# MULTIPLE SOLUTIONS FOR A KIRCHHOFF EQUATION WITH NONLINEARITY HAVING ARBITRARY GROWTH

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Abstract. We prove the existence of infinitely many solutions for the Kirchhoff equation

$$-\left(\alpha+\beta\int_\Omega |\nabla u|^2 dx\right)\Delta u = a(x)|u|^{q-1}u + \mu f(x,u), \text{ in } \Omega,$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $a(x)$ is a (possibly) sign-changing potential, $0 < q < 1$, $\alpha > 0$, $\beta \geq 0$, $\mu > 0$ and the function $f$ has arbitrary growth at infinity. In the proofs we apply variational methods together with a truncation argument.

## 1. Introduction

In this paper we consider a version of the problem

$$-\left(\alpha+\beta\int_\Omega |\nabla u|^2 dx\right)\Delta u = h(x,u), \quad \text{in } \Omega, \quad u \in H_0^1(\Omega),$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $\alpha > 0$ and $\beta \geq 0$. It is the stationary state of the hyperbolic equation

$$v_{tt} - \left(\alpha+\beta\int_\Omega |\nabla v|^2 dx\right)\Delta_x v = h(x,v), \quad \text{in } \Omega \times (0,T),$$

which was proposed, for $N = 1$, by Kirchhoff [10] as an extension of the classical d'Alembert wave equation for free vibrations of elastic strings. The main point in this model is that it allows changes on the length of the string during the vibration. Such type of problems are called nonlocal due to presence of the term $\int_\Omega |\nabla v|^2 dx$. After the paper of Lions [11], this kind of problem has been the subject of intensive research. In [2], the authors presented a variational approach to deal with the stationary equation. Since then, many authors applied Critical Point Theory to obtain existence and multiplicity of results for related problems.

We are interested here in the case that the right-hand side of the equation presents a sort of competition between concave and convex terms near the

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origin. More specifically, we shall consider

$$(P)\qquad \begin{cases} -\left(\alpha+\beta\displaystyle\int |\nabla u|^2 dx\right)\Delta u = a(x)|u|^{q-1}u+\mu f(x,u) \quad \text{in } \Omega,\\ u=0 \quad \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 3$ , is a bounded smooth domain, $0<q<1$, $\alpha>0$, $\beta \geq 0$ and $\mu>0$. The main assumptions on $f$ are

- $(f_0)$ $f \in C(\Omega \times \mathbb{R}, \mathbb{R})$ and there exists $\delta>0$ such that $f(x,s)$ is odd in $s$ for any $x \in \Omega$ and $|s| \leq \delta$;
- $(f_1)$ $f(x,s)=o(|s|^q)$, as $s \to 0$, uniformly in $\Omega$.

In order to introduce the regularity condition on the potential $a(x)$ we set $2^* := 2N/(N-2)$ and consider the sequence $(p_n) \subset \mathbb{R}$ defined as $p_1 := 2^*$ and

$$(1.1)\qquad p_{n+1} = \begin{cases} \dfrac{Np_n}{N-2p_n}, & \text{if } 2p_n < N,\\ p_n+1, & \text{if } \ 2p_n \geq N, \end{cases}$$

for each $n \in \mathbb{N}$. A straightforward calculation shows that $(p_n)$ is increasing and unbounded. Hence, it is well defined

$$m := \min\{n \in \mathbb{N} : 2p_n > N\}.$$

The main assumption on the potential $a(x)$ is

- $(a_0)$ $a \in L^{\sigma_q}(\Omega)$, with $\sigma_q := p_m/(1-q)$.

We denote by $H$ the Sobolev space $W_0^{1,2}(\Omega)$ endowed with the norm $\|u\| = (\int_\Omega |\nabla u|^2 \mathrm{d}x)^{1/2}$. From the variational point view, the equation in $(P)$ is the Euler-Lagrange equation of the energy functional

$$(1.2)\qquad I(u) = \frac{\alpha}{2}\|u\|^2 + \frac{\beta}{4}\|u\|^4 - \frac{1}{q+1}\int a(x)|u|^{q+1}\mathrm{d}x - \mu\int F(x,u)\mathrm{d}x\ ,$$

where $F(x,s) := \int_0^s f(x,t)\mathrm{d}t$. Since we have no control on the behaviour of $f$ at infinity, this functional is not well defined in the entire space $H$. However, in view of $(f_0)-(f_1)$, it is finite for every function $u \in H \cap L^\infty(\Omega)$ such that $\|u\|_{L^\infty(\Omega)}$ is sufficiently small.

In our first result we consider the definite case and prove the following.

