Positive and nodal solutions for an elliptic equation with critical growth

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Abstract

We consider the problem

$$-\mathrm{div}(p(x)\nabla u) = \lambda |u|^{q-2}u + |u|^{2^*-2}u, \;\; u \in H_0^1(\Omega),$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $N \geq 4$, $2^* = 2N/(N-2)$, $2 \leq q < 2^*$. Under some suitable conditions on the continuous potential $p(x)$ and on the parameter $\lambda > 0$, we obtain one nodal solution for $q = 2$ and one positive solution for $2 < q < 2^*$.

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1 Introduction

In this paper we address the existence of solutions for the semilinear problem

$$(P) \qquad \begin{cases} -\mathrm{div}(p(x)\nabla u) = \lambda |u|^{q-2}u + |u|^{2^*-2}u, & x \in \Omega, \\ u = 0, & \text{in } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $N \geq 4$, $2^* = 2N/(N-2)$ is the critical Sobolev exponent, $2 \leq q < 2^*$, $\lambda > 0$ and p is a continuous function. We are interested here in positive and sign changing solutions.

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In the case $p(x) \equiv 1$ the above problem becomes

$$-\Delta u = \lambda |u|^{q-2}u + |u|^{2^*-2}u, \ \ u \in H_0^1(\Omega).$$

In the seminal paper [4], Brezis and Nirenberg considered this the above problem and proved that, at least for for $q = 2$, the existence of positive solution is related with the interaction of λ with the spectrum of $-\Delta$. More specifically, if we denote by λ_1 the first eigenvalue of $(-\Delta, H_0^1(\Omega))$, they obtained:

- if $N \geq 3$ there is no positive solution if $\lambda \geq \lambda_1$;
- if $N \geq 4$ the problem has a positive solution provided $0 < \lambda < \lambda_1$;
- if $N = 3$ there exists $\lambda^* > 0$ such that the problem has a positive solution provided $\lambda^* < \lambda < \lambda_1$.

In the case $2 < q < 2^*$ they obtained positive solution for any $\lambda > 0$. The nonexistence result above holds only for positive solution. Actually, Capozzi, Fortunato and Palmieri proved in [7] that the above equation has a sign changing solution whenever $q = 2$, $N \geq 4$ and $\lambda \geq \lambda_1$. After these papers, problems with critical growth has been extensively studied. Since it is impossible to give a complete reference list we just cite [6, 5, 2, 1, 9, 11, 15, 10, 13, 16, 8] and references there in.

In [12], Hadiji and Yazidi consider the case of non constant potential. Assuming continuity for p, they proved that the existence of positive solutions depends, besides the parameter λ, on the behaviour of p near its minima. More specifically, they assumed that the function p satisfies the following conditions:

(p_1) $p \in H^1(\Omega) \cap C(\overline{\Omega})$;

(p_2) there exists $a \in \Omega$ such that

$$p(a) = p_0 := \min\{p(x) : x \in \overline{\Omega}\} > 0;$$

(p_3) there exist $k > 0$, $\beta_k > 0$ and θ such that, is a small neighborhood of a, there holds

$$p(x) = p_0 + \beta_k |x-a|^k + \theta(x)|x-a|^k,$$

with $\lim_{x \to a} \theta(x) = 0$.

As in the case of the Brezis and Nirenberg problem, it is important to consider the linearized problem

$$(LP) \qquad -\mathrm{div}(p(x)\nabla u) = \lambda u, \ \ u \in H_0^1(\Omega).$$

If we denote by $\lambda_{1,p} > 0$ the first eigenvalue of the above problem we can state some of the results of [12] as follows:

- if $N \geq 4$ and $k > 0$, there is no positive solution if $\lambda \geq \lambda_1$;

- if $N \geq 4$ and $k > 2$, the problem has a positive solution provided $0 < \lambda < \lambda_{1,p}$;
- if $N \geq 4$ and $k = 2$, there exists $\lambda^* = \lambda^*(N) > 0$ such that the problem has a positive solution provided $\lambda^* < \lambda < \lambda_{1,p}$.

The authors also present some existence results for the case $N = 3$. The above results shows that the critical value for k is exactly $k = 2$. All the results in [12] deals with the case $q = 2$ and they can viewed as the version of the Brezis and Nirenberg results for the non constant potential equation (P).

The aim of this paper is to complement the results of [12] in two senses: firstly considering the existence of sign changing solution for $q = 2$ and $\lambda \geq \lambda_{1,p}$ and secondly dealing with superlinear perturbations of the critical term. We stated in what follows the mains results of this paper.

Theorem 1.1. *Suppose that $q = 2$ and p satisfies $(p_1)-(p_3)$. Then the problem (P) has a sign changing solution if one of the below conditions holds*

1. *$N \geq 4$, $k > 2$ and $\lambda \geq \lambda_{1,p}$;*
2. *$N \geq 5$, $k = 2$ and $\lambda \geq \max\{\lambda_{1,p}, \vartheta(N)\}$ where*

$$\vartheta(N) = \begin{cases} 4\beta_2, & \text{if } N = 4, \\ \vartheta_0\beta_2, & \text{if } N \geq 5, \end{cases}$$

$\beta_2 > 0$ is given in (p_3) and

$$\vartheta_0 = (N-2)^2 \left(\int_{\mathbb{R}^N} \frac{|y|^4}{(1+|y|^2)^N}\, dy \right) \left(\int_{\mathbb{R}^N} \frac{1}{(1+|y|^2)^{N-2}}\, dy \right)^{-1}.$$

3. *$N = 4$, $k = 2$, $\lambda \geq \max\{\lambda_{1,p}, \vartheta(N)\}$ and there exists $l_0 > 0$ such that*

$$\int_{B(a,l_0)} \frac{\theta(x)}{|x-a|^4} dx < \infty.$$

Theorem 1.2. *Suppose that $2 < q < 2^*$ and p satisfies $(p_1)-(p_3)$. Then the problem (P) has a positive solution for any $\lambda > 0$.*

The technical restrictions on λ presented in Theorem 1.1 show that, even on the case of sign changing solution, the number $k = 2$ behaves like a critical value. The proof of the first result will be done as an application of the generalized Mountain Pass Theorem. The main difficult relies on some trick calculations to correct localize the minimax level of the associated functional. For the second result we apply the usual Mountain Pass Theorem. Again, we need to perform some careful estimates.

