Semilinear elliptic problems with combined nonlinearities on the boundary *

Marcelo F. Furtado, Ricardo Ruviaro,
Universidade de Brasília, Departamento de Matemática,
70910-900, Brasília-DF, Brazil

Edcarlos D. Silva
Universidade Federal de Goiás, Instituto de Matemática,
74000-000, Goiânia-GO, Brazil

Abstract

We prove the existence of two solutions for some elliptic equations with combined indefinite nonlinearities on the boundary. The main novelty is to consider variational methods together with a suitable split of the Sobolev space $W^{1,2}(\Omega)$.

2000 Mathematics Subject Classification : 35J20, 35J25, 35J60, 35J92, 58E05.

Key words: Concave-convex nonlinearities; aymptotically linear problems; indefinite nonlinearities; Steklov problems.

1 Introduction

Since the work of Steklov [25], the semilinear problem

$$\Delta u = 0, \text{ in } \Omega, \qquad \frac{\partial u}{\partial \eta} = g(x,u), \text{ on } \partial\Omega,$$

has been extensively studied with many different type of perturbations $g(x,u)$ appearing on the boundary of the open set $\Omega \subset \mathbb{R}^N$. For linear function g, it comes from physics, with the function u being the steady state temperature on Ω such that the flux on the boundary is proportional to the temperature. The Steklov was introduced in [25] where was considered a linear problem on the boundary. It is also

*Correponding author: Edcarlos D. Silva (eddomingos@hotmail.com)

important in conductivity and harmonic analysis [8] and some problems in conformal geometry [11]. There are many papers considering nonlinear perturbations on the boundary (see [5, 9, 12, 18, 21–24], for instance).

In the first part of the paper we suppose that the function $g(x,u)$ presents a competition between a concave and a convex term. More specifically, we shall study

$$\begin{cases} \Delta u = 0, & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \eta} = a(x)|u|^{p-2}u + \mu b(x)|u|^{q-2}u, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

{p1}

where $\Omega \subset \mathbb{R}^N$, $N \geq 3$, is a bounded smooth domain, $\mu > 0$, $1 < q < 2 < p < 2_* = 2(N-1)/(N-2)$ and the potentials a, b verify

(a_0) $a \in L^\infty(\partial\Omega)$ and $\displaystyle\int_{\partial\Omega} a(x)\,\mathrm{d}\sigma_x \neq 0$;

(a_1) there exists a non-empty set $\mathcal{A}^+ \subset \partial\Omega$ open in $\partial\Omega$ such that $a(x) > 0$ for a.e. $x \in \mathcal{A}^+$;

(b_0) $b \in L^\infty(\partial\Omega)$ and $\displaystyle\int_{\partial\Omega} b(x)\,\mathrm{d}\sigma_x < 0$;

(b_1) there exists a non-empty set $\mathcal{B}^+ \subset \partial\Omega$ open in $\partial\Omega$ such that $b(x) > 0$ for a.e. $x \in \mathcal{B}^+$.

In our first result, we obtain multiple positive solutions for small values of μ. More specifically, we prove the following.

{t1}

Theorem 1.1 *If $1 < q < 2 < p < 2_*$ and the potentials a and b satisfy (a_0), (a_1), (b_0) and (b_1), then there exists $\mu^* > 0$ such that the problem* (1.1) *admits at least two positive solutions if μ is small.*

Since the pioneer work of Ambrosetti, Brezis and Cerami [3], elliptic problems with concave-convex terms have been widely studied. It is impossible to give a complete list of references and therefore we quote the papers [1, 2, 4, 14, 16, 20], where the authors presented some related results for indefinite potentials.

In our proof, we use variational methods, by looking for critical points of the energy functional associated to (1.1). After obtaining a first solution with a minimization argument, we use a version of the Mountain Pass Theorem to get another solution. Although the main steps of the proof are standard, the calculation here becomes more involved since the first equation of the problem does not involve the operator $-\Delta + \mathrm{Id}$. Hence, the quadratic part of the energy functional, namely $\int_\Omega |\nabla u|^2 \mathrm{d}x$, is not a norm in the natural space $W^{1,2}(\Omega)$. In order to overcome this difficulty we follow [6], by making an appropriate decomposition of the space $W^{1,2}(\Omega)$.

In our second result we consider a different type of competition on the boundary. Actually, we replace the superlinear term in (1.1) by a function $f(x,u)$ and consider the problem

$$\begin{cases} \Delta u = 0, & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \eta} = f(x,u) + \mu b(x)|u|^{q-2}u, & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$ {p2}

where $\Omega \subset \mathbb{R}^N$, $N \geq 3$, is a bounded smooth domain, $\mu > 0$, $1 < q < 2$ and the function $f \in C(\partial\Omega \times \mathbb{R}, \mathbb{R})$ has subcritical growth, that is,

(f) there exist C_1, $C_2 > 0$ and $p \in (2, 2_* - 1)$ such that

$$|f(x,s)| \leq C_1 + C_2|s|^{p-1}, \quad \forall\, x \in \partial\Omega,\ s \in \mathbb{R}.$$

We are interested in the case that f is asymptotically linear at the origin and at the infinity. Since we shall consider some weighted eigenvalue problems, we introduce the set

$$\mathcal{F} := \left\{ k : \Omega \to \mathbb{R} : \begin{array}{l} k \in L^\sigma(\partial\Omega), \text{ for some } \sigma > N-1, \\ k^+ \neq 0,\ \int_{\partial\Omega} k(x)\, \mathrm{d}\sigma_x \neq 0 \end{array} \right\},$$

where $k^+(x) := \max\{k(x), 0\}$.

Setting $F(x,s) := \int_0^s f(x,t)\, \mathrm{d}t$, we suppose that

(f_0) there exists $K_0 \in \mathcal{F}$ such that

$$\lim_{s \to 0^+} \frac{2F(x,s)}{s^2} = K_0(x), \quad \text{uniformly for } x \in \partial\Omega;$$

(f_∞) there exists $k_\infty \in \mathcal{F}$ such that

$$\lim_{s \to +\infty} \frac{f(x,s)}{s} = k_\infty(x), \quad \text{uniformly for } x \in \partial\Omega.$$

For any $k \in \mathcal{F}$, we can prove (see Section 2) that the eigenvalue problem

$$\Delta u = 0, \text{ in } \Omega, \qquad \frac{\partial u}{\partial \nu} = \lambda k(x) u, \text{ on } \partial\Omega,$$

has a sequence of eigenvalues $(\lambda_j(k))_{j \in \mathbb{N}}$ such that $\lambda_j(k) \to \infty$ as $j \to \infty$. With this notation, we can state our second result in the following way

{t2}

Theorem 1.2 *Suppose that b satisfies $(b_0) - (b_1)$ and f satisfies (f), (f_∞) and (f_0) with $\int_{\partial\Omega} K_0(x) d\sigma_x < 0$. Assume also that*

$$\lambda_1(k_\infty) < 1 < \lambda_1(K_0) \quad \text{and} \quad \lambda_j(k_\infty) < 1 < \lambda_{j+1}(k_\infty),$$

for some $j \geq 1$. Then the problem (1.2) admits at least two nonzero solutions for any $\mu > 0$ small enough.

Although we can guarantee that one of the above solutions is positive, we do not have information about the sign of the second one. However, if we suppose that f is odd in the second variable, we can obtain two positive solutions as in Theorem 1.1 (see Remark 4.1). The condition $\lambda_j(k_\infty) < 1 < \lambda_{j+1}(k_\infty)$ is a sort of nonressonance condition. It is used only to show that the energy functional has some compactness properties (see Proposition 4.2).