**Theorem 1.1.** *Suppose that $0<q<1$, the function $f$ satisfies $(f_0)-(f_1)$ and the potential $a$ satisfies $(a_0)$ and*

- $(a_1)$ *there exists $a_0>0$ such that $a(x) \geq a_0$, for a.e. $x \in \Omega$.*

*Then, for any $\alpha>0$, $\beta \geq 0$ and $\mu>0$, the problem $(P)$ has a sequence of solutions $(u_k) \subset W_0^{1,2}(\Omega)$ such that $\|u_k\|_{L^\infty(\Omega)} \to 0$ as $k \to \infty$. Moreover, $I(u_k)<0$ and $I(u_k) \to 0$ as $k \to \infty$.*

We emphasize that the theorem holds independently of the growth of $f$ far way the origin. In order to be able to deal variationally, we use an argument borrowed from [12]. It consists in considering a modified functional $J_\theta$, defined in the entire space $H$, whose critical points with small $L^\infty$-norm are weak solutions of $(P)$. After obtaining infinitely many critical points for $J_\theta$, we use some kind of iteration regularity process to prove that these solutions go to zero in $L^\infty(\Omega)$.

In our second result we consider potentials which can be indefinite in sign. More specifically, we shall prove

**Theorem 1.2.** *Suppose that $0 < q < 1$, the function $f$ satisfies $(f_0) - (f_1)$ and the potential $a$ satisfies $(a_0)$ and*

$(\widetilde{a_1})$ *there exists $a_0 > 0$ and an open set $\widetilde{\Omega} \subset \Omega$ such that $a(x) \geq a_0$, for a.e. $x \in \widetilde{\Omega}$.*

*Then the problem $(P)$ has a sequence of solutions $(u_k) \subset W_0^{1,2}(\Omega)$ such that $I(u_k) < 0$ and $I(u_k) \to 0$ as $k \to \infty$, in each of the following cases*

(i) *$\alpha > 0$, $\beta \geq 0$ and $\mu \in (0, \mu^*)$, for some $\mu_* > 0$;*
(ii) *$\beta \geq 0$, $\mu > 0$ and $\alpha \in (\alpha^*, \infty)$, for some $\alpha^* > 0$.*

In our final result we present a version of Theorem 1.2 with no restriction on the size of the parameters. In this case, we need to replace the condition $(f_1)$ by a stronger one. The last result of the paper can be stated as follows.

**Theorem 1.3.** *Suppose that $0 < q < 1$, the function $f$ satisfies $(f_0)$ and*

$(\widetilde{f_1})$ *$f(x,s) = o(|s|)$, as $s \to 0$, uniformly in $\Omega$,*

*and the potential $a$ satisfies $(a_0)$ and $(\widetilde{a_1})$. Then the same conclusion of Theorem 1.1 holds.*

We recall that, in their celebrated paper [3], Ambrosetti, Brezis and Cerami studied the problem

$$-\Delta u = \lambda |u|^{q-2}u + |u|^{p-2}u \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

with $1 < q < 2$ and $2 < p < 2^*$. Among other results, the existence of two positive solutions is obtained for $\lambda > 0$ small. After this work, many authors have considered the effect of concave-convex terms in Dirichlet problems. Since it is impossible to give a complet list of references, we just cite the results which are closely related to ours.

In [12], the author considered the local case $\alpha = 1$, $\beta = 0$, $\mu = 1$ and $a(x) = \lambda > 0$. Under the conditions $(f_0) - (f_1)$, he obtained the existence of infinitely many solutions as in Theorem 1.1. The same result was proved in [8], by assuming that $a \in C(\overline{\Omega})$ has nonzero positive part and $f$ verifies $(\widetilde{f_1})$ instead of $(f_1)$. For the nonlocal problem, we can cite the paper [6], where the authors considered a more general nonlocal term, $a(x) \equiv \lambda > 0$, $\mu = 1$ and $f(x,s) = |s|^{p-1}s$, with $1 < p \leq (N+2)/(N-2)$, and obtained infinitely many solutions for low dimension $N \leq 3$ and some technical conditions on the size of the parameters $\lambda$ and $\beta$. We also refer to [4, 13] for some related results.

The main theorems of this paper extend and complement the aforementioned works in several senses: differently from [12, 8], we consider the case

$\beta > 0$; our potential $a(x)$ can be nonconstant, nonsmooth and indefinite in sign; there is no restrictions on the dimension; there is no restriction on the size of $\beta$. It is worthwhile to mention that, even in the local case $\beta = 0$, our results seem to be new. Finally, we notice that the same results hold for $N = 1$ and $N = 2$. In this case, it is sufficient to consider $2^* = +\infty$ and choose $p_1 \in (1, +\infty)$ in a arbitrary way.

