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ON A CLASS OF SEMILINEAR ELLIPTIC EIGENVALUE PROBLEMS IN $\mathbb{R}^2$

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Abstract We consider the semilinear problem

$$-\Delta u - \tfrac{1}{2}(x \cdot \nabla u) = \lambda f(u), \quad x \in \mathbb{R}^2,$$

where λ is a positive parameter and f has exponential critical growth. We first establish the existence of a non-zero weak solution. Then, by assuming that f is odd, we prove that the number of solutions increases when the parameter λ becomes large. In the proofs we apply variational methods in a suitable weighted Sobolev space consisting of functions with rapid decay at infinity.

Keywords: eigenvalue problem; Trudinger–Moser inequality; self-similar solutions; critical exponential growth equation

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1. Introduction

In this paper we consider the semilinear elliptic eigenvalue problem

$$-\Delta u - \tfrac{1}{2}(x \cdot \nabla u) = \lambda f(u), \quad x \in \mathbb{R}^2, \qquad (P_\lambda)$$

where $\lambda > 0$ is a parameter and f is a continuous function satisfying some suitable conditions. We shall look for solutions in a weighted Sobolev space of functions having a rapid decay at infinity.

As quoted by Haraux and Weissler [**10**] and Escobedo and Kavian [**8**], the above equation is closely related to the study of self-similar solutions for the heat equation. More specifically, if we consider the evolution equation

$$u_t - \Delta u = |u|^{p-1}u \quad \text{on } (0,\infty) \times \mathbb{R}^N$$

and we seek solutions of the form $u(t,x) = t^{-1/(p-1)}\omega(t^{-1/2}x)$, a straightforward calculation shows that $\omega\colon \mathbb{R}^N \to \mathbb{R}$ needs to satisfy

$$-\Delta\omega - \tfrac{1}{2}(x \cdot \nabla\omega) = \frac{1}{p-1}\omega + |\omega|^{p-1}\omega \quad \text{on } \mathbb{R}^N.$$

For higher dimensions, $N \geqslant 3$, there are some results concerning the above equation and its variants obtained by replacing the right-hand side of the equality by more general nonlinearities $g(\omega)$ (see [**3**, **5**, **8**, **9**, **13**, **14**] and references therein). In a large class of such results the authors used variational techniques in such a way that the range of the power p is limited from above by the critical Sobolev exponent $2N/(N-2)$.

In dimension 2 the role of the critical exponent is played by the limiting exponent α_0 for the Trudinger–Moser inequality in $H^1(\mathbb{R}^2)$, which is $\alpha_0 = 4\pi$. Here we are interested in the case in which the function f has the maximal growth, which allows us to deal with (P_λ) variationally. More precisely, in dimension 2 the maximal growth is related to the so-called Trudinger–Moser inequality, namely,

$$\sup_{\|u\|_{H_0^1(\Omega)\leqslant 1}} \int_\Omega \mathrm{e}^{\alpha u^2}\,\mathrm{d}x \leqslant C(\alpha)$$

for all $\alpha \leqslant 4\pi$ and $\Omega \subset \mathbb{R}^2$ bounded. Motivated by this inequality, in [**1**, **6**] the authors introduced the following notion of criticality in dimension 2: we say that $f\colon \mathbb{R} \to \mathbb{R}$ has exponential critical growth if there exists $\alpha_0 > 0$ such that

$$\lim_{|s|\to+\infty} \frac{f(s)}{\mathrm{e}^{\alpha s^2}} = \begin{cases} 0 & \text{if } \alpha > \alpha_0, \\ +\infty & \text{if } \alpha < \alpha_0. \end{cases} \tag{1.1}$$

We assume throughout this paper that the nonlinearity f is continuous and satisfies this growth condition.

Problems involving critical exponential growth in bounded domains $\Omega \subset \mathbb{R}^2$ have been studied by many authors (see [**6**, **7**, **15**, **16**] and references therein). When dealing with problems on the entire space we need a version of the Trudinger–Moser inequality for unbounded domains. It asserts that

$$\sup_{\|u\|_{H^1(\mathbb{R}^2)\leqslant 1}} \int_{\mathbb{R}^2} (\mathrm{e}^{\alpha u^2} - 1)\,\mathrm{d}x \leqslant C(\alpha)$$

for all $\alpha \leqslant 4\pi$ (see [**4**, **17**] and references therein).

Since we intend to apply variational methods, we first observe that (P_λ) can be rewritten in a divergence form. Indeed, if we set

$$k(x) := \exp\left(\frac{|x|^2}{4}\right), \quad x \in \mathbb{R}^2,$$

a direct calculation shows that (P_λ) is equivalent to

$$-\operatorname{div}(k(x)\nabla u) = \lambda k(x) f(u), \quad x \in \mathbb{R}^2. \tag{1.2}$$

Due to the presence of the weight $k(x)$ we are not able to use the usual Sobolev spaces. As quoted in [**8**], the natural space to look for rapid decay solutions is the space X defined as the closure of $C^\infty_{c,\mathrm{rad}}(\mathbb{R}^2)$ with respect to the norm

$$\|u\| := \left(\int_{\mathbb{R}^2} k(x)|\nabla u|^2\,\mathrm{d}x\right)^{1/2}.$$

As we will see, the space X is compactly immersed into the Lebesgue spaces with weight k. Furthermore, with the aid of the Trudinger–Moser inequality, we prove that the embedding of X into the Orlicz space $L_A(\mathbb{R}^2)$, where A is the N-function $A(t) = \mathrm{e}^{\alpha t^2} - 1$, is continuous, although it is not compact.

In order to perform our minimax approach we need to impose some suitable assumptions on the behaviour of f. More precisely, we shall assume the following conditions:

(f_1) $f(s) = o(s)$ as $s \to 0$,

(f_2) there exists $\theta > 2$ such that

$$0 \leqslant \theta F(s) := \theta \int_0^s f(t)\,\mathrm{d}t \leqslant sf(s) \quad \text{for all } s \in \mathbb{R},$$

(f_3) the following limit holds:

$$\lim_{|s|\to+\infty} \frac{f(s)s}{F(s)} = +\infty.$$

In our first result we establish the existence of a non-trivial weak solution for problem (P_λ) when $\lambda > 0$ is arbitrary. More precisely, we have the following theorem.

Theorem 1.1. *Suppose that f satisfies (f_1)–(f_3) and*

(f_4) *there exists $\beta_0 > 0$ such that*

$$\liminf_{|s|\to\infty} \frac{f(s)s}{\mathrm{e}^{\alpha_0 s^2}} \geqslant \beta_0 > \frac{4}{\lambda\alpha_0} \min\left\{\frac{1}{r^2}\exp\left(\frac{r^2}{4} + \frac{r^4}{256}\right) : r > 0\right\}.$$

Then, for any $\lambda > 0$, problem (P_λ) has at least one non-zero weak solution.

We remark that our first theorem can be applied for the model nonlinearity

$$f(s) = \frac{(\mathrm{e}^{s^2} - 1)s\beta_0}{1 + s^2}, \quad s \in \mathbb{R}.$$

Indeed, this function satisfies conditions (f_1)–(f_4) for $\alpha_0 = 1$ and $\lambda\beta_0 > 3$.