In the proof we follow the same lines of Theorem 1.1. Again, we need to consider some trick decomposition of the space $W^{1,2}(\Omega)$. In the asymptotically linear case, the eigenvalue problems associated to the asymptotic limits K_0 and k_∞ play an important rule in this feature.

Semilinear elliptic problems with concave-convex or asymptotically terms have been considered during the last years(see [2, 15, 19], for instance). Usually, they are treated under Dirichlet boundary conditions. In this case, the map $u \mapsto \int_\Omega |\nabla u|^2 \mathrm{d}x$ defines a norm in the working space $W_0^{1,2}(\Omega)$. The same does not occur in problems (1.1) and (1.2), since the natural space to look for solutions is $W^{1,2}(\Omega)$. For semilinear elliptic problems with nonlinear terms on the boundary we refer the reader to [4, 5, 10, 15, 21, 22]. In all these works the left hand side of the equation treated provides the norm $u \mapsto \int_\Omega (|\nabla u|^2 + c(x)u^2)\mathrm{d}x$ in $W^{1,2}(\Omega)$, since they considered $c(x) \geq c_0 > 0$ a.e. in Ω. The main feature of this paper is to consider the extremal case $c \equiv 0$. Hence, our results complement and/or generalize the aforementioned works. In particular, Theorem 1.1 complements the results of [15, 19] and Theorem 1.2 is closely related to [2, 4].

The paper is organized as follows: in the next section we give some preliminaries results and present the abstract framework to deal with our problems. Section 3 is devoted to the proof of Theorem 1.1 and in the final Section 4 we prove Theorem 1.2.

2 Preliminary results

In this section we present the variational framework to deal with our problems. The main difficulty is that its linear part induces the term $\int_\Omega |\nabla u|^2 \, \mathrm{d}x$ in the associated functional. Since we are going to work with the space $W^{1,2}(\Omega)$, this term is not a norm and therefore some of the usual arguments do not hold. In order to overcome this problem we adapt some ideas from [6] (see also [10]).

Here and throughout this work $C, C_1, C_2, \ldots,$ are positive constants. The element of area in the integrals over $\partial\Omega$ is denoted by $\mathrm{d}\sigma_x$, and the norm in $L^s(\Omega)$ by $\|\cdot\|_s$. For any $s > 1$, we denote by s' its conjugated exponent. We recall the set $\mathcal{F}$ defined in the introduction

$$\mathcal{F} := \left\{ k : \Omega \to \mathbb{R} : \begin{array}{l} k \in L^\sigma(\partial\Omega), \text{ for some } \sigma > N-1, \\ k^+ \not\equiv 0, \ \int_{\partial\Omega} k(x)\, \mathrm{d}\sigma_x \neq 0 \end{array} \right\}.$$

The next two results are versions of analogous results presented in [6].

{aux}

Lemma 2.1 *Let $k \in \mathcal{F}$ and define*

$$X_k := \left\{ u \in W^{1,2}(\Omega) : \int_{\partial\Omega} k(x) u \,\mathrm{d}\sigma_x = 0 \right\}. \tag{2.1}$$ {xk}

Then

(i) the Poincaré inequality holds in the subspace X_k, that is, for some $C > 0$ we have that

$$\|u\|_2 \le C \|\nabla u\|_2, \qquad \forall\, u \in X_k;$$

(ii) for any $u \in W^{1,2}(\Omega)$ there is (a unique) $t_u \in \mathbb{R}$ such that $u^{\perp} := (u - t_u) \in X_k$, and therefore $W^{1,2}(\Omega) = \langle 1 \rangle \oplus X_k$. Moreover, the expression

$$\langle (t_u + u^{\perp}), (t_v + v^{\perp}) \rangle = t_u \cdot t_v + \int_{\Omega} (\nabla u^{\perp} \cdot \nabla v^{\perp}) \,\mathrm{d}x, \tag{2.2}$$ {pii}

defines an inner product in $W^{1,2}(\Omega)$ with associated norm equivalent to the usual one.

Proof. Suppose, by contradition, that there exists $(u_n) \subset X_k$ such that $\|u_n\|_2 \ge n\|\nabla u_n\|_2$. If we define $v_n := u_n / \|u_n\|_2$, we have that $\|v_n\|_2 = 1$ and $\|\nabla v_n\|_2 \le 1/n$. Hence, (v_n) is bounded in $W^{1,2}(\Omega)$. Since $\sigma > (N-1) > 2(N-1)/N = (2_*)'$, up to a subsequence, $v_n \to v$ strongly in $L^2(\partial\Omega)$ and $L^{\sigma'}(\partial\Omega)$. Thus,

$$\left| \int_{\partial\Omega} k(x)(v_n - v) \,\mathrm{d}\sigma_x \right| \le \|k\|_{L^{\sigma}(\partial\Omega)} \|v_n - v\|_{L^{\sigma'}(\partial\Omega)}$$

and therefore $\|v\|_2 = 1$ and $\int_{\partial\Omega} k(x) v \,\mathrm{d}\sigma_x = 0$.

Since $\frac{\partial v_n}{\partial x_i} \to 0$ in $L^2(\Omega)$, for any $\varphi \in C_0^{\infty}(\Omega)$ we get

$$\int_{\Omega} v \frac{\partial \varphi}{\partial x_i} \,\mathrm{d}x = -\int_{\Omega} \frac{\partial v}{\partial x_i} \varphi \,\mathrm{d}x = -\lim_{n \to +\infty} \int_{\Omega} \frac{\partial v_n}{\partial x_i} \varphi \,\mathrm{d}x = 0.$$

Hence, $v \in W^{1,2}(\Omega)$ has null weak derivate and, for some $C \in \mathbb{R}$, $v(x) = C$ a.e. in Ω. This, $\int_{\partial\Omega} k(x) v \,\mathrm{d}\sigma_x = 0$ and $\int_{\partial\Omega} k(x) \,\mathrm{d}\sigma_x \ne 0$ imply that $v = 0$, which contradicts $\|v\|_2 = 1$. This proves the first item.

For item (ii) we take $u \in W^{1,2}(\Omega)$ and set

$$t_u := \frac{1}{\displaystyle\int_{\partial\Omega} k(x) \,\mathrm{d}\sigma_x} \int_{\partial\Omega} k(x) u \,\mathrm{d}\sigma_x.$$

A straightforward calculation shows that $u^{\perp} = u - t_u \in X_k$. Using (i), we can prove that (2.2) defines an inner product in $W^{1,2}(\Omega)$. If we denote by $\|\cdot\|$ the norm induced by this inner product we have that, for any $u \in W^{1,2}(\Omega)$,

$$\|u\|_{W^{1,2}}^2 = \int_{\Omega} (|\nabla u|^2 + u^2) \,\mathrm{d}x \le 2t_u^2 |\Omega| + \int_{\Omega} (|\nabla u^{\perp}|^2 + 2(u^{\perp})^2) \,\mathrm{d}x \le C\|u\|^2,$$

for some $C > 0$ independent of u. Using this inequality and item (i) again we can prove that $(W^{1,2}(\Omega), \|\cdot\|)$ is a Banach space. Hence, it follows from the Open Maping Theorem (see [7, Corollary 2.8]) that $\|\cdot\|$ is equivalent to $\|\cdot\|_{W^{1,2}}$. □

We present below a technical result which will be useful to control the behaviour of the energy functional near the origin. We borrow some ideas from [10].

