 6 we argue as in [9]. By supposing that $v \neq 0$ we shall consider two distinct cases. If $\int V(x)f^2(v) > 1$ we set $\xi_0 := \left(\int V(x)f^2(v)\right)^{-1} < 1$ and use the definition of $|v|_f$ and $(f_{10})$ to get

$$\begin{aligned} |v|_f &\leq \frac{1}{\xi_0}\left(1+\int V(x)f^2(\xi_0 v)\right) \\ &\leq \frac{1}{\xi_0}\left(1+\xi_0\int V(x)f^2(v)\right) = 2\int V(x)f^2(v). \end{aligned}$$

If $0 < \int V(x)f^2(v) \leq 1$ we set $\xi_0 := \left(\int V(x)f^2(v)\right)^{-1/2}$, use $(f_{11})$ and argue as above to conclude that $|v|_f \leq 2(\int V(x)f^2(v))^{1/2}$. This and the above expression finish the proof of item 6. The proposition is proved. □

## 3. The Mountain Pass geometry

In this section we prove that $J$ satisfies the geometry of a version of the Mountain Pass theorem. Before presenting it let us recall that, if $V$ is a real Banach space, we say that $I \in C^1(V, \mathbb{R})$ satisfies the Palais-Smale condition at level $c \in \mathbb{R}$, in short $(PS)_c$, if any sequence $(v_n) \subset V$ such that

$$\lim_{n\to+\infty} I(v_n) = c, \ \lim_{n\to\infty} \|I'(v_n)\|_{V^*} = 0$$

has a convergent subsequence. More generally, we say that $I$ satisfies the Palais-Smale condition, in short $(PS)$, when $I$ satisfies $(PS)_c$ for any level $c \in \mathbb{R}$.

We shall use the following version of the Mountain Pass Theorem.

**Theorem 3.1.** *Let $V$ be a real Banach space, $I \in C^1(V, \mathbb{R})$ and $S \subset V$ a closed subset which arcwise disconnect $V$ in connected components $V_1$ and $V_2$. Suppose further that $I(0) = 0$ and*

*$(I_1)$ $0 \in V_1$ and there exists $\alpha > 0$ such that $I(v) \geq \alpha$ for all $v \in S$ ;*

*$(I_2)$ there exists $e \in V_2$ such that $I(e) \leq 0$.*

*Let*

$$c_0 := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I(\gamma(t)) \geq \alpha, \tag{3.1}$$

*where $\Gamma := \{\gamma \in C([0, 1], E) : \gamma(0) = 0, \gamma(1) \in I^{-1}((-\infty, 0]) \cap V_2\}$. If $I$ satisfies $(PS)_{c_0}$ then $c_0$ is a critical level of $I$.*

We are intending to apply the above theorem with $V$ being the Orlicz-Sobolev space defined in the last section and $I = J$. We first verify that $J$ satisfies the geometric conditions $(I_1)$ and $(I_2)$ of Theorem 3.1.

For each $\rho > 0$ we define the set

$$S_\rho := \left\{ v \in E : \int |\nabla v|^2 + V(x)f^2(v) = \rho^2 \right\}.$$

Since $Q : E \to \mathbb{R}$ given by

$$Q(v) := \int |\nabla v|^2 + \int V(x)f^2(v), \tag{3.2}$$

is continuous we have that $S_\rho$ is a closed subset which disconnects the space $E$.

**Lemma 3.2.** *Suppose $g$ satisfies $(g_1)$ and $(G_0)$ with $\lambda_1(K_0) > 1$. Then there exist $\rho$, $\alpha > 0$ such that*

$$J(v) \geq \alpha, \ \textit{for all } v \in S_\rho.$$

*Proof.* We start by setting $q := 2 \cdot 2^*(\alpha_0 - 1)/\alpha_0$, where $\alpha_0$ comes from the growth condition $(g_1)$. Since $\alpha_0 > N/2$, a straightforward calculation provides $4 < q < 2 \cdot 2^*$. For any given given $\varepsilon > 0$, it follows from $(G_0)$ and $(g_1)$ that

