EXISTENCE OF SOLUTION FOR A GENERALIZED QUASILINEAR ELLIPTIC PROBLEM

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ABSTRACT. It is establish existence and multiplicity of solutions to the elliptic quasilinear Schrödinger equation

$$-\mathrm{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + V(x)u = h(x,u),\ x \in \mathbb{R}^N$$

where g, h, V are suitable smooth functions. The function g is asymptotically linear at infinity and, for each fixed $x \in \mathbb{R}^N$, the function $h(x,s)$ behaves like s at the origin and s^3 at infinity. In the proofs we apply variational methods.

1. Introduction

In this work we consider the quasilinear elliptic problem

$$(P) \qquad \begin{cases} -\mathrm{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + V(x)u = h(x,u),\ x \in \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N). \end{cases}$$

where $N \geq 3$, $V \in C(\mathbb{R}^N, \mathbb{R})$, $g \in C(\mathbb{R}, \mathbb{R})$ and $h \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ satisfy some suitable conditions. It is related with the existence of solitary wave solutions for the Schrödinger equation

$$i\partial_t w = -\Delta w + V(x)w - k(x,w) - l'(|w|^2)w\Delta l(|w|^2) \tag{1.1}$$

where $w : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}, V : \mathbb{R}^N \to \mathbb{R}$ is a given potential, $l : \mathbb{R} \to \mathbb{R}$ and $k : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ are fixed functions. It has been accepted as a model in many physical phenomena depending of the function l. For instance, if $l(s) = 1$ we have the classical stationary semilinear Schrödinger equation [17]. When $l(s) = s$, the equation arises from fluid mechanics, plasma physics and dissipative quantum mechanics, see [22, 21, 12, 15]. If $l(s) = \sqrt{1+s}$, the equation models the propagation of a high-irradiance laser in a plasma as well as the self-channeling of a high-power ultrashort laser in matter, see [16]. For further physical applications we also refer to [4, 14].

Generalized quasilinear elliptic problems in unbounded domains have been extensively considered in the literature (see [3, 24, 1, 18, 23, 19, 20]

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and references therein). One of the main difficulties is the lack of compactness which are inherent to elliptic problems defined in unbounded domains. Moreover, if $H(x,t) := \int_0^t h(x,\tau)\mathrm{d}\tau$, the equation in (P) is formally the Euler-Lagrange equation associated with the functional

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} \left(g(u)^2\,|\nabla u|^2 + V(x)u^2\right)\mathrm{d}x - \int_{\mathbb{R}^N} H(x,u)\mathrm{d}x.$$

Nevertheless, as quoted in [6], for some important examples of functions g this functional can take the value $+\infty$. Hence, a direct variational approach is not possible.

In order to overcame the lack of compactness, we suppose that the potential V satisfies the following:

(V_0) $V \in C(\mathbb{R}^N, \mathbb{R})$;
(V_1) $\inf_{x\in\mathbb{R}^N} V(x) \geq V_0 > 0$;
(V_2) for all $M > 0$, there holds

$$\text{measure}\left(\{x \in \mathbb{R}^N : V(x) \leq M\}\right) < +\infty.$$

The second difficult quoted above is more delicate. There are some solutions for particular cases of function g. For instance, if $g(s) = \sqrt{1+2s^2}$, the equation in (P) becomes

$$-\Delta u - u\Delta(u^2) + V(x)u = h(x,u), \qquad x \in \mathbb{R}^N. \tag{1.2}$$

It has been widely studied since the seminal papers [6, 18, 19]. In these works, the authors considered the change of variables $f : \mathbb{R} \to \mathbb{R}$ given by

$$f'(t) = \left(1 + 2f^2(t)\right)^{-1/2}, \quad f(0) = 0. \tag{1.3}$$

Under some growth restrictions on h, it provides a relation between the weak solutions of the problem and the critical points of the (well defined) functional

$$v \mapsto \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla v|^2 + V(x)f(v)^2)\mathrm{d}x - \int_{\mathbb{R}^N} H(x, f(v))\mathrm{d}x.$$

Thus, we can use all the machinery of the Critical Point Theory to obtain solutions for a large class of nonlinearities h (see [20, 7, 9, 26, 27, 28, 10, 5, 11] and their references). Although a similar approach can be done for some other particular g (see [8, 29]) there are some important functions $g : \mathbb{R} \to \mathbb{R}$ which arise from mathematical physics, biology and chemistry where is not possible to consider anymore the change of variables (1.3). We quote [25, 30] for some results with more general functions g.

In this paper, we consider a huge class of nonlinearities g, just assuming that

(g_0) $g \in C^1(\mathbb{R}, (0, +\infty))$ is even, non-decreasing in $[0, +\infty)$ and satisfies

$$g_\infty := \lim_{t\to\infty} \frac{g(t)}{t} \in (0, \infty). \tag{1.4}$$

and

$$\beta := \sup_{t\in\mathbb{R}} \frac{tg'(t)}{g(t)} \leq 1. \tag{1.5}$$

In order to correctly set our variational framework, we borrow an idea introduced in [25]. It consists in take the primitive $G(t) := \int_0^t g(\tau)\,\mathrm{d}\tau$, consider the change of variable $v = G(u)$ and set

$$X = \left\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)u^2\,\mathrm{d}x < \infty \right\}.$$

Consider also the Orlicz-Sobolev space given by

$$E := \left\{ v \in H^1(\mathbb{R}^N) : \int V(x)[G^{-1}(v)]^2 < \infty \right\}.$$

It can be proved that $J : E \to \mathbb{R}$ defined as $J(v) = I(G^{-1}(v))$, $v \in E$, is well defined and belongs to $C^1(E,\mathbb{R})$. Notice that $u = G^{-1}(v)$ belongs to X if, and only if, $v \in E$. Moreover, $v \in E$ is a critical point of J if, and only if, $u = G^{-1}(v)$ is a weak solution of (P). Hence, it is sufficient to look for critical points of J (see Section 2 for details).

In order to guarantee that the map $v \mapsto I(G^{-1}(v))$ is well defined, we need to impose some restrictions on the growth of the nonlinear term h. We suppose that

(h_0) $h \in C(\mathbb{R}^N \times \mathbb{R})$;

(h_1) there exist $\alpha_0 > N/2$ and a, $b \in L^{\alpha_0}(\mathbb{R}^N)$ such that

$$|h(x,t)| \leq a(x)|t| + b(x)g(t)|G(t)|, \text{ for all } (x,t) \in \mathbb{R}^N \times \mathbb{R};$$

(h_2) there exist $0 < q_1 < 2$, $(2 \cdot 2^*/q_1)' \leq \tau \leq (4/q_1)'$, $\Gamma_1 \in L^1(\mathbb{R}^N)$ and $\Gamma_2 \in L^\tau(\mathbb{R}^N)$ such that

$$h(x,t)t - 2(1+\beta)H(x,t) \geq -\Gamma_1(x) - \Gamma_2(x)|t|^{q_1}, \text{ for all } (x,t) \in \mathbb{R}^N \times \mathbb{R}.$$

where we define $1/s + 1/s' = 1$ for any $s > 1$.

