ELLIPTIC EQUATIONS WITH WEIGHT AND COMBINED NONLINEARITIES

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Abstract. We consider the equation

$$-\mathrm{div}(a(x)\nabla u) = b(x)|u|^{q-2}u + c(x)|u|^{p-2}u,\ u \in H_0^1(\Omega),$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain and $N \geq 4$. The functions a, b and c satisfy some hypotheses which provide a variational structure for the problem. For $1 < q < 2 < p \leq 2N/(N-2)$ we obtain the existence of two nonzero solutions if the function b has small Lebesgue norm. In the proofs we apply minimization arguments and the Mountain Pass Theorem.

1. Introduction

In this paper we are concerned with the existence of nonnegative solutions for the elliptic equation

$$(P) \qquad \begin{cases} -\mathrm{div}(a(x)\nabla u) = b(x)|u|^{q-2}u + c(x)|u|^{p-2}u, & x \in \Omega, \\ u = 0, & \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 4$ and $1 < q < 2 < p \leq 2^* := 2N/(N-2)$. The positive weight $a : \overline{\Omega} \to \mathbb{R}$ is such that

(a_1) $a \in H^1(\Omega) \cap C(\overline{\Omega})$;

(a_2) there exists $x_0 \in \Omega$ such that

$$a(x_0) = a_0 := \min\{a(x) : x \in \overline{\Omega}\} > 0.$$

Denoting by s' the conjugated exponent of $s > 1$, the basic assumptions on the potentials b and c are the following:

(b_1) $b \in L^{\sigma_q}(\Omega)$ for some $(p/q)' < \sigma_q \leq (2/q)'$;

(b_2) there exists a nonempty open subset $\Omega_b^+ \subset \Omega$ such that $b(x) > 0$ for a.e. $x \in \Omega_b^+$;

(c_1) $c \in L^\infty(\Omega)$, with $c \not\equiv 0$;

(c_2) there exists a nonempty open subset $\Omega_c^+ \subset \Omega$ such that $c(x) > 0$ for a.e. $x \in \Omega_c^+$.

In out first result we consider the subcritical case and prove the following

Theorem 1.1. *Suppose that $1 < q < 2 < p < 2^*$ and the potentials a, b and c satisfy $(a_1) - (a_2)$, $(b_1) - (b_2)$ and $(c_1) - (c_2)$, respectively. Then the problem (P) has at least two nonnegative nontrivial solutions if $|b|_{L_q^{\sigma}(\Omega)}$ is small.*

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In our second result we consider the critical version of (P), namely $p = 2^*$. In this setting we have some additional difficulties due to the lack of compactness of the embedding $H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$. We overcome it with the following technical assumptions

(a_3) there exist $k > 2$, $\beta_k > 0$ and θ such that, in a small neighborhood of x_0,

$$a(x) = a_0 + \beta_k |x - x_0|^k + \theta(x)|x - x_0|^k,$$

with $\lim_{x \to x_0} \theta(x) = 0$;

(c_3) there exist $\gamma > (N-2)/2$ and M, $\delta > 0$ such that

$$|c|_{L^\infty(\Omega)} - c(x) \leq M|x - x_0|^\gamma, \qquad \text{for a.e. } x \in B_\delta(x_0),$$

and both the potentials $b(x)$ and $c(x)$ are positive a.e. in $B_\delta(x_0)$.

In the main result of this paper we prove the following

Theorem 1.2. *Suppose that $1 < q < 2 < p = 2^*$ and the potentials a, b and c satisfy $(a_1) - (a_3)$, (b_1), (c_1) and (c_3). Then the problem (P) has at least two nonnegative nontrivial solutions if $|b|_{L_q^\sigma(\Omega)}$ is small and $k > (N-2)/2$.*

In the proof of Theorem 1.1 we apply variational methods. After introducing the energy functional associated to (P) we prove that, taking the L^{σ_q}-norm of b smaller if necessary, it achieves a negative infimum on a small ball centered at the origin. A second solution is obtained as an application of the Mountain Pass Theorem centered at the first solution.

Even though the proof of Theorem 1.2 follows the same lines, the arguments are more involved. Since the range of compactness is affected by the critical growth of the nonlinearity, we need to use the ideas introduced in [3] as well as some estimates proved in [9]. The assumption (c_3) plays a key role at this point. Actually, this condition is a version of one which already appeared in [7]. The assumption $k > (N-2)/2$ is also important in our trick calculations. Although we do not know if it is necessary, we would like to cite the paper [8], where the authors considered an analogous problem in the whole $\mathbb{R}^N$, but for the operator $u \mapsto \mathrm{div}(e^{|x|^\alpha/4} \nabla u)$. In that paper it was also imposed a condition relating α and the dimension N. In some sense, the notion of critical dimension for the problem is related with the behaviour of the potential $a(x)$ near its minima .