$$G(x, t) \leq \frac{(K_0(x) + \varepsilon)}{2} t^2 + d(x)|t|^q, \text{ for all } (x, t) \in \mathbb{R}^N \times \mathbb{R},$$

for some function $d \in L^{\alpha_0}(\mathbb{R}^N)$. Arguing as in (2.9) we get, for any $v \in E$,

$$\int d(x)|f(v)|^q \le \|d\|_{\alpha_0} \|f(v)\|_{2\cdot 2^*}^q \le c_1 \|d\|_{\alpha_0} \|\nabla v\|_2^{q/2} \le c_1 \|d\|_{\alpha_0} Q^{q/4}(v),$$

for some $c_1 > 0$. Moreover, in view of $(V_1)$, we have that

$$\int f^2(v) \le \frac{1}{V_0} \int V(x) f^2(v) \le \frac{1}{V_0} Q(v).$$

Item 3 of Proposition 2.2 implies that $f(v) \in L^2(\mathbb{R}^N)$. Furthermore, by $(f_2)$, we have that $|\nabla f(v)| \in L^2(\mathbb{R}^N)$. Hence $v \in X$ and the variational characterization of $\lambda_1(K_0)$ provides

$$\int K_0(x) f^2(v) \le \frac{1}{\lambda_1(K_0)} \int \left(|\nabla f(v)|^2 + V(x) f^2(v)\right) \le \frac{Q(v)}{\lambda_1(K_0)}.$$

The three inequalities above imply that, for any $v \in S_\rho$, there holds

$$J(v) \ge \frac{1}{2}\left(1 - \frac{1}{\lambda_1(K_0)} - \frac{\varepsilon}{V_0}\right)\rho^2 - c_1 \|d\|_{\alpha_0} \rho^{q/2}. \tag{3.3}$$

Since $2 < q/2$, if we choose $\varepsilon > 0$ small we obtain $\rho, \alpha > 0$ satisfying the statement of the lemma. The proof is finished. □

**Lemma 3.3.** *Suppose $g$ satisfies $(g_1)$ and $(G_\infty)$. Let $\varphi \in X$ be a positive solution of*

$$-\Delta\varphi + V(x)\varphi = \lambda_1(K_\infty) K_\infty(x)\varphi \ \text{ in } \mathbb{R}^N. \tag{3.4}$$

*If we set*

$$\eta := \frac{\int V(x)\varphi^2}{\int K_\infty(x)\varphi^2} > 0, \tag{3.5}$$

*then we have that*

$$\lim_{s\to+\infty} J(s\varphi) = -\infty$$

*provided $\lambda_1(K_\infty) < (1 + \eta)$.*

*Proof.* It follows from $(g_1)$, $(f_3)$ and $(f_7)$ that, for a.e. $x \in \mathbb{R}^N$, there holds

$$\frac{2|G(x, f(s\varphi))|}{s^2} \le (a(x) + b(x))\varphi^2 \ \in L^1(\mathbb{R}^N), \text{ for all } s \in \mathbb{R},$$

since $(a+b) \in L^{\alpha_0}(\mathbb{R}^N)$ with $\alpha_0 > N/2$. Hence, from Fatou's lemma, $\varphi > 0$, $(G_\infty)$ and $(f_5)$ we infer that

$$\begin{aligned}\liminf_{s\to+\infty} \int \frac{2G(x, f(s\varphi))}{s^2} &\ge \int \liminf_{s\to+\infty} \left(2\frac{G(x, f(s\varphi))}{f^4(s\varphi)} \left(\frac{f(s\varphi)}{\sqrt{s\varphi}}\right)^4 \varphi^2\right) \\ &= \int K_\infty(x)\varphi^2.\end{aligned}$$