{tech}

Lemma 2.2 *Let k and X_k as in the statement of Lemma 2.1. If (a_0) and (b_0) hold, then there exist $\eta > 0$ such that, for any $u = t_u + u^{\perp}$, with $t_u \in \langle 1 \rangle = \mathbb{R}$ and $u^{\perp} \in X_k$, we have that*

$$\int_{\partial\Omega} b(x)|t_u + u^{\perp}|^q \,\mathrm{d}\sigma_x \leq \frac{|t_u|^q}{2} \int_{\partial\Omega} b(x)\,\mathrm{d}\sigma_x, \ \textit{whenever}\ \|\nabla u^{\perp}\|_2 \leq \eta |t_u|.$$

Proof. Arguing by contradiction, we suppose that there exists a sequence $(u_n) \in H$ such that $\left(\int_{\Omega} |\nabla u_n^{\perp}|^2 \,\mathrm{d}x\right)^{1/2} \leq |t_{u_n}|/n$ and

$$\int_{\partial\Omega} b(x)|t_{u_n} + u_n^{\perp}|^q \,\mathrm{d}\sigma_x > \frac{|t_{u_n}|^q}{2} \int_{\partial\Omega} b(x)\,\mathrm{d}\sigma_x, \tag{2.3}$$ {100}

where $u_n = t_{u_n} + u_n^{\perp}$, $t_{u_n} \in \mathbb{R}$ and $u_n^{\perp} \in X_k$. If we set $w_n := u_n^{\perp}/t_{u_n}$, we conclude that $\int_{\Omega} |\nabla w_n|^2 \,\mathrm{d}x \to 0$. Since Poincare's inequality holds in X_k, see Lemma 2.1, we get $w_n \to 0$ in X_k. Moreover, the continuous embedding $X_k \subset L^q(\partial\Omega)$ shows that $w_n \to 0$ in $L^q(\partial\Omega)$ and $w_n \to 0$ a. e. in $\partial\Omega$. Dividing the inequality in (2.3) by $|t_{u_n}|^q$, we obtain

$$\int_{\partial\Omega} b(x)|1 + w_n|^q \,\mathrm{d}\sigma_x > \frac{1}{2} \int_{\partial\Omega} b(x)\,\mathrm{d}\sigma_x.$$

Taking the limit and using Lebesgue Theorem we conclude that $\int_{\partial\Omega} b(x)\,\mathrm{d}\sigma_x \geq 0$, which contradicts (b_0). This completes the proof. □

Remark 2.1 *A simple inspection of the proofs show that the two lemmas above remain true if we discard the hypothesis on k^+ and suppose only that $k \in L^{\sigma}(\partial\Omega)$ for some $\sigma > N - 1$.*

For any $k \in \mathcal{F}$, let us consider the eigenvalue problem

$$\left\{ \begin{array}{ll} \Delta u = 0, & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = \lambda k(x) u, & \text{on } \partial\Omega. \end{array} \right. \tag{2.4}$$ {pa}

Since $k \in L^{\sigma}(\Omega)$ with $\sigma > (N-1)$, we can use Hölder's inequality and the spectral theory of compact operators to obtain an unbounded sequence of eigenvalues $(\lambda_j(k))_{j \in \mathbb{N}}$ such that $\lambda_j(k) \to \infty$ as $j \to \infty$. Notice that $\lambda_0(k) = 0$ is an eigenvalue of (2.4) associated with the constant eigenfunctions. Since $\langle 1\rangle^{\perp} = X_k$, we have the following characterization

$$\lambda_1(k) = \inf_{u\in X_k}\left\{\int_\Omega |\nabla u|^2\,\mathrm{d}x,\ \int_{\partial\Omega} k(x)u^2\,\mathrm{d}\sigma_x = 1\right\} > 0.$$

Indeed, if we take $(u_n)\subset X_k$ such that $\int_{\partial\Omega} k(x)u_n^2\mathrm{d}\sigma_x = 1$ and $\|\nabla u_n\|_2 \to \lambda_1(k)$, we can use Lemma 2.1 to conclude that (u_n) is bounded. Hence, up to a subsequence, $u_n \rightharpoonup u$ weakly in $W^{1,2}(\Omega)$. Since $\sigma > N-1$, we can prove that $u\in X_k\setminus\{0\}$ and $\int_{\partial\Omega} k(x)u^2\,\mathrm{d}\sigma_x = 1$. Recalling that $v\mapsto (\int_\Omega |\nabla v|^2\,\mathrm{d}x)^{1/2}$ defines a norm in X_k, we can use a standard argument to show that $\lambda_1(k)$ is achieved by the function u. Since $u\neq 0$ and X_k does not contain constant function we conclude that $\lambda_1(k)>0$, as claimed.

Many other properties of the first eigenvalue could be proved depending on the sign of $\int_{\partial\Omega} k(x)\mathrm{d}\sigma_x$. We omit more details since just the above consideration is enough to our purposes. We refer the reader to [13, 26] for a full description of the first positive eigenvalue of a related eigenvalue problem.

3 The concave-convex case

In this section we consider the concave-convex case, proving that problem (1.1) admits two positive solutions. Along all this section we consider the function k of the previous section as being $k\equiv 1$. With this choice, the set X_1 defined in (2.1) is the subspace of the functions with zero average in Ω. More specifically, we denote by H the space $W^{1,2}(\Omega)$ with the following decomposition

$$H = \langle 1\rangle \oplus X_1 = \langle 1\rangle \oplus \left\{u\in W^{1,2}(\Omega) : \int_{\partial\Omega} u\,\mathrm{d}\sigma_x = 0\right\}.$$

Hence, any element $u\in H$ can be (uniquely) written as $u = t_u + u^{\perp}$, with $t_u\in\mathbb{R}$ and $u^{\perp}\in X_1$. In H, we consider the norm

$$\|t_u + u^{\perp}\| = \sqrt{t_u^2 + \|\nabla u^{\perp}\|_2^2}\,,$$

which is equivalent to the usual norm of $W^{1,2}(\Omega)$ (see Lemma 2.1).

The energy functional associated with the problem (1.1) is

$$J(u) := \frac{1}{2}\int_\Omega |\nabla u|^2\,\mathrm{d}x - \frac{1}{p}\int_{\partial\Omega} a(x)|u|^p\,\mathrm{d}\sigma_x - \frac{\mu}{q}\int_{\partial\Omega} b(x)|u|^q\,\mathrm{d}\sigma_x, \qquad u\in H.$$

A straightforward calculation show that $J\in C^1(H,\mathbb{R})$ and its critical points are weak solutions of (1.1).

We are going to obtain the solution by looking for the critical points f J. We first verify that the functional admits the Mountain Pass geometry.

{mountain}

Lemma 3.1 *Suppose that* (a_0), (a_1) *and* (b_0) *hold. Then there exists* $\mu^* > 0$ *such that, for any* $\mu \in (0, \mu^*)$, *we have the following*

(i) there exist α_μ, $r_\mu > 0$ *such that*

$$J(u) \geq \alpha_\mu, \qquad \forall\, u \in H,\ \|u\| = r_\mu;$$

(ii) there exists $e \in H^1(\Omega)$, *independent of* μ, *such that*

$$J(e) < 0, \qquad \|e\| > r_\mu.$$

Proof. Let η be given by Lemma 2.2 and consider $u = t_u + u^\perp$, with $t_u \in \mathbb{R}$ and $u^\perp \in X_1$. We split the proof in two distinct cases.

Case 1: $\|\nabla u^\perp\|_2 \leq \eta |t_u|$.