In the proof of Theorem 1.1 we shall use the mountain pass theorem. After having established all the properties of the space X we need to overcome the difficulty of dealing with the unboundedness of the domain. Actually, this implies some problems in handling with Palais–Smale sequences. Since the embedding of X in the Orlicz space $L_A(\mathbb{R}^2)$ is not compact, we need to prove some technical convergence results as well as perform some careful estimates of the minimax level of the associated functional. We also emphasize the necessity of establishing a version of a convergence result due to Lions [**11**] for our variational setting (see Lemma 2.6).

Condition (f_3), which essentially has already appeared in [**15**,**19**], is crucial in order to get some of the required convergence results. Although it appears to be quite restrictive, the model functions with critical exponential growth satisfy (f_3). Moreover, this condition is implied by

$(\widehat{f_3})$ there exist constants $R_0, M_0 > 0$ such that

$$0 < F(s) \leqslant M_0 f(s) \quad \text{for all } |s| \geqslant R_0,$$

which has been used in many papers (see [**1**, **6**, **7**], for instance).

Concerning the correct localization of the minimax level, we use the technical condition (f_4) and adapt some ideas presented in [**15**] by performing some careful estimates of a slight modification of the Green functions considered by Moser [**12**]. It is worthwhile mentioning that our condition (f_4) is more general than the analogous one considered in [**15**], since here the number β_0 may be finite. Actually, a numerical computation shows that the relation between α_0 and β_0 in condition (f_4) is satisfied if $\lambda\alpha_0\beta_0 > 9$.

In our last result we introduce more symmetry into the problem and show that the value of the parameter $\lambda > 0$ affects the number of solutions.

Theorem 1.2. *Suppose that the odd function f satisfies (f_1)–(f_3) and*

(f_5) *there are $\mu_0, s_0 > 0$ and $\theta_0 > 2$ such that*

$$F(s) \geqslant \mu_0 |s|^{\theta_0} \quad \text{for all } |s| \leqslant s_0.$$

Then, for any given $m \in \mathbb{N}$, there exists $\Lambda_m > 0$ such that problem (P_λ) has at least m pairs of non-zero weak solutions provided that $\lambda > \Lambda_m$.

The proof of this last theorem will be done as a byproduct of the compactness condition established in the proof of Theorem 1.1 and an application of the symmetric mountain pass theorem. Notice that, in this symmetric setting, we replace the condition at infinity (f_4) by (f_5), which provides the control of F at the origin. As far as we know, there are no multiplicity results concerning problems like (P_λ) when the nonlinearity has critical growth.

The paper is organized as follows. In § 2 we present some useful inequalities concerning the space X. In § 3 we establish the variational setting of our problem and prove a local compactness result for the associated functional. The main results are proved in § 4.

2. Some properties of the space X

In this section we present some technical results. Throughout the paper we write $\int u$ instead of $\int_{\mathbb{R}^2} u(x)\,\mathrm{d}x$.

We recall that X denotes the Hilbert space obtained as the completion of $C^\infty_{c,\mathrm{rad}}(\mathbb{R}^2)$ with respect to the norm

$$\|u\| := \left(\int k(x) |\nabla u|^2 \right)^{1/2},$$

which is induced by the inner product

$$\langle u, v \rangle := \int k(x)(\nabla u \cdot \nabla v).$$

For each $p \geqslant 2$ we also consider the weighted Lebesgue space $L^p(\mathbb{R}^2, k)$ of all radial functions $u\colon \mathbb{R}^2 \to \mathbb{R}$ such that

$$\|u\|_p := \left(\int k(x)|u|^p \right)^{1/p} < \infty.$$

Lemma 2.1. *The space X is compactly embedded into $L^p(\mathbb{R}^2, k)$ for any $p \in [2, +\infty)$.*

Proof. For any given $u \in C^\infty_{c,\mathrm{rad}}(\mathbb{R}^2)$ we have that

$$\int |\nabla(k(x)^{1/2}u)|^2 = \int k(x)|\nabla u|^2 + \int \nabla(k(x)^{1/2}u^2)\nabla(k(x)^{1/2}).$$

Integrating by parts we get

$$\int |\nabla(k(x)^{1/2}u)|^2 = \int k(x)|\nabla u|^2 - \frac{1}{2}\int k(x)u^2(\Delta\theta + \tfrac{1}{2}|\nabla\theta|^2), \tag{2.1}$$

where $\theta(x) := |x|^2/4$. Since

$$\Delta\theta(x) + \tfrac{1}{2}|\nabla\theta(x)|^2 = 1 + \tfrac{1}{8}|x|^2 \geqslant 1,$$

it follows that

$$\int k(x)u^2 \leqslant 2\int k(x)|\nabla u|^2. \tag{2.2}$$

By density, we have the same inequality for any $u \in X$. This establishes the continuous embedding $X \hookrightarrow L^2(\mathbb{R}^2, k)$.

If $p > 2$ and $u \in X$, we can use (2.1) and (2.2) to get

$$\int (|\nabla(k(x)^{1/2}u)|^2 + k(x)u^2) \leqslant \int k(x)|\nabla u|^2 + \frac{1}{2}\int k(x)u^2 \leqslant 2\|u\|^2.$$

Thus, we conclude that $k^{1/2}u \in H^1(\mathbb{R}^2)$. Hence, we can use (2.1) again to infer that

$$\begin{aligned}\int k(x)|\nabla u|^2 &\geqslant \int |\nabla(k(x)^{1/2}u)|^2 + \frac{1}{2}\int k(x)|u|^2 \\ &= \|k^{1/2}u\|^2_{H^1(\mathbb{R}^2)} - \frac{1}{2}\int k(x)|u|^2 \\ &\geqslant C_p\left(\int k(x)^{p/2}|u|^p\right)^{2/p} - \frac{1}{2}\int k(x)|u|^2,\end{aligned}$$

where $C_p > 0$ is related to the embedding $H^1(\mathbb{R}^2) \hookrightarrow L^p(\mathbb{R}^2)$. Since $k(x) \geqslant 1$ and $p \geqslant 2$, we have that $k(x)^{p/2} \geqslant k(x)$. It follows from (2.2) that

$$C_p\left(\int k(x)|u|^p\right)^{2/p} \leqslant C_p\left(\int k(x)^{p/2}|u|^p\right)^{2/p} \leqslant 2\int k(x)|\nabla u|^2, \tag{2.3}$$

and therefore $X \hookrightarrow L^p(\mathbb{R}^2, k)$ for $p \geqslant 2$.

