ASYMPTOTICALLY PERIODIC SUPERQUADRATIC HAMILTONIAN SYSTEMS

MARCELO F. FURTADO AND REINALDO DE MARCHI

Abstract. We obtain existence of solutions for the system

$$\begin{cases} -\Delta u + V(x)u = F_v(x,u,v), \ x \in \mathbb{R}^N, \\ -\Delta v + V(x)v = F_u(x,u,v), \ x \in \mathbb{R}^N, \end{cases}$$

where $N \geq 3$, $V \in C(\mathbb{R}^N,(0,\infty))$ is periodic and $F \in C^1(\mathbb{R}^N \times \mathbb{R}^2,\mathbb{R})$ is superquadratic at infinity. We consider the case that F is periodic and asymptotically periodic. In the proofs we apply variational techniques.

1. Introduction

In this paper we prove existence of solutions for the following Hamiltonian system

$$\begin{cases} -\Delta u + V(x)u = F_v(x,u,v), \ x \in \mathbb{R}^N, \\ -\Delta v + V(x)v = F_u(x,u,v), \ x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $N \geq 3$, $F \in C^1(\mathbb{R}^N \times \mathbb{R}^2,\mathbb{R})$ and F_u, F_v stand for the partial derivatives of F with respect to the second and third variable.

Recently, many authors have used variational techniques to study the system (1.1) and its variants. For the bounded domain case we refer the reader to [4, 6, 7, 11, 12] and references therein, while for the whole space $\mathbb{R}^N$ we quote the papers [2, 5, 8, 13, 18, 22], among others. In these works, a huge machinery is needed to obtain existence and multiplicity of solutions: fractional Sobolev spaces, reduction methods, the generalized Mountain Pass Theorem, radial approaches and many others.

One of the main difficulties in dealing with (1.1) relies on the lack of compactness due to the unboundedness of the domain. In some of the above quoted papers this difficulty was overcome by imposing periodicity both on the potential V and on the nonlinearity F. Here, we consider not only the periodic case for F, but also the asymptotically periodic case. The main assumption on $V \in C(\mathbb{R}^N,\mathbb{R})$ is the following

(V_0) $V(x) = V(x_1,\ldots,x_N)$ is positive and 1-periodic in the variables $x_1,\cdots,x_N$.

In our fist result, we consider the periodic problem. We start by assuming the following growth conditions on the nonlinearity F:

(F_0) $F(x,z)$ is 1-periodic in the variables $x_1,x_2,\cdots,x_N$;

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(F_1) there exist $c > 0$ and $p \in (2, 2N/(N-2))$ such that

$$|F_z(x,z)| \leq c(1 + |z|^{p-1}), \text{ for each } (x,z) \in \mathbb{R}^N \times \mathbb{R}^2,$$

where $F_z(x,z) = (F_u(x,z), F_v(x,z)) \in \mathbb{R}^2$;

(F_2) $F_z(x,z) = o(|z|)$ as $|z| \to 0$, uniformly for $x \in \mathbb{R}^N$.

We are interested in the case that F is superquadratic at infinity. Since the seminal work of Ambrosetti and Rabinowitz [1], superquadratic problems (at infinity) are subject to intensive studies. The superlinear condition introduced in [1] read as

(AR) there exists $\mu > 2$ such that, for each $x \in \mathbb{R}^N$, $z \in \mathbb{R}^2 \setminus \{0\}$, there holds

$$0 < \mu F(x,z) \leq F_z(x,z) \cdot z,$$

where the dot stands for the scalar product in $\mathbb{R}^2$. It is well known that is condition provides boundedness for the Palais-Smale sequences of the energy functional. A straightforward calculation shows that (AR) implies $F(x,z) \geq c|z|^\mu$ for large values of $|z|$, and therefore it is natural to consider the weaker assumption

(F_3) $\frac{F(x,z)}{|z|^2} \to \infty$ as $|z| \to \infty$, uniformly for $x \in \mathbb{R}^N$.

There are many results concerning the scalar version of (1.1) and considering the above superquadratic growth condition instead of (AR). Most of them use the Nehari approach for which the authors assume a monotonicity condition for $f(x,t)/t$. In our first result, we follow an analogous approach and therefore we need a version of this monotonicity condition for the nonlinearity F. We assume that

(F_4) there exists $g : \mathbb{R}^N \times \mathbb{R}^+ \to [0, +\infty)$ increasing in the second variable such that

$$F_z(x,z) = g(x,|z|)z, \text{ for each } (x,z) \in \mathbb{R}^N \times \mathbb{R}^2.$$

We recall that, for some suitable Banach space E (see Section 3), the weak solutions of problem (1.1) are critical points of the C^1-functional

$$I(z) = \int_{\mathbb{R}^N} (\nabla u \cdot \nabla v + V(x)uv)\, dx - \int_{\mathbb{R}^N} F(x,z) dx,$$

for any $z = (u,v) \in E$. We say that $z_0 \in E$ is a least energy solution if

$$I(z_0) = \inf \{I(z) \,:\, z \in E \setminus \{0\} \text{ is a weak solution of (1.1)}\}$$

In our first result, we prove the existence of this kind of solution.

Theorem 1.1. *Suppose that V satisfies (V_0) and F satisfies $(F_0)-(F_4)$. Then the problem* (1.1) *has a least energy solution.*

Due to the Hamiltonian nature of our system, the functional associated to the problem (1.1) is strongly indefinite. Hence, the usual Nehari approach can not be used. In [17], Pankov introduced a generalized Nehari manifold and imposed some technical assumptions on the nonlinearity in order to obtain regularity. However, with the weaker conditions imposed here, we cannot prove this regularity and therefore that argument does not work. Recently, Szulkin and Weth [19] developed a new approach based on a reduction method which allows one to prove that minimizers of the functional I on the the generalized Nehari manifold are critical points of the unconstrained functional. We follow the ideas of this last paper in the proof of Theorem 1.1.

Let us now give a concrete example which fits with Theorem 1.1. Let $a \in C(\mathbb{R}^N, \mathbb{R})$ be positive and 1-periodic in $x_1, \ldots, x_N$ and consider

$$F(x,z) := \begin{cases} \dfrac{a(x)}{4}\left[2(|z|^2-1)\ln(1+|z|)+2|z|-|z|^2+4\ln(2)-3\right], & \text{if } |z|\ge 1,\\ \dfrac{a(x)}{4}[-2|z|^2+2(1+|z|^2)\ln(1+|z|^2)], & \text{if } |z|<1. \end{cases}$$

We can see that this nonlinearity satisfies $(F_0)-(F_4)$ with

$$g(x,t) := \begin{cases} a(x)\ln(1+t), & \text{if } t\ge 1,\\ a(x)\ln(1+t^2), & \text{if } 0\le t<1. \end{cases}$$

However, it does not satisfy the hypothesis (AR).

Secondly, we consider the case in which F is non periodic. In this new setting we need to introduce the auxiliary function

$$\widehat{F}(x,z) := \frac{1}{2}F_z(x,z)\cdot z - F(x,z),$$

for $(x,z)\in\mathbb{R}^N\times\mathbb{R}^2$ and require the following assumptions:

(F_5) there exists $\delta_0>0$ such that, for any $r\in(0,\delta_0)$, there holds

$$q(r) := \inf\{\widehat{F}(x,z) : x\in\mathbb{R}^N,\ |z|\ge r\} > 0;$$

(F_6) there exist c_0, $R_0>0$ and $\tau>N/2$ such that

$$|F_z(x,z)|^\tau \le c_0|z|^\tau \widehat{F}(x,z) \text{ for each } x\in\mathbb{R}^N,\ |z|\ge R_0.$$

In order to make precise the definition of asymptotically periodic, let us denote by $\mathcal{F}$ the class of all functions $\varphi\in C(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$ such that, for any $\varepsilon>0$, the set $\{x\in\mathbb{R}^N : |\varphi(x)|\ge\varepsilon\}$ has finite Lebesgue measure. The next hypothesis provides the asymptotic behavior of the nonlinearity F:

(F_7) there exist $p_\infty\in(2,2^*)$, $\varphi\in\mathcal{F}$ and $F_\infty\in C^1(\mathbb{R}^N\times\mathbb{R}^2,\mathbb{R})$ such that
   (i) F_∞ satisfies $(F_0)-(F_4)$;
   (ii) $F(x,z)\ge F_\infty(x,z)$ for each $(x,z)\in\mathbb{R}^N\times\mathbb{R}^2$;
   (iii) $|F_z(x,z)-F_{\infty,z}(x,z)|\le\varphi(x)|z|^{p_\infty-1}$ for each $(x,z)\in\mathbb{R}^N\times\mathbb{R}^2$.