In this case, since $\|u\|^2 = t_u^2 + \|\nabla u^\perp\|_2^2 = r^2$, we have that

$$t_u \geq r/\sqrt{1+\eta^2}. \tag{3.1}$$ {13}

Hence, it follows from Lemma 2.2 that

$$\begin{aligned} J(u) &\geq \frac{1}{2}\int_\Omega |\nabla u^\perp|^2\,\mathrm{d}x - \frac{1}{p}\int_{\partial\Omega} a(x)|u|^p\,\mathrm{d}\sigma_x - \frac{\mu}{2q}|t_u|^q \int_{\partial\Omega} b(x)\,\mathrm{d}\sigma_x \\ &\geq -C_1\|u\|^p - \frac{\mu}{2q}|t_u|^q \int_{\partial\Omega} b(x)\,\mathrm{d}\sigma_x, \end{aligned}$$

for some $C_1 > 0$, where we have used the trace embedding $W^{1,2}(\Omega) \subset L^p(\partial\Omega)$ in the last inequality. Since $\int_{\partial\Omega} b(x)\,\mathrm{d}\sigma_x < 0$, we can use (3.1) to get

$$J(u) \geq r^q\left(-C_1 r^{p-q} + \mu C_2\right),$$

with $C_2 := -\int_{\partial\Omega} b(x)\,\mathrm{d}\sigma_x/(2q(1+\eta^2)^{q/2}) > 0$. Thus, if $r^{p-q} \leq (\mu C_2)/(2C_1)$, we have that

$$J(u) \geq \frac{\mu}{2} C_2 r^q, \tag{3.2}$$ {1111}

for any $u \in H$ such that $\|\nabla u^\perp\|_2 \leq \eta|t_u|$ and $\|u\| \leq r$. It is important to notice that the above inequality is verified for any $\mu > 0$.

Case 2: $\|\nabla u^\perp\|_2 > \eta|t_u|$.

In this case, for $\gamma := \sqrt{1+\eta^{-2}}$, we have that

$$\|u\|^2 \leq \gamma^2 \|\nabla u^\perp\|_2^2. \tag{3.3}$$ {14}

Hence

$$\left|\int_{\partial\Omega} b(x)|u|^q \,\mathrm{d}\sigma_x\right| \leq C_3\|u\|^q \leq C_3\gamma^q\|\nabla u^\perp\|_2^q$$

and we can use (3.3) to get

$$\begin{aligned} J(u) &\geq \frac{1}{2}\|\nabla u^\perp\|_2^2 - C_1\|u\|^p - \frac{\mu C_3}{q}\gamma^q\|\nabla u^\perp\|_2^q \\ &\geq \frac{1}{2}\|\nabla u^\perp\|_2^2\left\{1 - C_1\gamma^p\|\nabla u^\perp\|_2^{p-2} - \frac{\mu C_3}{q}\gamma^q\|\nabla u^\perp\|_2^{q-2}\right\}. \end{aligned}$$

It follows that, for some constants C_4, $C_5 > 0$ which are independent of $\mu > 0$, we have that

$$J(u) \geq \frac{1}{2}\|\nabla u^\perp\|_2^2\left\{1 - C_5\|\nabla u^\perp\|_2^{p-2} - \mu C_4\|\nabla u^\perp\|_2^{q-2}\right\}. \tag{3.4}$$ {calor}

We now set $r_1 := (4C_5)^{1/(2-p)}$ and notice that, if $\|u\| = r_1$, then $\|\nabla u^\perp\|_2 \leq r_1$, and therefore $C_5\|\nabla u^\perp\|_2^{p-2} \leq 1/4$. Moreover, since (3.3) implies that $\|\nabla u^\perp\|_2 \geq r_1/\gamma > 0$, there exists $\mu^* > 0$ such that $\mu C_4\|\nabla u^\perp\|_2^{q-2} \leq 1/4$, for any $\mu \in (0,\mu^*)$. Then, we can use (3.4) to deduce that

$$J(u) \geq \frac{1}{4}\|\nabla u^\perp\|_2^2 \geq \frac{1}{4\gamma^2}\|u\|^2 = \frac{\eta^2}{4(1+\eta^2)}r_1^2 > 0,$$

for any $\mu \in (0,\mu^*)$, $\|u\| = r_1$ and $\|\nabla u^\perp\|_2 > \eta|t_u|$. This and (3.2) show that the first statement of the lemma holds if we set

$$r_\mu := \min\left\{\left(\frac{\mu C_2}{2C_1}\right)^{1/(p-q}, (4C_5)^{1/(2-p)}\right\}$$

and

$$\alpha_\mu := \min\left\{\frac{\mu}{2}C_2 r_\mu^q, \frac{\eta^2}{4(1+\eta^2)}(4C_5)^{2/(2-p)}\right\}.$$

For the proof of (ii) we consider a positive smooth function $\phi_1 : \partial\Omega \to \mathbb{R}$ with support contained in the set $\mathcal{A}^+$ given by condition (a_1). According to [7, page 315], there exists an extension $\phi \in H$ such that $\phi_1 = \phi|_{\partial\Omega}$. We have that

$$J(t\phi) \leq \frac{t^2}{2}\|\nabla\phi\|_2^2 - \frac{t^p}{p}\int_{\mathcal{A}^+} a(x)\phi^p \,\mathrm{d}\sigma_x - \frac{\mu t^q}{q}\int_{\mathcal{A}^+} b(x)\phi^q \,\mathrm{d}\sigma_x.$$

Since the first integral above is positive and $q < 2 < p$, we get

$$\limsup_{t\to\infty}\frac{J(t\phi)}{t^p} \leq -\frac{1}{p}\int_{\mathcal{A}^+} a(x)\phi^p \,\mathrm{d}\sigma_x < 0,$$

and the second item follows for $e := t\phi$, with $t > 0$ large enough. □

We prove in the sequel that J admits at least one critical point with negative energy.

{ppp}

Proposition 3.1 *Suppose that* (a_0), (a_1), (b_0) *and* (b_1) *holds and let* μ^*, r_μ *be given by Lemma 3.1. If* $\mu \in (0, \mu^*)$, *then*

$$c_\mu := \inf_{u \in \overline{B_{r_\mu}(0)}} J(u) < 0$$

is achieved at some positive function $u_\mu \in B_{r_\mu}(0)$. *In particular,* $J'(u_\mu) = 0$.

Proof. We first prove that $c_\mu < 0$. For this purpose we pick $\psi_1 : \partial\Omega \to \mathbb{R}$ a positive smooth function with support contained in the set $\mathcal{B}^+$ given by condition (b_1). If we denote by ψ its extension to the whole set Ω, we can argue as in the proof of item (ii) of Lemma 3.1 to obtain

$$\liminf_{t \to 0^+} \frac{J(t\psi)}{t^q} \leq -\frac{\mu}{q} \int_{\mathcal{B}^+} b(x) \psi^q \, \mathrm{d}\sigma_x < 0.$$

For $t > 0$ small, we have that $\|t\psi\| < r_\mu$. This and the above inequality prove that $c_\mu < 0$.