The starting point of the study of problem (P) is the work of Ambrosetti, Brezis and Cerami [1], where the authors considered the case $a(x) \equiv 1$, $b(x) \equiv \lambda$, $c(x) \equiv 1$ and proved that, for some $\lambda^* > 0$ the following holds: the problem has two positive solutions if $\lambda \in (0, \lambda^*)$, one positive solution if $\lambda = \lambda^*$ and no positive solution if $\lambda > \lambda^*$. After this work, many results with combined nonlinearities have appeared. Since it impossible to give a complet list of reference we cite [2, 5, 4, 10, 12, 6, 11, 7, 13] and the references therein.

In [7], among other results, deFigueiredo, Gossez and Ubilla considered the case that $a(x) \equiv 1$ and b, c were sign changing potentials. They proved that the problem has two nonnegative nonzero solution if b has small norm. Concerning the case $a(x) \not\equiv 1$, we cite the paper of Hadiji and Yazidi [9], where they consider $p = 2^*$, $q = 2$, $b(x) \equiv \lambda$ and $c(x) \equiv 1$. They proved that the existence of positive solutions is related with the iteration of the parameter λ with the first eigenvalue of the operator $-\mathrm{div}(a(x)\nabla \cdot)$ in $H_0^1(\Omega)$.

In view of the aforementioned works it is natural to ask if you can extend some of the results of [1, 6] for the operator $\mathrm{div}(a(x)\nabla\cdot)$. The results of this paper provide a partial answer for this question. Hence, our results can be viewed as a complement of these two papers.

2. The subcritical case

For any $2 \le \tau \le \infty$ we denote by $|u|_\tau$ the L^τ-norm of a function $u \in L^\tau(\Omega)$. For $\tau \in [2, 2^*]$, we consider the constant

$$S_\tau := \inf\left\{ \int |\nabla u|^2 \ : \ |u|_\tau = 1 \right\} < +\infty \tag{2.1}$$

and set $S := S_{2^*}$. For any measurable function f we write only $\int f$ to indicate $\int_\Omega f(x)dx$. Throughout the paper we suppose that a satisfies $(a_1) - (a_2)$.

Let H be the space $H_0^1(\Omega)$ endowed with the norm $\|u\| := (\int_\Omega |\nabla u|^2)^{1/2}$. According to conditions $(a_1) - (a_2)$, the quantity $\|u\|_a := (\int a(x)|\nabla u|^2)^{1/2}$ is an equivalent norm in this space. For any $u \in H$, setting $u^+(x) := \max\{u(x), 0\}$, we obtain from Hölder's inequality

$$\left| \int b(x)(u^+)^q \right| \le |b|_{\sigma_q} \left(\int |u|^{q\sigma_q'} \right)^{1/\sigma_q'}. \tag{2.2}$$

Since $2 \le q\sigma_1 < p \le 2^*$, the right-hand side above is finite. Thus, by using some standard calculations we can show that the functional $I : H \to \mathbb{R}$ give by

$$I(u) := \frac{1}{2}\int a(x)|\nabla u|^2 - \frac{1}{q}\int b(x)(u^+)^q - \frac{1}{p}\int c(x)(u^+)^p$$

is well define and $I \in C^1(H, \mathbb{R})$. Moreover, if u is a critical point of I, then it is a weak solution of (P). If this is the case we have that $0 = I'(u)u^- = \|u^-\|_a^2$, and therefore $u \ge 0$ in Ω. Hence, in order to obtain nonnegative solutions for (P), we just need to find critical points of I.

We shall obtain our first critical point by applying a minimization procedure, as showed by the next two lemmas.

Lemma 2.1. *Suppose that b satisfies (b_1) and $|b|_{\sigma_q}$ is small enough. Then, there exist $\rho, \alpha > 0$ such that $I(u) \ge \alpha > 0$, for any $u \in H$ such that $\|u\| = \rho$.*

Proof. It follows from (a_2), (2.2) and (2.1) that

$$\begin{aligned} I(u) &\ge \frac{a_0}{2}\|u\|^2 - \frac{1}{q}|b|_{\sigma_q}|u|_{q\sigma_q'}^q - \frac{1}{p}|c|_\infty |u|_p^p \\ &\ge a_0\frac{\|u\|^q}{2}\left\{ \|u\|^{2-q} - \frac{2}{a_0 p}|c|_\infty S_p^{-p/2}\|u\|^{p-q} - \frac{2}{a_0 q}|b|_{\sigma_q} S_{q\sigma_q'}^{-q/2} \right\}. \end{aligned}$$