It was proved in [**8**, Proposition 11] that the embedding $X \hookrightarrow L^2(\mathbb{R}^2, k)$ is compact. For the $p > 2$ case, we take a sequence $(u_n) \subset X$ such that $u_n \rightharpoonup 0$ weakly in X. Fix $\tilde{p} > p$ and consider $\tau \in (0,1)$ such that $p = (1-\tau)2 + \tau\tilde{p}$. Hölder's inequality with exponents $1/(1-\tau)$ and $1/\tau$ provides

$$\begin{aligned} \int k(x)|u_n|^p &= \int k(x)^{(1-\tau)}|u_n|^{(1-\tau)2}k(x)^{\tau}|u_n|^{\tau\tilde{p}} \\ &\leqslant \biggl(\int k(x)|u_n|^2\biggr)^{1-\tau}\biggl(\int k(x)|u_n|^{\tilde{p}}\biggr)^{\tau} \\ &\leqslant c\|u_n\|_2^{2(1-\tau)}. \end{aligned}$$

Up to a subsequence, we have that $u_n \to 0$ in $L^2(\mathbb{R}^2, k)$. The above expression implies that $u_n \to 0$ in $L^p(\mathbb{R}^2, k)$. □

Remark 2.2. As a byproduct of the above calculations we see that $X \hookrightarrow H^1(\mathbb{R}^2)$. Indeed, for any $u \in X$, it holds that

$$\int (|\nabla u|^2 + |u|^2) \leqslant \int k(x)(|\nabla u|^2 + |u|^2) \leqslant c\|u\|^2.$$

We also quote for future reference that, in view of the second inequality of (2.3), for any $r \geqslant 1$ there exists $C = C(r)$ such that

$$\biggl(\int k(x)^r|u|^{2r}\biggr)^{1/r} \leqslant C\int k(x)|\nabla u|^2 \quad \text{for all } u \in X. \tag{2.4}$$

We shall need the following variant of a well-known radial lemma of Strauss [**18**].

Lemma 2.3. *There exists $c_0 > 0$ such that, for all $v \in X$, it holds that*

$$|v(x)| \leqslant c_0|x|^{-1/2}\mathrm{e}^{-|x|^2/8}\|v\| \quad \textit{for all } x \in \mathbb{R}^2 \setminus \{0\}.$$

Proof. It suffices to prove the lemma for $v \in C^{\infty}_{c,\mathrm{rad}}(\mathbb{R}^2)$. Let $r = |x|$ and $\varphi\colon [0,+\infty) \to \mathbb{R}$ be such that $\varphi(r) = v(|x|)$. We have that

$$\begin{aligned} \varphi(r)^2 &= -2\int_r^{\infty} \varphi(s)\varphi'(s)\,\mathrm{d}s \\ &\leqslant 2\int_r^{\infty} \mathrm{e}^{-s^2/4}s^{-1}|\varphi(s)||\varphi'(s)|\mathrm{e}^{s^2/4}s\,\mathrm{d}s \\ &\leqslant r^{-1}\mathrm{e}^{-r^2/4}\int_r^{\infty}(\varphi(s)^2 + \varphi'(s)^2)\mathrm{e}^{s^2/4}s\,\mathrm{d}s \\ &\leqslant c_1r^{-1}\mathrm{e}^{-r^2/4}\int (k(x)|\nabla v|^2 + k(x)v^2). \end{aligned}$$

Since $X \hookrightarrow L^2(\mathbb{R}^2, k)$, we get

$$\varphi(r)^2 \leqslant cr^{-1}\mathrm{e}^{-r^2/4}\|v\|^2,$$

and the lemma follows. □

In order to make the proof of our results more direct and effective, we state a technical lemma.

Lemma 2.4. *Suppose that $G \in C(\mathbb{R}, \mathbb{R})$ satisfies*

$$G(s) \leqslant c_1 s^4 (e^{\alpha s^2} - 1) \quad \text{for all } s \in \mathbb{R},$$

with $c_1, \alpha > 0$. Then there exists $c > 0$ such that, for any $R > 1$ and $u \in X$, it holds that

$$\int_{B_R(0)^c} k(x) G(u) \, dx \leqslant \frac{c}{R} \|u\|^4 (e^{\alpha c_0^2 \|u\|^2} - 1),$$

where $c_0 > 0$ comes from Lemma 2.3.

Proof. It follows from the monotone convergence theorem that

$$\begin{aligned} \int_{B_R(0)^c} k(x) G(u) \, dx &\leqslant c_1 \int_{B_R(0)^c} k(x) |u|^4 (e^{\alpha u^2} - 1) \, dx \\ &= c_1 \sum_{j=1}^{+\infty} \frac{\alpha^j}{j!} \int_{B_R(0)^c} k(x) |u|^{2j+4} \, dx. \end{aligned} \tag{2.5}$$

By using Lemma 2.3 we can estimate the last integral above as

$$\begin{aligned} \int_{B_R(0)^c} k(x) |u|^{2j+4} \, dx &\leqslant (c_0 \|u\|)^{2j+4} \int_{B_R(0)^c} e^{(|x|^2/4)(1-j-2)} |x|^{-j-2} \, dx \\ &\leqslant 2\pi (c_0 \|u\|)^{2j+4} \int_R^\infty s^{-j-2} s \, ds \\ &= 2\pi (c_0 \|u\|)^{2j+4} \frac{1}{j R^j} \\ &\leqslant \frac{2\pi}{R} (c_0 \|u\|)^{2j+4}, \end{aligned}$$

where we have used that $j \geqslant 1$ and $R > 1$. The above expression and (2.5) provide

$$\begin{aligned} \int_{B_R(0)^c} k(x) G(u) dx &\leqslant \frac{2\pi}{R} c_1 (c_0 \|u\|)^4 \sum_{j=1}^{\infty} \frac{(\alpha c_0^2 \|u\|^2)^j}{j!} \\ &= \frac{c}{R} \|u\|^4 (e^{\alpha c_0^2 \|u\|^2} - 1) \end{aligned}$$

with $c := 2\pi c_1 c_0^4 > 0$, and this completes the proof. □

We now recall the Trudinger–Moser inequality for the whole space $\mathbb{R}^2$, the proof of which can be found, for instance, in [**4**, **17**].

Lemma 2.5 (Trudinger–Moser inequality). *If $\alpha > 0$ and $v \in H^1(\mathbb{R}^2)$, then $(e^{\alpha v^2} - 1) \in L^1(\mathbb{R}^2)$. Moreover, if $\|\nabla v\|_{L^2(\mathbb{R}^2)} \leqslant 1$, $\|v\|_{L^2(\mathbb{R}^2)} \leqslant M < \infty$ and $\alpha \leqslant 4\pi$, then there exists a constant $C = C(M, \alpha)$ such that*

$$\int (e^{\alpha v^2} - 1) \leqslant C(M, \alpha). \tag{2.6}$$

In what follows we establish a version of a result due to Lions (see [**11**, Theorem I.6]) for the whole space $\mathbb{R}^2$ and considering our functional space.