Our second main result is the following:

Theorem 1.2. *Suppose that V satisfies (V_0) and F satisfies (F_2) and $(F_5)-(F_7)$. Then the problem (1.1) has a nonzero solution.*

In the proof we apply a version of the Linking Theorem due to Li and Szulkin [14] to obtain a Cerami sequence for the associated functional. After proving that the sequence is bounded, we prove that its weak limit is a solution of (1.1). The main difficulty is to prove that this weak limit is nonzero. Indeed, since we are not able to prove that I satisfies the usual compactness condition, we need to use an indirect argument and prove a local version of the Linking Theorem. To state this version, we need to introduce some notations and assumptions and we prefer to postpone its statement to Section 2 (see Theorem 2.3). We are confident that this new abstract result may be used in other problems related with functionals affected by a lack of compactness.

Condition (F_7) was introduced by Lins and Silva [15] in the study of a Schrödinger equation. As far we know, condition (F_6) was introduced by Ding and Lee [9] as an alternative to the (AR) condition. There the authors considered a Schrödinger equation with indefinite potential and showed that (AR) and (F_1) imply (F_6). This condition is also used in [20]. We also mention the paper of Zhang et al. [22] where the authors considered the periodic case of (1.1) with an indefinite potential. The technical condition (F_5) is important in the proof of the boundedness of Cerami sequences (see [10] for closely related results) and it is clearly satisfied in the periodic case.

2. Preliminary results

In this section we present some abstract results that will be used in our proofs. We will denote by E a real Hilbert space with scalar product $\langle \cdot, \cdot \rangle$ and associated norm $\| \cdot \|$. We assume that there is an orthogonal decomposition $E = E^- \oplus E^+$, in such way that any element $z \in E$ has a unique decomposition $z = z^+ + z^-$, with $z^\pm \in E^\pm$. Given $r > 0$, we define the sets

$$N_r := \{z \in E^+ : \|z\| = r\}, \quad S^+ := N_1 = \{z \in E^+ : \|z\| = 1\}, \tag{2.1}$$

$$E(z) := \mathbb{R}z \oplus E^- \equiv \mathbb{R}z^+ \oplus E^-,$$

and

$$\widehat{E}(z) := \mathbb{R}^+ z \oplus E^- \equiv \mathbb{R}^+ z^+ \oplus E^-.$$

Suppose that the functional $I \in C^1(E, \mathbb{R})$ verifies the following assumptions:

(N_1) I can be written as

$$I(z) = \frac{1}{2}\|z^+\|^2 - \frac{1}{2}\|z^-\|^2 - \mathcal{J}(z), \tag{2.2}$$

with $\mathcal{J} \in C^1(E, \mathbb{R})$ weakly lower semicontinuous, $\mathcal{J}(0) = 0$ and, for each $z \neq 0$, there holds

$$\mathcal{J}'(z)z > 2\mathcal{J}(z) > 0;$$

(N_2) for each $z \in E \backslash E^-$ the restriction of I to the set $\widehat{E}(z)$ has a unique nonzero critical point $\widehat{m}(z)$ which is a global maximum point of this restriction;

(N_3) there exists $\delta > 0$ such that $\|\widehat{m}(z)^+\| \geq \delta$, for each $z \in E \setminus E^-$. Moreover, if $\mathcal{K} \subset E \setminus E^-$ is compact, then there exists $c_\mathcal{K}$ such that $\|\widehat{m}(z)\| \leq c_\mathcal{K}$, for each $z \in \mathcal{K}$.

We introduce the generalized Nehari manifold of I by setting

$$\mathcal{M} := \{z \in E \setminus E^- : I'(z)z = 0 \text{ and } I'(z)w = 0 \text{ for all } w \in E^-\}.$$

Notice that, if $z \neq 0$ is a critical point of I, then we can use (N_1) to get

$$I(z) = I(z) - \frac{1}{2}I'(z)z = \frac{1}{2}\mathcal{J}'(z)z - \mathcal{J}(z) > 0.$$

Since $I \leq 0$ on E^-, the above inequality shows that $\mathcal{M}$ contains all nonzero critical points of I. Moreover, by using (N_2) and the definition of $\mathcal{M}$, we can construct the map

$$\widehat{m} : E \setminus E^- \to \mathcal{M}, \quad \widehat{m}(z) := \left\{ \text{the unique global maximum point of } I|_{\widehat{E}(z)} \right\}.$$

We shall denote by m the restriction of this map to the set S^+, that is,

$$m := \widehat{m}|_{S^+}.$$

We collect in the following result the main properties of the above maps, whose proofs can be found in a series of lemmas from [19].

Lemma 2.1. *If $I \in C^1(E, \mathbb{R})$ satisfies $(N_1) - (N_3)$ then* *(i)* $\widehat{m}$ *is continuous and* $m : S^+ \to \mathcal{M}$ *is an homeomorphism;*

(ii) *the functional* $\widehat{\Psi} : E^+ \setminus \{0\} \to \mathbb{R}$ *given by* $\widehat{\Psi}(z) := I(\widehat{m}(z))$ *is of class* C^1*. The same holds for* $\Psi := \widehat{\Psi}|_{S^+}$*, with*

$$\Psi'(z)w = \|m(z)^+\| I'(m(z))w, \ \ \textit{for each } w \in T_z(S^+);$$

(iii) *if* $(z_n) \subset S^+$ *is a Palais-Smale sequence for* Ψ*, then* $(m(z_n)) \subset \mathcal{M}$ *is a Palais-Smale sequence for* I*. If* $(w_n) \subset \mathcal{M}$ *is a bounded Palais-Smale sequence for* I*, then* $(m^{-1}(w_n)) \subset S^+$ *is a Palais-Smale sequence for* Ψ*;*

(iv) $\inf_{S^+} \Psi = \inf_{\mathcal{M}} I$*.*

In order to present our next abstract result we need to introduce a new topology in the space E. We take a total orthonormal sequence (e_k) in E^-, set

$$\|z\|_\tau := \max\left\{ \|z^+\|, \sum_{k=1}^{\infty} \frac{1}{2^k} |\langle z^-, e_k\rangle| \right\}$$

and call τ-topology the topology induced by this norm. It can be proved that, in bounded sets, its coincides with the topology of E that is weak is on E^- and strong on E^+. Hence, for a bounded sequence $(z_n) \subset (E, \tau)$, we have that $z_n \xrightarrow{\tau} z$ in E if, and only if $z_n^+ \to z^+$ and $z_n^- \rightharpoonup z^-$ weakly in E. We refer to [12, Section 2] for others properties of the τ-topology.

Given a set $M \subset E$, an homotopy $h : [0,1] \times M \to E$ is said to be admissible if

(i) h is τ-continuous, that is, if $t_n \to t$ and $z_n \xrightarrow{\tau} z$ then $h(t_n, z_n) \xrightarrow{\tau} h(t, z)$;

(ii) for each $(t, z) \in [0,1] \times M$ there is a neighborhood U of (t, z) in the product topology of $[0,1]$ and (E, τ) such that the set $\{w - h(t, w) : (t, w) \in U \cap ([0,1] \times M)\}$ is contained in a finite dimensional subspace of E.

The symbol Γ will denote the following class of admissible maps

$$\begin{aligned} \Gamma := \{ \quad & h \in C([0,1] \times M, E) \ : \ h \text{ is admissible}, h(0, \cdot) = \mathrm{Id}_M, \\ & I(h(t,z)) \le \max\{I(z), -1\} \text{ for all } (t,z) \in [0,1] \times M\}. \end{aligned}$$

The following version of the Linking Theorem was proved in [14, Theorem 2.1]:

Theorem 2.2. *Suppose that* $I \in C^1(E, \mathbb{R})$ *satisfies*

(L_1) *I can be written as in (2.2) with* $\mathcal{J}$ *bounded from below, weakly sequentially lower semicontinuous and* $\mathcal{J}'$ *weakly sequentially continuous;*

(L_2) *there exists* $z_0 \in E^+ \setminus \{0\}$*,* $\alpha > 0$ *and* $R > r > 0$ *such that*

$$\inf_{z \in N_r} I(z) \ge \alpha, \quad \sup_{z \in \partial M} I(z) \le 0,$$

where $M = M(z_0, R)$ *stands for*

$$M := \{z = tz_0 + z^- \ : \ z^- \in E^-, \ \|z\| \le R, \ t \ge 0\}$$

and ∂M *denotes the boundary of* M *relative to* $\mathbb{R}z_0 \oplus E^-$*. If we define*

$$c := \inf_{h \in \Gamma} \sup_{z \in M} I(h(1, z)),$$

then there exists $(z_n) \subset E$ *such that*

$$I(z_n) \to c \ge \alpha, \ \ (1 + \|z_n\|)\|I'(z_n)\| \to 0.$$

Actually, to prove Theorem 1.2, we can not directly apply Theorem 2.2. For this purpose, we need the following local version:

Theorem 2.3. *Under the same hypotheses of Theorem 2.2, suppose additionally that there exists* $h_0 \in \Gamma$ *such that*

$$c = \sup I(h_0(1, M)). \tag{2.3}$$

Then I *possesses a nonzero critical point* $z \in h_0(1, M)$ *such that* $I(z) = c$.