The paper contains three more sections: in the first one we fix some notation and present the variational framework to deal with the problem (P). In Section 3 we prove Theorem 1.1 and the final section is devoted to the proof of Theorem 1.2.

2 The variational setting

Throughout the paper we suppose that the function p satisfies $(p_1) - (p_3)$. For save notation, we write only $\int_\Omega u$ instead of $\int_\Omega u(x)\mathrm{d}x$. For any $1 \leq t \leq \infty$, $|u|_t$ denotes the norm in $L^t(\Omega)$.

Let H be the Hilbert space $H_0^1(\Omega)$ endowed with the norm $\|u\| = (|\nabla u|_2^2)^{1/2}$. In view of the conditions (p_1) and (p_2) we can also consider the norm

$$\|u\|_p = \left(\int p(x)|\nabla u|^2 \right)^{1/2}, \quad u \in H,$$

which is equivalent to $\| \cdot \|$.

The problem (P) is variational in nature. Standard calculations show that its weak solutions are precisely the critical points of the C^1-functional $I : H \to \mathbb{R}$ given by

$$I(u) := \frac{1}{2} \int p(x)|\nabla u|^2 - \frac{\lambda}{2} \int u^2 - \frac{1}{2^*} \int |u|^{2^*}, \quad u \in H.$$

We start this section with a local compactness result for the functional just defined. In what follows, the number S stands for the best constant of the Sobolev embedding $H \hookrightarrow L^{2^*}(\Omega)$, namely $S = \inf\{|\nabla u|_2^2 : |u|_{2^*} = 1\}$.

We say that I satisfies the $(PS)_c$ condition if any sequence $(u_n) \subset H$ such that $I(u_n) \to c$ and $I'(u_n) \to 0$ has a convergent subsequence.

Lemma 2.1. *I satisfies the $(PS)_c$ condition for any $c < c^* := (p_0 S)^{N/2}/N$.*

Proof. Suppose $(u_n) \subset H$ verifies $I'(u_n) \to 0$ and $I(u_n) \to c < c^*$. Let $\beta > 0$ be such that $(1/2^*) < \beta < (1/2)$. We have that

$$\begin{aligned} o_n(1) + c + o_n(1)\|u_n\| &= I(u_n) - \beta I'(u_n)u_n \\ &= \beta_0\|u_n\|_p^2 - \lambda\beta_0|u_n|_2^2 + \beta_1|u_n|_{2^*}^{2^*}, \end{aligned}$$

where $\beta_0 = (1/2-\beta)$, $\beta_1 = (\beta - 1/2^*)$ and $o_n(1)$ stands for a quantity approaching zero as $n \to +\infty$. Given $\varepsilon > 0$, it follows from Young's inequality and the Sobolev embedding that, for some c_1, $c_2 > 0$,

$$|u_n|_2 \leq c_1|u_n|_{2^*} \leq \varepsilon|u_n|_{2^*}^{2^*/2} + c_2.$$

Hence,

$$\begin{aligned} o_n(1) + c + o_n(1)\|u_n\| &\geq p_0\beta_0\|u_n\|^2 - \lambda\beta_0(\varepsilon|u_n|_{2^*}^{2^*/2} + c_2)^2 + \beta_1|u_n|_{2^*}^{2^*} \\ &\geq p_0\beta_0\|u_n\|^2 - \lambda\beta_0 c_3\varepsilon^2|u_n|_{2^*}^{2^*} - c_4 + \beta_1|u_n|_{2^*}^{2^*}, \end{aligned}$$

with c_3, $c_4 > 0$. By choosing $0 < \varepsilon < \sqrt{\beta_1}(\lambda\beta_0 c_3)^{-1/2}$, we obtain

$$1 + c + o_n(1)\|u_n\| \geq p_0\beta_0\|u_n\|^2 - c_4,$$

and therefore (u_n) is bounded in H.

Along a subsequence, we have that $u_n \rightharpoonup u$ weakly in H, $u_n \to u$ strongly in $L^s(\Omega)$ for any $2 \le s < 2^*$ and $u_n(x) \to u(x)$ a.e. in Ω. This, the Lebesgue Theorem and a standard calculation show that $I'(u) = 0$, and therefore

$$I(u) = I(u) - \frac{1}{2}I'(u)u = \frac{1}{N}|u|_{2^*}^{2^*} \ge 0. \tag{2.1}$$

If we set $v_n := u_n - u$, we can use the Brezis-Lieb's lemma [3] to get

$$\begin{aligned} o_n(1) = I'(u_n)u_n &= I'(u)u + \|v_n\|_p^2 - \lambda|v_n|_2^2 - |v_n|_{2^*}^{2^*} + o_n(1) \\ &= \|v_n\|_p^2 - |v_n|_{2^*}^{2^*} + o_n(1), \end{aligned}$$

and therefore, for some $b \ge 0$, we have that

$$\lim_{n\to+\infty} \|v_n\|_p^2 = \lim_{n\to+\infty} |v_n|_{2^*}^{2^*} = b. \tag{2.2}$$

It follows from the definition of S and p that

$$\int p(x)|\nabla v_n|^2 \ge p_0 \int |\nabla v_n|^2 \ge p_0 S |v_n|_{2^*}^2.$$

Taking the limit we conclude that $b \ge p_0 S b^{2/2^*}$. Suppose that $b \ne 0$. Then $b \ge (p_0 S)^{N/2} > 0$. Using the Brezis-Lieb's lemma again we can compute

$$c + o_n(1) = I(u_n) = I(u) + \frac{1}{2}\|v_n\|_p^2 - \frac{1}{2^*}|v_n|_{2^*}^{2^*} + o_n(1).$$

Letting $n \to +\infty$ and using (2.1) and (2.2) we obtain

$$c = I(u) + \left(\frac{1}{2} - \frac{1}{2^*}\right) \ge \frac{1}{N}b \ge \frac{1}{N}(p_0 S)^{N/2},$$

which does not make sense. This shows that $b = 0$, from which it follows that $\|u_n - u\|_p^2 = o_n(1)$. The lemma follows from the equivalence between $\|\cdot\|_p$ and the usual norm. □

3 Nodal solution for $q = 2$ and $\lambda \ge \lambda_{1,p}$

In this section we present the proof of Theorem 1.1 as an application of the generalized Moutain Pass Theorem [14] (see also [17, Theorem 2.12, pg. 43]).