For $B := 2(pa_0)^{-1}|c|_\infty S_p^{-p/2}$, the function $f : (0, +\infty) \to \mathbb{R}$, given by $f(t) := t^{2-q} - Bt^{p-q}$ achieves its maximum value at

$$t_0 := \left[\frac{(2-q)}{B(p-q)} \right]^{1/(p-2)} > 0.$$

For $M := f(t_0)$ and $\|u\| = t_0$, we have that

$$I(u) \ge a_0\frac{t_0^q}{2}\left\{ M - \frac{2}{qa_0}|b|_{\sigma_q} S_{q\sigma_q'}^{-q/2} \right\} \ge \frac{t_0^q}{2}\frac{M}{2} > 0,$$

whenever

$$|b|_{\sigma_q} \leq Mqa_0 S_{q\sigma_q'}^{-q/2}/4. \tag{2.3}$$

The lemma holds for $\alpha := t_0^q M/4$, $\rho := t_0$ and $|b|_{\sigma_q}$ as above. □

Lemma 2.2. *Suppose that b satisfies (b_1), (b_2) and (2.3). If $\rho > 0$ is given by Lemma 2.1, then*

$$-\infty < I_0 := \inf_{u \in \overline{B_\rho(0)}} I(u) < 0$$

is achieved at $u_0 \in B_\rho(0)$ which is a nonnegative solution of (P).

Proof. A straightforward calculation shows that I maps bounded sets into bounded sets, and therefore I_0 is finite. Since Ω_b^+ has nonempty interior there exists $\delta_1 > 0$ and $x_1 \in \Omega$ such that $B_{\delta_1}(x_1)$ is contained in the set Ω_b^+. Hence, we can take a nonnegative function $\varphi \in C_0^\infty(B_{\delta_1}(x_1))$ such that $\int b(x)\varphi^q > 0$. Since $q < 2 < p$, we have that

$$\limsup_{t \to 0^+} \frac{I(t\varphi)}{t^q} \leq -\frac{1}{q} \int b(x)\varphi^q < 0.$$

So, for $t > 0$ small, we have that $I(t\varphi) < 0$, and therefore $I_0 < 0$.

Let $(u_n) \subset \overline{B_\rho(0)}$ be a minimizing sequence for I_0. By Ekeland's Variational Principle we may assume that $I(u_n) \to I_0$ and $I'(u_n) \to 0$. Since (u_n) is bounded and $2 \leq q\sigma_q' < 2^*$ we have that, up to a subsequence,

$$\begin{aligned} &u_n \rightharpoonup u_0 \text{ weakly in } H, \\ &u_n \to u_0 \text{ strongly in } L^{q\sigma'_q}(\Omega), \\ &u_n^+(x) \to u_0^+(x),\ |u_n(x)| \leq \psi(x) \text{ for a.e. } x \in \Omega, \end{aligned} \tag{2.4}$$

for some $\psi \in L^{q\sigma'_q}(\Omega)$. Young's inequality provides

$$|b(x)(u_n^+)^q| \leq \frac{1}{\sigma_q} b(x)^{\sigma_q} + \frac{1}{\sigma_q'}\psi(x)^{q\sigma'_q}, \quad \text{for a.e. } x \in \Omega.$$

Since $\psi \in L^{q\sigma'_q}(\Omega)$ and $b \in L^{\sigma_q}(\Omega)$, it follows from (2.4) and the Lebesgue theorem that

$$\lim_{n\to\infty} \int b(x)(u_n^+)^q = \int b(x)(u_0^+)^q.$$

We now claim that $I'(u_0) = 0$. Assuming the claim, the above equality and the weak convergence of (u_n) provide

$$\begin{aligned} I_0 &= \liminf_{n\to\infty} \left(I(u_n) - \frac{1}{p} I'(u_n)u_n \right) \\ &= \liminf_{n\to\infty} \left\{ \left(\frac{1}{2} - \frac{1}{p}\right) \|u_n\|_a^2 + \left(\frac{1}{p} - \frac{1}{q}\right) \int b(x)(u_n^+)^q \right\} \\ &\geq \left(\frac{1}{2} - \frac{1}{p}\right)\|u_0\|_a^2 - \frac{1}{q}\int b(x)(u_0^+)^q = I(u_0) - \frac{1}{p}I'(u_0) = I(u_0), \end{aligned}$$

and therefore $I(u_0) = I_0 < 0$. By Lemma 2.1, $\|u_0\| \neq \rho$. Hence the infimum is achieved at $u_0 \in B_\rho(0)$. Since $I'(u_0) = 0$, the function u_0 is nonnegative.