Proof. Suppose, by contradiction, that $D := h_0(1, M)$ contains no critical points of I at level c. This implies that there exist $\varepsilon, \delta > 0$ such that

$$\|I'(z)\| \geq \frac{8\varepsilon}{\delta} \text{ for any } z \in I^{-1}([c - 2\varepsilon, c + 2\varepsilon]) \cap D^{\tau}_{2\delta}, \tag{2.4}$$

where $D^{\tau}_{2\delta}$ stands for the τ-closed set $\{z \in E \ : \ \|z - w\|_{\tau} \leq 2\delta, \text{ for any } w \in D\}$. Indeed, otherwise we obtain a sequence (z_n) such that $z_n \in D^{\tau}_{2/\sqrt{n}}$ and

$$c - \frac{2}{n} \leq I(z_n) \leq c + \frac{2}{n}, \quad \|I'(z_n)\| \leq \frac{2}{\sqrt{n}},$$

an therefore it follows that $I(z_n) \to c$ and $I'(z_n) \to 0$. The definition and boundedness of M, together with the convergence properties of the τ-topology, presented just before the definition of admissible homotopy, show that M is τ-compact. The same occurs with D, since h_0 is τ-continuous. It follows that D^{τ}_2 is τ-compact, and therefore we may assume that $z_n \xrightarrow{\tau} z \in D$. This implies that $z_n^+ \to z^+$ and $z_n^- \rightharpoonup z^-$ weakly in E. Hence, the regularity assumption on I' implies that $I'(z) = 0$. These last convergences, the lower semicontinuity of the norm, and the fact that $\mathcal{J}$ is weakly sequentially lower semicontinuous, imply that $I(z) \geq c$. Moreover, since $z \in D \subset I^c = \{w \in E : I(w) \leq c\}$, we also have that $I(z) \leq c$. Hence, we conclude that $z \in D$ is a critical point at level c, which is absurd.

In view of (2.4) and the regularity assumptions on I, we can use a version of the deformation lemma proved in [3, Lemma 8] (see also [21, Lemma 6.8]), to obtain an admissible homotopy $\eta : [0, 1] \times M \to E$ such that

$$\eta(1, D) \subset I^{c-\varepsilon} = \{w \in E : I(w) \leq c - \varepsilon\}.$$

We now define $h : [0, 1] \times M \to E$ by

$$h(t, z) := \begin{cases} h_0(2t, z), \\ \eta(2t - 1, h_0(1, z)). \end{cases}$$

Then $h \in \Gamma$ and, for any $z \in M$, there holds

$$I(h(1, z)) = I(\eta(1, h_0(1, z))) \leq c - \varepsilon,$$

since $h_0(1, z) \in D$. This inequality contradicts the definition of c and concludes the proof. $\square$

3. The periodoc case

In this section, we will apply Lemma 2.1 and a minimization argument to obtain a nonzero solution for problem (1.1). In what follows, we denote by E the Hilbert space $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ endowed with the norm

$$\|(u, v)\|^2 := \int (|\nabla u|^2 + V(x)u^2) + \int (|\nabla v|^2 + V(x)v^2).$$

We consider the following decomposition of the space E:

$$E^{\pm} := \{(u, \pm u) : u \in H^1(\mathbb{R}^N)\}.$$

We have that E^+ and E^- are orthogonal in E and also in $L^2(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$. For $z = (u, v) \in E$ we set

$$z^+ := \left(\frac{u+v}{2}, \frac{u+v}{2} \right) \text{ and } z^- := \left(\frac{u-v}{2}, -\frac{u-v}{2} \right).$$

Then $z^\pm \in E^\pm$ and $z = z^+ + z^-$. Thus, $E = E^+ \oplus E^-$ and we can compute

$$\int (\nabla u \nabla v + V(x)uv) = \frac{1}{2}(\|z^+\|^2 - \|z^-\|^2).$$

By using (F_1) and (F_2) we can check that, for any given $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\max\{|F(x,z)|, |F_z(x,z) \cdot z|\} \le \varepsilon |z|^2 + C_\varepsilon |z|^p \ , \tag{3.1}$$

for each $(x, z) \in \mathbb{R}^N \times \mathbb{R}^2$. Hence, it is well defined the associated functional

$$I(z) := \frac{1}{2}(\|z^+\|^2 - \|z^-\|^2) - \mathcal{J}(z), \tag{3.2}$$

where $\mathcal{J}(z) := \int F(x,z)$. Moreover, I belongs to $C^1(E, \mathbb{R})$ with

$$I'(z)w = \langle z^+, w^+ \rangle - \langle z^-, w^- \rangle - \int F_z(x,z) \cdot w, \qquad \forall \, z, w \in E.$$

Hence, the critical points of I are weak solutions of problem (1.1).

Since we aim to apply Lemma 2.1, we prove in the sequel that I satisfies conditions $(N_1) - (N_3)$.

Lemma 3.1. *Suppose that F satisfies (F_2) and (F_4). Then, for each $z \neq 0$ we have that*

$$\frac{1}{2} F_z(x,z) \cdot z > F(x,z) > 0.$$

Moreover, $\mathcal{J}(0) = 0$ and $\mathcal{J}$ is weakly lower semicontinuous.

Proof. By (F_2) we get $F(x, 0) = 0$ and therefore $\mathcal{J}(0) = 0$. Given $z \neq 0$, it follows from (F_4) that

$$F(x,z) = \int_0^1 \frac{d}{dt}[F(x,tz)]dt = \int_0^1 F_z(x,tz) \cdot z dt = |z|^2 \int_0^1 g(x, t|z|)t dt > 0. \tag{3.3}$$

This identity, (F_4) and the monotonicity of $g(x, \cdot)$ imply that

$$\frac{1}{2} F_z(x,z) \cdot z - F(x,z) = |z|^2 \left(\int_0^1 [g(x,|z|) - g(x,t|z|)] t dt \right) > 0.$$

In order to check the last statement, let $(z_n) \subset E$ be such that $z_n \rightharpoonup z$ weakly in E. Up to a subsequence, we have that $z_n(x) \to z(x)$ a.e. in $\mathbb{R}^N$. Since F is nonnegative, Fatou's lemma provides

$$\liminf_{n \to \infty} \mathcal{J}(z_n) = \liminf_{n \to \infty} \int F(x, z_n) \ge \int F(x,z) = \mathcal{J}(z)$$

and we have done. □

The next lemma is essential in order to get condition (N_2).

Lemma 3.2. *Suppose that F satisfies $(F_2) - (F_4)$. Let $s \ge -1$ and v, $z \in \mathbb{R}^2$ with $w = sz + v \neq 0$. Then, for each $x \in \mathbb{R}^N$, there holds*

$$F_z(x,z) \cdot \left(s(\frac{s}{2} + 1)z + (s+1)v \right) + F(x,z) - F(x, z+w) < 0.$$

Proof. Let $y = y(s) := w + z = (1 + s)z + v$ and define, for $s \geq -1$,

$$\beta(s) := F_z(x, z) \cdot \Big(s(\frac{s}{2} + 1)z + (s + 1)v\Big) + F(x, z) - F(x, z + w).$$

If $z = 0$, it follows from (F_2) and Lemma 3.1 that $\beta(s) = -F(x, y) < 0$. Hence, we may suppose that $z \neq 0$ and consider two distinct cases:

Case 1: $z \cdot y \leq 0$

In this case we notice that, by (F_4), $F_z(x, z)y = g(x, |z|)z \cdot y \leq 0$. Thus, recalling that $v = y - (1 + s)z$, using Lemma 3.1 and $s \geq -1$, we obtain

$$\begin{aligned} \beta(s) &= -\left(\frac{s^2}{2} + s + 1\right) F_z(x, z) \cdot z + (s + 1)F_z(x, z) \cdot y + F(x, z) - F(x, y) \\ &< -\frac{1}{2}(s + 1)^2 F_z(x, z) \cdot z + (s + 1)F_z(x, z) \cdot y - F(x, y) < 0. \end{aligned} \tag{3.4}$$

Case 2: $z \cdot y > 0$

Using Lemma 3.1 we get

$$\beta(-1) = -\frac{1}{2}F_z(x, z) \cdot z + F(x, z) - F(x, y) < -F(x, y) < 0.$$

It follows from (F_4) that $F_z(x, z)\cdot z = g(x, |z|)|z|^2 > 0$, and therefore, using (3.4) we get $\lim_{s\to\infty} \beta(s) = -\infty$. Hence, β attains its maximum at some point $s_0 \in [-1, \infty)$. If $s_0 = -1$ the result follows from the above inequality. If $s_0 > -1$, we have that

$$0 = \beta'(s_0) = F_z(x, z) \cdot y - F_z(x, y) \cdot z.$$

By using (F_4), we obtain $g(x, |z|)z \cdot y = g(x, |y|)y \cdot z$ and hence $|z| = |y|$. It follows from (3.3) that $F(x, z) = F(x, y)$. Moreover,

$$F_z(x, z) \cdot y = g(x, |z|)z \cdot y \leq g(x, |z|)|z|^2 = F_z(x, z) \cdot z,$$

and therefore

$$\beta(s) = -\frac{s^2}{2}F_z(x, z) \cdot z + (s + 1)(F_z(x, z) \cdot y - F_z(x, z) \cdot z) \leq -\frac{s^2}{2}F_z(x, z) \cdot z < 0.$$

This finishes the proof. □

Lemma 3.3. *Suppose that F satisfies $(F_1) - (F_4)$. If $z \in \mathcal{M}$, then for each $w \neq 0$ such that $z + w \in \widehat{E}(z)$ there holds*

$$I(z + w) < I(z).$$

In particular, z is the unique global maximum point of $I|_{\widehat{E}(z)}$.