On the other hand, using $(f_7)$ again, we have

$$\limsup_{s\to+\infty} \frac{V(x) f^2(s\varphi)}{s^2} \le \lim_{s\to+\infty} \frac{\sqrt{2}V(x)\varphi}{s} = 0,$$

for a.e. $x \in \mathbb{R}^N$. Moreover, from $(f_3)$,

$$\frac{V(x)f^2(s\varphi)}{s^2} \leq V(x)\varphi^2 \in L^1(\mathbb{R}^N),$$

and therefore the Lebesgue Theorem provides

$$\lim_{s\to\infty} \int \frac{V(x)f^2(s\varphi)}{s^2} \;=\; 0.$$

Hence,

$$\begin{aligned}
\limsup_{s\to+\infty} \frac{2J(s\varphi)}{s^2} &= \lim_{s\to\infty} \int \left( |\nabla\varphi|^2 + V(x)\frac{f^2(s\varphi)}{s^2} \right) - \liminf_{s\to+\infty} \int 2\frac{G(x,f(s\varphi))}{s^2} \\
&\leq \int |\nabla\varphi|^2 - \int K_\infty(x)\varphi^2 \\
&= (\lambda_1(K_\infty) - 1) \int K_\infty(x)\varphi^2 \;-\; \int V(x)\varphi^2 \\
&= [\lambda_1(K_\infty) - (1+\eta)] \int K_\infty(x)\varphi^2 \;<\; 0,
\end{aligned}$$

where we have used $\lambda_1(K_\infty) < (1+\eta)$ in the last inequality. The above estimate implies that $J(s\varphi) \to -\infty$ as $s \to +\infty$. □

## 4. Proof of Theorem 1.1

We present now the proof of Theorem 1.1. We first notice that, by Lemma 3.2 the functional $J$ satisfies the condition $(I_1)$ of Theorem 3.1. Lemma 3.3 shows that the condition $(I_2)$ also holds if we take $e := s\varphi$, for $s > 0$ large enough, where $\varphi \in X$ satisfies (3.4), $\eta$ comes from (3.5) and we are supposing that $\lambda_1(K_\infty) < (1+\eta)$. We need only to check that $J$ satisfies the Palais-Smale condition at the level $c_0$ defined in (3.1). This is the content of the Proposition 4.1 below. By assuming its validity for a moment we obtain a critical point $v_0 \in E$ for $J$ satisfying $J(v_0) = c_0 > 0$. Since $J(0) = 0$ we conclude that $v_0 \neq 0$. The growth condition $(g_1)$, the Sobolev embedding, standard bootstrap arguments and elliptic regularity results show that $v_0 \in C^{0,\alpha}_{loc}(\mathbb{R}^N)$ and so in $L^\infty_{loc}(\mathbb{R}^N)$ (cf. [5]). Thus, we conclude that $u_0 := f(v_0)$ is a nontrivial weak solution of the problem $(P)$.

We devote the rest of the paper for the proof of one compactness result and the proof of Theorem 1.2.

**Proposition 4.1.** *Suppose $g$ satisfies $(g_1)$ and $(g_2)$. Then $J$ satisfies the $(PS)_c$ condition for any $c \in \mathbb{R}$.*

*Proof.* Let $(v_n) \subset E$ be such that

$$\lim_{n\to+\infty} J(v_n) = c, \; \lim_{n\to+\infty} J'(v_n) = 0.$$

We first prove that $(v_n)$ is bounded in $E$.

In view of item 4 of Proposition 2.2 we have that $f(v_n)/f'(v_n) \in E$. Hence, we can use (2.7) to compute

$$J'(v_n) \cdot \frac{f(v_n)}{f'(v_n)} \le 2\int |\nabla v_n|^2 + \int V(x) f^2(v_n) - \int g(x, f(v_n)) f(v_n).$$