The rest of the paper is organized as follows: in the next section, after presenting some auxiliary results, we prove the first two theorems. In the final section 3, we prove Theorem 1.3.

## 2. The case $f(x, s) = o(|s|^q)$

For any $u \in L^1(\Omega)$, we write only $\int u$ to denote $\int u(x)\mathrm{d}x$. If $1 \le p \le \infty$, $\|u\|_p$ stands for the $L^p(\Omega)$-norm of the function $u \in L^p(\Omega)$. Hereafter, we assume that conditions $(a_0)$ and $(f_0)$ hold.

Let $H$ be the Sobolev space $W_0^{1,2}(\Omega)$ endowed with the norm

$$\|u\| := \left( \int |\nabla u|^2 \right)^{1/2}.$$

As quoted in the introduction, the functional $I$ given in (1.2) is not well defined in $H$. In order to overcome this difficult we use a truncation argument. So, we start by presenting a version of [12, Lemma 2.3].

**Lemma 2.1.** *Suppose that $f$ satisfies $(f_1)$. Then, for any given $\theta > 0$, there exist $0 < \xi < \delta/2$ and $g \in C(\Omega \times \mathbb{R}, \mathbb{R})$ odd in the second variable such that*

$(g_1)$ $g(x, s) = f(x, s),\ \forall\ (x, s) \in \Omega \times [-\xi, \xi]$.

*Moreover, if $G(x, s) := \int_0^s g(x, t)\mathrm{d}t$ then, for any $(x, s) \in \Omega \times \mathbb{R}$, there hold*

$(g_2)$ $g(x, s)s - 2G(x, s) \le \theta |s|^{q+1}$;

$(g_3)$ $g(x, s)s - (q + 1)G(x, s) \le \theta |s|^{q+1}$;

$(g_4)$ $|G(x, s)| \le \dfrac{\theta}{2} |s|^{q+1}$;

$(g_5)$ $|g(x, s)| \le \theta |s|^q$.

*Proof.* Given $0 < \varepsilon < \theta/14$, we obtain from $(f_1)$ a number $0 < \xi < \delta/2$ such that

$$\max\{|F(x, s)|, |f(x, s)s|\} \le \varepsilon |s|^{q+1}, \quad \forall\ (x, s) \in \Omega \times [-2\xi, 2\xi].$$

Let $\rho \in C^1(\mathbb{R}, [0, 1])$ be an even function satisfying, for any $s \in \mathbb{R}$,

$$\rho \equiv 1 \text{ in } [-\xi, \xi], \quad \rho \equiv 0 \text{ in } \mathbb{R} \setminus (-2\xi, 2\xi), \quad |\rho'(s)| \le 2/\xi, \quad \rho'(s)s \le 0,$$

pick $0 < \gamma < \theta/12$, consider $F_\infty(s) := \gamma |s|^{q+1}$ and define the function $g \in C(\Omega \times \mathbb{R}, \mathbb{R})$ by setting

$$g(x, s) := \rho'(s)F(x, s) + \rho(s)f(x, s) + (1 - \rho(s))F'_\infty(s) - \rho'(s)F_\infty(s).$$

A straightforward calculation shows that, for any $(x, s) \in \Omega \times \mathbb{R}$, we have that

$$G(x, s) = \rho(s)F(x, s) + (1 - \rho(s))F_\infty(s).$$

Using the properties of $\rho$, it is easy to see that $g$ is continuous, odd in the second variable, and verifies $(g_1)$, $(g_4)$ and $(g_5)$. In order to prove $(g_2)$, notice that

$$\begin{aligned} g(x,s)s - 2G(x,s) &= s\rho'(s)F(x,s) + s\rho(s)f(x,s) + s(1-\rho(s))F'_\infty(s) \\ &\quad - s\rho'(s)F_\infty(s) - 2\rho(s)F(x,s) - 2(1-\rho(s))F_\infty(s). \end{aligned}$$

Recalling that $sF'_\infty(s) = \gamma(q+1)|s|^{q+1}$ we get, for $|s| \le 2\xi$,

$$\begin{aligned} g(x,s)s - 2G(x,s) &\le 2\xi\frac{2}{\xi}|F(x,s)| + |sf(x,s)| + \gamma(q+1)|s|^{q+1} \\ &\quad + 2\xi\frac{2}{\xi}\gamma|s|^{q+1} + 2|F(x,s)| \\ &\le 6|F(x,s)| + |sf(x,s)| + 6\gamma|s|^{q+1} \\ &\le (7\varepsilon + 6\gamma)|s|^{q+1} < \theta|s|^{q+1}\ . \end{aligned}$$