Proof. Let $z \in \mathcal{M}$ and $w \neq 0$ with $z + w \in \widehat{E}(z)$. By the definition of $\widehat{E}(z)$ we can write $z + w = (1 + s)z + v$, with $s \geq -1$ and $v \in E^-$. Since $z \in \mathcal{M}$, if we define $\phi := s(\frac{s}{2} + 1)z + (s + 1)v \in E(z)$, we have that

$$0 = I'(z)\phi = s(\frac{s}{2} + 1)(\|z^+\|^2 - \|z^-\|^2) - (s + 1)\langle z^-, v\rangle - \int F_z(x, z) \cdot \phi.$$

Hence,

$$
\begin{aligned}
&I(z+w) - I(z) \\
&= s\left(\frac{s}{2}+1\right)(\|z^+\|^2 - \|z^-\|^2) - (s+1)\langle z^-, v\rangle - \frac{1}{2}\|v\|^2 + \int [F(x,z) - F(x,z+w)] \\
&= -\frac{1}{2}\|v\|^2 + \int \left[F_z(x,z)\cdot\left(s(\frac{s}{2}+1)z + (s+1)v\right) + F(x,z) - F(x,z+w)\right].
\end{aligned}
$$

Since $w \neq 0$, it follows from Lemma 3.2 that $I(z+w) < I(z)$.

Let us show that this last inequality implies that z is the unique maximum point of the restriction $I|_{\widehat{E}(z)}$. Indeed, given $tz + y \in \widehat{E}(z) \setminus \{z\}$, it is enough to consider $w = (t-1)z + y$ to obtain $tz + y = z + w$. Notice that, if $w = 0$, then $t = 1$ and $y = 0$, which cannot occurs since $tz + y \neq z$. Thus, $w \neq 0$ and we conclude that $I(tz + y) < I(z)$. □

Lemma 3.4. *Suppose that F satisfies $(F_1) - (F_2)$. Then, for each $z \in E \setminus E^-$, we have that $\widehat{E}(z) \cap \mathcal{M}$ has a unique element, which is exactly the unique global maximum of $I|_{\widehat{E}(z)}$.*

Proof. By Lemma 3.3, it suffices to prove that $\mathcal{M} \cap \widehat{E}(z) \neq \emptyset$ for each $z \in E \setminus E^-$. Moreover, since $\widehat{E}(z) = \widehat{E}(\frac{z^+}{\|z^+\|})$, we may assume that $z \in S^+$.

Claim: the exists $R > 0$ such that $I(w) \leq 0$, whenever $w \in \widehat{E}(z) \setminus B_R(0)$.

Indeed, if this is not the case, we can obtain a sequence $(w_n) \subset \widehat{E}(z)$ such that $\|w_n\| \to \infty$ and $I(w_n) > 0$. Setting $z_n := w_n/\|w_n\|$ we may assume that $z_n \rightharpoonup z_0$ weakly in E. If $z_0 \neq 0$, we infer from Fatou's lemma and (F_3) that

$$
0 \leq \frac{I(w_n)}{\|w_n\|^2} = \frac{1}{2}\|z_n^+\|^2 - \frac{1}{2}\|z_n^-\|^2 - \int \frac{F(x,w_n)}{|w_n|^2}|z_n|^2 \to -\infty,
$$

which is absurd. Thus, $z_0 = 0$. Since $F \geq 0$ we can use the above estimate to obtain $\|z_n^+\| \geq \|z_n^-\|$. Hence, recalling that $\|z_n\| = 1$, we conclude that $\|z_n^+\| \geq 1/\sqrt{2}$. Since $z \in S^+$, we can use this last inequality to write $z_n^+ = s_n z$, with $1/\sqrt{2} \leq s_n \leq 1$. Up to a subsequence, $z_n^+ \to sz$ in E with $s > 0$, which contradicts $z_n \rightharpoonup 0$. The claim is proved.

By using (F_2) and standard calculations we get $I(sz) = \frac{1}{2}s^2 + o(s^2)$ as $s \to 0$. This, the claim and (F_1) show that $0 < \sup_{\widehat{E}(z)} I < \infty$. Since I is lower weakly semicontinuous in $\widehat{E}(z) \cap B_R(0)$, we can use the fact that $I \leq 0$ in $\widehat{E}(z) \cap E^-$, to conclude that the maximum is attained in some point $\tilde{z} \in \widehat{E}(z)$ such that $\tilde{z}^+ \neq 0$. Thus, $\tilde{z} \in \mathcal{M}$ and we have done. □

We now define the following minimizer

$$
c := \inf_{z \in \mathcal{M}} I(z).
$$

Among other things, the next result relates the number c with the infimum of I on the set N_r defined in (2.1):

Lemma 3.5. *Suppose that F satisfies $(F_1) - (F_2)$. Then*

(i) there exists $r > 0$ such that

$$
c \geq \inf_{z \in N_r} I(z) > 0;
$$

(ii) for each $z \in \mathcal{M}$, there holds $\|z^+\| \geq \max\{\|z^-\|, \sqrt{2c}\}$.

Proof. If $z \in E^+$, then $I(z) = \frac{1}{2}\|z\|^2 - \int F(x,z)$. Hence, we can use $(F_1)-(F_2)$, to conclude that $\int F(x,z) = o(\|z\|^2)$ as $\|z\| \to 0$, from which it follows that $\inf_{N_r} I > 0$ for any $r > 0$ small. Moreover, if $z \in \mathcal{M}$, Lemma 3.3 provides

$$I(z) \geq I\left(r\frac{z^+}{\|z^+\|}\right) \geq \inf_{N_r} I,$$

and therefore $c \geq \inf_{N_r} I$. Finally, for any $z \in \mathcal{M}$, we get

$$c \leq \frac{1}{2}(\|z^+\|^2 - \|z^-\|^2) - \int F(x,z) \leq \frac{1}{2}(\|z^+\|^2 - \|z^-\|^2),$$

and this establishes (ii). □

Lemma 3.6. *Suppose that F satisfies $(F_1)-(F_2)$. If $\mathcal{K} \subset E\setminus E^-$ is compact, then there exists $c_{\mathcal{K}} > 0$ such that $\|\widehat{m}(z)\| \leq c_{\mathcal{K}}$, for each $z \in \mathcal{K}$.*

Proof. As in the proof of Lemma 3.4 we may assume that $\mathcal{K} \subset S^+$. Suppose, by contradiction, that there exists $(z_n) \subset \mathcal{K}$ such that $\|\widehat{m}(z_n)\| \to \infty$. Since $\widehat{m}(z_n) \in \widehat{E}(z_n)$, we can write $w_n := \widehat{m}(z_n)/\|\widehat{m}(z_n)\| = s_n z_n + w_n^-$. Arguing as in the proof of Lemma 3.4 we can show that $1/\sqrt{2} \leq s_n \leq 1$ and $s_n \geq \|w_n^-\|$. This and the compactness of $\mathcal{K}$ imply that, up to a subsequence, $s_n \to s > 0$, $z_n \to z \neq 0$ and $w_n^- \rightharpoonup w^-$ weakly in E. We may also assume that $w_n(x) \to w(x) := sz(x) + w^-(x)$ a.e. in $\mathbb{R}^N$, with $w \neq 0$ and

$$\lim_{n\to\infty} |\widehat{m}(z_n)(x)| = \infty \ \text{ a.e. in } \Omega := \{x \in \mathbb{R}^N : w(x) \neq 0\}.$$

By (F_3) and Fatou's lemma we obtain

$$\begin{aligned} 0 \leq \frac{I(\widehat{m}(z_n))}{\|\widehat{m}(z_n)\|^2} &= \frac{1}{2}\frac{s_n^2}{\|\widehat{m}(z_n)\|^2} - \frac{1}{2}\frac{\|w_n^-\|^2}{\|\widehat{m}(z_n)\|^2} - \int \frac{F(x,\widehat{m}(z_n))}{\|\widehat{m}(z_n)\|^2} \\ &\leq \frac{1}{2} - \int_\Omega \frac{F(x,\widehat{m}(z_n))}{|\widehat{m}(z_n)|^2}|w_n|^2 dx \to -\infty, \end{aligned}$$

which does not make sense. The lemma is proved. □

Lemma 3.7. *Suppose that F satisfies $(F_1)-(F_3)$. Then I is coercive on $\mathcal{M}$. In particular, any $(PS)_c$-sequence $(z_n) \subset \mathcal{M}$ is bounded.*

Proof. Suppose, by contradiction, that there exists $(z_n) \subset \mathcal{M}$ satisfying $I(z_n) \leq d$ e $\lim_{n\to\infty} \|z_n\| = \infty$. Setting $w_n := z_n/\|z_n\|$ we have that, up to a subsequence, $w_n \rightharpoonup w$ weakly in E and $w_n(x) \to w(x)$ a.e. in $\mathbb{R}^N$. The same argument of the last proof show that we cannot have $w \neq 0$. Hence $w = 0$.