We now consider $(u_n) \subset \overline{B_{r_\mu}(0)}$ such that $J(u_n) \to c_\mu$. Up to a subsequence, we have that $u_n \rightharpoonup u_\mu$ weakly in $W^{1,2}(\Omega)$, $u_n \to u_\mu$ strongly in $L^2(\Omega)$ and $L^s(\partial\Omega)$, for any $1 < s < 2_*$, and $u_n(x) \to u_\mu(x)$ a.e. in Ω. Hence, we have that

$$\int_{\partial\Omega} a(x)|u_n|^p \, \mathrm{d}\sigma_x \to \int_{\partial\Omega} a(x)|u_\mu|^p \, \mathrm{d}\sigma_x$$

and

$$\int_{\partial\Omega} b(x)|u_n|^q \, \mathrm{d}\sigma_x \to \int_{\partial\Omega} b(x)|u_\mu|^q \, \mathrm{d}\sigma_x.$$

Moreover, since the norm is weakly semicontinuous, we can use the strong convergence in $L^2(\Omega)$ to get

$$\int_\Omega |\nabla u_\mu|^2 \, \mathrm{d}x \leq \liminf_{n \to +\infty} \int_\Omega |\nabla u_n|^2 \, \mathrm{d}x.$$

All together, the above expressions provide

$$J(u_\mu) \leq \liminf_{n \to +\infty} J(u_n) = c_\mu < 0,$$

and therefore c_μ is attained at u_μ. Since $J(u_n) = J(|u_n|)$, we may suppose that the sequence (u_n) verifies $u_n \geq 0$, and therefore the pointwise convergence provides $u_\mu \geq 0$ a.e. in Ω. Moreover, since $J(u_\mu) < 0$ and item (i) of Lemma 3.1 imply that that $\|u_\mu\| < r_\mu$, we have that u_μ is a nonnegative critical point of J. Using classical results of elliptic regularity we obtain that $u_\mu \in W^{2,p}(\Omega)$ for any $1 < p < \infty$ (see [17, Corolary 9.25]). It follows from Harnack's inequality that $u_\mu > 0$ a.e. in Ω and we are done. □

We now prove that J verifies a classical compactness condition. The main novelty here is to ensure the compactness using the decomposition of the space $W^{1,2}(\Omega)$ presented in the last section.

{bound}

Proposition 3.2 *Suppose that* (a_0) *and* (b_0) *hold. If* $c \in \mathbb{R}$ *and* $(u_n) \subset H$ *is such that*

$$J(u_n) \to c \quad and \quad (1 + \|u_n\|)\|J'(u_n)\| \to 0,$$

as $n \to +\infty$*, then* (u_n) *has a convergent subsequence.*

Proof. We first show that (u_n) has a bounded subsequence. Suppose, by contradiction, that $\|u_n\| \to +\infty$ and define define $w_n := u_n/\|u_n\|$. Up to a subsequence, $w_n \rightharpoonup w$ weakly in H, $w_n \to w$ strongly in $L^p(\partial\Omega)$ and $L^q(\partial\Omega)$, for some $w \in H$.

Computing $J(u_n) - (1/2)J'(u_n)u_n$, we obtain

$$\left(\frac{1}{2} - \frac{1}{p}\right) \int_{\partial\Omega} a(x)|u_n|^p \, \mathrm{d}\sigma_x = \mu \left(\frac{1}{q} - \frac{1}{2}\right) \int_{\partial\Omega} b(x)|u_n|^q \, \mathrm{d}\sigma_x + c + o_n(1),$$

where $o_n(1)$ denotes a quantity going to zero as $n \to \infty$. Thus,

$$\int_{\partial\Omega} a(x)|u_n|^p \, \mathrm{d}\sigma_x \le C_1 + C_1\|u_n\|^q,$$

holds for some $C_1 > 0$. Recalling that $J(u_n) = c + o_n(1)$, we obtain

$$\begin{aligned} \frac{1}{2}\int_{\Omega} |\nabla u_n|^2 \, \mathrm{d}x &= \frac{1}{p}\int_{\partial\Omega} a(x)|u_n|^p \, \mathrm{d}\sigma_x + \frac{\mu}{q}\int_{\partial\Omega} b(x)|u_n|^q \, \mathrm{d}\sigma_x + c + o_n(1) \\ &\le C_2 + C_2\|u_n\|^q \end{aligned}$$

and therefore

$$\int_{\Omega} |\nabla w_n|^2 \, \mathrm{d}x \le o_n(1) + C_3\|u_n\|^{q-2}.$$

Since $q < 2$, we get $\|\nabla w_n\|_2 \to 0$. Hence, if we write $w_n = t_{w_n} + w_n^{\perp}$, with $t_{w_n} \in \mathbb{R}$ and $w_n^{\perp} \in X_1$, we conclude that $w_n^{\perp} \to 0$ in X_1. It follows that the sequence (t_{w_n}) is bounded, and therefore, up to a subsequence, $t_{w_n} \to t_w \in \mathbb{R}$ which satisfies

$$|t_w| = \|t_w\| = \|w\| = \lim_{n\to\infty} \|w_n\| = 1.$$

We conclude that $w_n \to 1$ strongly in H.

On the other hand, the equality $\|u_n\|^{1-p}J'(u_n)t_{w_n} = o_n(1)$ provides

$$\left|\int_{\partial\Omega} a(x)|w_n|^{p-2}w_n t_{w_n} \, \mathrm{d}\sigma_x\right| \le \frac{1}{\|u_n\|^{p-q}} \int |b(x)||w_n|^{q-1}|t_{w_n}| \, \mathrm{d}\sigma_x + o_n(1).$$

Since $p > q$ and $w_n \to 1$, the right-hand side above goes to zero. Thus, it follows from the Lebesgue Theorem that

$$0 = \lim_{n\to\infty} \int_{\partial\Omega} a(x)|w_n|^{p-2}w_n t_{w_n} \, \mathrm{d}\sigma_x = \int_{\partial\Omega} a(x) \, \mathrm{d}\sigma_x,$$

which contradicts (a_0). This contradiction proves that (u_n) is bounded.

Up to a subsequence, we have that $u_n \rightharpoonup u$ weakly in $W^{1,2}(\Omega)$, $u_n \to u$ strongly in $L^p(\partial\Omega)$ and $L^q(\Omega)$, for some $u \in H$. Hence,

$$\lim_{n\to\infty} \int_{\partial\Omega} a(x)|u_n|^{p-2}u_n(u_n - u)\,\mathrm{d}\sigma_x = 0 = \lim_{n\to\infty} \int_{\partial\Omega} b(x)|u_n|^{q-2}u_n(u_n - u)\,\mathrm{d}\sigma_x$$

and therefore

$$o_n(1) = J'(u_n)(u_n - u) = \int_\Omega \nabla u_n \cdot \nabla(u_n - u)\,\mathrm{d}x + o_n(1),$$

and we obtain $\nabla u_n \to \nabla u$ in $(L^2(\Omega))^N$. Since $u_n \to u$ strongly in $L^2(\Omega)$, we have that $u_n \to u$ strongly in $W^{1,2}(\Omega)$. By Lemma 2.1, $\|\cdot\|$ is equivalent to the usual norm of $W^{1,2}(\Omega)$ and therefore we conclude that $u_n \to u$ strongly in H. $\square$

We are ready to prove our first theorem.

Proof of Theorem 1.1. Let $\mu^* > 0$ be given in Lemma 3.1. In view of Proposition 3.1 the problem has a positive solution u_μ such that $J(u_\mu) < 0$. We shall obtain the second solution as an application of a variant of the Mountain Pass Theorem presented by Beresticky, Capuzzo-Dolcetta and Nirenberg in [6].