For any $w \in L^q(\mathbb{R}^N)$ we set $w^+(x) := \max\{w(x), 0\}$, define

$$\mathcal{F} := \left\{ w : \mathbb{R}^N \to \mathbb{R} : w^+ \not\equiv 0,\, w \in L^\alpha(\mathbb{R}^N) \text{ for some } \alpha > N/2 \right\}.$$

and suppose that

(H_0) there exists $K_0 \in \mathcal{F}$ such that

$$\limsup_{t\to 0} \frac{2H(x,t)}{G^2(t)} = K_0(x), \text{ uniformly for a.e. } x \in \mathbb{R}^N;$$

(H_∞) there exists $K_\infty \in \mathcal{F}$ such that

$$\liminf_{|t|\to +\infty} \frac{2H(x,t)}{G^2(t)} = K_\infty(x), \text{ uniformly for a.e. } x \in \mathbb{R}^N.$$

In view of conditions $(V_0)-(V_2)$, the space X is compactly embedded in $L^q(\mathbb{R}^N)$, for any $2 \le q < 2N/(N-2)$ (see [13]). Hence, for any $K \in \mathcal{F}$, the eigenvalue problem

$$-\Delta u + \frac{V(x)}{g(0)^2} u = \lambda K(x) u, \quad u \in X, \tag{1.6}$$

has a first positive eigenvalue $\lambda_1(K) > 0$. The same holds for the eigenvalue problem

$$-\Delta u = \mu K(x) u, \quad u \in X. \tag{1.7}$$

In our main result we prove the existence of a non-zero solution under some kind of crossing conditions on the eigenvalues associated to the asymptotic limits K_0 and K_∞. More specifically, we prove the following:

Theorem 1.1. *Suppose that V and g satisfy $(V_0)-(V_2)$ and (g_0), respectively. Suppose also that h satisfies $(h_0)-(h_2)$, (H_0) and (H_∞). Then the problem (P) has at least one non-zero solution, provided*

$$\mu_1(K_\infty) < 1 < \lambda_1(K_0).$$

Actually, as a product of our calculation and using an usual truncation argument, we are able to obtain a multiplicity result as stated in the next result.

Theorem 1.2. *Under the same hypotheses of Theorem 1.1 the problem (P) has at least two solutions. One of them is positive and the another one is negative.*

The above results complement the aforementioned works in two senses. First, because we consider a different class of functions g. Secondly, because in most of the papers the authors considered superlinear nonlinearities h. Here, we consider some sort of asymptotically linear conditions on h and use the nonquadraticity condition (h_2) to assure the existence of a solution. Notice that, in this condition, we have that $2(1+\beta) \le 4$, differently from the superlinear case where this constant is bigger than 4. It is worthwhile to mention that, if (1.4) holds and the function $g'(t)$ has limit at $+\infty$, then then number β is equal to 1, and therefore we have the limit case $2(1+\beta) = 4$.

Besides the generality of the function g, the main novel here is to consider the ratio $h(\cdot, s)/s^3$ being bounded as $s \to +\infty$. As far we know, this is the first result where functions $h(\cdot, s)$ behaving like s^3 at infinity are considered for the general equation in (P). As it is well known, even if you have compact embeddings on the Lebesgue spaces, this kind of problem present many difficulties in the proof of the compactness properties required by the usual minimax theorems. We believe that the proof of the Palais-Smale condition contained in Proposition 3.1 can be used in many other variations of the problem. The main difficult there is that, differently from [25], our potential V is unbounded, and therefore our working space E is a proper subset of $H^1(\mathbb{R}^N)$ from which we have no information about reflexivity

Throughout this work we use the eigenvalues problems (1.6) and (1.7). In our context is natural to consider different eigenvalues problems due the fact that G^{-1} behaves like s at the origin and $\sqrt{|s|}$ at infinity. Furthermore, if we consider the parametric eigenvalue problem

$$-\Delta u+\frac{V(x)}{g(s)^2}u=\lambda K(x)u,\quad u\in X,$$

we have that, for $s=0$, it becomes (1.6). On the other hand, if $s=+\infty$, we have formally the equation in (1.7), since (1.4) implies that $g(s)\to+\infty$, as $s\to+\infty$. In other words, there is no effect of the potential V in the linearized problem at infinity.

The paper is organized as follows: in the next section we present the variational framework to deal with the problem as well as the main properties of the function g. In Section 3, we prove the Palais-Smale condition and the final Section 4 is devoted to the proof of the main results.

2. The variational framework

Hereafter, we write $\int u$ instead of $\int_{\mathbb{R}^N}u(x)\mathrm{d}x$ and denote by $\|\cdot\|_p$ the $L^p(\mathbb{R}^N)$-norm, for $p\geq 1$.

As quoted in the introduction, the problem (P) is formally the Euler-Lagrange equation associated with the functional

$$u\mapsto\frac{1}{2}\int g(u)^2\,|\nabla u|^2+\frac{1}{2}\int V(x)u^2-\int H(x,u). \tag{2.1}$$

Since it is not well defined in $H^1(\mathbb{R}^N)$, we shall follow [25] and use the change of variables $v=G(u)$, where the function G is defined as $G(t):=\int_0^t g(\tau)\mathrm{d}\tau$. For an easy reference we list below the main properties of the function G^{-1}. They will be extensively used in the rest of the paper.

Lemma 2.1. *The function $G^{-1}\in C^2(\mathbb{R},\mathbb{R})$ satisfies the following properties:*

- (g_1) G^{-1} *is increasing;*
- (g_2) $0<\frac{d}{dt}\left(G^{-1}(t)\right)=\frac{1}{g(G^{-1}(t))}\leq\frac{1}{g(0)}$, *for all* $t\in\mathbb{R}$;
- (g_3) $\left|G^{-1}(t)\right|\leq\frac{|t|}{g(0)}$, *for all* $t\in\mathbb{R}$;
- (g_4) $\lim\limits_{t\to 0}\frac{G^{-1}(t)}{t}=\frac{1}{g(0)}$;
- (g_5) $\lim\limits_{t\to\pm\infty}\frac{G^{-1}(t)}{g(G^{-1}(t))}=\pm\frac{1}{g_\infty}$;
- (g_6) $1\leq\frac{tg(t)}{G(t)}\leq 2$ *and* $1\leq\frac{G^{-1}(t)\,g\left(G^{-1}(t)\right)}{t}\leq 2$, *for all* $t\neq 0$;
- (g_7) $\frac{G^{-1}(t)}{\sqrt{t}}$ *is non-decreasing in* $(0,+\infty)$ *and* $|G^{-1}(t)|\leq(2/g_\infty)^{1/2}\sqrt{|t|}$, *for all* $t\in\mathbb{R}$;
- (g_8) $\left|G^{-1}(t)\right|\geq\begin{cases}G^{-1}(1)\,|t|, & \text{for all } |t|\leq 1,\\ G^{-1}(1)\sqrt{|t|}, & \text{for all } |t|\geq 1;\end{cases}$