We claim that $w_n^+ \not\to 0$ in $L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N)$, where $p \in (2, 2^*)$ comes from (F_1). If this is true, we can use a lemma due to Lions [16, Lemma I.1] to obtain the existence of $\beta > 0$ and a sequence $(y_n) \subset \mathbb{R}^N$ satisfying

$$\int_{B_1(y_n)} |w_n^+|^2 dx \geq \beta > 0. \tag{3.5}$$

With no loss of generality, we may assume that $(y_n) \in \mathbb{Z}^N$ and by translation if necessary, that (y_n) is bounded. Taking the limit in (3.5) we conclude that $w^+ \neq 0$, which does not make sense.

It remains to check the claim. If $w_n^+ \to 0$ in $L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N)$, we can use (3.1) to conclude that $\int F(x, sw_n^+) \to 0$. So, recalling that $sw_n^+ \in \widehat{E}(z_n)$, Lemma 3.3 implies

$$d \geq I(z_n) \geq I(sw_n^+) \geq \frac{1}{4}s^2 - \int F(x, sw_n^+) = \frac{1}{4}s^2 + o(1),$$

which is absurd, since $s > 0$ is arbitrary. The lemma is proved. □

We are now in the position to prove our first result.

Proof of Theorem 1.1: All together, the above lemmas show that the functional I verifies $(N_1) - (N_3)$ and therefore the conclusions of Lemma 2.1 hold. With that notation, let $(w_n) \subset S^+$ be such that $\Psi(w_n) \to \inf_{S^+} \Psi$. By the Ekeland Variational Principle we may suppose that $\Psi'(w_n) \to 0$. So, by Lemma 2.1(iii) we have that $I'(z_n) \to 0$, where $z_n = m(w_n) \in \mathcal{M}$. In view of Lemma 3.7 we may assume that $z_n \rightharpoonup z$ weakly in E and $z_n(x) \to z(x)$ a.e. in $\mathbb{R}^N$. Condition (F_1) and straightforward calculations show that $I'(z) = 0$.

If $z_n \to 0$ in $L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N)$ we can use (3.1) and (F_1) to conclude that $\int F_z(x, z_n) z_n = o(1)$. Hence

$$o(\|z_n\|) = I'(z_n)z_n = \|z_n^+\|^2 - \|z_n^-\|^2 - \int F_z(x, z_n) \cdot z_n \leq \|z_n^+\|^2 + o(1).$$

Thus $z_n^+ \to 0$. But this contradicts Lemma 3.5(ii), and therefore $z_n \not\to 0$ in $L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N)$. This implies that

$$\int_{B_1(y_n)} |z_n|^2 dx \geq \beta > 0,$$

for some $\beta > 0$ and $(y_n) \subset \mathbb{Z}^N$. Since I is invariant by integer translations the sequence $\widetilde{z}_n := z_n(\cdot - y_n)$ weakly converges to a nonzero critical point $\widetilde{z}$. Recalling that I has no nonzero critical points in E^-, we conclude that $\widetilde{z} \in \mathcal{M}$.

It remains to prove that $I(\widetilde{z}) = c = \inf_{\mathcal{M}} I$. For simplicity, we denote by z_n the Palais-Smale sequence which weakly converges to a nonzero critical point $z \in \mathcal{M}$. Since we may suppose that $z_n(x) \to z(x)$ a.e. in $\mathbb{R}^N$, it follows from Lemma 3.1 and Fatou's lemma that

$$\begin{aligned} c + o(1) = I(z_n) - \frac{1}{2}I'(z_n)z_n &= \int \widehat{F}(x, z_n) \\ &\geq \int \widehat{F}(x, z) + o(1) = I(z) - \frac{1}{2}I'(z)z + o(1) \\ &= I(z) + o(1), \end{aligned}$$

which implies $c \geq I(z)$. The reverse inequality follows from $z \in \mathcal{M}$, and therefore $I(z) = c$ and the theorem is proved. □

4. The asymptotically periodic case

In this section, we consider the case when F is asymptotically periodic. We start by introducing the limit functional $I_\infty : E \to \mathbb{R}$ given by

$$I_\infty(z) := \frac{1}{2}\|z^+\|^2 - \frac{1}{2}\|z^-\|^2 - \int F_\infty(x, z),$$

where F_∞ is the asymptotic limit of the function F provided by condition (F_7). Since F_∞ satisfies $(F_0)-(F_4)$, we can use Theorem 1.1 to guarantee the existence of a least energy solution $z_\infty \in E$ of the periodic problem

$$\begin{cases} -\Delta u + V(x)u = F_{\infty,v}(x,u,v), \ x \in \mathbb{R}^N, \\ -\Delta v + V(x)v = F_{\infty,u}(x,u,v), \ x \in \mathbb{R}^N. \end{cases}$$

We use this solution to define our link set in the following way

$$M_{R,z_0} := \{z = tz_0 + z^- \ : \ z^- \in E^-, \ \|z\| \le R, \ t \ge 0\},$$

where $z_0 := z_\infty^+$. Since $M_{R,z_0} \subset \widehat{E}(z_0) = \widehat{E}(z_\infty)$, it follows from Lemma 3.3 that

$$\sup_{z \in M_{R,z_0}} I_\infty(z) \le I_\infty(z_0). \tag{4.1}$$

In order to obtain a solution for the problem (1.1), we need to find a nonzero critical point for the function I defined in (3.2). We notice that, since we do not directly impose subcritical growth for F, we need first to show that the functional is well defined. This is a consequence of the following lemma.

Lemma 4.1. *Suppose that F satisfies (F_2), (F_6) and (F_7). Then, for any given $\varepsilon > 0$, there exist $C_\varepsilon > 0$ and $q \in (2, 2^*)$ such that*

$$|F_z(x,z)| \le \varepsilon|z| + C_\varepsilon |z|^{q-1}, \quad |F(x,z)| \le \varepsilon|z|^2 + C_\varepsilon|z|^q, \tag{4.2}$$

for each $(x,z) \in \mathbb{R}^N \times \mathbb{R}^2$.

Proof. For $\varepsilon > 0$, we can use (F_2) to obtain $\delta > 0$ such that

$$|F_z(x,z)| \le \varepsilon|z|, \ \forall\, x \in \mathbb{R}^N, \ |z| \le \delta. \tag{4.3}$$

By (F_6), there exists $R_0 > 0$ satisfying

$$|F_z(x,z)|^\tau \le c_0|z|^\tau \widehat{F}(x,z) \le \frac{c_0}{2}|z|^{\tau+1}|F_z(x,z)|, \ \forall\, x \in \mathbb{R}^N, \ |z| \ge R_0.$$

Setting $q := 2\tau/(\tau-1)$, we can use $\tau > N/2$ to conclude that $2 < q < 2^*$. Furthermore,

$$|F_z(x,z)| \le C|z|^{\frac{\tau+1}{\tau-1}} = C|z|^{q-1}, \ \forall\, x \in \mathbb{R}^N, \ |z| \ge R. \tag{4.4}$$

Since $F_{\infty,z}$ is continuous and periodic, there exists $M > 0$ such that

$$|F_{\infty,z}(x,z)| \le M, \ x \in \mathbb{R}^N, \ \delta \le |z| \le R_0.$$

By using (F_7), we get

$$|F_z(x,z)| \le \left(|\varphi|_\infty + \frac{M}{\delta^{p_\infty - 1}}\right)|z|^{p_\infty - 1}, \ \forall\, x \in \mathbb{R}^N, \ \delta \le |z| \le R_0.$$

This, (4.3) and (4.4) prove the first inequality in (4.2). The second one follows from the Mean Value Theorem. □

In the next result, we obtain the geometry of the Linking Theorem.