Hence, we can use item 4 of Proposition 2.2 again to obtain

$$\begin{aligned} c + o_n(1)\|v_n\| &\ge J(v_n) - \frac{1}{4} J'(v_n) \cdot \tfrac{f(v_n)}{f'(v_n)} \\ &\ge \frac{1}{4}\int V(x) f^2(v_n) + \frac{1}{4}\int \left(g(x, f(v_n)) f(v_n) - 4G(x, f(v_n))\right), \end{aligned}$$

and therefore it follows from $(g_2)$ and Hölder inequality that

$$\begin{aligned} \int V(x) f^2(v_n) &\le 4c + o_n(1)\|v_n\| + \int \left(4G(x, f(v_n) - g(x, f(v_n)) f(v_n))\right) \\ &\le 4c + o_n(1)\|v_n\| + \|\Gamma_1\|_1 + \|\Gamma_2\|_\tau \left( \int |f(v_n)|^{q_1 \tau'} \right)^{1/\tau'} \\ &\le 4c + o_n(1)\|v_n\| + \|\Gamma_1\|_1 + 2^{q_1\tau'/4} \|\Gamma_2\|_\tau \|v_n\|_{q_1\tau'/2}^{q_1/2}. \end{aligned}$$

Since $4 \le q_1\tau' \le 2 \cdot 2^*$ we infer from the choice of $\tau$ in $(g_2)$, the embedding $E \hookrightarrow L^{q_1\tau'/2}(\mathbb{R}^N)$ and the above expression that, for some constant $c_1 > 0$, we have that

$$\int V(x) f^2(v_n) \le 4c + o_n(1)\|v_n\| + \|\Gamma_1\|_1 + c_1 \|\Gamma_2\|_\tau \|v_n\|^{q_1/2}. \tag{4.1}$$

Arguing by contradiction we suppose that, up to a subsequence, $\|v_n\| \to +\infty$ as $n \to +\infty$. We define $w_n := v_n/\|v_n\|$ and notice that, since we may suppose that $\|v_n\| \ge 1$, the above inequality, $(f_{10})$ and $q_1 < 2$ provide

$$\int V(x) f^2(w_n) = \int V(x) f^2\left(\frac{v_n}{\|v_n\|}\right) \le \frac{1}{\|v_n\|}\int V(x) f^2(v_n) \to 0.$$

Since $(w_n)$ is bounded in $\mathcal{D}^{1,2}(\mathbb{R}^N)$, up to a subsequence we have that $w_n \rightharpoonup w$ weakly in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ and $w_n(x) \to w(x)$ a.e. in $\mathbb{R}^N$. By Fatou's lemma and the last estimate it follows that $\int V(x) f^2(w) \le \liminf_{n\to\infty} \int V(x) f^2(w_n) = 0$, and therefore $w = 0$. We infer from item 1 of Proposition 2.2 that

$$|w_n|_f \to 0. \tag{4.2}$$

We now claim that

$$\lim_{n\to+\infty} \frac{1}{\|v_n\|^2}\int G(x, f(v_n)) = 0. \tag{4.3}$$

If this is true we can finish the proof of the boundedness of $(v_n)$ by noticing that

$$\int |\nabla w_n|^2 = \frac{2J(v_n)}{\|v_n\|^2} - \frac{1}{\|v_n\|^2}\int V(x) f^2(v_n) + \frac{2}{\|v_n\|^2}\int G(x, f(v_n)) \to 0,$$

where we have used $J(v_n) \to c$, (4.1) and (4.3). This convergence and (4.2) imply that $1 = \|w_n\| = \|w_n\|_2^2 + |w_n|_f \to 0$, which does not make sense. This contradiction shows that $(v_n)$ is bounded.

In the sequel we prove (4.3). We first notice that, since $v_n = \|v_n\|w_n$, we can use $(g_1), (f_3)$ and $(f_7)$ to get

$$\begin{aligned}\frac{|G(x,f(v_n))|}{\|v_n\|^2} &\le \frac{a(x)}{2}\frac{f^2(\|v_n\|w_n)}{\|v_n\|^2} + \frac{b(x)}{4}\frac{f^4(\|v_n\|w_n)}{\|v_n\|^2} \\ &\le \frac{1}{2}\left(a(x)+b(x)\right)w_n^2.\end{aligned}$$

It follows from $(f_8)$ that

$$|t| \le \frac{1}{\kappa}|f(t)| + \frac{1}{k^2}f^2(t), \text{ for all } t \in \mathbb{R}. \tag{4.4}$$

Hence

$$\int \frac{|G(x,f(v_n))|}{\|v_n\|^2} \le c_1 \int (a(x)+b(x))(f^2(w_n)+f^4(w_n)). \tag{4.5}$$