It remains to prove that $I'(u_0) = 0$. Let us denote by A the support of $\varphi \in C_0^\infty(\Omega)$. Since $\sigma_q > (p/q)' = p/(p-q)$, we can choose $q_0 \in (2, p)$ such that

$$\sigma_q > \frac{q_0}{q_0 - q} > \frac{q_0}{(q_0+1) - q}.$$

Thus

$$\frac{1}{\sigma_q} + \frac{q-1}{q_0} < \frac{(q_0+1)-q}{q_0} + \frac{q-1}{q_0} = 1$$

and there exists $\theta > 1$ satisfying

$$\frac{1}{\sigma_q} + \frac{1}{q_0/(q-1)} + \frac{1}{\theta} = 1.$$

The above inequality implies that $u_n \to u_0$ in $L^{q_0}(\Omega)$ and provides ψ_{q_0} such that $|u_n(x)| \leq \psi_{q_0}(x)$ a.e. in Ω. By Young's inequality there is $C > 0$ such that

$$|b(x)(u_n^+)^{q-1}\varphi| \leq C(|b(x)|^{\sigma_q} + |\psi_{q_0}(x)|^{q_0} + |\varphi|^{\theta}), \quad \text{for a.e. } x \in A.$$

It follows from the Lebesgue theorem that

$$\lim_{n\to+\infty} \int b(x)(u_n^+)^{q-1}\varphi = \int b(x)(u_0^+)^{q-1}\varphi.$$

An analogous argument holds for $\int c(x)(u_n^+)^{r-1}\varphi$, and therefore we conclude that $0 = \lim_{n\to+\infty} I'(u_n)\varphi = I'(u_0)\varphi$, for all $\varphi \in C_0^\infty(\Omega)$. The result follows by density. □

Lemma 2.3. *Suppose that b and c satisfy (b_1) and $(c_1)-(c_2)$, respectively, and let $B_{\delta_1}(x_1) \subset \Omega_c^+$. If u_0 is given by Lemma 2.2 and $\varphi \in C_0^\infty(B_{\delta_1}(x_1)) \setminus \{0\}$ is nonnegative, then*

$$\lim_{t\to+\infty} I(u_0 + t\varphi) = -\infty.$$

Proof. Since $\varphi = 0$ outside $B_{\delta_1}(x_1) \subset \Omega_c^+$ and $u_0 \geq 0$ a.e. in Ω, we can easily compute

$$\begin{aligned} I(u_0+t\varphi) &\leq O(t^2) + O(t^q) + O(1) - \int_{\{c>0\}} c(x)(u_0+t\varphi)^p dx \\ &\leq O(t^2) + O(1) - \frac{t^p}{p}\int_{B_{\delta_1}(x_1)} c(x)\varphi^p dx. \end{aligned}$$

Since $p > 2$, the result follows from the positivity of the last integral above. □

We recall that $I \in C^1(H, \mathbb{R})$ satisfies the Palais-Smale condition at level $d \in \mathbb{R}$ ($(PS)_d$ for short), if any sequence $(u_n) \subset H$ such that $I'(u_n) \to 0$ and $I(u_n) \to d$ has a convergent subsequence.

Lemma 2.4. *If $2 < p < 2^*$ then the functional I satisfies the $(PS)_d$ condition for any $d \in \mathbb{R}$.*

Proof. Let $(u_n) \subset H$ be such that $I(u_n) \to d$ and $I'(u_n) \to 0$. We have that

$$d + \|u_n\| + o(1) \geq a_0\left(\frac{1}{2} - \frac{1}{p}\right)\|u_n\|^2 - \left(\frac{1}{q} - \frac{1}{p}\right)\int b(x)(u_n^+)^q.$$

Hölder's inequality and the embedding $H \hookrightarrow L^\tau(\Omega)$ provide $C_1 > 0$ such that

$$d + \|u_n\| + o(1) \geq a_0\left(\frac{1}{2} - \frac{1}{p}\right)\|u_n\|^2 - \left(\frac{1}{q} - \frac{1}{p}\right)C_1\|u_n\|^q,$$

and therefore (u_n) is bounded in H. Up to a subsequence, we have that $u_n \rightharpoonup u$ weakly in H and $u_n \to u$ strongly in $L^\tau(\Omega)$, for $2 \leq \tau < 2^*$. By definition of σ_q, there exists $2 \leq p_0 < p$ such that $\sigma_q = (p_0/q)' = p_0/(p_0 - q)$. So

$$\frac{1}{\sigma_q} + \frac{1}{p_0(q-1)} + \frac{1}{p_0} = \frac{p_0 - q}{p_0} + \frac{q-1}{p_0} + \frac{1}{p_0} = 1.$$