On the other hand, for $|s| > 2\xi$, we have that

$$g(x,s)s - 2G(x,s) = sF'_\infty(s) - 2F_\infty(s) < sF'_\infty(s) < \theta|s|^{q+1}\ .$$

So, we conclude that $(g_2)$ holds. The property $(g_3)$ can be proved with an analogous argument. □

For any $\theta > 0$, it follows from $(g_4) - (g_5)$ that the functional

$$J_\theta(u) := \frac{\alpha}{2}\|u\|^2 + \frac{\beta}{4}\|u\|^4 - \frac{1}{q+1}\int a(x)|u|^{q+1} - \mu\int G(x,u)\ ,$$

belongs to $C^1(H,\mathbb{R})$ and, for any $u$, $v \in H$, we have that

$$J'_\theta(u)v = (\alpha + \beta\|u\|^2)\int(\nabla u\cdot\nabla v) - \int a(x)|u|^{q-1}uv - \mu\int g(x,u)v.$$

Thus, if $u \in H \cap L^\infty(\Omega)$ is such that $\|u\|_\infty < \xi$, it follows from $(g_1)$ that $g(x,u(x)) = f(x,u(x))$ a.e. in $\Omega$. We then conclude that any critical point of $J_\theta$ with small $L^\infty$-norm is a weak solution of $(P)$.

Now, we prove a technical result.

**Lemma 2.2.** *Suppose that the functions $a$ and $f$ satisfy $(a_1)$ and $(f_1)$, respectively. If*

$$0 < \theta < \frac{(1-q)a_0}{(1+q)\mu}, \tag{2.1}$$

*then $J_\theta(u) = J'_\theta(u)u = 0$ if, and only if, $u = 0$.*

*Proof.* It is obvious that $J_\theta(0) = J'_\theta(0)0 = 0$ independently of $\theta$. On the other hand, if $J_\theta(u) = J'_\theta(u)u = 0$, then we can use $(a_1)$, $J'_\theta(u)u - 2J_\theta(u) = 0$

and $(g_2)$ to get

$$\begin{aligned}\frac{(1-q)a_0}{q+1}\int|u|^{q+1} &\le \frac{\beta}{2}\|u\|^4+\frac{1-q}{q+1}\int a(x)|u|^{q+1}\\ &= \mu\int(g(x,u)u-2G(x,u))\\ &\le \mu\theta\int|u|^{q+1}.\end{aligned}$$

The above inequality and (2.1) imply that $u=0$. □

Notice that the positivity of the potential $a$ was essential in the above proof. If we are in the setting of the local condition $(\widetilde{a_1})$, we can obtain a priori estimates for the functions verifying $J_\theta(u)=J_\theta'(u)u=0$. More precisely, if we set

$$S_{q+1}:=\inf_{\substack{u\in H\\ u\neq 0}}\frac{\int|\nabla u|^2}{\left(\int|u|^{q+1}\right)^{\frac{2}{q+1}}}>0,$$

we have the following.

**Lemma 2.3.** *Suppose that the functions $a$ and $f$ satisfy $(\widetilde{a_1})$ and $(f_1)$, respectively. If*

$$0<\theta<\frac{(1-q)}{2}S_{q+1}^{(q+1)/2}, \tag{2.2}$$

*then $J_\theta(u)=J_\theta'(u)u=0$ implies $\|u\|\le(\mu/\alpha)^{1/(1-q)}$.*

*Proof.* If $J_\theta(u)=J_\theta'(u)u=0$, then the equality $J_\theta'(u)u-(q+1)J_\theta(u)=0$ implies

$$\frac{\alpha(1-q)}{2}\|u\|^2+\frac{\beta(3-q)}{4}\|u\|^4=\mu\int(g(x,u)u-(q+1)G(x,u)). \tag{2.3}$$

It follows from $(g_3)$, the embedding $H\hookrightarrow L^{q+1}(\Omega)$ and (2.2), that

$$\frac{\alpha(1-q)}{2}\|u\|^2\le\mu\theta S_{q+1}^{-(q+1)/2}\|u\|^{q+1}\le\frac{\mu(1-q)}{2}\|u\|^{q+1}$$

and the result follows. □

**Lemma 2.4.** *For any $\theta>0$ the functional $J_\theta$ is coercive and satisfies the Palais-Smale condition.*

*Proof.* Since the sequence $(p_n)$ defined in the introduction is increasing, we have that

$$\sigma_q=\frac{p_m}{1-q}\ge\frac{2^*}{1-q}>\frac{2^*}{2^*-(q+1)}=\left(\frac{2^*}{q+1}\right)', \tag{2.4}$$