On the other hand, item 5 of Proposition 2.2 implies that, up to a subsequence,

$$f(w_n) \to 0 \text{ strongly in } L^q(\mathbb{R}^N) \text{ for any } 2 \le q < 2\cdot 2^*. \tag{4.6}$$

Recalling that $b \in L^{\alpha_0}(\mathbb{R}^N)$ with $\alpha_0 > N/2$, we can use Hölder's inequality to get

$$\int b(x)f^4(w_n) \le \|b\|_{\alpha_0}\|f(w_n)\|^4_{4\alpha_0/(\alpha_0-1)} \to 0,$$

where we have used (4.6) and the fact that $4 < 4\alpha_0/(\alpha_0-1) < 2\cdot 2^*$. The same argument shows that

$$\max\left\{\int a(x)f^4(w_n), \int a(x)f^2(w_n), \int b(x)f^2(w_n)\right\} \to 0.$$

The proof of (4.3) follows from the above expression and (4.5). Hence, we conclude that $(v_n)$ is bounded.

The boundedness of $(v_n)$ implies that, for some $v \in \mathcal{D}^{1,2}(\mathbb{R}^N)$, we have that $v_n \rightharpoonup v$ weakly in $\mathcal{D}^{1,2}(\mathbb{R}^N)$. Since we also have pointwise convergence we can use (4.1) and Fatou's lemma to get

$$\int V(x)f^2(v) \le \liminf_{n\to+\infty}\int V(x)f^2(v_n) < \infty, \tag{4.7}$$

and therefore the weak limit $v$ belongs to $E$. In the sequel we shall prove that $\|v_n - v\| \to 0$.

We start by noticing that, since $f^2$ is convex, the function $Q$ defined (3.2) is also convex. Hence,

(4.8)

$$\begin{aligned}Q(v)-Q(v_n) &\ge Q'(v_n)\cdot(v-v_n) \\ &= 2J'(v_n)\cdot(v-v_n) + 2\int g(x,f(v_n))f'(v_n)(v-v_n).\end{aligned}$$

We claim that

$$\lim_{n\to+\infty}\int g(x,f(v_n))f'(v_n)(v-v_n)=0. \tag{4.9}$$

Assuming the claim, recalling that $J'(v_n)\to 0$ and taking the limit in (4.8) we get

$$\limsup_{n\to+\infty} Q(v_n)\le Q(v).$$

On the other hand, the weak converge of $(v_n)$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ provides

$$\int |\nabla v|^2 \le \liminf_{n\to+\infty}\int |\nabla v_n|^2. \tag{4.10}$$

Hence, we infer from (4.7) that $Q(v)\le \liminf_{n\to+\infty} Q(v_n)$, and therefore

$$\lim_{n\to+\infty} Q(v_n)=Q(v). \tag{4.11}$$

Before continuing the proof we justify the convergence in (4.9). From $(f_{12})$ and (4.7) we conclude that $(\int V(x)f^2(v_n-v))$ is a bounded sequence. Hence, the weak convergence $(v_n-v)\rightharpoonup 0$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ and item 5 of Proposition 2.2 imply that

$$f(v_n-v)\to 0 \text{ strongly in } L^q(\mathbb{R}^N) \text{ for all } 2\le q<2\cdot 2^*. \tag{4.12}$$

On the other hand, from $(g_1)$, $(f_2)$, $(f_9)$, we get

$$|g(x,f(v_n))f'(v_n)| \;\le\; c_1(a(x)+b(x))(|f(v_n)|+f^2(v_n)).$$

The above expression and inequality (4.4) provide $c_2>0$ such that

$$|g(x,f(v_n))f'(v_n)||v_n-v|\le c_2\psi(x)h_n(x)(|f(v_n-v)|+f^2(v_n-v)), \tag{4.13}$$

