Solutions for a Schrödinger-Kirchoff equation with indefinite potentials

Alex M. Batista and Marcelo F. Furtado

Abstract. We deal with the equation

$$-\Big(1+\int_{\mathbb{R}^3}|\nabla u|^2dx\Big)\Delta u + V(x)u = a(x)|u|^{p-1}u \quad x\in\mathbb{R}^3,$$

with $p\in(3,5)$. Under some conditions on the sign-changing potentials V and a we obain a nonnegative ground state solution. In the radial case we also obtain a nodal solution.

Mathematics Subject Classification (2010). Primary 35J20; Secondary 35J25, 35J60.

Keywords. Kirchoff equation; nonlocal problems; ground state solution; nodal solution.

1. Introduction

We consider a version of the equation

$$-\Big(\alpha+\beta\int_{\mathbb{R}^N}|\nabla u|^2dx\Big)\Delta u + V(x)u = f(x,u), \quad x\in\mathbb{R}^3,$$

with α, $\beta\in\mathbb{R}$, V and f satisfying some suitable conditions. Due to the presence of the term $\int_{\mathbb{R}^N}|\nabla u|^2dx$ the equation is not a pointwise identity and therefore the problem is called nonlocal. The main interest in this kind of operator relies on the fact that it arises in the following physical context: if we set $V\equiv 0$ and replace the entire space by $\Omega\subset\mathbb{R}^N$, then we get the problem

$$-\Big(\alpha+\beta\int_{\Omega}|\nabla u|^2dx\Big)\Delta u = f(x,u),\ x\in\Omega, \quad u\in H_0^1(\Omega), \tag{1.1}$$

The authors were partially supported by CNPq/Brazil. The second author was also supported by FAPDF/Brazil.

which is related to the stationary analogue of the equation

$$\rho\frac{\partial^2 u}{\partial t^2}-\Big(\frac{P_0}{h}+\frac{E}{2L}\int_0^L\Big|\frac{\partial u}{\partial x}\Big|^2dx\Big)\frac{\partial^2 u}{\partial x^2}=0,$$

proposed by Kirchoff in [12]. This is an extension of the classical d'Alembert wave equation which considers the effects of the changes on the length of the string during vibrations. Actually, in the physical model, the parameters has the following meaning: L is the length of the string, h is the area of cross-section, E is the Young modulus of the material, ρ is the mass density and P_0 is the initial tension. After J.L.Lions [14] presented an abstract functional analysis framework to the evolution equation related with (1.1), these kind of problem has been extensively studied (see [1, 5, 3, 13, 4] and references there in).

In this paper we assume, with no loss of generality, that $\alpha=\beta=1$ and consider the problem

$$-\Big(1+\int_{\mathbb{R}^3}|\nabla u|^2dx\Big)\Delta u+V(x)u=a(x)|u|^{p-1}u \quad x\in\mathbb{R}^3, \qquad (P)$$

with $p\in(3,5)$. In order to present the assumptions on the potentials we set $V^-(x):=\max\{-V(x),0\}$,

$$S:=\inf\Big\{\|\nabla u\|^2_{L^2(\mathbb{R}^3)};u\in\mathcal{D}^{1,2}(\mathbb{R}^3),\ \|u\|_{L^6(\mathbb{R}^3)}=1\Big\},$$

and assume the following conditions:

(V_0) $V\in L^t_{loc}(\mathbb{R}^3)$ for some $t>3/2$;

(V_1) $V^-\in L^{3/2}(\mathbb{R}^3)$ and $\|V^-\|_{L^{3/2}(\mathbb{R}^3)}<S$;

(V_2) there are constants c_V, $\mu>0$ such that

$$V(x)\le V_\infty-c_Ve^{-\mu|x|},\quad \text{for a.e. } x\in\mathbb{R}^3,$$

with

$$V_\infty:=\lim_{|x|\to+\infty}V(x)>0;$$

(a_0) $a\in L^\infty(\mathbb{R}^3)$

(a_1) there are constants c_a, $\gamma>0$ such that

$$a(x)\ge a_\infty-c_ae^{-\gamma|x|},\quad \text{for a.e. } x\in\mathbb{R}^3,$$

with

$$a_\infty:=\lim_{|x|\to+\infty}a(x)>0.$$

The equation in (P) is the Euler-Lagrange equation of the functional $I:H^1(\mathbb{R}^3)\to\mathbb{R}$ given by

$$I(u)=\frac{1}{2}\|\nabla u\|^2_{L^2(\mathbb{R}^3)}+\frac{1}{2}\int_{\mathbb{R}^3}V(x)u^2dx+\frac{1}{4}\|\nabla u\|^4_{L^2(\mathbb{R}^3)}-\frac{1}{p+1}\int_{\mathbb{R}^3}a(x)|u|^{p+1}dx.$$

We say that a solution u is a ground state solution if it has minimal energy between all the non zero solutions.