Theorem 3.1. *Let X be a real Banach space with $X = Y \oplus Z$ and $\dim Y < \infty$. Suppose $I \in C^1(X, \mathbb{R})$ satisfies*

(I_1) *there exist ρ, $\sigma > 0$ such that $I|_{\partial B_\rho(0)\cap Z} \ge \sigma$;*

(I_2) *there exist $e \in \partial B_1(0) \cap Z$ and $R > \rho$ such that,*

$$I|_{\partial Q} \le 0,$$

with

$$Q := (\overline{B_R(0)} \cap Y) \oplus \{te : 0 < t < R\}.$$

Let

$$c := \inf_{\gamma\in\Gamma}\max_{u\in Q} I(\gamma(u)) \geq \sigma > 0, \tag{3.1}$$

where $\Gamma := \left\{\gamma \in C(\overline{Q}, X) : \gamma \equiv \mathrm{Id} \text{ on } \partial Q\right\}$. *If* I *satisfies the* $(PS)_c$ *condition, then* c *is a critical value of* I.

In order to present the decomposition of the space H we recall that, from the spectral theory for compact operators, the eigenvalue problem

$$-\mathrm{div}(p(x)\nabla u) = \lambda u, \;\; u \in H_0^1(\Omega),$$

has a sequence of eigenvalues

$$0 < \lambda_{1,p} < \lambda_{2,p} \leq \lambda_{3,p} \leq ... \leq \lambda_{n,p} \leq \lambda_{n+1,p} \to \infty,$$

and associated eigenfunctions $(\varphi_{k,p})_{k\in\mathbb{N}}$. Let $n \in \mathbb{N}$ be such that $\lambda_{n,p} \leq \lambda < \lambda_{n+1,p}$ and define

$$Y := \mathrm{span}\,\{\varphi_{1,p}, \varphi_{2,p}, ..., \varphi_{n,p}\}\,, \quad Z := Y^{\perp}. \tag{3.2}$$

The variational characterization of eigenvalues provides the following inequalities

$$\int p(x)|\nabla u|^2 \leq \lambda_{n,p}\int u^2, \;\; \forall u \in Y \tag{3.3}$$

and

$$\int p(x)|\nabla u|^2 \geq \lambda_{n+1,p}\int u^2, \;\; \forall u \in Z. \tag{3.4}$$

The next proposition plays a crucial rule in the proof of Theorem 1.1. We postpone its proof to the end of the section.

Proposition 3.2. *If* $\lambda \in [\lambda_{n,p}, \lambda_{n+1,p})$ *satisfies the hypotheses of Theorem 1.1, then there exists* $z \in Z \setminus \{0\}$ *such that*

$$\max_{u\in Y\oplus\mathbb{R}z} I(u) < c^* = \frac{1}{N}(p_0 S)^{N/2}.$$

Assuming the above result we can prove our first theorem as follows:

Proof of Theorem 1.1. We shall apply Theorem 3.1 with the decomposition $H = Y \oplus Z$, the spaces Y and Z defined above and $e = z/\|z\|$, where the function $z \in Z \setminus \{0\}$ comes from Proposition 3.2.

It follows from (3.4), $p(x) \geq p_0 > 0$ and the embedding $H \hookrightarrow L^{2^*}(\Omega)$ that, for any $u \in Z$, there holds

$$I(u) \geq \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_{n+1,p}}\right)p_0\|u\|^2 - \frac{1}{2^*}\int |u|^{2^*} \geq c_1\|u\|^2 - c_2\|u\|^{2^*}$$

with c_1, $c_2 > 0$. Since $2^* > 2$ the above equation implies that the condition (I_1) of Theorem 3.1 holds for some ρ, $\sigma > 0$.

In order to check (I_2) we first consider $u \in Y$ and use (3.3) to get

$$I(u) \leq \frac{(\lambda_{n,p} - \lambda)}{2} \int u^2 - \frac{1}{2^*} \int |u|^{2^*} \leq 0.$$

Moreover, since $Y \oplus \mathbb{R}z$ has finite dimension, we have that all norms in this space are equivalent. So, there exist c_2, c_3, $c_4 > 0$ such that, for any $u \in Y \oplus \mathbb{R}z$, there holds

$$I(u) \leq \frac{1}{2} c_2 \|u\|^2 - \frac{\lambda}{2} c_3 \|u\|^2 - \frac{1}{2^*} c_4 \|u\|^{2^*},$$

and therefore $I(u) \to -\infty$ whenever $\|u\| \to +\infty$, with $u \in Y \oplus \mathbb{R}z$. Thus, the condition (I_2) holds for some $R > \rho$ sufficiently large.

All the geometric conditions of the generalized Moutain Pass Theorem are satisfied. Moreover, Proposition 3.2 and the definition of c in (3.1) imply that $c < c^*$. Hence I satisfies the $(PS)_c$ condition and it follows from Theorem 3.1 that there exists $u \in H$ such that $I'(u) = 0$ and $I(u) = c \geq \sigma > 0$.

It remains to check that u is a sign changing solution. Indeed, if this is not the case, it follows from $u \not\equiv 0$ and the Maximum Principle that $u > 0$ (or $u < 0$) in Ω. But this contradicts the nonexistence result proved in [12, Theorem 1.1], since $\lambda \geq \lambda_{1,p}$. The theorem is proved. □

We devote the rest of this section to the proof of Proposition 3.2. For any $\varepsilon > 0$ we consider the function

$$w_\varepsilon(x) := \frac{\psi(x) \varepsilon^{\frac{N-2}{4}}}{[\varepsilon + |x-a|^2]^{(N-2)/2}}, \tag{3.5}$$

where $\psi \in C_0^\infty(\Omega)$ is such that $0 \leq \psi \leq 1$, $\psi \equiv 1$ in $B(a,l)$, $\psi \equiv 0$ in $\Omega \setminus \overline{B(a,2l)}$ and $l > 0$ satisfies $\overline{B(a,2l)} \subset \Omega$. We also consider the ratio

$$Q_{\lambda,p}(u) := \frac{\|u\|_p^2 - \lambda |u|_2^2}{|u|_{2^*}^2}, \quad u \in H \setminus \{0\}.$$

The following estimates are proved in [12, Lemma 3.2].