Lemma 2.6. *Let (v_n) in X with $\|v_n\| = 1$ and suppose that $v_n \rightharpoonup v$ weakly in X with $\|v\| < 1$. Then for each $0 < p < 4\pi(1 - \|v\|^2)^{-1}$, up to a subsequence, we have that*

$$\sup_{n\in\mathbb{N}} \int (\mathrm{e}^{pv_n^2} - 1) < \infty.$$

Proof. We first notice that, if $a, b, \varepsilon > 0$, Young's inequality implies that

$$a^2 = (a-b)^2 + b^2 + 2\varepsilon(a-b)\frac{b}{\varepsilon} \leqslant (1+\varepsilon^2)(a-b)^2 + c\left(1 + \frac{1}{\varepsilon^2}\right)b^2.$$

Hence, we can use Young's inequality again to get

$$\begin{aligned}\int (\mathrm{e}^{pv_n^2} - 1) &\leqslant \int \left(\mathrm{e}^{p(1+\varepsilon^2)(v_n - v)^2}\mathrm{e}^{p(1+1/\varepsilon^2)v^2} - \frac{1}{\gamma} - \frac{1}{\gamma'}\right)\\ &\leqslant \frac{1}{\gamma}\int (\mathrm{e}^{\gamma p(1+\varepsilon^2)(v_n-v)^2} - 1) + \frac{1}{\gamma'}\int (\mathrm{e}^{\gamma' p(1+1/\varepsilon^2)v^2} - 1),\end{aligned}$$

where $\gamma > 1$ and $1/\gamma + 1/\gamma' = 1$. Since the last integral above is finite, it suffices to prove that

$$\sup_{n\in\mathbb{N}} \int (\mathrm{e}^{\gamma p(1+\varepsilon^2)(v_n-v)^2} - 1) < \infty.$$

Since $v_n \rightharpoonup v$ and $\|v_n\| = 1$, we conclude that

$$\lim_{n\to\infty} \|v_n - v\|^2 = 1 - \|v\|^2 < \frac{4\pi}{p}.$$

Thus, we can take $0 < \alpha < 4\pi$ and choose $\gamma > 1$ and $\varepsilon > 0$ small in such a way that

$$\gamma p(1+\varepsilon^2)\|v_n - v\|^2 < \alpha < 4\pi. \tag{2.7}$$

We now set $u_n := (v_n - v)/\|v_n - v\|$ and notice that, since $\int |\nabla u_n|^2 \leqslant \|u_n\|^2 = 1$, we have that $\|\nabla u_n\|_{L^2(\mathbb{R}^2)} \leqslant 1$. Moreover, it follows from (2.2) that $\|u_n\|_{L^2(\mathbb{R}^2)} \leqslant 2$. Hence, we can invoke Lemma 2.5 and (2.7) to obtain a positive constant $C(M, \beta)$ such that

$$\begin{aligned}\int (\mathrm{e}^{\gamma p(1+\varepsilon^2)(v_n-v)^2} - 1) &= \int (\mathrm{e}^{\gamma p(1+\varepsilon^2)\|v_n - v\|^2 u_n^2} - 1)\\ &\leqslant \int (\mathrm{e}^{\alpha u_n^2} - 1)\\ &\leqslant C(M, \beta),\end{aligned}$$

and the lemma is proved. □

3. A local compactness condition

Let $\alpha > \alpha_0$ and let $q \geqslant 1$. By using the critical growth of f we obtain

$$\lim_{|s|\to+\infty} \frac{f(s)}{|s|^{q-1}(\mathrm{e}^{\alpha s^2}-1)} = 0.$$

This and (f_1) imply that, for any given $\varepsilon > 0$, it holds that

$$\max\{|f(s)s|, |F(s)|\} \leqslant \tfrac{1}{2}\varepsilon s^2 + c_1|s|^q(\mathrm{e}^{\alpha s^2}-1) \quad \text{for all } s \in \mathbb{R}. \tag{3.1}$$

Given $u \in X$ we can set $q = 2$ to obtain

$$\int k(x)F(u) \leqslant c_2 \int k(x)|u|^2 + c_3 \int k(x)|u|^2(\mathrm{e}^{\alpha u^2}-1).$$

If we apply the inequality $(1+t)^r \geqslant 1 + t^r$ with $t = \mathrm{e}^s - 1 \geqslant 0$, we get

$$(\mathrm{e}^s - 1)^r \leqslant (\mathrm{e}^{rs} - 1) \quad \text{for all } r \geqslant 1,\ s \geqslant 0. \tag{3.2}$$

It follows from Hölder's inequality that

$$\begin{aligned}\int k(x)|u|^2(\mathrm{e}^{\alpha u^2}-1) &\leqslant \biggl(\int k(x)^2|u|^4\biggr)^{1/2}\biggl(\int (\mathrm{e}^{\alpha u^2}-1)^2\biggr)^{1/2} \\ &\leqslant c_4\|u\|^2\biggl(\int (\mathrm{e}^{2\alpha u^2}-1)\biggr)^{1/2} \\ &< +\infty,\end{aligned}$$

where we have used (2.4) and Lemma 2.5 to conclude that the last term is finite.

All together the above estimates imply that the functional

$$I_\lambda(u) := \tfrac{1}{2}\|u\|^2 - \lambda \int k(x)F(u), \quad u \in X,$$

is well defined. Standard calculations show that $I_\lambda \in C^1(X, \mathbb{R})$ with derivative given by

$$I'_\lambda(u)\varphi = \int k(x)\nabla u \cdot \nabla\varphi - \lambda \int k(x)f(u)\varphi \quad \text{for all } u, \varphi \in X,$$

and therefore the critical points of I_λ are precisely the weak solutions of (1.2).

Remark 3.1. Since the value of $\lambda > 0$ is not relevant in Theorem 1.1, we shall assume, until § 4.2, that $\lambda = 1$. To simplify notation we write simply I to denote I_1.

We devote the rest of this section to the proof of a compactness condition for the functional I. We follow here some arguments developed in [**15**]. First, we recall that $(u_n) \subset X$ is said to be a $(\mathrm{PS})_c$ sequence (where 'PS' stands for 'Palais–Smale') for the functional I if $I(u_n) \to c$ and $I'(u_n) \to 0$. We say that I satisfies the $(\mathrm{PS})_c$ condition if any $(\mathrm{PS})_c$ sequence has a convergent subsequence.

Lemma 3.2. *Suppose that f satisfies (f_2) and (f_3). If $(u_n) \subset X$ is a $(PS)_c$ sequence for I, then, up to a subsequence, we have that*

(i) $u_n \rightharpoonup u$ *weakly in* X *with* $I'(u) = 0$,

(ii) $\displaystyle\lim_{n\to\infty} \int k(x)F(u_n) = \int k(x)F(u)$,

(iii) $\displaystyle\limsup_{R\to\infty} \int_{B_R(0)^c} k(x)f(u_n)u_n = 0$.

Proof. Let $(u_n) \subset X$ be such that $I'(u_n) \to 0$ and $I(u_n) \to c$. By using (f_2) and a standard argument we can check that (u_n) is bounded in X. Hence, up to a subsequence, $u_n \rightharpoonup u$ weakly in X and $u_n(x) \to u(x)$ almost everywhere (a.e.) in $\mathbb{R}^2$. Recalling that $I'(u_n)u_n = o_n(1)$, using the above expression, (f_2) and the boundedness of (u_n), we obtain $c_1 > 0$ such that

$$\int k(x)f(u_n)u_n \leqslant c_1, \qquad \int k(x)\left(\frac{1}{\theta}f(u_n)u_n - F(u_n)\right) \leqslant c_1. \tag{3.3}$$

The first estimate above, $k(x) \geqslant 1$ and condition (f_2) again imply that

$$\int |f(u_n)u_n| = \int f(u_n)u_n \leqslant \int k(x)f(u_n)u_n \leqslant c_1.$$

Thus, it follows from a convergence result due to de Figueiredo *et al*. (see [**6**, Lemma 2.1]) that $f(u_n) \to f(u)$ in $L^1_{\mathrm{loc}}(\mathbb{R}^2)$. Hence, given $\varphi \in C^\infty_{c,\mathrm{rad}}(\mathbb{R}^2)$, we can take the limit in $I'(u_n)\varphi$ to conclude that $I'(u)\varphi = 0$. By density, $I'(u) = 0$.