According to Propositions 3.1 and 3.2 the functional J admits the Mountain Pass geometry and satisfies the Cerami condition. If we define $p : H^1(\Omega) \to H^1(\Omega)$ by $p(u) = |u|$, we easily see that $J(p(u)) = J(u)$ for any $u \in H^1(\Omega)$. Then, applying [6, Theorem 10], we guarantee the existence of $u \in \overline{p(H^1(\Omega))}$ such that $J(u) > 0$ and $J'(u) = 0$. In particular, the last assertion give us a nonzero critical point $u \in H$ such that $u \geq 0$ a.e. in Ω. As before, this critical point is positive a.e. in Ω and the theorem is proved. $\square$

4 The asymptotically linear case

In this section we prove Theorem 1.2. From now on, we shall consider a different splitting of the space $W^{1,2}(\Omega)$. Actually, considering the function $K_0 \in \mathcal{F}$ given by condition (f_0), we denote by H the space $W^{1,2}(\Omega)$ with the following decomposition

$$H := \langle 1 \rangle \oplus X_{K_0} = \langle 1 \rangle \oplus \left\{ v \in W^{1,2}(\Omega) : \int_{\partial\Omega} K_0(x) v\,\mathrm{d}\sigma_x = 0 \right\}.$$

Recalling the results of Lemma 2.1, from now on we write $u = t_u + u^\perp$, with $t_u \in \mathbb{R}$ and $u^\perp \in X_{K_0}$. As before, the norm of such element is $\|u\| = \sqrt{t_u^2 + \|\nabla u^\perp\|_2^2}$.

The energy functional associated to the problem (1.2) is $J : H \to \mathbb{R}$ given by

$$J(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 \mathrm{d}x - \int_{\partial\Omega} F(x, u)\mathrm{d}\sigma_x - \frac{\mu}{q}\int_{\partial\Omega} b(x)|u|^q \mathrm{d}\sigma_x,$$

where $F(x,s) = \int_0^s f(x,t)\mathrm{d}t$. Condition (f) imply that $J \in C^1(H,\mathbb{R})$ and the critical points of J are weak solutions of (1.2).

In what follows we prove a version of Lemma 3.1 for this new functional.

{mountain2}

Lemma 4.1 *Suppose that* (b_0), (f), (f_∞) *and* (f_0) *hold with* $\int_{\partial\Omega} K_0(x)\mathrm{d}\sigma_x < 0$. *If*

$$\lambda_1(k_\infty) < 1 < \lambda_1(K_0),$$

then there exists $\mu^* > 0$ *such that, for any* $\mu \in (0,\mu^*)$, *we have the following*

(i) there are α_μ, $r_\mu > 0$ *such that*

$$J(u) \geq \alpha_\mu, \qquad \forall\, u \in H,\ \|u\| = r_\mu;$$

(ii) there exists $e \in H^1(\Omega)$, *independent of* μ, *such that*

$$J(e) < 0, \qquad \|e\| > r_\mu.$$

Proof. We first notice that, for any given $\varepsilon > 0$, it follows from (f) and (f_0) that, for some $C_\varepsilon > 0$ there holds

$$F(x,s) \leq \frac{1}{2}(K_0(x)+\varepsilon)s^2 + C_\varepsilon|s|^p, \quad \text{for a.e. } x \in \partial\Omega, \forall\, s \in \mathbb{R}. \qquad (4.1)$$ {go}

As before, we consider η given by Lemma 2.2, take $u = t_u + u^\perp \in H$ and consider two cases.

Case 1: $\|\nabla u^\perp\|_2 \leq \eta|t_u|$.

In this case, we can use (4.1) and Lemma 2.2 to get

$$J(u) \geq -C_1\|u\|^p - \frac{\mu}{2q}|t_u|^q \int_{\partial\Omega} b(x)\,\mathrm{d}\sigma_x + \mathcal{J}(u) \qquad (4.2)$$ {pequi1}

where

$$\mathcal{J}(u) := \frac{1}{2}\|\nabla u^\perp\|_2^2 - \frac{1}{2}\int_{\partial\Omega} K_0(x)(t_u+u^\perp)^2\mathrm{d}\sigma_x - \frac{\varepsilon}{2}\int_{\partial\Omega} u^2\,\mathrm{d}\sigma_x.$$

Recalling that $\int_{\partial\Omega} K_0(x)t_u u^\perp \mathrm{d}\sigma_x = 0$, we can use the definition of $\lambda_1(K_0)$ to obtain

$$\int_{\partial\Omega} K_0(x)(t_u+u^\perp)^2\,\mathrm{d}\sigma_x \leq t_u^2\int_{\partial\Omega} K_0(x)\,\mathrm{d}\sigma_x + \frac{1}{\lambda_1(K_0)}\|\nabla u^\perp\|_2^2.$$

Since $2ab \leq (a^2+b^2)$, the trace embedding $H \hookrightarrow L^2(\partial\Omega)$ and Poincare's inequality provide

$$\int_{\partial\Omega} u^2\,\mathrm{d}\sigma_x \leq 2\int_{\partial\Omega}\left(t_u^2 + (u^\perp)^2\right)\mathrm{d}\sigma_x \leq C_2(t_u^2 + \|\nabla u^\perp\|_2^2).$$

All together, the above inequalities imply that

$$\mathcal{J}(u) \geq \frac{1}{2}\left(1 - \frac{1}{\lambda_1(K_0)} - \varepsilon C_3\right)\|\nabla u^{\perp}\|_2^2 - \frac{t_u^2}{2}\left(\int_{\partial\Omega} K_0(x)\,\mathrm{d}\sigma_x + \varepsilon C_3\right). \qquad (4.3)$$ {go2}

Since $\lambda_1(K_0) > 1$ and $\int_{\partial\Omega} K_0(x)\,\mathrm{d}\sigma_x < 0$, we can choose $\varepsilon > 0$ small in such way that $\mathcal{J}(u) \geq 0$. Thus, it follows from (4.2) and the same argument used in Case 1 of Lemma 3.1 that the inequality (3.2) holds for the new functional J.

Case 2: $\|\nabla u^{\perp}\|_2 > \eta|t_u|$.

In this case we start by picking $\varepsilon > 0$ in such way that the last term into the parenthesis in (4.3) is negative and the first one is equal to $C_4 > 0$. Arguing as above, using (4.1), (b_0) and the trace embedding we obtain

$$\begin{aligned} J(u) &\geq \frac{C_4}{2}\|\nabla u^{\perp}\|_2^2 - C_1\int_{\partial\Omega}|u|^p\,\mathrm{d}\sigma_x - \frac{\mu}{q}\int_{\partial\Omega} b(x)|u|^q\,\mathrm{d}\sigma_x \\ &\geq C_5\|\nabla u^{\perp}\|_2^2 - C_6\|u\|^p - \mu C_7\|u\|^q. \end{aligned}$$

Recalling that, by (3.3), $\|u\|^2 \leq (1+\eta^{-2})\|\nabla u^{\perp}\|_2^2$, we obtain

$$J(u) \geq \|\nabla u^{\perp}\|_2^2\left\{C_5 - C_6\|\nabla u^{\perp}\|_2^{p-2} - \mu C_7\|\nabla u^{\perp}\|_2^{q-2}\right\}$$

with all the constants independent of μ. Now, we can argue as in the proof of Lemma 3.1 to conclude that item (i) holds.