Lemma 3.3. *As $\varepsilon \to 0^+$, we have that*

$$Q_{\lambda,p}(w_\varepsilon) \leq \begin{cases} p_0 S - \lambda \frac{K_3}{K_2} \varepsilon + o(\varepsilon), \quad N \geq 5,\ k > 2; \\ p_0 S - \left(\lambda - \frac{A_2}{K_3} \beta_2\right) \frac{K_3}{K_2} \varepsilon + o(\varepsilon), \quad N \geq 5 \ e\ k = 2; \\ p_0 S - \lambda \frac{\omega_4}{2K_2} \varepsilon |\log \varepsilon| + o(\varepsilon |\log \varepsilon|), \quad N = 4 \ e\ k > 2; \\ p_0 S - (\lambda - 4\beta_2) \frac{\omega_4}{2K_2} \varepsilon |\log \varepsilon| + o(\varepsilon |\log \varepsilon|), \quad N = 4, k = 2, \end{cases}$$

where ω_4 is the area of the unitary sphere of $\mathbb{R}^4$,

$$K_2 := \frac{1}{S}(N-2)^2 \int_{\mathbb{R}^N} \frac{|y|^2}{[1+|y|^2]} dy, \quad K_3 := \int_{\mathbb{R}^N} \frac{1}{[1+|y|^2]^{N-2}} dy$$

and

$$A_2 := (N-2)^2 \int_{\mathbb{R}^N} \frac{|y|^4}{(1+|y|^2)^N}\, dy.$$

We divide the proof of Proposition 3.2 in two distinct cases depending on λ be or not to be an eigenvalue.

Case 1. $\lambda_{n,p} < \lambda < \lambda_{n+1,p}$

In this first case we define

$$z_\varepsilon := w_\varepsilon - \sum_{i=1}^{n} \left(\int w_\varepsilon \varphi_{i,p} \right) \varphi_{i,p},$$

where $\varphi_{i,p}$ are the eigenfunctions of (LP). We shall prove Proposition 3.2 for $z = z_\varepsilon$ with ε small.

For any $u \in H \setminus \{0\}$, a straightforward computation shows that

$$\max_{t\geq 0} I(tu) \leq \frac{1}{N} Q_{\lambda,p}(u)^{N/2}.$$

Since $Y \oplus \mathbb{R}w_\varepsilon = Y \oplus \mathbb{R}z_\varepsilon$ it suffices to prove that

$$m_\varepsilon := \max_{u\in\Sigma_\varepsilon} \left(\|u\|_p^2 - \lambda |u|_2^2 \right) < p_0 S$$

where the set Σ_ε is defined as below

$$\Sigma_\varepsilon := \{u = y + tw_\varepsilon;\ y \in Y,\ t \in \mathbb{R},\ |u|_{2^*} = 1\}. \tag{3.6}$$

In what follows we present some estimates which will provide the desired inequality $m_\varepsilon < p_0 S$.

Lemma 3.4. *As $\varepsilon \to 0^+$, we have the following*

$$|w_\varepsilon|_{2^*-1}^{2^*-1} = O(\varepsilon^{\frac{N-2}{4}}), \quad |w_\varepsilon|_1 = O(\varepsilon^{\frac{N-2}{4}}) \tag{3.7}$$

and

$$\max\left\{ |\langle y, w_\varepsilon\rangle_p|\ ,\ \left| \int y w_\varepsilon \right| \right\} = |y|_2 O(\varepsilon^{\frac{N-2}{4}}), \quad \forall\, y \in Y. \tag{3.8}$$

Proof. Since $0 \leq \psi \leq 1$ and $\psi \equiv 0$ outside $B(a, 2l)$, it follows that

$$\begin{aligned} |w_\varepsilon|_{2^*-1}^{2^*-1} &\leq \varepsilon^{(N+2)/4} \int_{B(a,2l)} \frac{1}{(\varepsilon + |x-a|^2)^{(N+2)/2}} \mathrm{d}x \\ &= \varepsilon^{(N-2)/4} \int_{B_{2l/\sqrt{\varepsilon}}(0)} \frac{1}{[1+|y|^2]^{(N+2)/2}} \,\mathrm{d}y = O(\varepsilon^{\frac{N-2}{4}}). \end{aligned}$$

Moreover,

$$
\begin{aligned}
|w_\varepsilon|_1 &\le \varepsilon^{(N-2)/4}\int_{B(a,2l)} \frac{1}{[\varepsilon+|x-a|^2]^{(N-2)/2}}\mathrm{d}x\\
&\le \varepsilon^{(N-2)/4}\int_{B(a,2l)} \frac{1}{|x-a|^{(N-2)}}\mathrm{d}x = O(\varepsilon^{\frac{N-2}{4}}),
\end{aligned}
$$

since the last integral is finite.

In order to verify (3.8) we take $y=\sum_{i=1}^n \beta_i\varphi_{i,p}\in Y$. It follows from (3.7) that

$$
\begin{aligned}
|\langle y, w_\varepsilon\rangle_p| &= \left|\left\langle \sum_{i=1}^n \beta_i\varphi_{i,p}, w_\varepsilon\right\rangle_p\right| = \left|\sum_{i=1}^n \lambda_{i,p}\beta_i\int \varphi_{i,p}w_\varepsilon\right|\\
&\le c_3\left(\sum_{i=1}^n|\beta_i|\right)\int w_\varepsilon \le \left(\sum_{i=1}^n|\beta_i|\right)O(\varepsilon^{\frac{N-2}{4}}),
\end{aligned}
\tag{3.9}
$$

with $c_3=\lambda_{n,p}\max\{|\varphi_{1,p}|_\infty,\ldots,|\varphi_{n,p}|_\infty\}$. The equivalence on the norms in Y implies that $\sum_{i=1}^n|\beta_i|\le c_4|y|_2$. This and (3.9) provides

$$
|\langle y, w_\varepsilon\rangle_p|\le |y|_2 O(\varepsilon^{\frac{N-2}{4}}).
$$