(g_9) $\frac{t}{g(t)}$ *is increasing and* $\left|\frac{t}{g(t)}\right| \leq \dfrac{1}{g_\infty}$*, for all* $t \in \mathbb{R}$*;*

(g_{10}) *the function* $[G^{-1}(t)]^2$ *is convex. In particular,* $[G^{-1}(st)]^2 \leq s[G^{-1}(t)]^2$*, for all* $t \in \mathbb{R}$*,* $s \in [0,1]$*;*

(g_{11}) $[G^{-1}(st)]^2 \leq s^2[G^{-1}(t)]^2$*, for all* $t \in \mathbb{R}$*,* $s \geq 1$*;*

(g_{12}) $[G^{-1}(s-t)]^2 \leq 4([G^{-1}(s)]^2 + [G^{-1}(t)]^2)$*, for all* s*,* $t \in \mathbb{R}$*.*

(g_{13}) *the function* $G^{-1}(t)$ *is concave. In particular,* $G^{-1}(st) \leq sG^{-1}(t)$*, for all* $t \in \mathbb{R}$*,* $s \in [1,\infty)$*;*

(g_{14}) $G^{-1}(st) \geq sG^{-1}(t)$*, for all* $t \in \mathbb{R}$*,* $0 \leq s \leq 1$*;*

Proof. The proof of the first item follows from the monotonicity of g. For the second one, it is sufficient to compute the derivative on both sides of the equality $G(G^{-1}(t)) = t$ and use (g_1). Now, for each $t > 0$, the Mean Value Theorem provides $\xi \in [0,t]$ such that $G^{-1}(t) - G^{-1}(0) = (G^{-1})'(\xi)t$. Therefore, the proof of (g_3) follows from (g_2). The statement (g_4) is a consequence of L'Hopital rule, while (g_5) follows from (1.4) and the fact that G^{-1} is an odd function. Now, integrating the inequality $tg'(t) - g(t) \leq 0$ by parts, we conclude that $tg(t) \leq 2G(t)$. The Mean Value Theorem and the monotonicity of g provide $G(t) \leq g(t)t$. Hence, the the first inequality in (g_6) follows. For the second one, it is sufficient to use the change of variables $t = G^{-1}(s), t \geq 0$. The first statement in (g_7) is a simple consequence of (g_6), while the second one follows from the following limit $\lim_{t\to+\infty} G^{-1}(t)^2/t = (2/g_\infty)$. Using ($g_6$) again we can prove that $G^{-1}(t)/t$ is non-decreasing. This fact and (g_7) imply property (g_8). The item (g_9) is an easy consequence of (g_1).

It remains to prove $(g_{10}), (g_{11})$ and (g_{12}). Using (1.5) and a straightforward calculation we get, for $z = G^{-1}(t)$,

$$([G^{-1}(t)]^2)'' = \frac{2}{g(z)^2}\left(1 - \frac{zg'(z)}{g(z)}\right) = \frac{2}{g(z)}\frac{d}{dz}\left(\frac{z}{g(z)}\right).$$

Item (g_{10}) follows from this equality and (g_9). In order to check (g_{11}) we first notice that $(G^{-1}(t))'' = -g'(G^{-1}(t))g(G^{-1}(t))^{-3} \leq 0$. Hence, the function $[G^{-1}(t)]'$ is non-increasing. For any $t \geq 0$ fixed, we consider the function $\psi(s) := G^{-1}(st) - sG^{-1}(t)$, for $s \geq 1$. Since $g\left(G^{-1}(t)\right) \leq g\left(G^{-1}(st)\right)$, we can use ($g_6$) to get

$$\psi'(s) \leq G^{-1}(t)\left(\frac{t}{G^{-1}(t)g(G^{-1}(t))} - 1\right) \leq 0.$$

Therefore $\psi(s) \leq \psi(1) = 0$ holds for all $s \geq 1$. Thus $G^{-1}(st) \leq sG^{-1}(t)$ for all $t \geq 0$ and $s \geq 1$. Since $[G^{-1}]^2$ is even the proof of (g_{11}) is concluded. Finally, to establish (g_{12}), we use the fact that $[G^{-1}]^2$ is even and non-decreasing in $(0,+\infty)$ together with (g_{10}) and (g_{11}) to get, for all s, $t \in \mathbb{R}$,

$$\begin{aligned}
[G^{-1}(s-t)]^2 &= [G^{-1}(|s-t|)]^2 \leq [G^{-1}(|s|+|t|)]^2 \\
&\leq [G^{-1}(2\max\{|s|,|t|\})]^2 \leq 4([G^{-1}(s)]^2 + [G^{-1}(t)]^2).
\end{aligned}$$

Items (g_{13}) and (g_{14}) can be proved with similar arguments. □

Now we consider the Hilbert space X defined as

$$X = \left\{ u \in H^1(\mathbb{R}^N) : \int V(x)u^2 < \infty \right\},$$

endowed with the inner product

$$\langle u, v\rangle = \int \nabla u \nabla v + V(x)uv, \quad \forall\, u, v \in X.$$

It is well known that X is continuous embedding into $L^q(\mathbb{R}^N)$ for $q \in [2, 2^*]$. Furthermore, X is compactly embedded in $L^q(\mathbb{R}^N)$ for any $q \in [2, 2^*)$ (see [13]).

We also define the Orlicz-Sobolev space

$$E := \left\{ v \in H^1(\mathbb{R}^N) : \int V(x)[G^{-1}(v)]^2 < \infty \right\}.$$

Since, by (g_{10}), $[G^{-1}]^2$ is a convex function, we can argue as in [20, 19] to conclude that E is a Banach space when endowed with the norm

$$\|v\| := \|\nabla v\|_2 + |v|_g, \quad \forall\, v \in E,$$

where

$$|v|_g := \inf_{\xi>0} \frac{1}{\xi} \left\{ 1 + \int V(x)[G^{-1}(\xi v)]^2 \right\}.$$

By a weak solution of (P) we mean a function $u \in H^1(\mathbb{R}^N) \cap L^\infty_{loc}(\mathbb{R}^N)$ such that, for all $\varphi \in C_0^\infty(\mathbb{R}^N)$, there holds

$$\int [g^2(u)\nabla u \nabla \varphi + g(u)g'(u)|\nabla u|^2 \varphi + V(x)u\varphi] = \int h(x, u)\varphi.$$

After the change of variables $u = G^{-1}(v)$ in the map given in (2.1), we obtain the following functional

$$J(v) := \frac{1}{2} \int \left(|\nabla v|^2 + V(x)[G^{-1}(v)]^2 \right) - \int H(x, G^{-1}(v)), \quad v \in E.$$

Under the growth conditions (1.5), (1.4) and (h_1), we have that $J \in C^1(E, \mathbb{R})$ and its critical points are weak solutions of the problem

$$-\Delta v + V(x) \frac{G^{-1}(v)}{g\left(G^{-1}(v)\right)} = \frac{h(x, G^{-1}(v))}{g\left(G^{-1}(v)\right)}, \; v \in E.$$

Moreover, if $v \in E \cap C^2(\mathbb{R}^N)$ is a critical point of J, then the function $u = G^{-1}(v)$ is a classical solution of (P) (see [6]).