Lemma 4.2. *Suppose that F satisfies (F_2) and $(F_5)-(F_7)$. Then*

(i) there exist $r, \alpha > 0$, such that $I|_{N_r} \ge \alpha$;
(ii) there exists $R > r$ such that $I|_{\partial M_{R,z_0}} \le 0$.

Proof. The first statement is an easy consequence of (F_2) and Lemma 4.1. For the second one, we take $z = \rho z_0 + z^- \in \partial M_{R,z_0}$. If $\|z\| \le R$ and $\rho = 0$, we have that $z = z^- \in E^-$, and therefore we can use (F_7) to get

$$I(z) = I(z^-) = -\frac{1}{2}\|z^-\|^2 - \int F(x, z^-) \le 0.$$

So, it is suffices to consider the case $\|z\| = R$ and $\rho > 0$. We argue by contradiction. If the result is false, there exists a sequence (z_n) such that $z_n = \rho_n z_0 + z_n^-$, $\rho_n > 0$, $\|z_n\| = R_n \to \infty$ and $I(z_n) > 0$. Then

$$\frac{I(z_n)}{\|z_n\|^2} = \frac{1}{2}\left(\frac{\rho_n^2\|z_0\|^2}{\|z_n\|^2} - \frac{\|z_n^-\|^2}{\|z_n\|^2}\right) - \int \frac{F(x, z_n)}{\|z_n\|^2} > 0$$

Since F is nonnegative, we have that $\rho_n\|z_0\| \ge \|z_n^-\|$. Noticing that

$$\frac{\rho_n^2\|z_0\|^2}{\|z_n\|^2} + \frac{\|z_n^-\|^2}{\|z_n\|^2} = 1,$$

it follows that $\frac{1}{\sqrt{2}\|z_0\|} \le \frac{\rho_n}{\|z_n\|} \le \frac{1}{\|z_0\|}$ and $z_n^-/\|z_n\|$ is bounded. Hence, going to a subsequence, we may assume that

$$\frac{\rho_n}{\|z_n\|} \to \rho > 0, \quad \frac{z_n^-}{\|z_n\|} \rightharpoonup w \in E^- \quad \text{and} \quad \frac{z_n^-(x)}{\|z_n\|} \to w(x) \text{ for a.e. } \ x \in \mathbb{R}^N,$$

where we have used the fact that E^- is weakly closed. Since $\|z_n\| \to \infty$, this implies that $\rho_n \to \infty$ and therefore,

$$\lim |z_n(x)| = \infty \ \text{ a.e. in } \Omega := \{x \in \mathbb{R}^N : \rho z_0(x) + w(x) \ne 0\}.$$

Since $\rho > 0$ and $w \in E^-$, we have that Ω has positive measure. Hence, taking the lim sup as $n \to +\infty$ in the inequality

$$0 < \frac{I(z_n)}{\|z_n\|^2} \le \frac{1}{2}\left(\frac{\rho_n^2\|z_0\|^2}{\|z_n\|^2} - \frac{\|z_n^-\|^2}{\|z_n\|^2}\right) - \int_\Omega \frac{F(x, z_n)}{|z_n|^2}\frac{|z_n|^2}{\|z_n\|^2}dx,$$

using Fatou's lemma and (F_7), we conclude that

$$0 \le \frac{1}{2}\left(\rho^2\|z_0\|^2 - \|w\|^2\right) - \int_\Omega \liminf_{n\to\infty} \frac{F(x, z_n)}{|z_n|^2}\frac{|z_n|^2}{\|z_n\|^2}dx = -\infty.$$

This contradiction concludes the proof. □

We recall that $(z_n) \subset E$ is called a $(Ce)_c$-sequence for I if $I(z_n) \to c$ and $(1 + \|z_n\|)\|I'(z_n)\| \to 0$. In the next result, we prove the boundedness of such sequences.

Lemma 4.3. *Suppose that F satisfies (F_2) and $(F_5) - (F_7)$. If $(z_n) \subset E$ is a $(Ce)_c$ sequence, then it is bounded.*

Proof. If $(z_n) \subset E$ is a $(Ce)_c$-sequence for I then

$$c + o(1) = I(z_n) - \frac{1}{2}I'(z_n)z_n = \int \widehat{F}(x, z_n). \tag{4.5}$$

Arguing by contradiction, we suppose that, up to a subsequence, $\|z_n\| \to \infty$. Then

$$o(1) = \frac{I'(z_n)(z_n^+ - z_n^-)}{\|z_n\|^2} = 1 - \int \frac{F_z(x, z_n) \cdot (z_n^+ - z_n^-)}{\|z_n\|^2}.$$

Setting $w_n := z_n/\|z_n\|$, it follows that

$$\lim_{n\to+\infty}\int \frac{F_z(x,z_n)\cdot(w_n^+ - w_n^-)}{\|z_n\|} = 1. \tag{4.6}$$

If $R_0 > 0$ is given by condition (F_6), for any $x \in \mathbb{R}^N$ and $|z| > R_0$ there holds

$$c_0\widehat{F}(x,z) \geq \left(\frac{|F_z(x,z)|}{|z|}\right)^{\tau} \geq \left(\frac{2F(x,z)}{|z|^2}\right)^{\tau}.$$

In the last inequality we have used the positivity of $\widehat{F}(x,z)$, which comes from (F_6), and Cauchy-Schwarz inequality. The above relation and the first two statements in (F_7) imply that $\widehat{F}(x,z) \to \infty$ as $|z| \to \infty$, uniformly for $x \in \mathbb{R}^N$. Hence, recalling that $q(r) = \inf\{\widehat{F}(x,z) : x \in \mathbb{R}^N, |z| \geq r\}$ and noticing that q is nondecreasing, we can use (F_5) to conclude that $q(r) > 0$ for each $r > 0$ and $q(r) \to \infty$ as $r \to \infty$.

For $0 \leq a < b$, we set

$$\Omega_n(a,b) := \{x \in \mathbb{R}^N : a \leq |z_n(x)| < b\}.$$

By using (4.5) we obtain

$$\begin{aligned}
c + o(1) &= \int\limits_{\Omega_n(0,a)} \widehat{F}(x,z_n)dx + \int\limits_{\Omega_n(a,b)} \frac{\widehat{F}(x,z_n)}{|z_n|^2}|z_n|^2dx + \int\limits_{\Omega_n(b,\infty)} \widehat{F}(x,z_n)dx \\
&\geq \int\limits_{\Omega_n(0,a)} \widehat{F}(x,z_n)dx + \frac{q(a)}{b^2}\int\limits_{\Omega_n(a,b)} |z_n|^2dx + q(b)|\Omega_n(b,\infty)|,
\end{aligned}$$

and therefore there exists $C_1 > 0$ such that

$$\max\left\{\int\limits_{\Omega_n(0,a)} \widehat{F}(x,z_n)dx,\ \frac{q(a)}{b^2}\int\limits_{\Omega_n(a,b)} |z_n|^2dx,\ q(b)|\Omega_n(b,\infty)|\right\} \leq C_1. \tag{4.7}$$

The above inequality implies that $|\Omega_n(b,\infty)| \leq C_1/q(b)$. Since $q(b) \to +\infty$ as $b \to +\infty$, we conclude that

$$\lim_{b\to+\infty} |\Omega_n(b,\infty)| = 0. \tag{4.8}$$

Let $\mu \in [2, 2^*)$. By Hölder's inequality and the Sobolev embeddings, we obtain the existence of a constant $C_2 > 0$ satisfying

$$\begin{aligned}
\int_{\Omega_n(b,\infty)} |w_n|^\mu dx &\leq \left(\int_{\Omega_n(b,\infty)} |w_n|^{2^*}dx\right)^{\mu/2^*} |\Omega_n(b,\infty)|^{(2^*-\mu)/2^*} \\
&\leq C_2\|w_n\|^\mu|\Omega_n(b,\infty)|^{(2^*-\mu)/2^*} = C_2|\Omega_n(b,\infty)|^{(2^*-\mu)/2^*}.
\end{aligned}$$

Since $2^* - \mu > 0$, we infer from (4.8) that

$$\lim_{b\to+\infty}\int_{\Omega_n(b,\infty)} |w_n|^\mu dx = 0. \tag{4.9}$$

By using the definition of w_n, we get

$$\int\limits_{\Omega_n(0,a_\varepsilon)} \frac{F_z(x,z_n)\cdot(w_n^+ - w_n^-)}{\|z_n\|}dx \leq \int\limits_{\Omega_n(0,a_\varepsilon)} \frac{|F_z(x,z_n)|}{|z_n|}|w_n||w_n^+ - w_n^-|dx.$$