and therefore $1 < \sigma_q'(q+1) < 2^*$. Hence, we can use Hölder's inequality, $(g_4)$ and the Sobolev embeddings to get

$$\begin{aligned} J_\theta(u) &\geq \frac{\alpha}{2}\|u\|^2 + \frac{\beta}{4}\|u\|^4 - \frac{1}{q+1}\|a\|_{\sigma_q}\|u\|_{\sigma_q'(q+1)}^{q+1} - \frac{\mu\theta}{2}\|u\|_{q+1}^{q+1} \\ &\geq \frac{\alpha}{2}\|u\|^2 + \frac{\beta}{4}\|u\|^4 - C\|u\|^{q+1} \ , \end{aligned}$$

for some constant $C > 0$. Recalling that $(q+1) < 2$, we conclude that $J_\theta(u) \to \infty$ if $\|u\| \to +\infty$, that is, $J_\theta$ is coercive.

Suppose now that $(u_n) \subset H$ is such that $J_\theta(u_n) \to c$ and $J_\theta'(u_n) \to 0$. By the above considerations, $(u_n)$ is bounded. Hence, up to a subsequence, for some $A \geq 0$ and $u \in H$, we have that

$$\|u_n\| \to A, \quad u_n \rightharpoonup u \text{ weakly in } H, \quad u_n \to u \text{ strongly in } L^p(\Omega),$$

for any $p \in [1, 2^*)$. By (2.4), there exists $p_0 \in (q+1, 2^*)$ such that $\sigma_q = (\frac{p_0}{q+1})'$. Hölder's inequality and the above convergences provide

$$\left| \int a(x)|u_n|^{q-1}u_n(u_n - u) \right| \leq \|a\|_{\sigma_q}\|u_n\|_{p_0}^q \|u_n - u\|_{p_0} \to 0 \ ,$$

as $n \to \infty$. Moreover, by $(g_5)$ and Hölder's inequality again

$$\left| \int g(x, u_n)(u_n - u) \right| \leq \theta \|u_n\|_{q+1}^q \|u_n - u\|_{q+1} \to 0 \ .$$

Thus

$$o_n(1) = J_\theta'(u_n)(u_n - u) = (\alpha + \beta\|u_n\|^2)\left(\|u_n\|^2 - \int \nabla u_n \cdot \nabla u\right) + o_n(1) \ .$$

Taking the limit we obtain $(\alpha + \beta A^2)(A^2 - \|u\|^2) = 0$, which implies that $\|u\| = A$. It follows from the weak convergence that $u_n \to u$ strongly in $H$. □

In order to prove our result we shall apply the following variant of a result due to Clark [5] (see [9, Theorem 2.1, Proposition 2.2]).

**Theorem 2.1.** *Let $X$ be a Banach space and $J \in C^1(X, \mathbb{R})$ an even functional bounded from below which satisfies the Palais-Smale condition and $J(0) = 0$. If, for each $k \in \mathbb{N}$, there exists a $k$-dimensional subspace $X^k \subset X$ and $\rho_k > 0$ such that*

$$\sup_{u \in X^k \cap S_{\rho_k}} J(u) < 0 \ , \tag{2.5}$$

*where $S_\rho := \{u \in X : \|u\|_X = \rho\}$, then $J$ has a sequence of critical values $(c_k) \subset (-\infty, 0)$ such that $c_k \to 0$ as $k \to +\infty$.*

We are ready to prove our first result.

*Proof of Theorem 1.1.* Let $\xi > 0$ be given by Lemma 2.1 with $\theta > 0$ satisfying (2.1). As quoted before, any critical point of $J_\theta$ such that $\|u\|_\infty <$

$\xi$ is a weak solution of $(P)$. We are going to apply the last abstract result for this modified functional and show that the obtained solutions have small $L^\infty$-norm.

It is clear that the even functional $J_\theta$ satisfies $J_\theta(0) = 0$. Moreover, by Lemma 2.4, it also satisfies the Palais-Smale condition. Since $J_\theta$ is bounded on bounded sets of $H$, we also conclude from the same lemma that $J_\theta$ is bounded from below.