We list below the mains properties of the space E.

Proposition 2.2. *Suppose that V satisfies $(V_0) - (V_2)$. Then the space E has the following properties:*

(1) *if $(v_n) \subset E$ is such that $v_n(x) \to v(x)$ a.e. in $\mathbb{R}^N$ and*

$$\lim_{n \to +\infty} \int V(x)[G^{-1}(v_n)]^2 = \int V(x)[G^{-1}(v)]^2,$$

then

$$\lim_{n\to+\infty} |v_n - v|_g = 0$$

(2) *the embeddings* $E \hookrightarrow \mathcal{D}^{1,2}(\mathbb{R}^N)$, $E \hookrightarrow H^1(\mathbb{R}^N)$ *and* $X \hookrightarrow E$ *are continuous.*
(3) *the map* $v \to G^{-1}(v)$ *from* E *to* $L^q(\mathbb{R}^N)$ *is continuous for each* $q \in [2, 2\cdot 2^*]$*, and it is compact for each* $q \in [2, 2\cdot 2^*)$*;*
(4) *if* $v \in E$ *and* $u = G^{-1}(v)$*, then*

$$\|ug(u)\| \le 4\|v\|;$$

(5) *If* $v_n \rightharpoonup 0$ *in* $\mathcal{D}^{1,2}(\mathbb{R}^N)$ *and* $\left(\int V(x)[G^{-1}(v_n)]^2\right)$ *is bounded then, up to a subsequence,* $G^{-1}(v_n) \to 0$ *strongly in* $L^q(\mathbb{R}^N)$ *for any* $2 \le q < 2\cdot 2^*$*;*
(6) *if* $v \in E$*, then*

$$|v|_g \le 2\max\left\{\int V(x)[G^{-1}(v)]^2, \left(\int V(x)[G^{-1}(v)]^2\right)^{1/2}\right\}.$$

(7) *if* $v \in E$*, then*

$$|v|_g \ge \frac{1}{4}\min\left\{\int V(x)[G^{-1}(v)]^2, \left(\int V(x)[G^{-1}(v)]^2\right)^{1/2}\right\}.$$

Proof. The first three items can be proved along the same lines discussed in [7, 20]. In order to prove the fourth, we fix $v \in E$ and notice that, by using the definition of G^{-1} and a straightforward calculation, we get

$$\nabla\left[ug(u)\right] = \left(1 + \frac{ug'(u)}{g(u)}\right)\nabla v, \tag{2.2}$$

and therefore, by (1.5),

$$\|\nabla(ug(u))\|_2 \le 2\|\nabla v\|_2. \tag{2.3}$$

For $t \neq 0$ and $s = G^{-1}(t)$, it follows from the first inequality in (g_6) and (g_{11}) that

$$\left[G^{-1}\left(\frac{sg(s)}{G(s)}\xi G(s)\right)\right]^2 \le \left(\frac{sg(s)}{G(s)}\right)^2\left[G^{-1}(\xi G(s))\right]^2 \le 4\left[G^{-1}(\xi t)\right]^2,$$

for any $\xi > 0$. Thus, for $\psi = G^{-1}(v)g\left(G^{-1}(v)\right)$, we get

$$|\psi|_g = \inf_{\xi>0}\left\{\frac{1}{\xi}\left(1 + \int V(x)\left[G^{-1}(\xi\psi)\right]^2\right)\right\} \le 4|v|_g.$$

The statement 4 follows from the above inequality and (2.3).

For proving item 5, we may suppose that $v_n(x) \to 0$ a.e. in $\mathbb{R}^N$. Since

$$\begin{aligned}\|G^{-1}(v_n)\|_X^2 &= \int\left(\frac{|\nabla v_n|^2}{g\left(G^{-1}(v_n)\right)} + V(x)[G^{-1}(v_n)]^2\right)\\ &\le \max\{1, g(0)^{-2}\}\int\left(|\nabla v_n|^2 + V(x)[G^{-1}(v_n)]^2\right),\end{aligned}$$

and $(\int V(x)[G^{-1}(v_n)]^2)$ is bounded, up to a subsequence, $(G^{-1}(v_n))$ weakly converges in X. The compactness of the embedding $X \hookrightarrow L^q(\mathbb{R}^N)$, for $2 \leq q < 2^*$, and the pointwise convergence $G^{-1}(v_n(x)) \to G^{-1}(0) = 0$ a.e. in $\mathbb{R}^N$, imply that the weak limit is zero. So, $G^{-1}(v_n) \to 0$ strongly in $L^q(\mathbb{R}^N)$, whenever $2 \leq q < 2^*$.

For $u_n := G^{-1}(v_n)$, it follows from (g_9) that

$$\left|\nabla[G^{-1}(v_n)]^2\right|^2 = 4\left|\frac{u_n}{g(u_n)}\nabla v_n\right|^2 \leq \frac{4}{g_\infty^2}|\nabla v_n|^2.$$

Hence, we can use the Sobolev inequality to get, for some $c_1 > 0$,

$$\begin{aligned}\|G^{-1}(v_n)\|_{2\cdot 2^*} &= \|[G^{-1}(v_n)]^2\|_{2^*}^{1/2} \leq c_1\|\nabla([G^{-1}(v_n)]^2)\|_2^{1/2} \\ &\leq c_1\sqrt{2/g_\infty}\left(\int |\nabla v_n|^2\right)^{1/4} < \infty.\end{aligned}$$

It follows from the interpolation inequality that $G^{-1}(v_n) \to 0$ in $L^q(\mathbb{R}^N)$ for any $2 \leq q < 2 \cdot 2^*$.