For proving (ii), we first notice that, for any given $\varepsilon > 0$, we can use (f_∞) and (f_0) to obtain $C_\varepsilon > 0$ such that

$$F(x,s) \geq \frac{1}{2}(k_\infty(x) - \varepsilon)s^2 - C_\varepsilon, \quad \text{for a.e. } x \in \partial\Omega, \forall\, s \in \mathbb{R}.$$

If we take $\phi_1 \in X_{k_\infty} \setminus \{0\}$ such that (see Section 2)

$$\Delta\phi_1 = 0, \text{ in } \Omega, \qquad \frac{\partial\phi_1}{\partial\nu} = \lambda_1(k_\infty)k_\infty(x)\phi_1, \text{ on } \partial\Omega,$$

it follows from the above inequality for F that

$$J(t\phi_1) \leq \frac{t^2}{2}\int_\Omega |\nabla\phi_1|^2\,\mathrm{d}x - \frac{t^2}{2}\int_{\partial\Omega}(k_\infty(x) - \varepsilon)\,\phi_1^2\,\mathrm{d}\sigma_x - \mu C_8 t^q + C_9.$$

Since $\lambda_1(k_\infty)\int_\Omega |\nabla\phi_1|^2\,\mathrm{d}x = \int_{\partial\Omega} k_\infty(x)\phi_1^2\,\mathrm{d}\sigma_x$, we can use the above expression and the trace embedding $H \hookrightarrow L^2(\partial\Omega)$ to get

$$\frac{J(t\phi_1)}{t^2} \leq \frac{1}{2}\left(1 - \frac{1}{\lambda_1(k_\infty)} + \varepsilon C_{10}\right)\int_\Omega |\nabla\phi_1|^2\,\mathrm{d}x - \mu C_8 t^{q-2} + C_9 t^{-2}.$$

If we choose $\varepsilon > 0$ small, we conclude that

$$\limsup_{t\to+\infty} \frac{J(t\phi_1)}{t^2} < 0$$

and therefore item (ii) holds for $e := t\phi_1$, with $t > 0$ large enough. □

Now we shall ensure that J has a critical point with negative energy using the behavior of f at the origin.

{ppp2}

Proposition 4.1 *Suppose that* (f_0), (f), (b_0) *and* (b_1) *hold and let* μ^*, r_μ *be given by Lemma 4.1. If* $\mu \in (0, \mu^*)$, *then*

$$c_\mu := \inf_{u \in \overline{B_{r_\mu}(0)}} J(u) < 0$$

is achieved at some positive function $u_\mu \in B_{r_\mu}(0)$. *In particular,* $J'(u_\mu) = 0$.

Proof. Given $\varepsilon > 0$, it follows from (f) and (f_0) that, for some $C_\varepsilon > 0$ there holds

$$F(x,s) \geq \frac{1}{2}(K_0(x) - \varepsilon)s^2 - C_\varepsilon |s|^p, \quad \text{for a.e. } x \in \partial\Omega, \forall\, s \in \mathbb{R}.$$

Let $\psi_1 : \partial\Omega \to \mathbb{R}$ be a positive smooth function with support contained in the set $\mathcal{B}^+$ given by condition (b_1). If we denote by ψ its extension to the whole set Ω we can use the above inequality to get

$$\begin{aligned} J(t\psi) &\leq \frac{t^2}{2}\left(\int_\Omega |\nabla\psi|^2 \,\mathrm{d}x - \int_{\partial\Omega} (K_0(x) - \varepsilon)\psi^2 \,\mathrm{d}\sigma_x\right) \\ &\quad + C_\varepsilon t^p \int_{\partial\Omega} \psi^p \,\mathrm{d}\sigma_x - \frac{\mu t^q}{q}\int_{\mathcal{B}^+} b(x)\psi^q \,\mathrm{d}\sigma_x. \end{aligned}$$

Since $q < 2 < p$, we infer that

$$\liminf_{t\to 0^+} \frac{J(t\psi)}{t^q} = -\int_{\mathcal{B}^+} b(x)\psi^q \,\mathrm{d}\sigma_x < 0,$$

and therefore $c_\mu < 0$. Now we can use (f_0) and the same argument of Proposition 3.1 to conclude the proof. We omit the details. □

{bound2}

Proposition 4.2 *Suppose that* (b_0), (f), (f_∞) *hold and, for some* $j \in \mathbb{N}$, *we have that*

$$\lambda_j(k_\infty) < 1 < \lambda_{j+1}(k_\infty). \tag{4.4}$$ {hotel3}

If $c \in \mathbb{R}$ *and* $(u_n) \subset H$ *is such that*

$$J(u_n) \to c \quad \text{and} \quad (1 + \|u_n\|)\|J'(u_n)\| \to 0, \tag{4.5}$$ {go4}

as $n \to +\infty$, *then* (u_n) *has a convergent subsequence.*

Proof. As quoted before, the usual norm of $W^{1,2}(\Omega)$ is equivalent to that induced by the inner product (2.2) for any function k verifying the hypotheses of Lemma 2.1. Hence, the sequence $(u_n) \subset \langle 1\rangle \oplus X_{k_\infty}$ satisfies the same conditions on (4.5) with the norm induced by this former decomposition. Moreover, it is sufficient to prove the result with this new norm. Along all this proof we shall write $u = t_u + u^\perp$, with $t_u \in \mathbb{R}$ and $u^\perp \in X_{k_\infty}$, that is, we use the decomposition $\langle 1\rangle \oplus X_{k_\infty}$.

We start by claiming that (u_n) has a bounded subsequence. Indeed, suppose by contradiction that $\|u_n\| \to +\infty$. If we set $v_n := u_n/\|u_n\|$ we may suppose that, up to a subsequence,

$$\begin{cases} v_n \rightharpoonup v_\infty, & \text{weakly in } \langle 1\rangle \oplus X_{k_\infty}, \\ v_n(x) \to v_\infty(x), & \text{for a.e. in } x \in \Omega, \\ v_n \to v_\infty, & \text{strongly in } L^s(\partial\Omega),\ 2 \le s < 2_*, \\ |v_n(x)| \le h_s(x), & \text{for a.e. } x \in \partial\Omega, \end{cases} \tag{4.6}$$

{hotel0}

for some $v_\infty \in \langle 1\rangle \oplus X_{k_\infty}$ and $h_s \in L^s(\partial\Omega)$, with $2 \le s < 2_*$.

For any $\varphi \in \langle 1\rangle \oplus X_{k_\infty}$, we have that

$$\begin{aligned} o_n(1) = J'(u_n)(\varphi/\|u_n\|) \quad = \quad & \int_\Omega (\nabla v_n \cdot \nabla\varphi)\,\mathrm{d}x - \int_{\partial\Omega} \frac{f(x,u_n)}{u_n} v_n \varphi \,\mathrm{d}\sigma_x \\ & - \frac{\mu}{\|u_n\|^{2-q}} \int_{\partial\Omega} b(x)|v_n|^{q-1}\varphi\,\mathrm{d}\sigma_x. \end{aligned} \tag{4.7}$$

{hotel1}

Since $\lim_{|s|\to+\infty} f(x,s)/s = k_\infty(x)$ for any $x \in \Omega$, we can use (f_0) and the Lebesgue theorem to get

$$\int_{\partial\Omega} \frac{f(x,u_n)}{u_n} v_n\varphi\,\mathrm{d}\sigma_x = o_n(1).$$

Hence, taking the limit into (4.7), using $q < 2$, (4.6), (f_∞) and Lebesgue Theorem again we conclude that

$$\int_\Omega (\nabla v_\infty \cdot \nabla\varphi)\mathrm{d}x = \int_{\partial\Omega} k_\infty(x) v_\infty \varphi\,\mathrm{d}\sigma_x, \qquad \forall\, \varphi \in \langle 1\rangle \oplus X_{k_\infty}. \tag{4.8}$$