It follows from Hölder's inequality that

$$\left| \int b(x)(u_n^+)^{q-1}(u_n - u) \right| \leq |b|_{\sigma_q} |u_n|_{p_0}^{q-1} |u_n - u|_{p_0} = o(1),$$

as $n \to +\infty$. Since an analogous argument holds for $\int c(x)(u_n^+)^{p-1}(u_n - u)$, we obtain $o(1) = I'(u_n)(u_n - u) = \|u_n\|_a^2 - \|u\|_a^2 + o(1)$. Thus $\|u_n\|^2 \to \|u\|^2$ and it follows from the weak convergence of (u_n) that, along a subsequence, it converges. □

Proof of Theorem 1.1. If $|b|_{\sigma_q}$ is small, we can use Lemma 2.2 to obtain a nonnegative solution u_0 such that $I(u_0) < 0$. For the second one, we take $\rho > 0$ and φ as in Lemmas 2.1 and 2.3, respectively. Let $t_0 > 0$ be such that $e := u_0 + t_0\varphi$ satisfies $I(e) \leq I(u_0)$. If we define

$$\widetilde{d} := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)),$$

where $\Gamma := \{\gamma \in C([0,1], H) \ : \ \gamma(0) = u_0, \ \gamma(1) = e\}$, by Lemma 2.1 we get $\widetilde{d} \geq \alpha > 0$. Moreover, the Mountain Pass Theorem provide $(u_n) \subset H$ such that $I(u_n) \to \widetilde{d}$ and $I'(u_n) \to 0$. By Lemma 2.4, along a subsequence, $u_n \to u_1$ in H. Hence $I(u_1) = \widetilde{d} > 0$ and $I'(u_1) = 0$, in such way that we have obtained a second (nonnegative) solution. □

3. The critical case

We deal in this section with the critical case $p = 2^*$. Since (c_3) implies (b_2) and (c_2), a simple inspection of the proof of Lemma 2.2 shows that it remains true under the hypotheses of Theorem 1.2. So, hereafter we denote by u_0 a solution of (P) with negative energy. In order to obtain a second solution we need to modify the argument, since the embedding $H \hookrightarrow L^{2^*}(\Omega)$ is no longer compact. Firstly, we follow [3] to obtain the following local compactness result.

Lemma 3.1. *If $p = 2^*$ and u_0 is the only nontrivial critical point I, then I satisfies $(PS)_d$ for*

$$d < d^* := I(u_0) + \frac{1}{N} \frac{(a_0 S)^{N/2}}{|c|_\infty^{(N-2)/2}}.$$

Proof. Let $(u_n) \subset H$ be such that $I'(u_n) \to 0$ and $I(u_n) \to d$. As in the proof of Lemma 2.4 we can show that it is bounded. Hence, along a subsequence, we have that $u_n \rightharpoonup u$ weakly in H and $u_n \to u$ strongly in $L^{q\sigma'_q}(\Omega)$. Setting $v_n := u_n - u$, we can use the last convergence and the Brezies-Lieb lemma to get

$$\begin{aligned} o(1) = I'(u_n)u_n &= \|u_n\|_a^2 - \int b(x)(u_n^+)^q - \int c(x)(u_n^+)^{2^*} \\ &= \|u\|_a^2 + \|v_n\|_a^2 - \int b(x)(u^+)^q + o(1) \\ &- \int c(x)(u^+)^{2^*} - \int c(x)(v_n^+)^{2^*} \\ &= I'(u)u + \|v_n\|_a^2 - \int c(x)(v_n^+)^{2^*} + o(1). \end{aligned}$$

As in the proof of Lemma 2.2, we have that $I'(u) = 0$. Hence, there exists $l \geq 0$, such that

$$\lim_{n\to\infty} \|v_n\|_a^2 = l = \lim_{n\to\infty} \int c(x)(v_n^+)^{2^*}.$$

If $l = 0$ then $u_n \to u$ strongly in H and we have done. So, we may suppose that $l > 0$. It follows from the definition of S and a_0 that

$$\left(\int c(x)(v_n^+)^{2^*}\right)^{2/2^*} \leq \frac{|c|_\infty^{2/2^*}}{a_0 S} \int a(x)|\nabla v_n|^2.$$

Taking the limit we obtain

$$l \geq \frac{(a_0 S)^{N/2}}{|c|_\infty^{(N-2)/2}}. \tag{3.1}$$

On the other hand, arguing as in the beginning of the proof, we obtain

$$d + o(1) = I(u_n) = I(u) + \frac{1}{2}\|v_n\|_a^2 - \frac{1}{2^*}\int c(x)(v_n^+)^{2^*} + o(1).$$

Taking the limit again and using (3.1), we get

$$d = I(u) + \left(\frac{1}{2} - \frac{1}{2^*}\right) l = I(u) + \frac{l}{N} \geq I(u) + \frac{1}{N}\frac{(a_0 S)^{N/2}}{|c|_\infty^{(N-2)/2}}.$$

But we are assuming that the only critical points are $u = 0$ and $u = u_0$. Since $\max\{I(0), I(u_0)\} \leq 0$, the above inequality contradicts $d < d^*$. □

We are now ready to present the proof of our main result.