Let $v \neq 0$ and suppose first that $\int V(x)[G^{-1}(v)]^2 > 1$. Setting $\xi_0 := \left(\int V(x)[G^{-1}(v)]^2\right)^{-1} < 1$, using the definition of $|v|_g$ and (g_{10}), we get

$$\begin{aligned}|v|_g &\leq \frac{1}{\xi_0}\left(1 + \int V(x)[G^{-1}(\xi_0 v)]^2\right) \\ &\leq \frac{1}{\xi_0}\left(1 + \xi_0\int V(x)[G^{-1}(v)]^2\right) = 2\int V(x)[G^{-1}(v)]^2.\end{aligned}$$

If $0 < \int V(x)[G^{-1}(v)]^2 \leq 1$, we set $\xi_0 := \left(\int V(x)[G^{-1}(v)]^2\right)^{-1/2} \geq 1$. Since, by (g_{13}), G^{-1} is a concave function, we car argue as above to conclude that $|v|_g \leq 2(\int V(x)[G^{-1}(v)]^2)^{1/2}$. This and the above expression finish the proof of item 6. The last item can de proved with an analogous argument. We omit the details. □

3. A compactness condition

If V is a real Banach space, we say that $\mathcal{J} \in C^1(V, \mathbb{R})$ satisfies the Palais-Smale condition at level $c \in \mathbb{R}$, in short $(PS)_c$, if any sequence $(v_n) \subset V$ such that

$$\lim_{n\to+\infty} \mathcal{J}(v_n) = c, \ \lim_{n\to\infty} \mathcal{J}'(v_n) = 0$$

has a convergent subsequence. We devote all this section to the proof of the following result.

Proposition 3.1. *Suppose that V, g and h satisfy $(V_0) - (V_2)$, (g_0) and $(h_0) - (h_2)$. If $(v_n) \subset E$ is such that $J(v_n) \to c$ and $J'(v_n) \to 0$, then (v_n) has a convergent subsequece.*

Proof. We first prove that (v_n) is bounded. Indeed, in view of item 4 of Proposition 2.2, we have that $\psi_n := G^{-1}(v_n)\, g\left(G^{-1}(v_n)\right) \in E$. Hence, we can use (2.2) and (1.5) to compute

$$J'(v_n)\psi_n \leq (1+\beta)\int |\nabla v_n|^2 + \int V(x)\left[G^{-1}(v_n)\right]^2 - \int h(x, G^{-1}(v_n))G^{-1}(v_n).$$

This and the embedding $E \hookrightarrow L^s(\mathbb{R}^N)$, for $2 \leq s \leq 2^*$, imply that $J'(v_n)\psi_n = o_n(1)\|v_n\|$. Hence, we can use (h_2) to get

$$\begin{aligned} c + o_n(1)\|v_n\| &\geq J(v_n) - \frac{1}{2(\beta+1)}J'(v_n)\psi_n \\ &\geq \frac{\beta}{2(\beta+1)}\int V(x)[G^{-1}(v_n)]^2 - \frac{1}{2(\beta+1)}\int \Gamma_1, \\ &-\frac{\beta}{2(\beta+1)}\int \Gamma_2(x)|G^{-1}(v_n)|^{q_1}. \end{aligned}$$

Since the number τ given in (h_2) verifies $2 \leq q_1\tau'/2 \leq 2^*$, we can use the above expression, (1.5), (g_7), Holder's inequality and the embedding $E \hookrightarrow L^{q_1\tau'/2}(\mathbb{R}^N)$, to obtain

$$\begin{aligned} \int V(x)[G^{-1}(v_n)]^2 &\leq o_n(1)\|v_n\| + c_1\|\Gamma_1\|_1 + c_1\|\Gamma_2\|_\tau \|v_n\|^{q_1/2}_{q_1\tau'/2} \\ &\leq o_n(1)\|v_n\| + c_1\|\Gamma_1\|_1 + c_1\|\Gamma_2\|_\tau \|v_n\|^{q_1/2}. \end{aligned} \tag{3.1}$$

Arguing by contradiction we suppose that, up to a subsequence, $\|v_n\| \to +\infty$ as $n \to +\infty$. Set $w_n := v_n/\|v_n\|$ and notice that the above inequality and (g_{11}) provide

$$\int V(x)[G^{-1}(w_n)]^2 = \int V(x)\left[G^{-1}\left(\frac{v_n}{\|v_n\|}\right)\right]^2 \leq \frac{1}{\|v_n\|}\int V(x)[G^{-1}(v_n)]^2 \to 0.$$

Since (w_n) is bounded in $\mathcal{D}^{1,2}(\mathbb{R}^N)$, we may assume that $w_n \rightharpoonup w$ weakly in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ and $w_n(x) \to w(x)$ a.e. in $\mathbb{R}^N$. From Fatou's lemma and the last estimate we get $\int V(x)[G^{-1}(w)]^2 \leq \liminf_{n\to\infty} \int V(x)[G^{-1}(w_n)]^2 = 0$, and therefore $w = 0$. We infer from item 1 of Proposition 2.2 that $|w_n|_g \to 0$.

We now claim that

$$\lim_{n\to+\infty} \frac{1}{\|v_n\|^2}\int H(x, G^{-1}(v_n)) = 0. \tag{3.2}$$

If this is true, we can finish the proof of the boundedness of (v_n) by noticing that

$$\int |\nabla w_n|^2 = \frac{2J(v_n)}{\|v_n\|^2} - \frac{1}{\|v_n\|^2}\int V(x)[G^{-1}(v_n)]^2 + \frac{2}{\|v_n\|^2}\int H(x, G^{-1}(v_n)) \to 0,$$

where we have used $J(v_n) \to c$, (3.1) and (3.2). This and $|w_n|_g \to 0$ imply that $1 = \|w_n\| = \|w_n\|_2^2 + |w_n|_g \to 0$, which does not make sense. This contradiction shows that (v_n) is bounded.

In order to verify (3.2), we first notice that, by (g_8),

$$|t| \le \frac{1}{G^{-1}(0)}|G^{-1}(t)| + \frac{1}{G^{-1}(0)^2}[G^{-1}(t)]^2, \quad \forall\, t \in \mathbb{R}. \tag{3.3}$$

Thus, since $v_n = \|v_n\| w_n$, we can use (h_1) and (g_3) to get

$$\begin{aligned} \frac{|H(x,G^{-1}(v_n))|}{\|v_n\|^2} &\le \frac{a(x)}{2}\frac{\left[G^{-1}(\|v_n\|w_n)\right]^2}{\|v_n\|^2} + \frac{b(x)}{2}\frac{[\|v_n\|w_n]^2}{\|v_n\|^2} \\ &\le \frac{1}{2}\left(\frac{a(x)}{g^2(0)} + b(x)\right) w_n^2 \\ &\le c_1(a(x)+b(x))([G^{-1}(w_n)]^2 + [G^{-1}(w_n)]^4). \end{aligned} \tag{3.4}$$

On the other hand, item 5 of Proposition 2.2 implies that, up to a subsequence,

$$G^{-1}(w_n) \to 0 \text{ strongly in } L^q(\mathbb{R}^N), \quad \text{for any } 2 \le q < 2\cdot 2^*. \tag{3.5}$$

Recalling that $b \in L^{\alpha_0}(\mathbb{R}^N)$ with $\alpha_0 > N/2$, we can use Hölder's inequality to get

$$\int b(x)[G^{-1}(w_n)]^4 \le \|b\|_{\alpha_0}\|G^{-1}(w_n)\|^4_{4\alpha_0/(\alpha_0-1)} \to 0,$$

where we have used (3.5) and the fact that $4 < 4\alpha_0/(\alpha_0-1) < 2\cdot 2^*$. The same argument shows that

$$\max\left\{\int a(x)[G^{-1}(w_n)]^4, \int a(x)[G^{-1}(w_n)]^2, \int b(x)[G^{-1}(w_n)]^2\right\} \to 0.$$

The proof of (3.2) follows from the above expression and (3.4). Hence, we conclude that (v_n) is bounded.