The argument for $|\int y w_\varepsilon|$ is analogous and we omit it. □

Lemma 3.5. *If $u=y+tw_\varepsilon\in\Sigma_\varepsilon$, then $t=O(1)$ as $\varepsilon\to 0^+$.*

Proof. Let

$$
A(u):=|u|_{2^*}^{2^*}-|y|_{2^*}^{2^*}-|tw_\varepsilon|_{2^*}^{2^*}.
$$

Since Y is finite dimensional and the eigenfunctions of (LP) are regular, we can use the Mean Value Theorem to get

$$
\begin{aligned}
A(u) &= 2^*\int\int_0^1(|tw_\varepsilon+sy|^{2^*-2}(tw_\varepsilon+sy)-|sy|^{2^*-2}sy)ds\\
&= 2^*(2^*-1)\int\int_0^1|sy+tw_\varepsilon\eta(x)|^{2^*-2}tw_\varepsilon y ds,
\end{aligned}
$$

where $0\le\eta(x)\le 1$ is measurable, and therefore

$$
|A(u)|\le c_1 2^*(2^*-1)\left(\int\int_0^1|s|^{2^*-2}|y|^{2^*-1}|tw_\varepsilon|ds+\int\int_0^1|tw_\varepsilon\eta|^{2^*-2}tw_\varepsilon y ds\right).
$$

The equivalence of the norms in Y and the previous lemma provide

$$
\begin{aligned}
|A(u)| &\le c_2\left\{|y|_\infty^{2^*-1}|t||w_\varepsilon|_1+|y|_\infty|t|^{2^*-1}|w_\varepsilon|_{2^*-1}^{2^*-1}\right\}\\
&\le c_3\left\{|y|_{2^*}^{2^*-1}|t|O(\varepsilon^{\frac{N-2}{4}})+|y|_{2^*}|t|^{2^*-1}O(\varepsilon^{\frac{N-2}{4}})\right\}.
\end{aligned}
\tag{3.10}
$$

For $\xi > 0$, we can use Young's inequality to get

$$|y|_{2^*}^{2^*-1}|t|O(\varepsilon^{\frac{N-2}{4}}) \le \xi |y|_{2^*}^{2^*} + c_4|t|^{2^*}O(\varepsilon^{\frac{N-2}{4}})^{\frac{2N}{N-2}}$$

and

$$|y|_{2^*}|t|^{2^*-1}O(\varepsilon^{\frac{N-2}{4}}) \le \xi |y|_{2^*}^{2^*} + c_5|t|^{2^*}O(\varepsilon^{\frac{N-2}{4}})^{\frac{2N}{N+2}}.$$

Picking $\xi < 1/(4c_3)$, we can use (3.10) and the inequalities above to assure that

$$|A(u)| \le \frac{1}{2}|y|_{2^*}^{2^*} + |t|^{2^*}\left\{O(\varepsilon^{\frac{N}{2}}) + O(\varepsilon^{\frac{N(N-2)}{2(N+2)}})\right\}. \tag{3.11}$$

Recalling that (see [4] or [17, pg. 35])

$$|w_\varepsilon|_{2^*}^{2^*} = [N(N-2)]^{-2^*}S^{N/2} + O(\varepsilon^{\frac{N}{2}}),$$

using the definition of $A(u)$ and (3.11), it follows that

$$\begin{aligned} 1 &= |u|_{2^*}^{2^*} \ge |tw_\varepsilon|_{2^*}^{2^*} + \frac{1}{2}|y|_{2^*}^{2^*} + |t|^{2^*}\left\{O(\varepsilon^{\frac{N}{2}}) + O(\varepsilon^{\frac{N(N-2)}{2(N+2)}})\right\} \\ &\ge |t|^{2^*}\left\{[N(N-2)]^{-2^*}S^{N/2} + O(\varepsilon^{\frac{N}{2}}) + O(\varepsilon^{\frac{N(N-2)}{2(N+2)}})\right\}, \end{aligned}$$

and therefore $t = O(1)$ as $\varepsilon \to 0^+$. □

We are now able to present the

Proof of Proposition 3.2 (for $\lambda_{n,p} < \lambda < \lambda_{n+1,p}$). Given $u = y + tw_\varepsilon \in \Sigma_\varepsilon$, we can use (3.8) and Lemma 3.5 to get

$$\|u\|_p^2 = \|y + tw_\varepsilon\|_p^2 \le \|y\|_p^2 + |y|_2 O(\varepsilon^{\frac{N-2}{4}}) + \|tw_\varepsilon\|_p^2.$$

The same argument for $|u|_2^2 = |y + tw_\varepsilon|_2^2$, the above inequality and (3.3) imply that

$$\|u\|_p^2 - \lambda|u|_2^2 \le (\lambda_{n,p} - \lambda)|y|_2^2 + |y|_2 O(\varepsilon^{\frac{N-2}{4}}) + Q_{\lambda,p}(tw_\varepsilon)|tw_\varepsilon|_{2^*}^2.$$

For $a < 0$, we have that $ar^2 + br \le -b^2/4a$, for any b, $r \in \mathbb{R}$. Thus, recalling that $(\lambda_{n,p} - \lambda) < 0$, we have that

$$\|u\|_p^2 - \lambda|u|_2^2 \le \frac{1}{4(\lambda - \lambda_{n,p})}O(\varepsilon^{\frac{N-2}{2}}) + Q_{\lambda,p}(w_\varepsilon)|tw_\varepsilon|_{2^*}^2. \tag{3.12}$$