Proof of Theorem 1.2. We know that the problem has a nontrivial solution u_0 such that $I(u_0) < 0$. Arguing by contradiction, we suppose that this the only nontrivial critical point of I. As in the proof of Theorem 1.1, we set $e := u_0 + tv_\varepsilon$, with $t > 0$ large in such way that $I(e) \leq I(u_0)$, and define the Mountain Pass level

$$\widetilde{d} := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I(\gamma(t)), \tag{3.2}$$

where $\Gamma := \{\gamma \in C([0,1], H) \ : \ \gamma(0) = u_0,\ \gamma(1) = e\}$. We obtain $(u_n) \subset H$ satisfying $I'(u_n) \to 0$ and $I(u_n) \to \widetilde{d}$. It follows from Proposition 3.2, which we state and prove in the sequel, that $\widetilde{d} < d^*$. Hence, Lemma 3.1 implies that, along a subsequence, (u_n) strongly converges to a solution with positive energy. But this is a contradiction with the assumption that 0 and u_0 are the only critical points of I. Hence, we conclude that there exist a nonzero solution u_1 such that $u_1 \neq u_0$. As before, we have that $u_1 \geq 0$ a.e. in Ω. □

We finish the paper proving that the level $\widetilde{d}$ defined above is smaller than d^*.

Proposition 3.2. *Suppose that* b *and* c *satisfy* (b_1), (c_1) *and* (c_3). *If* $N < (2k+2)$, *then*

$$\max_{t>0} I(u_0 + tv_\varepsilon) < d^* = I(u_0) + \frac{1}{N}\frac{(a_0 S)^{N/2}}{|c|_\infty^{(N-2)/2}},$$

for $\varepsilon > 0$ *small enough. In particular, the minimax level defined in* (3.2) *satisfies* $\widetilde{d} < d^*$.

Proof. Let $\delta > 0$ be given by hypothesis (c_3) and consider $\psi \in C_0^\infty(\Omega)$ satisfying $\psi(x) = 1$ if $x \in B_{l/2}(x_0)$ and $\psi(x) = 0$ if $x \in \Omega \setminus B_l(x_0)$, where $0 < l < \delta$. Given $\varepsilon > 0$, we define

$$u_\varepsilon(x) := \frac{\psi(x)}{[\varepsilon + |x - x_0|^2]^{\frac{N-2}{2}}} \text{ and } v_\varepsilon(x) := \frac{u_\varepsilon(x)}{|u_\varepsilon|_{2^*}}.$$

For any $\varepsilon > 0$, Lemma 2.3 implies that the function $t \to I(u_0 + tv_\varepsilon)$ achieves its maximum at $t_\varepsilon > 0$. It follows from $I'(u_0)v_\varepsilon = 0$ that

$$m_\varepsilon := I(u_0 + t_\varepsilon v_\varepsilon) = I(u_0) + \frac{t_\varepsilon^2}{2}\|v_\varepsilon\|_a^2 - \frac{1}{q}A_\varepsilon - \frac{1}{2^*}D_\varepsilon, \tag{3.3}$$

for

$$A_\varepsilon := \int_{B_l(x_0)} b(x)\left[(u_0 + t_\varepsilon v_\varepsilon)^q - u_0^q - qt_\varepsilon u_0^{q-1}v_\varepsilon\right]dx.$$

and

$$D_\varepsilon := \int_{B_l(x_0)} c(x)\left[(u_0 + t_\varepsilon v_\varepsilon)^{2^*} - u_0^{2^*} - 2^* t_\varepsilon u_0^{2^*-1}v_\varepsilon\right]dx,$$

where we also have used that $v_\varepsilon \equiv 0$ outside $B_l(x_0)$.

Since $u_0 \geq 0$ a.e. in $B_l(x_0)$, we can apply the Mean Value Theorem to obtain $\eta(x) \in [0,1]$ such that

$$\begin{aligned}(u_0(x) + t_\varepsilon v_\varepsilon(x))^q - u_0(x)^q &= q(u_0(x) + \eta(x)t_\varepsilon v_\varepsilon(x))^{q-1}t_\varepsilon v_\varepsilon \\ &\geq qt_\varepsilon u_0(x)^{q-1}v_\varepsilon(x),\end{aligned}$$

for a.e. $x \in B_l(x_0)$. Since $b(x) \geq 0$ a.e. in $B_l(x_0)$, we get $A_\varepsilon \geq 0$.