with $\psi(x):=c_1(a(x)+b(x))\in L^{\alpha_0}(\mathbb{R}^N)$ and $h_n(x):=|f(v_n(x))|+f^2(v_n(x))$. If we set $q:=2\alpha_0/(\alpha_0-1)$ we can use $\alpha_0>N/2$ to conclude that $2<q<2^*$. Hence, the embedding $E\hookrightarrow L^q(\mathbb{R}^N)$, $(f_3)$ and $(f_7)$ imply that the sequence $h_n$ is bounded in $L^q(\mathbb{R}^N)$. It follows from Hölder's inequality that

$$\int \psi(x)h_n(x)f^2(v_n-v)\le \|\psi\|_{\alpha_0}\|h_n\|_q\|f(v_n-v)\|_{2q}^2\to 0,$$

where we have used $4<2q<2\cdot 2^*$ and (4.12). Analogously,

$$\int \psi(x)h_n(x)|f(v_n-v)|\to 0.$$

The statement (4.9) is a consequence of inequality (4.13) and the two convergences above.

By using (4.11) we obtain

$$\begin{aligned} Q(v) &= \liminf_{n\to+\infty} Q(v_n)\\ &\ge \liminf_{n\to+\infty}\int |\nabla v_n|^2+\liminf_{n\to+\infty}\int V(x)f^2(v_n)\\ &\ge \int |\nabla v|^2+\int V(x)f^2(v)=Q(v) \end{aligned}$$

We infer from the above inequality, (4.7) and (4.10) that
(4.14)
$$\liminf_{n\to+\infty}\int|\nabla v_n|^2=\int|\nabla v|^2,\quad \liminf_{n\to+\infty}\int V(x)f^2(v_n)=\int V(x)f^2(v).$$
Hence
$$\begin{aligned}Q(v) &= \limsup_{n\to+\infty}\left(\int|\nabla v_n|^2+\int V(x)f^2(v_n)\right)\\ &\geq \limsup_{n\to+\infty}\int|\nabla v_n|^2+\liminf_{n\to+\infty}\int V(x)f^2(v_n)\\ &\geq \liminf_{n\to+\infty}\left(\int|\nabla v_n|^2+\int V(x)f^2(v_n)\right)=Q(v),\end{aligned}$$
and therefore we conclude that
$$\limsup_{n\to+\infty}\int|\nabla v_n|^2=\int|\nabla v|^2.$$
This and (4.14) imply that $\|v_n\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)}\to\|v\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)}$. So, the weak convergence of $(v_n)$ imply that $v_n\to v$ strongly in $\mathcal{D}^{1,2}(\mathbb{R}^N)$, that is

$$\lim_{n\to+\infty}\|\nabla(v_n-v)\|_2=0. \tag{4.15}$$

Arguing as above we can also conclude that
$$\limsup_{n\to+\infty}\int V(x)f^2(v_n)=\int V(x)f^2(v).$$
and therefore we have that $\sqrt{V(x)f(v_n)}\to\sqrt{V(x)f(v)}$ strongly in $L^2(\mathbb{R}^N)$. Thus, up to a subsequence, we have that $\sqrt{V(x)f(v_n)}\leq\varphi(x)$ a.e. in $\mathbb{R}^n$ for some $\varphi\in L^2(\mathbb{R}^N)$. Thus, we can use $(f_{12})$ to obtain
$$V(x)f^2(v_n-v)\leq 4(V(x)f^2(v_n)+V(x)f^2(v))\leq 4(\varphi(x)^2+V(x)f^2(v)).$$
Since the right-hand side above belongs to $L^1(\mathbb{R}^N)$ it follows from the Lebesgue Theorem that $\int V(x)f^2(v_n-v)\to 0$. Thus, the item 1 of Proposition 2.2 implies that
$$\lim_{n\to+\infty}|v_n-v|_f=0.$$
By using this equality and (4.15) we conclude that
$$\lim_{n\to+\infty}\|v_n-v\|=\lim_{n\to+\infty}(\|\nabla(v_n-u)\|_2+|v_n-v|_f)=0$$
and the proposition is proved. □

## 5. Proof of Theorem 1.2

In order to obtain a positive solution we define $g_+:\mathbb{R}^N\times\mathbb{R}^N\to\mathbb{R}$ by setting
$$g_+(x,t):=\begin{cases}g(x,t), & \text{if } t\geq 0,\\ 0, & \text{if } t<0,\end{cases}$$

and consider the new functional

$$J_+(v) := \frac{1}{2}\int(|\nabla u|^2 + V(x)f^2(u)) - \int G_+(x, f(v)),\ v \in E,$$

where $G_+(x,t) := \int_0^t g_+(x,\tau)\mathrm{d}\tau$ is a primitive of $g_+$.