For any given $k \in \mathbb{N}$, we set $X^k := \mathrm{span}\{\phi_1, \dots, \phi_k\}$, where $\{\phi_n\}_{n\in\mathbb{N}}$ is a Hilbertian basis of $H$. Since all the norm in $X^k$ are equivalent, we can use $0 < \theta < \frac{(1-q)a_0}{(1+q)\mu} < \frac{a_0}{(1+q)\mu}$, $(a_1)$ and $(g_4)$ to get, for any $u \in X^k$,

$$
\begin{aligned}
J_\theta(u) &\leq \frac{\alpha}{2}\|u\|^2 + \frac{\beta}{4}\|u\|^4 - \frac{a_0}{q+1}\|u\|_{q+1}^{q+1} + \frac{\mu}{2}\theta\|u\|_{q+1}^{q+1} \\
&\leq \frac{\alpha}{2}\|u\|^2 + \frac{\beta}{4}\|u\|^4 - \frac{a_0}{2(q+1)}C\|u\|^{q+1} \quad ,
\end{aligned}
\tag{2.6}
$$

for some constant $C > 0$ independent of $u$. Recalling that $(q+1) < 2$, we can choose $\rho_k > 0$ small in such way that $J_\theta$ verifies (2.5).

Theorem 2.1 provides a sequence of critical points $(u_k) \subset H$ such that $J_\theta(u_k) = c_k \to 0$ as $k \to \infty$. Since $(u_k)$ is a Palais-Smale sequence at level $c = 0$, by Lemma 2.4, we may suppose that $u_k \to u$ strongly in $H$. Hence, $J_\theta(u) = J'_\theta(u)u = 0$ and we infer from Lemma 2.2 that $u = 0$, that is, $u_k \to 0$ strongly in $H$.

Notice that each function $u_k$ is a weak solution of

$$
-\Delta u = \frac{a(x)|u_k(x)|^{q-1}u_k(x) + \mu g(x, u_k(x))}{\alpha + \beta\|u_k\|^2} \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega.
$$

If we denote by $h_k$ the right-hand side of the first equation above we have, by $(g_5)$, $|h_k(x)| \leq \alpha^{-1}(|a(x)||u_k(x)|^q + \mu\theta|u_k(x)|^q)$ a.e. in $\Omega$. Hence,

$$
\int |h_k(x)|^{2^*} \leq C_1\alpha^{-2^*}\left(\int |a(x)|^{2^*}|u_k|^{q2^*} + (\mu\theta)^{2^*}\int |u_k|^{q2^*}\right),
$$

with $C_1 := 2^{2^*-1}$. Since $\sigma_q \geq 2^*/(1-q) > 2^*$, we have that

$$
\tau_q := q\left(\frac{\sigma_q}{2^*}\right)' = \frac{q\sigma_q}{\sigma_q - 2^*} \leq 1.
$$

Thus, Hölder's inequality implies that

$$
\begin{aligned}
\int |a(x)|^{2^*}|u_k|^{q2^*} &\leq \|a\|_{\sigma_q}^{2^*}\left(\int |u_k|^{\tau_q 2^*}\right)^{q/\tau_q} \\
&\leq C_2\|a\|_{\sigma_q}^{2^*}\|u_k\|_{2^*}^{q2^*}
\end{aligned}
$$

and

$$
\int |u_k(x)|^{q2^*} \leq C_3\|u_k\|_{2^*}^{q2^*} \quad ,
$$

with $C_2 := |\Omega|^{q(1-\tau_q)/\tau_q}$ and $C_3 := |\Omega|^{1-q}$. We conclude that $h_k \in L^{2^*}(\Omega)$ and therefore, by the Agmon-Douglis-Nirenberg result [1] (see also [7, Lemma 9.17]), $u_k \in W^{2,2^*}(\Omega)$ and there exists $C_4 = C_4(\Omega)$ such that

$$\|u_k\|_{W^{2,2^*}} \leq C_4 \|h_k\|_{2^*} \leq C_5 \|u_k\|_{2^*}^q \leq \widehat{C}_1 \|u_k\|^q, \tag{2.7}$$

for each $k \in \mathbb{N}$, where $C_5 = C_4\alpha^{-1}\{C_1(C_2 \|a\|_{\sigma_q}^{2^*} + C_3(\mu\theta)^{2^*})\}^{1/2^*}$ and $\widehat{C}_1 = C_5 S^{-q/2}$. Since $u_k \to 0$ strongly in $H$, we have that $\|u_k\|_{W^{2,2^*}} \to 0$. If $m = 1$, that is, $2p_1 = 2 \cdot 2^* > N$ (see the definition of $m$ and of the sequence $(p_n)$ in (1.1)), the continuous embedding $W^{2,2^*}(\Omega) \hookrightarrow C(\overline{\Omega})$ implies that $\|u_k\|_\infty \to 0$.