In order to estimate D_ε, we shall use the following inequality (see [4]): for $m, n \geq 0$, $s > 2$ and $1 < \mu < s - 1$,

$$(m+n)^s \geq m^s + n^s + sm^{s-1}n + smn^{s-1} - C_\mu n^\mu m^{s-\mu},$$

for some $C_\mu > 0$. If we choose $m = u_0$, $n = t_\varepsilon v_\varepsilon$ and $s = 2^*$, we obtain

$$(u_0 + t_\varepsilon v_\varepsilon)^{2^*} - u_0^{2^*} - 2^* t_\varepsilon u_0^{2^*-1}v_\varepsilon \geq t_\varepsilon^{2^*}v_\varepsilon^{2^*} + 2^* t_\varepsilon^{2^*-1}u_0 v_\varepsilon^{2^*-1} - C_\mu t_\varepsilon^\mu v_\varepsilon^\mu u_0^{2^*-\mu}.$$

Since $c(x) \geq 0$ a.e. in $B_l(x_0)$, we obtain

$$D_\varepsilon \geq \int_{B_l(x_0)} c(x)\left[t_\varepsilon^{2^*}v_\varepsilon^{2^*} + 2^* t_\varepsilon^{2^*-1}u_0 v_\varepsilon^{2^*-1} - C_\mu t_\varepsilon^\mu v_\varepsilon^\mu u_0^{2^*-\mu}\right]dx.$$

Hence, we can use (3.3) to get

$$\begin{aligned} m_\varepsilon &\leq I(u_0) + \left(\frac{t_\varepsilon^2}{2}\|v_\varepsilon\|_a^2 - \frac{t_\varepsilon^{2^*}}{2^*}|c|_\infty\right) + \frac{t_\varepsilon^{2^*}}{2^*}\int_{B_l(x_0)}(|c|_\infty - c(x))v_\varepsilon^{2^*}dx \\ &\quad - t_\varepsilon^{2^*-1}\int_{B_l(x_0)} c(x)u_0 v_\varepsilon^{2^*-1}dx + C_\mu\frac{t_\varepsilon^\mu}{2^*}\int_{B_l(x_0)} c(x)u_0^{2^*-\mu}v_\varepsilon^\mu dx, \end{aligned} \tag{3.4}$$

where we have used $\int_{B_l(x_0)} v_\varepsilon^{2^*}\,dx = 1$.

In what follows we shall suppose that (t_ε) is bounded. The other case will be considered later. We compute

$$\max_{t\geq 0}\left(\frac{t^2}{2}\|v_\varepsilon\|_a^2 - \frac{t^{2^*}}{2^*}|c|_\infty\right) = \frac{1}{N}\frac{\|v_\varepsilon\|_a^N}{|c|_\infty^{(N-2)/2}}. \tag{3.5}$$

As proved in the equation (5.10) of [5], the L^σ-norms of v_ε are such that

$$\int |v_\varepsilon|^\sigma = O(\varepsilon^{(N(2-\sigma)+2\sigma)/4}), \qquad N/(N-2) < \sigma < 2^*,$$

as $\varepsilon \to 0^+$. Recalling that the functions u_0 and c are bounded in $B_l(x_0)$, we can choose $\mu = (N+1)/(N-2)$ to get

$$C_\mu \frac{t_\varepsilon^\mu}{2^*}\int_{B_l(x_0)} c(x) u_0^{2^*-\mu} v_\varepsilon^\mu\,dx = O(\varepsilon^{(N-1)/4}). \tag{3.6}$$

In the same way

$$t_\varepsilon^{2^*-1}\int c(x) u_0 v_\varepsilon^{2^*-1} = A_0 O(\varepsilon^{(N-2)/4}), \tag{3.7}$$

for some $A_0 > 0$.

On the other hand, a known estimate from the paper of Brezis and Nirenberg [3] state that, for some $A_1 > 0$, there holds

$$|u_\varepsilon|_{2^*}^{2^*} = \varepsilon^{-N/2}A_1 + O(1).$$

Thus, we can use the inequality in (c_3) to obtain

$$\begin{aligned}\int_{B_l(x_0)} (|c|_\infty - c(x)) v_\varepsilon^{2^*}\,dx &= \frac{1}{|u_\varepsilon|_{2^*}^{2^*}}\int_{B_l(x_0)} (|c|_\infty - c(x)) u_\varepsilon^{2^*}\,dx \\ &\leq O(\varepsilon^{N/2})\int_{B_l(x_0)} \frac{|x-x_0|^\gamma}{(\varepsilon + |x-x_0|^2)^N}\,dx.\end{aligned}$$