Let $M > 0$ be such that $\|u_n\|, \|u\| \leqslant M$. Given $R, \varepsilon > 0$, we can use (3.1) with $q = 4$, Lemma 2.4 and the embedding $X \hookrightarrow L^2(\mathbb{R}^2, k)$ to get

$$\int_{B_R(0)^c} k(x)F(u_n) \leqslant \frac{\varepsilon}{2}\int k(x)|u_n|^2 + \frac{c}{R}\|u_n\|^4(\mathrm{e}^{\alpha c_0^2\|u_n\|^2} - 1) \leqslant c_2\varepsilon + \frac{c_3}{R},$$

with $c_2, c_3 > 0$ depending only on M. It follows that

$$\limsup_{R\to\infty} \int_{B_R(0)^c} k(x)F(u_n) \leqslant c_2\varepsilon. \tag{3.4}$$

Moreover, since $k(x)F(u) \in L^1(\mathbb{R}^2)$, it holds that

$$\limsup_{R\to\infty} \int_{B_R(0)^c} k(x)F(u) = 0. \tag{3.5}$$

For any fixed $R > 0$ we claim that

$$\lim_{n\to\infty} \int_{B_R(0)} k(x)F(u_n)\,\mathrm{d}x = \int_{B_R(0)} k(x)F(u)\,\mathrm{d}x. \tag{3.6}$$

If this is true, we can use the above convergence, (3.4) and (3.5) to get (ii).

In order to prove (3.6) we first notice that, by (f_3), we can obtain $\theta_0, R_0 > 0$ such that $\theta_0 F(s) \leqslant f(s)s$ for any $|s| > R_0$. Moreover, we can suppose that θ_0 also satisfies $(\theta c_1)/(\theta_0 - \theta) < \varepsilon$. If $A_n := \{|u_n| \geqslant R_0\}$, we can use the second inequality in (3.3) and (f_2) to obtain

$$\begin{aligned}\theta c_1 &\geqslant \int_{A_n} k(x)(f(u_n)u_n - \theta F(u_n))\,\mathrm{d}x \\ &= (\theta_0 - \theta)\int_{A_n} k(x)F(u_n)\,\mathrm{d}x + \int_{A_n} k(x)(f(u_n)u_n - \theta_0 F(u_n))\,\mathrm{d}x,\end{aligned}$$

and therefore it follows that

$$\int_{A_n} k(x)F(u_n)\,\mathrm{d}x \leqslant \frac{\theta c_1}{\theta_0 - \theta} < \varepsilon. \tag{3.7}$$

Applying Egoroff's theorem we obtain a measurable set $A \subset B_R(0)$ such that $|A| < \varepsilon$ and $u_n(x) \to u(x)$ uniformly on $B_R(0) \setminus A$. Hence,

$$\left|\int_{B_R(0)} k(x)(F(u_n) - F(u))\,\mathrm{d}x\right| \leqslant \int_A k(x)F(u_n)\,\mathrm{d}x + \int_A k(x)F(u)\,\mathrm{d}x + o_n(1). \tag{3.8}$$

Given $\alpha > \alpha_0$, it follows from the growth condition of f that $F(s) \leqslant c_4 \mathrm{e}^{\alpha s^2}$ for all $s \in \mathbb{R}$ and some $c_4 > 0$. Thus, given $\gamma > 1$, we can use Hölder's inequality to get

$$\int_A k(x)F(u)\,\mathrm{d}x \leqslant M_1 c_4 |A|^{1/\gamma}\left(\int \mathrm{e}^{\alpha\gamma' u^2}\right)^{1/\gamma'} \leqslant c_5 \varepsilon^{1/\gamma} \tag{3.9}$$

with $1/\gamma + 1/\gamma' = 1$. On the other hand, using (3.7), Lebesgue's theorem and the above expression, we obtain

$$\begin{aligned}\int_A k(x)F(u_n)\,\mathrm{d}x &\leqslant \int_{A\cap A_n} k(x)F(u_n)\,\mathrm{d}x + \int_{A\cap\{|u_n|<R_\theta\}} k(x)F(u_n)\,\mathrm{d}x \\ &\leqslant \varepsilon + \int_{A\cap\{|u_n|<R_\theta\}} k(x)F(u)\,\mathrm{d}x + o_n(1) \\ &\leqslant \varepsilon + c_5\varepsilon^{1/\gamma} + o_n(1).\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, the convergence in (3.6) follows from (3.8), (3.9) and the above inequality.

In order to prove (iii) it suffices to use the estimate for $|f(s)s|$ in (3.1) with $q = 4$ and proceed as in the proof of (3.4). □

Now we are ready to prove our compactness result.

Proposition 3.3. *Suppose that f satisfies (f_2) and (f_3). Then the functional I satisfies the (PS)$_d$ condition for any $0 < d < (2\pi)/\alpha_0$.*

Proof. Let $(u_n) \subset X$ be such that $I'(u_n) \to 0$ and $I(u_n) \to d < 2\pi/\alpha_0$. According to the previous lemma we have that $u_n \rightharpoonup u$ weakly in X, with $I'(u) = 0$, and moreover $\int k(x)F(u_n) \to \int k(x)F(u)$. We shall consider two possible cases.

Case 1 ($u = 0$). In this case we have that $\int k(x)F(u) = 0$, and therefore

$$\lim_{n\to\infty} \|u_n\|^2 = 2\left(d + \int k(x)F(u)\right) = 2d < \frac{4\pi}{\alpha_0}.$$

Since $I'(u_n)u_n = o_n(1)$, in order to prove that $u_n \to 0$ it is enough to check that

$$\lim_{n\to\infty} \int k(x)f(u_n)u_n = 0.$$

By using (3.1) with $q = 3$, we get

$$\left| \int k(x)f(u_n)u_n \right| \leqslant \frac{\varepsilon}{2}\|u_n\|_2^2 + c_1 D_n,$$

where

$$D_n := \int k(x)|u_n|^2|u_n|(\mathrm{e}^{\alpha u_n^2} - 1).$$

Since the embedding $X \hookrightarrow L^2(\mathbb{R}^2, k)$ is compact, we have that $\|u_n\|_2 \to 0$. Hence, it is enough to verify that $D_n \to 0$. By taking $r_i > 1$, $i = 1, 2, 3$, such that $1/r_1 + 1/r_2 + 1/r_3 = 1$ and $r_2 > 2$, we can use Hölder's inequality to get

$$\begin{aligned} D_n &\leqslant c_2\left(\int k(x)^{r_1}|u_n|^{2r_1}\right)^{1/r_1} \|u_n\|_{L^{r_2}(\mathbb{R}^2)} \left(\int (\mathrm{e}^{\alpha r_3\|u_n\|^2(u_n/\|u_n\|)^2} - 1)\right)^{1/r_3} \\ &\leqslant c_3\|u_n\|^2 o_n(1)\left(\int (\mathrm{e}^{\alpha r_3\|u_n\|^2(u_n/\|u_n\|)^2} - 1)\right)^{1/r_3}, \end{aligned} \tag{3.10}$$

where we have used (2.4), (3.2) and the compactness of $X \hookrightarrow L^{r_2}(\mathbb{R}^2)$. Since $\|u_n\|^2 \to \gamma < 4\pi/\alpha_0$, we can choose r_3 close to 1 and $\alpha > \alpha_0$ close to α_0 in such a way that $r_3\alpha\|u_n\|^2 \leqslant \tilde{\gamma} < 4\pi$. It follows from Lemma 2.5 that the last term in the parentheses in (3.10) is bounded. Hence, $D_n \to 0$ and the proposition is proved in the first case.