{l1}

Picking $\varphi \equiv 1$, recalling that $\int_{\partial\Omega} k_\infty(x) v_\infty^\perp\,\mathrm{d}\sigma_x = 0$ and $\int_{\partial\Omega} k_\infty(x)\,\mathrm{d}\sigma_x < 0$, we conclude that $t_{v_\infty} = 0$, and therefore $v_\infty = v_\infty^\perp \in X_{k_\infty}$. Since $J'(u_n)(u_n/\|u_n\|^2) = o_n(1)$, we can proceed as above to conclude that $\int_\Omega |\nabla v_n|^2\,\mathrm{d}x \to \int_{\partial\Omega} k_\infty(x) v_\infty^2\,\mathrm{d}\sigma_x$. On the other hand, recalling that $v_\infty \in X_{k_\infty}$, we conclude that $t_{v_n} \to 0$. Thus,

$$1 = \lim_{n\to+\infty} \left\{ \left(t_{v_n}^2 + \int |\nabla v_n|^2\,\mathrm{d}x \right) - t_{v_n}^2 \right\} = \int_{\partial\Omega} k_\infty(x) v_\infty^2\,\mathrm{d}\sigma_x,$$

and therefore $v_\infty \not\equiv 0$. This and (4.8) show that v_∞ an eigenfunction of the problem (2.4), with $k = k_\infty$. But this contradicts (4.4). Thus, we conclude that (u_n) is bounded.

Since all the nonlinear terms in the functional J has subcritical growth, we can use a standard procedure to conclude that, for some subsequence, $\nabla u_n \to \nabla u$ in $(L^2(\Omega))^N$. The result easily follows from this fact. $\square$

We are ready to finish the paper proving our second theorem.

Proof of Theorem 1.2. Arguing as in the proof of the first theorem we obtain a positive solution by minimizing the functional J. The existence of the second solution is a consequence of all the above auxiliary results and the classical Mountain Pass Theorem. $\square$

{f-impar}

Remark 4.1 *We notice that, in the last proof, we cannot guarantee that the second solution is positive. However, if f is odd in the second variable, we have that $J(u) = J(|u|)$. Therefore, the same argument of Theorem 1.1 applies and we can obtain a positive mountain pass solution.*

References

[1] S. Alama, M. del Pino, *Solutions of elliptic equations with indefinite non linearities via Morse theory and linking.* Ann. Inst. Henri Poincar **13** (1), (1996), 95–115.

[2] S. Alama, G. Tarantello, *On semilinear elliptic equations with indefinite nonlinearities.* Calc. Var. Partial Differ. Equ. **1** (1993), 439–475.

[3] A. Ambrosetti, H. Brezis, G. Cerami, *Combined effects of concave and convex nonlinearities in some elliptic problems.* J. Funct. Anal. **122** (1994), no. 2, 519–543.

[4] J. G. Azorero, I. Peral, J. D. Rossi, *A convex-concave problem with a nonlinear boundary condition.* J. Differential Equations **198** (2004), no. 1, 91–128.

[5] E. Berchio, F. Gazzola, D. Pierotti, *Gelfand type elliptic problems under Steklov boundary conditions.* Ann. Inst. H. Poincar Anal. Non Linaire **27** (2010), no. 1, 315–335.

[6] H. Berestycki, I. Capuzzo-Dolcetta, L. Nirenberg, *Variational methods for indefinite superlinear homogeneous elliptic problems.* NoDEA Nonlinear Differential Equations Appl. **2** (1995), no. 4, 553–572.

[7] H. Brezis, *Functional analysis, Sobolev spaces and partial differential equations.* Universitext. Springer, New York, 2011.

[8] A. P. Calderón, *On a inverse boundary value problem*, in "Seminar in Numerical Analysis and Its Applications to Cotinuum Physics", pp. 65–73, Soc. Brasileira de Matemática, Rio de Janeiro, 1980.

[9] M. Cuesta; H. R. Quoirin, *A weighted eigenvalue problem for the p-Laplacian plus a potential*. NoDEA Nonlinear Differential Equations Appl., pp 469-491, 16 (2009).

[10] J. Chabrowski, C. Tintarev, *An elliptic problem with an indefinite nonlinearity and a parameter in the boundary condition*. NoDEA Nonlinear Differential Equations Appl. **21** (2014), no. 4, 519–540.

[11] J. F. Escobar, *The geometry of the first non-zero Steklov eigenvalue*. J. Funct. Analysis **150** (1997), 544–556.

[12] J. F. Escobar, *A comparison theorem for the first non-zero Steklov eigenvalue*. J. Funct. Analysis **178** (2000), 143–155.

[13] E. D. da Silva, F. O. V. de Paiva, *Landesman-Lazer type conditions and multiplicity results for nonlinear elliptic problems with Neumann boundary values*. Acta Math. Sin. (Engl. Ser.) **30** (2014), no. 2, 229–250.

[14] D. G. de Figueiredo, J. P. Gossez, P. Ubilla, *Local superlinearity and sublinearity for indefinite semilinear elliptic problems*. J. Funct. Analysis **199** (2003), no. 2, 452–467.

[15] M. F. Furtado, R. Ruviaro, *Multiple solutions for a semilinear problem with combined terms and nonlinear boundary conditions*. Nonlinear Anal. 74 (2011), no. 14, 4820–4830.

[16] M. F. Furtado, R. Ruviaro, J. P. P. da Silva, *Two solutions for an elliptic equation with fast increasing weight and concave-convex nonlinearities*. J. Math. Anal. Appl. **416** (2014), no. 2, 698–709.

[17] D. Gilbarg, N. S. Trudinger, *Elliptic partial differential equations of second order*. Reprint of the 1998 edition. Classics in Mathematics. Springer-Verlag, Berlin, 2001.

[18] L. Leadi, A. Marcos, *A weighted eigencurve for Steklov problems with a potential*. NoDEA Nonlinear Differential Equations Appl. **20** (2013), no. 3, 687–713.

[19] S. Li, S. Wu, H.S. Zhou, *Solutions to semilinear elliptic problems with combined nonlinearities*, J. Differential Equations **185** (2002), 200–224.

[20] J. L. Lions, E. Magenes, E. *Non-homogeneous boundary value problems and applications*. Vol. II. Translated from the French by P. Kenneth. Die Grundlehren der mathematischen Wissenschaften, Band 182. Springer-Verlag, New York-Heidelberg, 1972.

[21] N. Mavinga, M.N. Nkashama, *Steklov-Neumann eigenproblems and nonlinear elliptic equations with nonlinear boundary conditions*. J. Differential Equations **248** (2010), no. 5, 1212–1229.

[22] N. Mavinga, *Generalized eigenproblem and nonlinear elliptic equations with nonlinear boundary conditions.* Proc. Roy. Soc. Edinburgh Sect. A **142** (2012), no. 1, 137–153.

[23] C. D. Pagani, D. Pierotti, *Multiple variational solutions to nonlinear Steklov problems.* NoDEA Nonlinear Differential Equations Appl. **19** (2012), no. 4, 417–436.

[24] C. D. Pagani, D. Pierotti, *Variational methods for nonlinear Steklov eigenvalue problems with an indefinite weight function.* Calc. Var. Partial Differential Equations **39** (2010), no. 1-2, 35–58.

[25] M. W. Steklov, *Sur les problmes fondamentaux de la physique mathmatique.* Ann. Sci. Ecole Normale Sup. **19**, (1902), 455–490.

[26] O. Torné, *Steklov problem with an indefinite weight for the p-lapacian.* Electronic J. of Dif. Eq., (2005) no. 87, 1–8.