On the other hand, if $2 \cdot 2^* \leq N$ (equivalently, $m > 1$), the embedding $W^{2,2^*}(\Omega) \hookrightarrow L^{p_2}(\Omega)$ and (2.7) imply that, for some $C_6 = C_6(\Omega) > 0$,

$$\|u_k\|_{p_2} \leq C_6 \|u_k\|_{W^{2,2^*}} \leq C_6\widehat{C}_1 \|u_k\|^q ,$$

where $p_2$ is the second term of the sequence defined in (1.1). Furthermore, since in this case $\sigma_q \geq p_2/(1-q) > p_2$, we can argue as above to conclude that $h_k \in L^{p_2}(\Omega)$. Hence $u_k \in W^{2,p_2}(\Omega)$ and

$$\|u_k\|_{W^{2,p_2}} \leq C_7 \|u_k\|_{p_2}^q \leq \widehat{C}_2 \|u_k\|^{q^2} ,$$

where $C_7 > 0$ is independent of $k$ and $\widehat{C}_2 = C_7(C_6\widehat{C}_1)^q$.

Since $\sigma_q \geq p_n/(1-q) > p_n$ for $n = 1, \dots, m$, we can repeat this argument until we get $u_k \in W^{2,p_m}(\Omega)$ and

$$\|u_k\|_{W^{2,p_m}} \leq \widehat{C}_m \|u_k\|^{q^m} , \tag{2.8}$$

with $\widehat{C}_m > 0$ independent of $k$. Then, $\|u_k\|_{W^{2,p_m}} \to 0$ as $k \to \infty$. But $2p_m > N$ provides $W^{2,p_m}(\Omega) \hookrightarrow C(\overline{\Omega})$, and therefore we conclude that $\|u_k\|_\infty \to 0$. So, there exists $k_0 \in \mathbb{N}$ such that

$$\|u_k\|_\infty \leq \frac{\xi}{2}, \quad \forall\, k \geq k_0$$

and the theorem is proved. □

*Proof of Theorem 1.2.* Under the setting of item (i), that is, $\alpha > 0$ and $\beta \geq 0$ fixed, we suppose that $\mu \leq 1$ and choose

$$0 < \theta < \min\left\{\frac{1-q}{2} S_{q+1}^{(q+1)/2}, \frac{a_0}{1+q}\right\}.$$

In the setting of the second item ($\beta \geq 0$ and $\mu > 0$ fixed) we choose

$$0 < \theta < \min\left\{\frac{1-q}{2} S_{q+1}^{(q+1)/2}, \frac{a_0}{\mu(1+q)}\right\}.$$

In order to obtain a sequence of critical points for $J_\theta$ we argue as in Theorem 1.1. The first difference appears when we try to prove (2.5). Indeed, since $a$ is no longer positive, we need an alternative construction for the finite-dimensional subspace. For any given $k \in \mathbb{N}$, we choose $k$ linearly independents functions $\phi_1, \ldots, \phi_k \in C_0^\infty(\widetilde{\Omega})$, where the open set $\widetilde{\Omega} \subset \Omega$ comes from the condition $(\widetilde{a_1})$. Since $a(x) \geq a_0$ a.e. in $\widetilde{\Omega}$, the inequality (2.6) still holds for $u \in X^k$. Hence, there is a sequence of critical points $(u_k) \subset H$ such that $J_\theta(u_k) = c_k \to 0$ as $k \to \infty$. Again, there exists $u \in H$ such that $u_k \to u$ strongly in $H$. Though we cannot guarantee that $u = 0$, it follows from Lemma 2.3 that

$$\|u\| \leq \left(\frac{\mu}{\alpha}\right)^{1/(1-q)} . \tag{2.9}$$

As in the proof of Theorem 1.1, we have that $u_k \in W^{2,p_m}(\Omega)$ and inequality (2.8) holds. Hence, we can use the embedding $W^{2,p_m}(\Omega) \hookrightarrow C(\overline{\Omega})$ to get

$$\|u_k\|_\infty \leq C_0 \|u_k\|_{W^{2,p_m}} \leq C_0 \widehat{C}_m \|u_k\|^{q^m} ,$$

for some constant $C_0 = C_0(\Omega) > 0$. Since $\|u_k\| \to \|u\|$, it follows from (2.9) that there exists $k_0 \in \mathbb{N}$ such that

$$\|u_k\|_\infty \leq C_0 \widehat{C}_m 2^{q^m} \left(\frac{\mu}{\alpha}\right)^{q^m/(1-q)} , \quad \forall\, k \geq k_0.$$

A simple inspection of the proof of Theorem 1.1 show that the constant $\widehat{C}_m = \widehat{C}_m(\mu, \alpha)$ is directly proportional to both $\mu$ and $\alpha^{-1}$. Thus, if $\alpha > 0$ is fixed (item (i)) the $L^\infty$-norm of the function $u_k$ becomes small if $\mu$ is close to zero. On the other hand, if $\mu > 0$ is fixed (item (ii)) the same occurs if $\alpha > 0$ is large. In any case, the approximated solutions are weak solutions of $(P)$. □

## 3. The case $f(x, s) = o(|s|)$

In this section we prove Theorem 1.3. The ideas are analogous to that used in the previous section. We need only to adapt the auxiliary results.