On the other hand, the Mean Value Theorem provides a (bounded) measurable function ξ, such that

$$\begin{aligned} 1 &= \int \left(|tw_\varepsilon|^{2^*} + 2^*|tw_\varepsilon + \xi(x)y|^{2^*-2}(tw_\varepsilon + \xi(x)y)y\right) \\ &\ge |tw_\varepsilon|_{2^*}^{2^*} + 2^*\int |tw_\varepsilon|^{2^*-1}|y| \ge |tw_\varepsilon|_{2^*}^{2^*} - |y|_2 O(\varepsilon^{\frac{N-2}{4}}), \end{aligned}$$

from which we conclude that

$$|tw_\varepsilon|_{2^*}^{2^*} \le 1 + |y|_2 O(\varepsilon^{\frac{N-2}{4}}).$$

If we now suppose that $N \ge 5$ and $k > 2$, we can use use the first inequality in Lemma 3.3, the above expression and (3.12) to obtain $d < 0$ such that

$$\|u\|_p^2 - \lambda |u|_2^2 \le p_0 S + \varepsilon \left(d + \frac{1}{4(\lambda - \lambda_{n,p})} O(\varepsilon^{\frac{N-2}{4}}) + o(1) \right),$$

and therefore, for $\varepsilon > 0$ small, we have that

$$\|u\|_p^2 - \lambda |u|_2^2 \le \gamma < p_0 S, \quad \forall\, u \in \Sigma_\varepsilon.$$

The other three cases presented in the estimate of Lemma 3.3 can be handled with similar arguments. We omit the details and finish the proof of Proposition 3.2 in the case $\lambda_{n,p} < \lambda < \lambda_{n+1,p}$. □

We consider now the complementary case:

Case 2. $\lambda = \lambda_{n,p}$.

We define

$$\widetilde{w}_\varepsilon := w_\varepsilon - \langle w_\varepsilon, \varphi_{n,p} \rangle_p \varphi_{n,p},$$

and we shall verify that

$$\widetilde{m}_\varepsilon = \max_{u \in \widetilde{\Sigma}_\varepsilon} \left(\|u\|_p^2 - \lambda_{n,p} |u|_2^2 \right) < p_0 S,$$

where

$$\widetilde{\Sigma}_\varepsilon = \{ u = y + t\widetilde{w}_\varepsilon;\ y \in Y,\ t \in \mathbb{R},\ |u|_{2^*} = 1 \}.$$

Lemma 3.6. *As $\varepsilon \to 0^+$, we have that*

$$|\widetilde{w}_\varepsilon|_{2^*-1}^{2^*-1} = O(\varepsilon^{\frac{N-2}{4}}), \qquad |\widetilde{w}_\varepsilon|_1 = O(\varepsilon^{\frac{N-2}{4}}),$$

$$\max \left\{ |\langle y, \widetilde{w}_\varepsilon \rangle_p|, \left| \int y \widetilde{w}_\varepsilon \right| \right\} \le |y|_2 O(\varepsilon^{\frac{N-2}{4}}),$$

and

$$Q_{\lambda,p}(\widetilde{w}_\varepsilon) = Q_{\lambda,p}(w_\varepsilon) + \frac{O(\varepsilon^{\frac{N-2}{2}})}{|w_\varepsilon|_{2^*}^2 + O(\varepsilon^{\frac{N-2}{2}})}.$$

Proof. By using Lemma 3.4 we have that

$$\begin{aligned} |\widetilde{w}_\varepsilon|_{2^*-1}^{2^*-1} &\le c_1 \int |w_\varepsilon|^{2^*-1} + c_2 |\langle w_\varepsilon, \varphi_{n,p} \rangle_p|^{2^*-1} \int |\varphi_{n,p}|^{2^*-1} \\ &= O(\varepsilon^{\frac{N-2}{4}}) + O(\varepsilon^{\frac{N-2}{4}})^{\frac{N+2}{N-2}} = O(\varepsilon^{\frac{N-2}{4}}). \end{aligned}$$

The second and third inequality follows from a similar argument. The proof of the last one is more delicate. We first notice that

$$\|\widetilde{w}_\varepsilon\|_p^2 = \|w_\varepsilon\|_p^2 + \langle w_\varepsilon, \varphi_{n,p}\rangle_p^2 \|\varphi_{n,p}\|_p^2 - 2\langle w_\varepsilon, \varphi_{n,p}\rangle_p^2 = \|w_\varepsilon\|_p^2 + O(\varepsilon^{\frac{N-2}{2}}), \quad (3.13)$$

and also

$$|\widetilde{w}_\varepsilon|_2^2 \le |w_\varepsilon|_2^2 + O(\varepsilon^{\frac{N-2}{2}}). \quad (3.14)$$

Moreover,

$$\begin{aligned} |\widetilde{w}_\varepsilon|_{2^*}^{2^*} - |w_\varepsilon|_{2^*}^{2^*} &= \int\int_0^1 \frac{d}{ds}|w_\varepsilon - s\langle w_\varepsilon, \varphi_{n,p}\rangle_p \varphi_{n,p}|^{2^*} ds \\ &\le c_2|\langle w_\varepsilon, \varphi_{n,p}\rangle_p| \int |w_\varepsilon|^{2^*-1} + c_3|\langle w_\varepsilon, \varphi_{n,p}\rangle_p|^{2^*} \int |\varphi_{n,p}|^{2^*} \\ &= O(\varepsilon^{\frac{N-2}{4}})O(\varepsilon^{\frac{N-2}{4}}) + O(\varepsilon^{\frac{N-2}{4}})^{2^*} = O(\varepsilon^{\frac{N-2}{2}}), \end{aligned}$$

where we have used Lema 3.4 again. For some $\xi \in (0,1)$ it holds

$$\begin{aligned} |\widetilde{w}_\varepsilon|_{2^*}^2 &= (|\widetilde{w}_\varepsilon|_{2^*}^{2^*})^{2/2^*} = \left(|w_\varepsilon|_{2^*}^{2^*} + O(\varepsilon^{\frac{N-2}{2}})\right)^{2/2^*} \\ &= (|w_\varepsilon|_{2^*}^{2^*})^{2/2^*} + \frac{2}{2^*}\left(|w_\varepsilon|_{2^*}^{2^*} + \xi O(\varepsilon^{\frac{N-2}{2}})\right)^{\frac{2}{2^*}-1} O(\varepsilon^{\frac{N-2}{2}}). \end{aligned}$$

Since $0 < \lim_{\varepsilon\to 0^+} |w_\varepsilon|_{2^*} < \infty$, we conclude that

$$|\widetilde{w}_\varepsilon|_{2^*}^2 = |w_\varepsilon|_{2^*}^2 + O(\varepsilon^{\frac{N-2}{2}}).$$