Case 2 ($u \neq 0$). We are going to verify that

$$\lim_{n\to\infty} \int k(x)f(u_n)u_n = \int k(x)f(u)u. \tag{3.11}$$

If this is true, we obtain

$$\begin{aligned} o_n(1) = I'(u_n)u_n &= \|u_n\|^2 - \int k(x)f(u)u + o_n(1) \\ &= \|u_n\|^2 - \|u\|^2 + I'(u)u + o_n(1). \end{aligned}$$

Since $I'(u) = 0$, we conclude that $\|u_n\| \to \|u\|$ and the proposition follows from the weak convergence of u_n.

It remains to check (3.11). In view of item (iii) of the previous lemma, and since $\limsup_{R\to\infty}\int_{B_R(0)^c} k(x)f(u)u = 0$, we need only verify that, for any $R > 0$, the following holds:

$$\lim_{n\to\infty}\int_{B_R(0)} k(x)f(u_n)u_n\,\mathrm{d}x = \int_{B_R(0)} k(x)f(u)u\,\mathrm{d}x. \tag{3.12}$$

With this aim we first notice that, as in the first case, we have

$$\lim_{n\to\infty}\|u_n\|^2 = 2\biggl(d + \int k(x)F(u)\biggr) = 2(d + d_0) > 0, \tag{3.13}$$

where $d_0 := \int k(x)F(u)$. We may suppose that $\|u_n\| \neq 0$ for all $n \geqslant n_0$, and therefore $v_n := u_n/\|u_n\|$ is well defined. The weak convergence of (u_n) and (3.13) imply that

$$v_n \rightharpoonup v := \frac{u}{\sqrt{2(d+d_0)}} \quad \text{weakly in } X.$$

It follows from (f_2) that $I(u) \geqslant 0$. Thus, we can take $\alpha > \alpha_0$ such that $d < I(u) + 2\pi/\alpha$. Hence,

$$1 - \|v\|^2 < \frac{4\pi/\alpha}{2(d+d_0)}$$

and we can use (3.13) to obtain $p_0 > 0$ such that $\alpha\|u_n\|^2 < p_0 < (4\pi)/(1 - \|v\|^2)$. We now choose $q > 1$ sufficiently close to 1 in such way that

$$\alpha q\|u_n\|^2 < p < \frac{4\pi}{1 - \|v\|^2},$$

with $p = p_0 q$. It follows from Lemma 2.6 that

$$\sup_{n\in\mathbb{N}}\int (\mathrm{e}^{\alpha q\|u_n\|^2 v_n^2} - 1) \leqslant \sup_{n\in\mathbb{N}}\int (\mathrm{e}^{p v_n^2} - 1) = c_4 < \infty.$$

Up to a subsequence, we have that $u_n \to u$ strongly in $L^s(B_R(0))$ for any $s \geqslant 1$. Hence, there exists $\Psi_s \in L^1(B_R(0))$ such that $|u_n(x)|^s \leqslant \Psi_s(x)$ a.e. in $B_R(0)$. Hence, by using that $k \in L^\infty(B_R(0))$, (3.1) with $q = 1$, Hölder's inequality and the above expression, for any measurable subset $A \subset B_R(0)$, we get

$$\begin{aligned}
\int_A k(x)f(u_n)u_n\,\mathrm{d}x &\leqslant c_5\int_A |u_n|^2\,\mathrm{d}x + c_6\biggl(\int_A |u_n|^{q'}\,\mathrm{d}x\biggr)^{1/q'}\biggl(\int_A (\mathrm{e}^{\alpha u_n^2} - 1)^q\,\mathrm{d}x\biggr)^{1/q} \\
&\leqslant c_5\int_A \Psi_2\,\mathrm{d}x + c_7\biggl(\int_A \Psi_{q'}\,\mathrm{d}x\biggr)^{1/q'}\biggl(\int (\mathrm{e}^{\alpha q\|u_n\|^2 v_n^2} - 1)\biggr)^{1/q} \\
&\leqslant c_5\int_A \Psi_2\,\mathrm{d}x + c_8\biggl(\int_A \Psi_{q'}\,\mathrm{d}x\biggr)^{1/q'}.
\end{aligned}$$

Since $\Psi_s \in L^1(B_R(0))$ and the set $A \subset B_R(0)$ is arbitrary, we conclude that the first integral above is uniformly small provided that the measure of A is small. Hence, the set $\{k(x)f(u_n)u_n\}$ is uniformly integrable, and Vitali's theorem implies that $k(x)f(u_n)u_n \to k(x)f(u)u$ in $L^1(B_R(0))$. This establishes (3.12) and concludes the proof. □

4. Proof of the main results

We start this section with the following technical result.

Lemma 4.1. *If $u \in X$, $\beta > 0$, $q > 0$ and $\|u\| \leqslant M$ with $\beta M^2 < 4\pi$, then there exists $C = C(\beta, M, q) > 0$ such that*

$$\int k(x)|u|^{2+q}(e^{\beta u^2} - 1) \leqslant C\|u\|^{2+q}.$$

Proof. Let $r_i > 1$, $i = 1, 2, 3$, be such that $1/r_1 + 1/r_2 + 1/r_3 = 1$ and $qr_2 \geqslant 2$. Hölder's inequality implies that

$$\int k(x)|u|^{2+q}(e^{\beta u^2} - 1) \leqslant \left(\int k(x)^{r_1}|u|^{2r_1}\right)^{1/r_1} \|u\|^q_{L^{qr_2}(\mathbb{R}^2)} \left(\int (e^{\beta u^2} - 1)^{r_3}\right)^{1/r_3}.$$

Using the embedding $X \hookrightarrow L^{qr_2}(\mathbb{R}^2)$, (2.4) and (3.2) we get

$$\int k(x)|u|^{2+q}(e^{\beta u^2} - 1) \leqslant C(q)\|u\|^{2+q}\left(\int (e^{\beta r_3 u^2} - 1)\right)^{1/r_3}. \tag{4.1}$$

By choosing r_3 close to 1, we can suppose that $\alpha := \beta r_3 M^2 < 4\pi$. Thus,

$$\int (e^{\beta r_3 u^2} - 1) \leqslant \int (e^{\beta r_3 M^2 (u/\|u\|)^2} - 1) = \int (e^{\alpha v^2} - 1),$$

with $v := u/\|u\|$. Arguing as in the proof of Lemma 2.6, we obtain $C(M, \beta) > 0$ such that

$$\int (e^{\beta r_3 u^2} - 1) \leqslant C(M, \beta).$$

This and (4.1) conclude the proof. □

4.1. Proof of Theorem 1.1

In this section we prove our existence result. Since we have already proved a local compactness result, the key point is the correct localization of the mountain pass level. This will be done using the following result, the proof of which will be postponed to the last section of the paper.