Now, for some $v \in \mathcal{D}^{1,2}(\mathbb{R}^N)$, we have that $v_n \rightharpoonup v$ weakly in $\mathcal{D}^{1,2}(\mathbb{R}^N)$. Since we also have pointwise convergence, we can use (3.1) and Fatou's lemma to get

$$\int V(x)[G^{-1}(v)]^2 \le \liminf_{n\to+\infty}\int V(x)[G^{-1}(v_n)]^2 < \infty, \tag{3.6}$$

and therefore the weak limit v belongs to E.

Notice that, since $[G^{-1}]^2$ is convex, the function Q defined by

$$Q(v) := \int |\nabla v|^2 + \int V(x)[G^{-1}(v)]^2, \tag{3.7}$$

is also convex. Hence,

$$\begin{aligned} Q(v) - Q(v_n) &\ge Q'(v_n)\cdot(v-v_n) \\ &= 2J'(v_n)\cdot(v-v_n) + 2\int \frac{h(x,G^{-1}(v_n))}{g(G^{-1}(v_n))}\,(v-v_n). \end{aligned} \tag{3.8}$$

We claim that

$$\lim_{n\to+\infty}\int \frac{h(x,G^{-1}(v_n))}{g(G^{-1}(v_n))}(v-v_n) = 0. \tag{3.9}$$

Assuming the claim, recalling that $J'(v_n) \to 0$ and taking the limit in (3.8), we get

$$\limsup_{n\to+\infty} Q(v_n) \le Q(v).$$

On the other hand, the weak converge of (v_n) in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ provides

$$\int |\nabla v|^2 \le \liminf_{n\to+\infty} \int |\nabla v_n|^2. \tag{3.10}$$

Hence, we infer from (3.6) that $Q(v) \le \liminf_{n\to+\infty} Q(v_n)$, and therefore

$$\lim_{n\to+\infty} Q(v_n) = Q(v). \tag{3.11}$$

Before continuing the proof we justify equation (3.9). From (h_1), (g_2), (g_3) and (3.3), we obtain

$$\left|\frac{h(x,G^{-1}(v_n))}{g(G^{-1}(v_n))}\right| \le \left(\frac{a(x)}{g(0)^2} + b(x)\right)|v_n| \le \psi(x)(|G^{-1}(v_n)| + [G^{-1}(v_n)]^2),$$

with $\psi(x) := c_1(a(x)+b(x)) \in L^{\alpha_0}(\mathbb{R}^N)$. This and (3.3) imply that

$$\left|\frac{h(x,G^{-1}(v_n))}{g(G^{-1}(v_n))}\right| |v_n - v| \le c_2\psi(x)M_n(x)(|G^{-1}(v_n-v)| + [G^{-1}(v_n-v)]^2), \tag{3.12}$$

with $M_n(x) := |G^{-1}(v_n(x))| + [G^{-1}(v_n(x))]^2$. If we set $q := 2\alpha_0/(\alpha_0-1)$ we can use $\alpha_0 > N/2$ to conclude that $2 < q < 2^*$. Hence, the embedding $E \hookrightarrow L^q(\mathbb{R}^N)$, (g_3) and (g_7) imply that the sequence h_n is bounded in $L^q(\mathbb{R}^N)$. It follows from Hölder's inequality that

$$\int \psi(x)h_n(x)[G^{-1}(v_n-v)]^2 \le \|\psi\|_{\alpha_0}\|M_n\|_q\|G^{-1}(v_n-v)\|_{2q}^2 \to 0,$$

where we have used $4 < 2q < 2\cdot 2^*$ and item 5 of Proposition 2.2. An analogous argument provides $\int \psi(x)M_n(x)|G^{-1}(v_n-v)| \to 0$ and therefore the statement (3.9) is a consequence of (3.12).

By using (3.11) we obtain

$$\begin{aligned} Q(v) &= \liminf_{n\to+\infty} Q(v_n) \\ &\ge \liminf_{n\to+\infty} \int |\nabla v_n|^2 + \liminf_{n\to+\infty} \int V(x)[G^{-1}(v_n)]^2 \\ &\ge \int |\nabla v|^2 + \int V(x)[G^{-1}(v)]^2 = Q(v) \end{aligned}$$

We infer from the above inequality, (3.6) and (3.10) that

$$\liminf_{n\to+\infty} \int |\nabla v_n|^2 = \int |\nabla v|^2, \quad \liminf_{n\to+\infty} \int V(x)[G^{-1}(v_n)]^2 = \int V(x)[G^{-1}(v)]^2. \tag{3.13}$$

Hence

$$\begin{aligned} Q(v) &= \limsup_{n\to+\infty}\left(\int |\nabla v_n|^2 + \int V(x)[G^{-1}(v_n)]^2\right) \\ &\geq \limsup_{n\to+\infty}\int |\nabla v_n|^2 + \liminf_{n\to+\infty}\int V(x)[G^{-1}(v_n)]^2 \\ &\geq \liminf_{n\to+\infty}\left(\int |\nabla v_n|^2 + \int V(x)[G^{-1}(v_n)]^2\right) = Q(v), \end{aligned}$$

and therefore we conclude that $\limsup_{n\to+\infty}\int |\nabla v_n|^2 = \int |\nabla v|^2$. This and (3.13) imply that $\|v_n\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)} \to \|v\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)}$. So, the weak convergence of (v_n) imply that $v_n \to v$ strongly in $\mathcal{D}^{1,2}(\mathbb{R}^N)$.

Arguing as above we can also conclude that

$$\limsup_{n\to+\infty}\int V(x)[G^{-1}(v_n)]^2 = \int V(x)[G^{-1}(v)]^2.$$

and therefore $\sqrt{V(x)[G^{-1}(v_n)]^2} \to \sqrt{V(x)[G^{-1}(v)]^2}$ strongly in $L^2(\mathbb{R}^N)$. Thus, up to a subsequence, we have that $\sqrt{V(x)[G^{-1}(v_n)]^2} \leq \varphi(x)$ a.e. in $\mathbb{R}^n$, for some $\varphi \in L^2(\mathbb{R}^N)$. Hence, we can use (g_{12}) to obtain

$$V(x)[G^{-1}(v_n - v)]^2 \leq 4(\varphi(x)^2 + V(x)[G^{-1}(v)]^2).$$

Since the right-hand side above belongs to $L^1(\mathbb{R}^N)$, it follows from the Lebesgue Theorem that $\int V(x)[G^{-1}(v_n - v)]^2 \to 0$. Thus, item 1 of Proposition 2.2 implies that $|v_n - v|_g \to 0$. This fact and $\|\nabla(v_n - v)\|_2 \to 0$ show that $v_n \to v$ strongly in E. The proposition is proved. □

4. Proof of the main theorems

In this setion we present the proofs of our main results. We shall use the following version of the Moutain Pass Theorem.