This, (3.13) and (3.14) provide

$$Q_{\lambda,p}(\widetilde{w}_\varepsilon) \le \frac{\|w_\varepsilon\|_p^2 - \lambda|w_\varepsilon|_2^2 + O(\varepsilon^{\frac{N-2}{2}})}{|w_\varepsilon|_{2^*}^2 + O(\varepsilon^{\frac{N-2}{2}})} = Q_{\lambda,p}(w_\varepsilon) + \frac{O(\varepsilon^{\frac{N-2}{2}})}{|w_\varepsilon|_{2^*}^2 + O(\varepsilon^{\frac{N-2}{2}})},$$

and the proof is finished. □

Proof of Proposition 3.2 (for $\lambda = \lambda_{n,p}$). Given $u = y + t\widetilde{w}_\varepsilon \in \widetilde{\Sigma}_\varepsilon$, we can rewrite $y \in Y$ as

$$y = \widetilde{y} + \langle y, \varphi_{n,p}\rangle_p \varphi_{n,p}.$$

In this way $\langle \varphi_{n,p}, \widetilde{w}_\varepsilon\rangle_p = \int \varphi_{n,p}\widetilde{w}_\varepsilon = 0$ and $\|\varphi_{n,p}\|_p^2 = \lambda_{n,p}|\varphi_{n,p}|_2^2$. Hence

$$\|u\|_p^2 - \lambda_{n,p}|u|_2^2 = \|\widetilde{y}\|_p^2 - \lambda_{n,p}|\widetilde{y}|_2^2 + 2\langle \widetilde{y}, t\widetilde{w}_\varepsilon\rangle_p - 2\lambda_{n,p}\int t\widetilde{y}\widetilde{w}_\varepsilon + Q_\lambda(t\widetilde{w}_\varepsilon)|t\widetilde{w}_\varepsilon|_{2^*}^2.$$

It follows from Lemma 3.6 that

$$\|u\|_p^2 - \lambda_{n,p}|u|_2^2 \le \frac{1}{4(\lambda_{n-1,p} - \lambda_{n,p})} O(\varepsilon^{\frac{N-2}{2}}) + Q_\lambda(\widetilde{w}_\varepsilon)|t\widetilde{w}_\varepsilon|_{2^*}^2.$$

Since $t = O(1)$ as $\varepsilon \to 0^+$, we can use Lemma 3.5 and argue as in the first case to obtain, for $\varepsilon > 0$ small enough,

$$\|u\|_p^2 - \lambda|u|_2^2 < p_0 S, \quad \forall\, u \in \widetilde{\Sigma}_\varepsilon,$$

and this concludes the proof. □

4 Positive solution for $2 < q < 2^*$ and $\lambda > 0$

We prove in this section Theorem 1.2. Since we are looking for positive solution we redefine the functional I as follows

$$I(u) := \frac{1}{2}\int p(x)|\nabla u|^2 - \frac{\lambda}{q}\int (u^+)^q - \frac{1}{2^*}\int (u^+)^{2^*},$$

where $u^+(x) := \max\{u(x), 0\}$.

It is standard to check that there exist $\rho, \sigma > 0$ such that $I\mid_{\partial B_\rho(0)} \geq \sigma$ as well as $e \in H$ verifying $\|e\| \geq \rho$ and $I(e) < 0$. Hence, if we define

$$c := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I(\gamma(t)),$$

with

$$\Gamma := \{\gamma \in C([0,1], H) : \gamma(0) = 0,\ \gamma(1) = e\}.$$

Arguing as in the proof of Lemma 2.1 we can check that I satisfies $(PS)_d$ for any $d < c^*$. Hence, if

$$c < c^* = \frac{1}{N}(p_0 S)^{N/2}, \tag{4.1}$$

we can apply the Mountain Pass Theorem to obtain a nonzero critical point $u \in H$. Testing the derivative $I'(u)$ with $u^- = \min\{u(x), 0\}$ we conclude that $u \geq 0$. By the Maximum Principle the solution is positive.

We devote the rest of this section to prove that (4.1) holds. We first recall that the minimax level c can also be characterized by (see [17, Theorem 4.2]),

$$c = \inf_{u\in H\setminus\{0\}} \max_{t\geq 0} I(tu).$$

Hence, it suffices to prove the following

Proposition 4.1. *There exists $v \in H \setminus \{0\}$ such that*

$$\sup_{t\geq 0} I(tv) < \frac{1}{N}(p_0 S)^{N/2}.$$

Proof. Let $\psi \in C_0^\infty(\Omega)$ be such that $0 \leq \psi \leq 1$, $\psi \equiv 1$ in $B(a, l)$, $\psi \equiv 0$ in $\Omega \setminus \overline{B(a, 2l)}$ and $l > 0$ verifies $\overline{B(a, 2l)} \subset \Omega$. For $\varepsilon > 0$ we set

$$u_\varepsilon(x) := \frac{\psi(x)}{[\varepsilon + |x-a|^2]^{\frac{N-2}{2}}},\quad v_\varepsilon := \frac{u_\varepsilon}{|u_\varepsilon|_{2^*}}.$$

We define, for $t \geq 0$, the function

$$h(t) := I(tv_\varepsilon).$$

Let $t_\varepsilon > 0$ be the unique critical point of h in $(0, +\infty)$. Since $v_\varepsilon \geq 0$, we have that

$$0 = h'(t_\varepsilon) = t_\varepsilon\|v_\varepsilon\|_p^2 - \lambda t_\varepsilon^{q-1}|v_\varepsilon|_q^q - t_\varepsilon^{2^*-1}|v_\varepsilon|_{2^*}^{2^*}.$$

Moreover, since $|v_\varepsilon|_{2^*}^{2^*} = 1$ and $\lambda > 0$, we get

$$\|v_\varepsilon\|_p^2 = \lambda t_\varepsilon^{q-2}|v_\varepsilon|_q^q + t_\varepsilon^{2^*-2} \geq t_\varepsilon^{2^*-2},$$

We set

$$\widehat{t} := \|v_\varepsilon\|_p^{\frac{2}{2^*-2}} \geq t_\varepsilon$$

and

$$g(t) := \frac{1}{2}t^2\,\widehat{t}^{\,2^*-2} - \frac{1}{2^*}t^{2^*}, \;\; t \geq 0.$$