Let $C_3 > 0$ be such that $\|z\|_{L^2}^2 \le C_3\|z\|^2$ for each $z \in E$ and consider $\varepsilon > 0$. By (F_2), there exists $a_\varepsilon > 0$ such that $|F_z(x,z)| \le \varepsilon|z|/C_3$ for each $|z| \le a_\varepsilon$. It follows from the inequality quoted above that, for any $n \in \mathbb{N}$, there holds

$$\int\limits_{\Omega_n(0,a_\varepsilon)} \frac{F_z(x,z_n)\cdot(w_n^+ - w_n^-)}{\|z_n\|}dx \le \frac{\varepsilon}{C_3} \int\limits_{\Omega_n(0,a_\varepsilon)} |w_n|^2 dx \le \varepsilon\|w_n\|^2 = \varepsilon. \tag{4.10}$$

Let $b_\varepsilon > a_\varepsilon$ to be chosen later during the proof. Since $2\tau' = 2\tau/(\tau-1) \in (2, 2^*)$, we can use Hölder's inequality to obtain

$$\begin{aligned}\int\limits_{\Omega_n(b_\varepsilon,\infty)} \frac{F_z(x,z_n)\cdot(w_n^+ - w_n^-)}{\|z_n\|}dx &= \int\limits_{\Omega_n(b_\varepsilon,\infty)} \frac{F_z(x,z_n)\cdot(w_n^+ - w_n^-)|w_n|}{|z_n|}dx \\ &\le \left(\int\limits_{\Omega_n(b_\varepsilon,\infty)} \frac{|F_z(x,z_n)|^\tau}{|z_n|^\tau}dx\right)^{1/\tau} \left(\int\limits_{\Omega_n(b_\varepsilon,\infty)} (|w_n||w_n^+ - w_n^-|)^{\tau'}dx\right)^{1/\tau'} \\ &= A_n^1 \cdot A_n^2\end{aligned}$$

By (F_6) and (4.7) the term A_n^1 is uniformly bounded. Moreover, by Hölder's inequality, we have that

$$\begin{aligned}A_n^2 &\le \left(\int\limits_{\Omega_n(b_\varepsilon,\infty)} |w_n|^{2\tau'}dx\right)^{1/2\tau'} \left(\int\limits_{\Omega_n(b_\varepsilon,\infty)} |w_n^+ - w_n^-|^{2\tau'}dx\right)^{1/2\tau'} \\ &\le C_4\left(\int_{\Omega_n(b_\varepsilon,\infty)} |w_n|^{2\tau'}dx\right)^{1/2\tau'}.\end{aligned}$$

All together, the above estimates provide

$$\int_{\Omega_n(b_\varepsilon,\infty)} \frac{F_z(x,z_n)\cdot(w_n^+ - w_n^-)}{\|z_n\|}dx \le C_5\left(\int_{\Omega_n(b_\varepsilon,\infty)} |w_n|^{2\tau'}dx\right)^{1/2\tau'}.$$

This inequality and (4.9) provide the existence of $b_\varepsilon > 0$ (sufficiently large) satisfying, for $n \ge n_0$,

$$\int_{\Omega_n(b_\varepsilon,\infty)} \frac{F_z(x,z_n)\cdot(w_n^+ - w_n^-)}{\|z_n\|}dx < \varepsilon. \tag{4.11}$$

Recalling that $\varphi \in L^\infty(\mathbb{R}^N)$ and using (F_7) we obtain $C_6 > 0$ such that $|F_z(x,z_n)| \le C_6|z_n|$ for each $x \in \Omega_n(a_\varepsilon, b_\varepsilon)$. Hence, we can argue as above and use the definition of w_n to obtain

$$\int\limits_{\Omega_n(a_\varepsilon,b_\varepsilon)} \frac{F_z(x,z_n)\cdot(w_n^+ - w_n^-)}{\|z_n\|}dx \le \frac{C_6}{\|z_n\|^2} \int\limits_{\Omega_n(a_\varepsilon,b_\varepsilon)} |z_n|^2 dx.$$

Thus, we infer from (4.7) that, taking n_0 larger if necessary, there holds

$$\int\limits_{\Omega_n(a_\varepsilon,b_\varepsilon)} \frac{F_z(x,z_n)\cdot(w_n^+ - w_n^-)}{\|z_n\|}dx \le \frac{C_6}{\|z_n\|^2}\frac{C_1 b_\varepsilon^2}{q(a_\varepsilon)} < \varepsilon,$$

for $n \ge n_0$.

The above inequality, (4.10) and (4.11) imply that

$$\int \frac{F_z(x,z_n)\cdot(w_n^+ - w_n^-)}{\|z_n\|} \leq 3\varepsilon, \qquad \forall\, n \geq n_0.$$

But this contraditcs (4.6) , since $\varepsilon > 0$ is arbitrary. This contradiction proves the boundedness of (z_n). □

Lemma 4.4. *Suppose that F satisfies (F_2), (F_6) and (F_7). Let $c > 0$ and $(z_n) \subset E$ be a $(Ce)_c$ sequence for I. If $z_n \rightharpoonup 0$ weakly in E, then there exists a sequence $(y_n) \subset \mathbb{R}^N$, $R > 0$ and $\beta > 0$ such that $|y_n| \to \infty$ and*

$$\limsup_{n\to\infty} \int_{B_R(y_n)} |z_n|^2 dx \geq \beta > 0$$

Proof. Suppose, by contradiction, that the lemma is false. Then, for any $R > 0$, we have that

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} |z_n|^2 dx = 0,$$

and therefore Lion's lemma implies that $z_n \to 0$ in $L^s(\mathbb{R}^N) \times L^s(\mathbb{R}^N)$ for each $s \in (2, 2^*)$. It follows from (4.2) that $\int F(x, z_n) \to 0$ as $n \to +\infty$. In the same way $\int F_z(x, z_n)\cdot z_n \to 0$. On the other hand,

$$c = \lim_{n\to\infty} \left[I(z_n) - \frac{1}{2} I'(z_n) z_n \right] = \lim_{n\to\infty} \int \left(\frac{1}{2} F_z(x,z_n)\cdot z_n - F(x,z_n) \right) = 0,$$

which contradicts the hypothesis $c > 0$ and finishes the proof. □

Next, we state two technical convergence results. The proofs can be done arguing along the same lines of [15, Lemmas 5.1 and 5.2], respectively.

Lemma 4.5. *Suppose that F satisfies (F_7). Let $(z_n) \subset E$ be a bounded sequence and $w_n(x) = w(x - y_n)$, with $w \in E$ and $(y_n) \subset \mathbb{R}^N$. If $|y_n| \to \infty$, then*

$$[F_{\infty,z}(x,z_n) - F_z(x,z_n)]\cdot w_n \to 0,$$

strongly in $L^1(\mathbb{R}^N)$ as $n \to \infty$.

Lemma 4.6. *Suppose that $\varphi \in \mathcal{F}$ and $s \in [2, 2^*]$. If $w_n \rightharpoonup w$, then*

$$\lim_{n\to+\infty} \int \varphi |w_n|^s = \int \varphi |w|^s.$$

We are now ready to prove our last theorem.

Proof of Theorem 1.2: We first notice that, by (F_7), Lemma 4.1, and Fatou's lemma, the condition (L_1) of Theorem 2.2 holds and from Lemma 4.2, we see that (L_2) is also verified. So, by Theorem 2.2 we obtain a $(Ce)_c$-sequence $(z_n) \subset E$ for I at level $c > 0$. By Lemma 4.3, we may suppose that $z_n \rightharpoonup z$ weakly in E. We claim that $I'(z) = 0$. Indeed, it is sufficient to show that $I'(z)\eta = 0$ for each $\eta \in C_0^\infty(\mathbb{R}^N) \times C_0^\infty(\mathbb{R}^N)$. We have that

$$I'(z_n)\eta - I'(z)\eta = \langle z_n - z, \eta \rangle - \int [F_z(x,z_n) - F_z(x,z)]\cdot \eta. \tag{4.12}$$

Up to a subsequence, $z_n \to z$ in $L^s_{loc}(\mathbb{R}^N) \times L^s_{loc}(\mathbb{R}^N)$ for each $s \in [1, 2^*)$ and

$$z_n(x) \to z(x) \text{ a.e. in } K,$$
$$|z_n(x)| \leq w_s(x) \in L^s(K), \text{ a.e. in } K,$$

where K stands for the support of η. Thus $F_z(x, z_n) \to F_z(x, z)$ a.e. in K. Furthermore, using (4.2) and Hölder's inequality, we get

$$|F_z(x, z_n)\eta| \leq \varepsilon |w_2||\eta| + C_\varepsilon |w_{q-1}||\eta| \in L^1(K).$$

Using the Lebesgue Theorem and the weak convergence of z_n, we can take the limit in (4.12) to conclude that

$$I'(z)\eta = \lim_{n\to\infty} I'(z_n)\eta = 0.$$

If $z \neq 0$ we have done. So, we consider in what follows the case $z = 0$. By Lemma 4.4, there exist a sequence $(y_n) \subset \mathbb{R}^N$, $R > 0$ and $\beta > 0$ such that $|y_n| \to \infty$ as $n \to \infty$, and

$$\limsup_{n\to\infty} \int_{B_R(y_n)} |z_n|^2 dx \geq \beta > 0.$$

Furthermore, we can suppose that $(y_n) \subset \mathbb{Z}^N$. Setting $\widetilde{z}_n(x) := z_n(x + y_n)$ and noticing that $\|\widetilde{z}_n\| = \|z_n\|$, along a subsequence, we have that $\widetilde{z}_n \rightharpoonup \widetilde{z}$ weakly in E. By the last inequality above we have that $\widetilde{z} \neq 0$.