Theorem 4.1. *Let V be a real Banach space, $\mathcal{J} \in C^1(V,\mathbb{R})$ and $S \subset V$ a closed subset which arcwise disconnect V in connected components V_1 and V_2. Suppose further that $\mathcal{J}(0) = 0$ and*

$(\mathcal{J}_1)$ $0 \in V_1$ and there exists $\alpha > 0$ such that $\mathcal{J}(v) \geq \alpha$ for all $v \in S$;
$(\mathcal{J}_2)$ there exists $e \in V_2$ such that $\mathcal{J}(e) \leq 0$.

Let

$$c_0 := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} \mathcal{J}(\gamma(t)) \geq \alpha, \tag{4.1}$$

where $\Gamma := \{\gamma \in C([0,1],V) : \gamma(0) = 0,\ \gamma(1) \in \mathcal{J}^{-1}((-\infty,0]) \cap V_2\}$. If $\mathcal{J}$ satisfies $(PS)_{c_0}$, then $\mathcal{J}$ has a critical point at level $c_0 > 0$.

We are intending to apply the above theorem with V being the Orlicz-Sobolev space E defined in the last section and $\mathcal{J} = J$. We first verify that J satisfies the geometric conditions $(\mathcal{J}_1)$ and $(\mathcal{J}_2)$ of Theorem 4.1.

For each $\rho > 0$, we define the set

$$S_\rho := \left\{ v \in E : \int |\nabla v|^2 + V(x)[G^{-1}(v)]^2 = \rho^2 \right\}.$$

Since $Q : E \to \mathbb{R}$ given in (3.7) is continuous, we have that S_ρ is a closed subset which disconnects the space E.

Recall that $\lambda_1(K_0)$ denotes the first positive eigenvalue problem (1.6) with K replaced by K_0. Hence, we have that

$$\int K_0(x)u^2 \leq \frac{1}{\lambda_1(K_0)} \int \left(|\nabla u|^2 + \frac{V(x)}{g(0)^2} u^2 \right), \qquad \forall\, u \in X.$$

We shall see in the next result that this inequality provides the mountain geometry near the origin.

Lemma 4.2. *Suppose that V, g and h satisfy $(V_0) - (V_2)$, (g_0) and $(h_0) - (h_1)$, respectively. Assume also that (H_0) holds and $\lambda_1(K_0) > 1$. Then there exist ρ, $\alpha > 0$ such that*

$$J(v) \geq \alpha, \text{ for all } v \in S_\rho.$$

Proof. We start noticing that, by (H_0) and (g_4),

$$\limsup_{t \to 0} \frac{2H(x, G^{-1}(t))}{[G^{-1}(t)]^2} = \limsup_{t \to 0} \frac{2H(x, G^{-1}(t))}{t^2} \left(\frac{t}{G^{-1}(t)} \right)^2 = g(0)^2 K_0(x),$$

uniformly for a.e. $x \in \mathbb{R}^N$. Setting $q := 2^*/\alpha_0' > 2$, we can use (h_1) to obtain, for any given $\varepsilon > 0$, a function $d \in L^{\alpha_0}(\mathbb{R}^N)$ such that

$$|H(x, G^{-1}(t))| \leq g^2(0) \frac{(K_0(x) + \varepsilon)}{2} [G^{-1}(t)]^2 + d(x)|t|^q,$$

for a.e. $x \in \mathbb{R}^N$ and $t \in \mathbb{R}$. Hence, if we fix $v \in S_\rho$ such that $Q(v) = \rho$ and using hypothesis (V_1), we get

$$J(v) \geq \frac{1}{2}\rho^2 - \frac{g(0)^2}{2} \int K_0(x)[G^{-1}(v)]^2 - \frac{\varepsilon g(0)^2}{2V_0} \int V(x)[G^{-1}(v)]^2 - \int d(x)|v|^q.$$

Using Hölder and Gagliardo-Nirenberg inequalities we can write

$$\int d(x)|v|^{2^*/\alpha_0'} \leq \|d\|_{\alpha_0} \|v\|_{2^*}^q \leq c_1 \|d\|_{\alpha_0} \|\nabla v\|_2^q \leq c_1 \|d\|_{\alpha_0} Q^{q/2}(v) \leq c_1 \|d\|_{\alpha_0} \rho^q,$$

for some $c_1 > 0$. Moreover, since $\nabla G^{-1}(v) = g(G^{-1}(v))^{-1} \nabla v$, we can use the variational characterization of $\lambda_1(K_0)$ and (g_2) to obtain

$$\begin{aligned} \int K_0(x)[G^{-1}(v)]^2 &\leq \frac{1}{\lambda_1(K_0)} \left(\int |\nabla [G^{-1}(v)]|^2 + V(x) \frac{[G^{-1}(v)]^2}{g(0)^2} \right) \\ &\leq \frac{g(0)^{-2}}{\lambda_1(K_0)} \left(\int |\nabla v|^2 + V(x)[G^{-1}(v)]^2 \right) = \frac{g(0)^{-2}}{\lambda_1(K_0)} \rho^2. \end{aligned}$$

Putting all these estimates together, we obtain

$$J(v) \geq \frac{1}{2} \left(1 - \frac{1}{\lambda_1(K_0)} - \frac{\varepsilon g(0)^2}{V_0} - C\rho^{q-2} \right) \rho^2.$$

Choosing $\varepsilon > 0$ small and recalling that $q > 2$ and $\lambda_1(K_0) > 1$ we obtain the desired result. □

At this stage we shall prove that J has negative energy in some specific directions proving the mountain pass geometry for the functional J. This can be read in the following form

Lemma 4.3. *Suppose that V, g and h satisfy $(V_0) - (V_2)$, (g_0) and $(h_0) - (h_1)$, respectively. Assume also that (H_∞) holds and $\mu_1(K_\infty) < 1$. If $\varphi \in X$ is a positive eigenfunction associated to $\mu_1(K_\infty)$, then*

$$\lim_{t\to+\infty} J(t\varphi) = -\infty.$$

Proof. We start noticing that, by (h_1) and (g_3), for $t \in \mathbb{R}$ and a.e. $x \in \mathbb{R}^N$ there holds

$$\frac{2|H(x, G^{-1}(t\varphi))|}{t^2} \leq \left(\frac{a(x)}{g(0)^2} + b(x)\right)\varphi(x)^2.$$