Since $g'(t) = t(\widehat{t}^{\,2^*-2} - t^{2^*-2})$, the function g is increasing in $[0, \widehat{t}\,]$. Recalling that $|v_\varepsilon|_{2^*} = 1$ we can use the definition of $\widehat{t}$ to compute

$$I(t_\varepsilon v_\varepsilon) = g(t_\varepsilon) - \frac{\lambda}{q}t_\varepsilon^q|v_\varepsilon|_q^q \leq g(\widehat{t}) - \frac{\lambda}{q}t_\varepsilon^q|v_\varepsilon|_q^q = \frac{1}{N}(\|v_\varepsilon\|_p^2)^{N/2} - \frac{t_\varepsilon^q}{q}\lambda|v_\varepsilon|_q^q. \quad (4.2)$$

As $\varepsilon \to 0^+$, we have that

$$\begin{aligned}
|u_\varepsilon|_q^q &= \int_{B(a,l)} \frac{1}{(\varepsilon + |x-a|^2)^{q(N-2)/2}}dx + O(1) \\
&= \int_{B(l/\sqrt{\varepsilon},0)} \frac{\varepsilon^{\frac{N}{2}}}{(\varepsilon + \varepsilon|y|^2)^{q(N-2)/2}}dy + O(1) \\
&= \varepsilon^{\frac{N}{2}-\frac{q(N-2)}{2}} \int_{B(l/\sqrt{\varepsilon},0)} \frac{1}{(1 + |y|^2)^{q(N-2)/2}}dy + O(1).
\end{aligned}$$

Since $q > 2$ we have that $-q(N-2) + N < 0$, and therefore the last integral above is $O(1)$, in such way that

$$|u_\varepsilon|_q^q = O\left(\varepsilon^{\frac{N}{2}-\frac{q(N-2)}{2}}\right) + O(1). \quad (4.3)$$

On the other hand, as proved in (cf. [2, pg 444]), we have that

$$|u_\varepsilon|_{2^*}^2 = K_2\varepsilon^{-\frac{N-2}{2}} + O(\varepsilon),$$

with $K_2 = K_2(N) > 0$. Thus,

$$|u_\varepsilon|_{2^*}^q = \left(|u_\varepsilon|_{2^*}^2\right)^{\frac{q}{2}} = \left(K_2\varepsilon^{-\frac{(N-2)}{2}} + O(\varepsilon)\right)^{\frac{q}{2}} = O(\varepsilon^{-\frac{q(N-2)}{4}}) + O(1). \quad (4.4)$$

Since $u_\varepsilon = |u_\varepsilon|_{2^*}v_\varepsilon$, and $Q_{0,p}(u_\varepsilon) = Q_{0,p}(|u_\varepsilon|_{2^*}v_\varepsilon) = \|v_\varepsilon\|_p^2$, it follows from the calculations presented in the proof of [12, Lemma 3.2] that

$$\|v_\varepsilon\|_p^2 = \begin{cases} p_0S + O(\varepsilon), & N \geq 5; \\ p_0S + O(\varepsilon|\log\varepsilon|), & N = 4. \end{cases} \quad (4.5)$$

Hence $\|v_\varepsilon\|_p^2 = p_0S + O(\varepsilon|\log\varepsilon|)$ and we conclude that

$$\left(\|v_\varepsilon\|_p^2\right)^{N/2} = (p_0S)^{N/2} + O(\varepsilon|\log\varepsilon|).$$

We claim that there exists $C_0 > 0$ such that $t_\varepsilon^q \geq qC_0$ for any $\varepsilon > 0$. Indeed, suppose by contradiction that, for some sequence $\varepsilon_n \to 0^+$, we have that $t_{\varepsilon_n} \to 0$. Then

$$\|t_{\varepsilon_n} v_{\varepsilon_n}\|_p^2 = t_{\varepsilon_n}^2 \|v_{\varepsilon_n}\|_p^2 = t_{\varepsilon_n}^2 (p_0 S + O(\varepsilon_n |\log \varepsilon_n|)) = o_n(1),$$

and therefore $t_{\varepsilon_n} v_{\varepsilon_n} \to 0$ in H. This would imply

$$0 < c \leq \sup_{t\geq 0} I(t v_{\varepsilon_n}) = I(t_{\varepsilon_n} v_{\varepsilon_n}) \to I(0) = 0,$$

which is absurd.

The boundedness of (t_n), (4.2) and (4.5) imply that

$$\begin{aligned} I(t_\varepsilon v_\varepsilon) &\leq \frac{1}{N}(p_0 S)^{\frac{N}{2}} + O(\varepsilon |\log \varepsilon|) - C_0 \lambda |v_\varepsilon|_q^q \\ &= \frac{1}{N}(p_0 S)^{\frac{N}{2}} + \varepsilon |\log \varepsilon| \left(O(1) - \lambda C_0 \frac{|v_\varepsilon|_q^q}{\varepsilon |\log \varepsilon|} \right). \end{aligned}$$

It suffices now to prove that

$$\lim_{\varepsilon \to 0^+} \frac{|v_\varepsilon|_q^q}{\varepsilon |\log \varepsilon|} = +\infty. \tag{4.6}$$

By (4.3) and (4.4) we get

$$|v_\varepsilon|_q^q = \frac{|u_\varepsilon|_q^q}{|u_\varepsilon|_{2^*}^q} = \frac{O\left(\varepsilon^{\frac{N}{2} - \frac{q(N-2)}{4}}\right)}{1 + O\left(\varepsilon^{\frac{q(N-2)}{4}}\right)} + O(1).$$

or, equivalently,

$$\frac{|v_\varepsilon|_q^q}{\varepsilon |\log \varepsilon|} = \frac{O\left(\varepsilon^{1 - \frac{N}{2} + \frac{q(N-2)}{4}} |\log \varepsilon|\right)^{-1}}{1 + O\left(\varepsilon^{\frac{q(N-2)}{4}}\right)} + O(1).$$

Since $q > 2$, a direct calculation provides $1 - (N/2) + q(N-2)/4 > 0$ and therefore (4.6) holds. The proposition is proved. □

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