For any $v \in E$ we have that $\int G_+(x, f(v)) = \int G(x, f(v^+))$ and therefore the same conclusion of Lemma 3.2 holds for $J_+$. Moreover, since the function $\varphi$ in Lemma 3.3 is positive, it is easy to check that the lemma also holds for the functional $J_+$. In the sequel we focus on the proof of the Palais-Smale condition.

Let $(v_n) \subset E$ be a $(\mathrm{PS})_c$ sequence for $J_+$. We set $v_n^-(x) := \min\{v_n(x), 0\}$ and claim that $\|v_n^-\| \to 0$ as $n \to +\infty$. Indeed, if this is not true, there exists $\beta > 0$ such that, up to a subsequence, $\|v_n^-\| \geq \beta > 0$. It follows from item 4 of Proposition 2.2, $f(0) = 0$, $f'(0) = 1$ and the definition of $g_+$ that

$$\begin{aligned} o_n(1)\|v_n^-\| &\geq J_+'(v_n)\cdot\frac{f(v_n^-)}{f'(v_n^-)} \\ &= \int\left(1 + \frac{2f^2(v_n^-)}{1+2f^2(v_n^-)}\right)|\nabla v_n^-|^2 + \int V(x)f(v_n)f'(v_n)\frac{f(v_n^-)}{f'(v_n^-)} \\ &\geq \int |\nabla v_n^-|^2 + V(x)f^2(v_n^-). \end{aligned}$$

Since $\|v_n^-\| \geq \beta > 0$, it is well defined $w_n := \frac{v_n^-}{\|v_n^-\|}$. The above inequalities provide

$$\max\left\{\int|\nabla w_n|^2, \int V(x)\frac{f^2(v_n^-)}{\|v_n^-\|}, \int V(x)\frac{f^2(v_n^-)}{\|v_n^-\|^2}\right\} \to 0,$$

as $n \to +\infty$. Hence, we infer from $(f_{10})$ and $(f_{11})$ that

$$\int V(x)f^2(w_n) = \int V(x)f^2\left(\frac{v_n^-}{\|v_n^-\|}\right) \leq \left(\frac{1}{\|v_n^-\|} + \frac{1}{\|v_n^-\|^2}\right)\int V(x)f^2(v_n^-) \to 0.$$

It follows from item 6 of Proposition 2.2 that $|w_n|_f \to 0$ as $n \to +\infty$. Thus, $1 = \|w_n\| = \|\nabla w_n\|_2 + |w_n|_f \to 0$, which does not make sense. Since $v_n^- \to 0$ as $n \to +\infty$, replacing $(v_n)$ by $(v_n^+)$ if necessary, we may suppose that $v_n \geq 0$. Thus, we can argue along the same lines of Proposition 4.1 to conclude that $(v_n)$ has a convergent subsequence, that is, the truncated functional $J_+$ satisfies the Palais-Smale condition.

As in the proof of Theorem 1.1 we obtain a nonzero critical point $v_0$ for $J_+$. Since $J_+'(v_0)v_0^- = 0$ we can argue as above to conclude that $v_0^- \equiv 0$, and therefore $v_0 \geq 0$ a.e. in $\mathbb{R}^N$. Elliptic regularity theory and the maximum principle imply that $v_0 > 0$ in $\mathbb{R}^N$.

The existence of a negative solution can be proved in the same way just considering $g_- : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ given by

$$g_-(x,t) = \begin{cases} 0, & \text{if } t > 0, \\ g(x,t), & \text{if } t \leq 0 \end{cases}$$

We omit the details.

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