**Lemma 3.1.** *Suppose that $f$ satisfies $(\widetilde{f_1})$. Then, for any given $\theta > 0$, there exist $0 < \xi < \delta/2$ and $g \in C(\Omega \times \mathbb{R}, \mathbb{R})$ odd in the second variable such that $(g_1)$ (see Lemma 2.1) holds. Moreover, if $G(x, s) := \int_0^s g(x, t)\mathrm{d}t$ then, for any $(x, s) \in \Omega \times \mathbb{R}$, we have that*

- $(\widetilde{g_3})$ $g(x, s)s - (q + 1)G(x, s) \leq \theta|s|^2$;
- $(\widetilde{g_4})$ $|G(x, s)| \leq \dfrac{\theta}{2}|s|^2$;
- $(\widetilde{g_5})$ $|g(x, s)| \leq \theta|s|$.

*Proof.* It follows from $(\widetilde{f_1})$ that, for any given $0 < \varepsilon < \theta/14$, there exists $0 < \xi < \delta/2$ such that

$$\max\{|F(x, s)|, |f(x, s)s|\} \leq \varepsilon|s|^2, \quad \forall\ (x, s) \in \Omega \times [-2\xi, 2\xi].$$

The argument now is analogous to that presented in the proof of Lemma 2.1. We omit the details. □

As in the previous section, for any $\theta > 0$, we consider the function $g$ given by the above lemma and define $J_\theta : H \to \mathbb{R}$ by setting

$$J_\theta(u) := \frac{\alpha}{2}\|u\|^2 + \frac{\beta}{4}\|u\|^4 - \frac{1}{q+1}\int a(x)|u|^{q+1} - \mu\int G(x,u).$$

In the next result, we denote by $\lambda_1 > 0$ the first eigenvalue of $(-\Delta, W_0^{1,2}(\Omega))$.

**Lemma 3.2.** *Suppose that the functions $a$ and $f$ satisfy $(\widetilde{a_1})$ and $(\widetilde{f_1})$, respectively. If*

$$0 < \theta < \frac{(1-q)\alpha\lambda_1}{2\mu}, \tag{3.1}$$

*then $J_\theta(u) = J'_\theta(u)u = 0$ if, and only if, $u = 0$.*

*Proof.* If $J_\theta(u) = J'_\theta(u)u = 0$, it follows from (2.3), $(\widetilde{g_3})$ and Poincaré's inequality that

$$\frac{\alpha(1-q)}{2}\|u\|^2 \le \mu\int_\Omega (g(x,u)u - (q+1)G(x,u)) \le \mu\theta\int |u|^2 \le \frac{\mu\theta}{\lambda_1}\|u\|^2.$$

The result follows from (3.1). $\square$

**Lemma 3.3.** *For any $\theta > 0$ satisfying* (3.1) *the functional $J_\theta$ is coercive and satisfies the Palais-Smale condition.*

*Proof.* As in the proof of Lemma 2.4, $1 < \sigma'_q(q+1) < 2^*$. Hence, we can use Hölder's inequality and $(\widetilde{g_4})$ to obtain

$$\begin{aligned} J_\theta(u) &\ge \frac{\alpha}{2}\|u\|^2 + \frac{\beta}{4}\|u\|^4 - \frac{1}{q+1}\|a\|_{\sigma_q}\|u\|^{q+1}_{\sigma'_q(q+1)} - \frac{\mu\theta}{2}\|u\|_2^2 \\ &\ge \frac{\alpha}{2}\|u\|^2 + \frac{\beta}{4}\|u\|^4 - \frac{1}{q+1}\|a\|_{\sigma_q}\|u\|^{q+1}_{\sigma'_q(q+1)} - \frac{(1-q)\alpha\lambda_1}{4}\|u\|_2^2 \\ &\ge \frac{(1+q)\alpha}{4}\|u\|^2 + \frac{\beta}{4}\|u\|^4 - C\|u\|^{q+1}, \end{aligned}$$

for some constant $C > 0$. It is sufficient now to argue as in the proof of Lemma 2.4. $\square$

We present now the proof of our last result.

*Proof of Theorem 1.3.* The proof is a consequence of the above lemmas and the same argument used in the proof of Theorems 1.1 and 1.2. $\square$

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