Hence, setting $y := (x-x_0)/\sqrt{\varepsilon}$, we get

$$\begin{aligned}\int_{B_l(x_0)} \frac{|x-x_0|^\gamma}{(\varepsilon + |x-x_0|^2)^N}\,dx &= \varepsilon^{(\gamma-N)/2}\int_{B(l/\sqrt{\varepsilon},0)} \frac{|y|^\gamma}{(1+|y|^2)^N}\,dy \\ &\leq \varepsilon^{(\gamma-N)/2}\omega_N\left(\int_0^1 \frac{r^\gamma r^{N-1}}{(1+r^2)^N}\,dr + \int_1^{l\sqrt{\varepsilon}} r^{\gamma-2N+N-1}\,dr\right) \\ &= O(\varepsilon^{(\gamma-N)/2}) + O(1),\end{aligned}$$

where ω_N is the area of the unit sphere in $\mathbb{R}^N$. All together, the last estimates provide

$$\int_{B_l(x_0)} (|c|_\infty - c(x)) v_\varepsilon^{2^*}\,dx = O(\varepsilon^{\frac{\gamma}{2}}) + O(\varepsilon^{\frac{N}{2}}). \tag{3.8}$$

If we now replace (3.5)-(3.8) in the inequality (3.4), we obtain

$$m_\varepsilon \leq I(u_0) + \frac{1}{N}\frac{\|v_\varepsilon\|_a^N}{|c|_\infty^{(N-2)/2}} + O(\varepsilon^{(N-1)/4}) + O(\varepsilon^{\gamma/2}) + O(\varepsilon^{N/2}) - O(\varepsilon^{(N-2)/4}).$$

Without loss of generality we may suppose $\gamma/2 < (N-1)/4$, and therefore the above expression becomes

$$m_\varepsilon \le I(u_0) + \frac{1}{N}\frac{\|v_\varepsilon\|_a^N}{|c|_\infty^{(N-2)/2}} + O(\varepsilon^{\gamma/2}) - A_0 O(\varepsilon^{(N-2)/4}).$$

We now recall a key estimate, which is a consequence of (3.13) in [9]:

$$\|v_\varepsilon\|_a^2 \le \begin{cases} a_0S + O(\varepsilon), & N=4 \text{ and } k>2; \\ a_0S + O(\varepsilon^{\frac{N-2}{2}}), & N\ge 5 \text{ and } N<k+2; \\ a_0S + O(\varepsilon^{\frac{N-2}{2}}|\log\varepsilon|), & N\ge 5 \text{ and } N=k+2; \\ a_0S + O(\varepsilon^{\frac{k}{2}}), & N\ge 5 \text{ and } N>k+2. \end{cases} \tag{3.9}$$

We first consider the last case, that is, $N\ge 5$ and $N>k+2$. By using the Mean Value Theorem we have that

$$\|v_\varepsilon\|_a^N = (\|v_\varepsilon\|_a^2)^{N/2} = (a_0S)^{N/2} + O(\varepsilon^{k/2}).$$

Hence

$$\begin{aligned} m_\varepsilon &\le I(u_0) + \frac{1}{N}\frac{(a_0S)^{N/2}}{|c|_\infty^{(N-2)/2}} + O(\varepsilon^{k/2}) + O(\varepsilon^{\gamma/2}) - A_0O(\varepsilon^{(N-2)/4}) \\ &= d^* + O(\varepsilon^{(N-2)/4})(O(\varepsilon^{\frac{k}{2}-\frac{(N-2)}{4}}) + O(\varepsilon^{\frac{\gamma}{2}-\frac{(N-2)}{4}}) - A_0) \end{aligned}$$

Since we are supposing that $\max\{\gamma,k\} > (N-2)/2$, the above expression implies that $m_\varepsilon < d^*$, if $\varepsilon>0$ is small enough. The other three cases in (3.9) can be handled with the same kind of argument. Actually, in all of them that are no extra restrictions on k.

It remains to consder the case that $\limsup_{\varepsilon\to 0^+} t_\varepsilon = +\infty$. Recalling that the support of v_ε is contained in $B_l(x_0)$ and arguing as in the proof of Lemma 2.3, we obtain

$$m_\varepsilon \le \frac{t_\varepsilon^2}{2}\|v_\varepsilon\|_a^2 + t_\varepsilon\int p(x)(\nabla u_0\cdot\nabla v_\varepsilon) - \frac{t_\varepsilon^{2^*}}{2^*}\int_{B_l(x_0)} c(x)v_\varepsilon^{2^*}\,dx + O(1).$$

Since $\|v_\varepsilon\|_a^2 = a_0S + o(1)$, $B_l(x_0)\subset\Omega_c^+$ and $\int_{B_l(x_0)} v_\varepsilon^{2^*}dx = 1$, we infer from the above inequality that $m_\varepsilon \to -\infty$ as $\varepsilon\to 0^+$. Hence, for $\varepsilon>0$ small, we have that $m_\varepsilon < d^*$ and we are done. □

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