Proposition 4.2. *Suppose that f satisfies (f_2) and (f_4). Then there exists $v \in X$ with compact support such that*

$$\max_{t \geqslant 0} I(tv) < \frac{2\pi}{\alpha_0}. \tag{4.2}$$

If we assume the above proposition, we can prove Theorem 1.1 as follows: by using (3.1) with $q > 2$, Lemma 4.1 and (2.2), we obtain

$$\begin{aligned} I(u) &\geqslant \tfrac{1}{2}\|u\|^2 - \frac{\varepsilon}{2}\int k(x)|u|^2 - c_1 \int k(x)|u|^{2+(q-2)}(e^{\alpha u^2} - 1) \\ &\geqslant \tfrac{1}{2}(1 - c_2\varepsilon)\|u\|^2 - c_3\|u\|^q, \end{aligned} \tag{4.3}$$

with $c_1, c_2, c_3 > 0$. Since $q > 2$ and $\varepsilon > 0$ is arbitrary, it is standard to obtain $\alpha, \rho > 0$ such that

$$I(u) \geqslant \alpha > 0 \quad \text{for all } u \in \partial B_\rho(0).$$

Moreover, from (f_2) we obtain $c_4, c_5 > 0$ such that

$$F(s) \geqslant c_4 |s|^\theta - c_5 \quad \text{for all } s \in \mathbb{R}. \tag{4.4}$$

If v is given by Proposition 4.2 and $A \subset \mathbb{R}^2$ is its support, it follows from the above inequality that, for any $t > 0$,

$$I(tv) \leqslant \frac{t^2}{2}\|v\|^2 - c_4 t^\theta \int_A k(x)|v|^\theta \, \mathrm{d}x - c_5|A|.$$

Since $\theta > 2$, we conclude that $I(tv) \to -\infty$ as $t \to \infty$. Hence, $I(t_0 v) < 0$ for some $t_0 > 0$ large enough.

The above calculations show that I has the mountain pass geometry, and therefore we can define the minimax level

$$c_M := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)),$$

where $\Gamma := \{\gamma \in C([0,1], X) \colon \gamma(0) = 0,\ \gamma(1) = t_0 v\}$. The definition of c_M and (4.2) imply that

$$c_M \leqslant \max_{t \geqslant 0} I(tv) < \frac{2\pi}{\alpha_0}.$$

It follows from Proposition 3.3 and the mountain pass theorem that I has a non-zero critical point, and the theorem is proved. □

4.2. Proof of Theorem 1.2

In order to prove our multiplicity result we shall use the following version of the symmetric mountain pass theorem (see [**2**]).

Theorem 4.3. *Let E be a real Banach space, and let $J \in C^1(E, \mathbb{R})$ be an even functional satisfying $J(0) = 0$ and*

- (J_1) *there are constants $\rho, \alpha > 0$ such that $J|_{\partial B_\rho(0)} \geqslant \alpha$;*
- (J_2) *there is $A > 0$ and a finite-dimensional subspace V of E such that*

$$\max_{u \in V} J(u) \leqslant A.$$

If the functional J satisfies the $(PS)_d$ condition for $0 < d < A$, then it possesses at least $\dim V$ pairs of non-zero critical points.

Given $m \in \mathbb{N}$, we are going to apply this abstract result with $E = X$ and $J = I_\lambda$ and

$$V := \text{span}\{\psi_1, \dots, \psi_m\},$$

where $\{\psi_i\}_{i=1}^m \subset C_0^\infty(\mathbb{R}^2)$ is a collection of smooth functions with disjoint supports.

We notice that, for any $\lambda > 0$, all the results proved in the last sections for the functional $I = I_1$ also hold for I_λ. Thus, arguing as in the proof of Theorem 1.1 we can easily check that the even functional I_λ satisfies (J_1). By using the local condition (f_5) and (f_3) we can obtain $\mu > 0$ such that

$$F(s) \geqslant \mu |s|^{\theta_0} \quad \text{for all } s \in \mathbb{R}.$$

Since V is finite dimensional, all the norms in these spaces are equivalent. This fact and the above inequality yield, for any $u \in V$,

$$I_\lambda(u) \leqslant \tfrac{1}{2}\|u\|^2 - \lambda c_m \|u\|^{\theta_0},$$

where $c_m > 0$ depends only on m. If we consider the maximum value of the map $t \mapsto (1/2)t^2 - \lambda c_m t^{\theta_0}$ on $[0, +\infty)$, the above expression, $\theta_0 > 2$ and a straightforward calculation imply that

$$\max_{u \in V} I(u) \leqslant A_m := \left\{ \frac{1}{2} \left(\frac{1}{\theta_0 c_m} \right)^{2/(\theta_0 - 2)} - c_m \left(\frac{1}{\theta_0 c_m} \right)^{\theta_0/(\theta_0 - 2)} \right\} \lambda^{2/(2-\theta_0)},$$

and therefore I_λ verifies (J_2). Since $2/(2-\theta_0) < 0$, we have that $\lim_{\lambda \to +\infty} A_m = 0$. Hence, there exists $\Lambda_m > 0$ such that $A_m < (2\pi)/\alpha_0$ for any $\lambda > \Lambda_m$. In view of Proposition 3.3, for any $\lambda > \Lambda_m$ we can apply Theorem 4.3 to obtain m pairs of non-zero critical points of I_λ. This finishes the proof. □

4.3. Proof of Proposition 4.2

We devote this section to the proof of Proposition 4.2 by supposing, once again, that $\lambda = 1$ and $I = I_1$. The general case follows from obvious modifications.

We consider a small modification to the sequence of scaled truncated Green functions considered by Moser (see [**12**]). More specifically, we define for $n > 1$,

$$\tilde{M}_n(x) := \frac{1}{\sqrt{2\pi \log n}} \begin{cases} k\left(\dfrac{r}{n}\right)^{-1/2} \log n & \text{if } |x| \leqslant r/n, \\ k(x)^{-1/2} \log\left(\dfrac{r}{|x|}\right) & \text{if } r/n \leqslant |x| < r, \\ 0 & \text{if } |x| \geqslant r, \end{cases}$$

with $r > 0$ fixed. Notice that $\tilde{M}_n \in H^1(\mathbb{R}^2)$ and $\text{supp}(\tilde{M}_n) = \bar{B}_r(0)$. Moreover, the following lemma holds.