Claim 1: $I'_\infty(\widetilde{z}) = 0$

In order to prove this claim we fix $\eta \in C_0^\infty(\mathbb{R}^N) \times C_0^\infty(\mathbb{R}^N)$ and define, for each $n \in \mathbb{N}$, the translation $\eta_n(x) = \eta(x - y_n)$. Arguing as above and using the periodicity of F_∞ we get

$$I'_\infty(\widetilde{z}_n)\eta = I'_\infty(z_n)\eta_n = I'_\infty(\widetilde{z})\eta + o(1).$$

Hence, we need only to show that $I'_\infty(z_n)\eta_n = o(1)$. But, Lemma 4.5 provides

$$I'_\infty(z_n)\eta_n = I'(z_n)\eta_n - \int [F_z(x, z_n) - F_{\infty,z}(x, z)] \cdot \eta_n = I'(z_n)\eta_n + o_n(1),$$

and the claim follows from the fact that (z_n) is a bounded Cerami sequence.

Claim 2: If we set $\widehat{F}_\infty(x, z) := (1/2)F_{\infty,z}(x, z) \cdot z - F_\infty(x, z)$, then

$$\liminf_{n\to\infty} \int \widehat{F}(x, z_n) \geq \int \widehat{F}_\infty(x, \widetilde{z})$$

Indeed, using the definition of $\widehat{F}$, $\widehat{F}_\infty$, the first part of (3.3) and (F_7), we obtain

$$\begin{aligned}
|\widehat{F}(x, z_n) - \widehat{F}_\infty(x, z_n)| &\leq \frac{1}{2}|F_z(x, z_n) - F_{\infty,z}(x, z_n)||z_n| \\
&\quad + \int_0^1 |F_z(x, tz_n) - F_{\infty,z}(x, tz_n)||z_n| dt \\
&\leq \frac{1}{2}\varphi(x)|z_n|^{p_\infty} + \int_0^1 \varphi(x) t^{p_\infty - 1}|z_n|^{p_\infty} dt \\
&= \left(\frac{1}{2} + \frac{1}{p_\infty}\right)\varphi(x)|z_n|^{p_\infty}.
\end{aligned}$$

This inequality and Lemma 4.6 enable us to use Fatou's lemma and the periodicity of $\widehat{F}_\infty$ to obtain

$$\begin{aligned}\liminf_{n\to\infty}\int \widehat{F}(x,z_n) &= \liminf_{n\to\infty}\int \widehat{F}_\infty(x,z_n)\\ &= \liminf_{n\to\infty}\int \widehat{F}_\infty(x,\widetilde{z}_n) \geq \int \widehat{F}_\infty(x,\widetilde{z}),\end{aligned}$$

which proves the second claim.

The two claims and the periodicity of $\widehat{F}$ provide

$$\begin{aligned}c &= \lim_{n\to\infty}[I(z_n) - \frac{1}{2}I'(z_n)z_n] = \liminf_{n\to\infty}\int \widehat{F}(x,z_n)\\ &\geq \int \widehat{F}_\infty(x,\widetilde{z}) = I_\infty(\widetilde{z}) - \frac{1}{2}I'_\infty(\widetilde{z})\widetilde{z} = I_\infty(\widetilde{z}).\end{aligned}$$

Using the definition of c given in Theorem 2.2, the inequality $F \geq F_\infty$ and (4.1) we obtain

$$c \leq \sup_{z\in M_{R,z_0}} I(z) \leq \sup_{z\in M_{R,z_0}} I_\infty(z) \leq I_\infty(z_0) \leq I_\infty(\widetilde{z}) \leq c.$$

Thus, if we define $h_0 : [0,1]\times M_{R,z_0} \to E$ by $h_0(t,z) := z$ for any $(t,z) \in [0,1]\times M_{R,z_0}$, the above inequality implies that

$$\sup_{z\in M_{R,z_0}} I(h_0(z,1)) = c > 0.$$

If follows from Theorem 2.3 that I possesses a nonzero critical point. The theorem is proved. $\square$

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References

[1] A. Ambrosetti and P. H. Rabinowitz. Dual variational methods in critical point theory and applications. *Journal of Functional Analysis*, 14(4):349–381, 1973.

[2] T. Bartsch and D. G. De Figueiredo. Infinitely many solutions of nonlinear elliptic systems. *Progress in Nonlinear Differential Equations and Their Applications*, 35:51–67. Birkhäuser, Basel/Switzerland, 1999.

[3] C. J. Batkam and F. Colin. Generalized fountain theorem and applications to strongly indefinite semilinear problems. *Journal of Mathematical Analysis and Applications*, 405(2):438–452, 2013.

[4] V. Benci and P. H. Rabinowitz. Critical point theorems for indefinite functionals. *Inventiones Mathematicae*, 52(3):241–273, 1979.

[5] D. Bonheure, E. M. dos Santos, and M. Ramos. Ground state and non-ground state solutions of some strongly coupled elliptic systems. *Transactions of the American Mathematical Society*, 364(1):447–491, 2012.

[6] D. G. De Figueiredo and Y. Ding. Strongly indefinite functional and multiple solutions of elliptic systems. *Transactions of the American Mathematical Society*, 355:2973–2989, 2003.

[7] D. G. De Figueiredo and P. L. Felmer. On superquadratic elliptic systems. *Transactions of the American Mathematical Society*, 343:99–116, 1994.

[8] D. G. De Figueiredo and Y. Jianfu. Decay, symmetry and existence of solutions of semilinear elliptic systems. *Nonlinear Analysis: Theory, Methods & Applications*, 33(3):211–234, 1998.

[9] Y. Ding and C. Lee. Multiple solutions of Schrödinger equations with indefinite linear part and super or asymptotically linear terms. *Journal of Differential Equations*, 222(1):137–163, 2006.

[10] H. He. Nonlinear Schrödinger equations with sign-changing potential. *Advanced Nonlinear Studies*, 12(2):237–253, 2012.

[11] J. Hulshof and R. Vandervorst. Differential systems with strongly indefinite variational structure. *Journal of Functional analysis*, 114(1):32–58, 1993.

[12] W. Kryszewski and A. Szulkin. Generalized linking theorem with an application to a semilinear Schrödinger equation. *Advances in Differential Equations*, 3(3):441–472, 1998.

[13] G. Li and J. Yang. Asymptotically linear elliptic systems. *Communications in Partial Differential Equations*, 29:025–954, 2004.

[14] G. Li and A. Szulkin. An asymptotically periodic Schrödinger equation with indefinite linear part. *Communications in Contemporary Mathematics*, 4(04):763–776, 2002.

[15] H. F. Lins and E. A. B. Silva. Quasilinear asymptotically periodic elliptic equations with critical growth. *Nonlinear Analysis: Theory, Methods & Applications*, 71(7):2890–2905, 2009.

[16] P.-L. Lions. The concentration-compactness principle in the calculus of variations. The locally compact case, part 2. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 1(4):223–283, 1984.

[17] A. Pankov. Periodic nonlinear Schrödinger equation with application to photonic crystals. *Milan Journal of Mathematics*, 73(1):259–287, 2005.

[18] B. Sirakov. On the existence of solutions of elliptic systems in $\mathbb{R}^n$. *Advances in Differential Equations*, 5:1445–1464, 2000.

[19] A. Szulkin and T. Weth. The method of Nehari manifold. *Handbook of nonconvex analysis and applications*, pages 597–632, 2010.

[20] J. Wang, J. Xu, and F. Zhang. The existence of solutions for superquadratic Hamiltonian elliptic systems on $\mathbb{R}^n$. *Nonlinear Analysis: Theory, Methods & Applications*, 74(3):909–921, 2011.

[21] M. Willem. *Minimax theorems*, volume 24. Birkhäuser Boston, 1997.

[22] R. Zhang, J. Chen, and F. Zhao. Multiple solutions for superlinear elliptic systems of Hamiltonian type. *Discrete and Continuous Dynamical Systems (DCDS-A)*, 30(4):1249–1262, 2011.

Universidade de Brasília, Departamento de Matemática, 70910-900 Brasília-DF, Brazil
E-mail address: mfurtado@unb.br

Universidade Federal de Mato Grosso, Departamento de Matemática, 78060-900 Cuiabá-MT, Brazil
E-mail address: reinaldodemarchi@ufmt.br