Since $a, b \in L^{\alpha_0}(\mathbb{R}^N)$ with $\alpha_0 > N/2$, the right-hand side above belongs to $L^1(\mathbb{R}^N)$. Thus, we can set $s = G^{-1}(t\varphi)$, use $\varphi > 0$, Fatou's lemma and (H_∞) to get

$$\liminf_{t\to+\infty} \int \frac{2H(x, G^{-1}(t\varphi))}{t^2} \geq \int \liminf_{s\to+\infty} \left(2\frac{H(x,s)}{G^2(s)}\varphi^2\right) = \int K_\infty(x)\varphi^2.$$

From (g_3), we have that for $t \in \mathbb{R}$ and a.e. $x \in \mathbb{R}^N$

$$\frac{V(x)[G^{-1}(t\varphi(x))]^2}{t^2} \leq g(0)^{-2}V(x)\varphi(x)^2.$$

Moreover, by (g_7),

$$\limsup_{t\to+\infty} \frac{V(x)[G^{-1}(t\varphi)]^2}{t^2} \leq \lim_{t\to+\infty} \frac{2}{k}\frac{V(x)\varphi(x)}{t} = 0.$$

Since $V\varphi^2 \in L^1(\mathbb{R}^N)$, it follows from the Lebesgue Theorem that

$$\lim_{t\to\infty} \int \left(|\nabla\varphi|^2 + \frac{V(x)[G^{-1}(t\varphi)]^2}{t^2}\right) = \int |\nabla\varphi|^2.$$

The above estimates, the equality $\int |\nabla\varphi|^2 = \mu_1(K_\infty)\int K_\infty(x)\varphi^2$ and $\mu_1(K_\infty) < 1$ imply that

$$\limsup_{t\to+\infty} \frac{2J(t\varphi)}{t^2} \leq \int |\nabla\varphi|^2 - \int K_\infty(x)\varphi^2 = (\mu_1(K_\infty) - 1)\int K_\infty(x)\varphi^2 < 0,$$

and we are done. □

We are ready to prove our main results.

Proof of Theorems 1.1 and 1.2. In view of the above lemmas and Proposition 3.1, we can apply Theorem 4.1 with $V = E$ and $\mathcal{J} = J$ to obtain $v_0 \in E$ such that $J'(v_0) = 0$ and $J(v_0) = c_0 \geq \alpha > 0$. Since $J(0) = 0$, we conclude that v_0 is a non-zero critical point of J. As quoted before, the function $u_0 := G^{-1}(v_0)$ is a solution of the problem (P) proving Theorem 1.1.

In order to prove the second theorem we define the function

$$h_+(x,t) = \begin{cases} h(x,t), & \text{if } t \geq 0, \\ 0, & \text{if } t \leq 0, \end{cases}$$

and consider the functional

$$J_+(v) := \frac{1}{2}\int \left(|\nabla v|^2 + V(x)[G^{-1}(v)]^2\right) - \int H_+(x, G^{-1}(v)), \quad v \in E.$$

where $H_+(x,t) = \int_0^t h(x,\tau)\mathrm{d}\tau$. Since $\int H_+(x, G^{-1}(v)) = \int H(x, G^{-1}(v^+)$, we can easily check that the two lemmas of this section also holds for J_+. If the same holds for Proposition 3.1, we can argue as in the first theorem to obtain $v_1 \in E$ such that $J_+'(v_1) = 0$. Setting $v_1^- := \min\{u(x), 0\}$, recalling that $J_+'(v_1)v_1^- = 0$ and the definition of h_+, we conclude that $v_1^- \equiv 0$, that is, $v_1(x) \geq 0$ for a.e. $x \in \mathbb{R}^N$. It follows from the Maximum Principle that v_1 is a positive in $\mathbb{R}^N$, and therefore $u_1 := G^{-1}(v_1)$ is a positive solution of (P). The negative solution can be obtained in a similar way, just truncating h on the other side of the real line.

It remains to check that J_+ verifies the $(PS)_c$ condition for any $c \in \mathbb{R}$. Let $(v_n) \subset E$ be a $(\mathrm{PS})_c$ sequence for the functional J_+. Now we define $v_n^-(x) := \min\{v_n(x), 0\}$ and claim that $\|v_n^-\| \to 0$ as $n \to +\infty$. Indeed, if this is not true, there exists $\gamma > 0$ in such way that, up to a subsequence, $\|v_n^-\| \geq \gamma > 0$. It follows from item 4 of Proposition 2.2, $G^{-1}(0) = 0$, and the definition of h_+ that

$$\begin{aligned} o_n(1)\|v_n^-\| &\geq J_+'(v_n) \cdot G^{-1}(v_n^-)g(G^{-1}(v_n^-)) \\ &= \int \left(1 + \frac{G^{-1}(v_n^-)g'(G^{-1}(v_n^-))}{g(G^{-1}(v_n^-))}\right)|\nabla v_n^-|^2 \\ &+ \int V(x)\frac{G^{-1}(v_n)}{g(G^{-1}(v_n))}G^{-1}(v_n^-)g(G^{-1}(v_n^-)) \\ &\geq \int |\nabla v_n^-|^2 + V(x)G^{-1}(v_n^-)^2. \end{aligned}$$

Using the fact that $\|v_n^-\| \geq \gamma > 0$ follows that $w_n := \frac{v_n^-}{\|v_n^-\|}$ is well defined. The above inequalities provide

$$\max\left\{\int |\nabla w_n|^2, \int V(x)\frac{G^{-1}(v_n^-)^2}{\|v_n^-\|}, \int V(x)\frac{G^{-1}(v_n^-)^2}{\|v_n^-\|^2}\right\} \to 0,$$

as $n \to +\infty$. Hence, we infer from (g_{10}) and (g_{11}) that

$$\begin{aligned} \int V(x)[G^{-1}]^2(w_n) &= \int V(x)[G^{-1}]^2\left(\frac{v_n^-}{\|v_n^-\|}\right) \\ &\leq \left(\frac{1}{\|v_n^-\|}+\frac{1}{\|v_n^-\|^2}\right)\int V(x)[G^{-1}]^2(v_n^-) \to 0. \end{aligned}$$

It follows from item 6 of Proposition 2.2 that $|w_n|_g \to 0$ as $n \to +\infty$. Thus, $1 = \|w_n\| = \|\nabla w_n\|_2 + |w_n|_g \to 0$, which is an absurd. Hence $\|v_n^-\| \to 0$ as $n \to +\infty$, as claimed. Using this fact and replacing (v_n) by (v_n^+) if necessary, we may suppose that $v_n \geq 0$. Arguing along the same lines of Proposition 3.1 we conclude that (v_n) has a convergent subsequence, that is, the truncated functional J_+ satisfies the Palais-Smale condition. □

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