Lemma 4.4. *There exist $D = D(r) > 0$ and a sequence $(d_n) \subset \mathbb{R}$, which also depends on r, such that*

$$\|\tilde{M}_n\|^2 = 1 + \frac{D}{\log n} - d_n,$$

with $\lim_{n\to\infty} d_n \log n = 0$. In particular,

$$\lim_{n\to\infty} \|\tilde{M}_n\|^2 = 1. \tag{4.5}$$

Proof. We set

$$A_n := B_r(0) \setminus B_{r/n}(0)$$

and notice that $\nabla \tilde{M}_n$ is zero outside the set A_n, and

$$\nabla \tilde{M}_n(x) = -\mathrm{e}^{-|x|^2/8}(2\pi \log n)^{-1/2}\left(\frac{x}{|x|^2} + \tfrac{1}{4}x\log(r/|x|)\right), \quad x \in A_n.$$

Hence, we can compute

$$\begin{aligned}\int k(x)|\nabla \tilde{M}_n|^2 &= \frac{1}{2\pi \log n}\int_{A_n}\left(\frac{1}{|x|^2} + \frac{|x|^2}{16}\log^2(r/|x|) + \tfrac{1}{2}\log(r/|x|)\right)\mathrm{d}x \\ &= \frac{1}{\log n}\int_{r/n}^{r}\left(\frac{1}{s} + \frac{s^3}{16}\log^2(r/s) + \tfrac{1}{2}s\log(r/s)\right)\mathrm{d}s \\ &= \frac{1}{\log n}\left(\log n + \frac{r^2}{8} + \frac{r^4}{512} - \Gamma_{r,n,1} - \Gamma_{r,n,2}\right)\end{aligned}$$

with

$$\Gamma_{r,n,1} := \frac{r^2}{8}\left(\frac{2\log n}{n^2} + \frac{1}{n^2}\right), \qquad \Gamma_{r,n,2} := \frac{r^4}{512}\left(\frac{8\log^2 n}{n^4} + \frac{4\log n}{n^4} + \frac{1}{n^4}\right).$$

If we now set

$$D := \frac{r^2}{8} + \frac{r^4}{512}, \qquad d_n := (\log n)^{-1}(\Gamma_{r,n,1} + \Gamma_{r,n,2}), \tag{4.6}$$

we get the conclusions of the lemma. □

We now normalize the Green function and consider the function M_n defined by

$$M_n := \frac{\tilde{M}_n}{\|\tilde{M}_n\|}.$$

Since $\|M_n\| = 1$, Proposition 4.2 is a direct consequence of the following lemma.

Lemma 4.5. *Suppose that f satisfies (f_2) and (f_4). Then there exists $n \in \mathbb{N}$ such that*

$$\max_{t\geqslant 0}\left\{\frac{t^2}{2} - \int k(x)F(tM_n)\right\} < \frac{2\pi}{\alpha_0}. \tag{4.7}$$

Proof. Since M_n has compact support, we can argue as in the proof of Theorem 1.1 to conclude that the function

$$g_n(t) := \frac{t^2}{2} - \int k(x)F(tM_n), \quad t \geqslant 0,$$

goes to $-\infty$ as $t \to +\infty$. Hence, it attains its global maximum at a point $t_n > 0$ such that $g_n'(t_n) = 0$, that is,

$$t_n^2 = \int_{B_r(0)} k(x) t_n M_n f(t_n M_n)\,\mathrm{d}x. \tag{4.8}$$

Suppose, by contradiction, that the lemma is false. Then for any $n \in \mathbb{N}$ it holds that

$$\frac{t_n^2}{2} - \int k(x)F(t_n M_n) \geqslant \frac{2\pi}{\alpha_0},$$

and therefore

$$t_n^2 \geqslant \frac{4\pi}{\alpha_0} \quad \text{for all } n \in \mathbb{N}. \tag{4.9}$$

We claim that $(t_n) \subset \mathbb{R}$ is bounded. Indeed, let $\beta_0 > 0$ be given by (f_4) and let $0 < \varepsilon < \beta_0$. Condition (f_4) provides $R = R(\varepsilon) > 0$ such that

$$sf(s) \geqslant (\beta_0 - \varepsilon)\exp(\alpha_0 s^2) \quad \text{for all } |s| \geqslant R. \tag{4.10}$$

The definition of M_n, (4.9) and $\|\tilde{M}_n\| \to 1$ imply that, for any large values of n, it holds that

$$t_n M_n(x) \geqslant \mathrm{e}^{-r^2/8}\sqrt{\frac{2\log n}{\alpha_0}} \geqslant R \quad \text{for all } x \in B_{r/n}(0).$$

It follows from (4.8), (4.10), the above expression, $k(x) \geqslant 1$ and the definition of M_n that

$$\begin{aligned} t_n^2 &\geqslant \int_{B_{r/n}(0)} k(x) t_n M_n f(t_n M_n)\,\mathrm{d}x \\ &\geqslant (\beta_0 - \varepsilon)\int_{B_{r/n}(0)} \exp(\alpha_0 (t_n M_n)^2)\,\mathrm{d}x \\ &= (\beta_0 - \varepsilon)\int_{B_{r/n}(0)} \exp\left(\alpha_0 t_n^2 \frac{\log n}{2\pi}\frac{\mathrm{e}^{-r^2/(4n^2)}}{\|\tilde{M}_n\|^2}\right)\mathrm{d}x. \end{aligned}$$

This, (4.5), the equation $1/n^2 = \exp(-2\log n)$ and direct calculation provide

$$t_n^2 \geqslant (\beta_0 - \varepsilon)\pi r^2 \exp\left(2\left(\frac{\mathrm{e}^{-r^2/(4n^2)}}{\|\tilde{M}_n\|^2}\frac{\alpha_0}{4\pi}t_n^2 - 1\right)\log n\right). \tag{4.11}$$

Since $\exp(s) \geqslant s$, we can invoke Lemma 4.4 and the above expression to conclude that (t_n) is bounded.

By going to a subsequence, we may use (4.9) to get $t_n^2 \to \gamma \geqslant 4\pi/\alpha_0$. Since

$$\mathrm{e}^{-r^2/(4n^2)\|\tilde{M}_n\|^{-2}} \to 1,$$

we can take the limit in (4.11) to conclude that $\gamma > 4\pi/\alpha_0$ cannot occur. Hence,

$$\lim_{n\to\infty} t_n^2 = \frac{4\pi}{\alpha_0}. \tag{4.12}$$

By using (4.9) and (4.11) again, we get

$$t_n^2 \geqslant (\beta_0 - \varepsilon)\pi r^2 \exp\left(\frac{-2}{\|\tilde{M}_n\|^2}(\|\tilde{M}_n\|^2 - \mathrm{e}^{-r^2/(4n^2)})\log n\right). \tag{4.13}$$

It follows from Lemma 4.4 and L'Hopital's rule that

$$(\|\tilde{M}_n\|^2 - \mathrm{e}^{-r^2/(4n^2)})\log n = (1 - \mathrm{e}^{-r^2/(4n^2)})\log n + D - d_n \log n = D + o_n(1).$$

Hence, recalling that $\|\tilde{M}_n\|^2 \to 1$, we can take the limit in (4.13) and use (4.12) to obtain

$$\frac{4\pi}{\alpha_0} \geqslant (\beta_0 - \varepsilon)\pi r^2 \mathrm{e}^{-2D}.$$

Letting $\varepsilon \to 0$ and using the expression of $D = D(r)$ given in (4.6), we conclude that

$$\alpha_0\beta_0 \leqslant \frac{4}{r^2}\exp\left(\frac{r^2}{4} + \frac{r^4}{256}\right).$$

Since $r > 0$ is arbitrary, the above expression contradicts (f_4) and the lemma is proved. □

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