As we shall see, it is important to consider the limit problem associated with (P), namely

$$-\left(1+\|\nabla u\|_{L^2(\mathbb{R}^3)}^2\right)\Delta u + V_\infty u = a_\infty |u|^{p-1}u, \quad x\in\mathbb{R}^3.$$

Arguing as in [4], we can prove that it has a positive ground state solution $\omega \in H^1(\mathbb{R}^3)$ (see Lemma 2.1 in Section 2). Thus, if we set

$$h^* := \left(1+\|\nabla\omega\|_2^2\right)^{1/2},$$

we can state our first result in the following way.

Theorem 1.1. *Suppose that $p\in(3,5)$ and the potentials V and a satisfy $(V_0)-(V_2)$ and $(a_0)-(a_1)$. If*

$$\mu<\gamma<\frac{\sqrt{V_\infty}(p+1)}{h^*}, \tag{1.2}$$

then the problem (P) has a nonnegative ground state solution.

In our second result we look for a nodal solution. The main inspiration is the paper [16], where the authors obtained a sign changing solution for the local case $\beta=0$, under conditions analogous to those of Theorem 1.1. Unfortunately, in the nonlocal case we are not able to use the decay estimates of the potentials to localize the minimax level of the functional in the correct compactness range. So, we shall restrict our attention to radial functions and prove the following:

Theorem 1.2. *Suppose that $p\in(3,5)$ and the potentials V and a are radial and satisfy (V_1) and (a_0). If*

$(\widetilde{V_2})$ $V_\infty = \lim_{|x|\to+\infty} V(x) > 0$;

$(\widetilde{a_1})$ $a_\infty = \lim_{|x|\to+\infty} a(x) > 0$,

then the problem (P) has a sign-changing radial solution. This solution has small energy in the class of sign-changing radial solutions.

In the proof we apply variational methods. Although this is rather standard we need to overcome the lack of compactness of $H^1(\mathbb{R}^3)$ into the Lebesgue spaces. This is done by using some comparison arguments of the minimax level of I and the minimax level of the limit problem. Another difficult in our proof relies on the fact that the potential a changes sign. Hence, it is not true that any $u\in H^1(\mathbb{R}^3)\setminus\{0\}$ can be projected in the Nehari manifold. For proving the second theorem we follow [2], by using the set $\{u\in H^1(\mathbb{R}^3) : I'(u)u^+ = 0 = I'(u)u^-\}$. Since it is not a smooth manifold, we need to do some trick calculations for obtaining minimize sequences. Although the second theorem holds for radial potentials the other assumptions are very weak.

The case $a\equiv 1$ and $p\in(3,5)$ was treated in [13], where the authors obtained a positive ground state solution. Also for positive potentials, the authors in [11] obtained some multiplicity and concentration results for (P).

In [9], the authors obtained existence and multiplicity of solutions for an equation with potentials vanishing at infinity. In [8], the authors considered the symmetric case for a sign-changing radial potential V. We finnaly quote the recent paper [6], where a Schrödinger-Poisson system is considered under conditions closely related to that of Theorem 1.1. Actually, it is proved there that the positive solution has exponential decay and therefore a second (nodal) solution is obtained under almost the same conditions. Here, we are not able to prove that the positive solution decays exponentially and therefore we use a slight different argument in the space of radial functions. In this setting we do not need to control the decay rate of the potentials. The main results of this paper can be viewed as a nonlocal version of those proved in [10, 16] and complement the aforementioned works.

The paper has two more sections. In the first one we prove Theorem 1.1 and in the other we present the proof of Theorem 1.2.

2. The nonnegative solution

For any $2 \le q \le \infty$, we denote by $\|u\|_q$ the L^q-norm of a function $u \in L^q(\mathbb{R}^3)$. For saving notation, we write only $\int u$ to denote $\int_{\mathbb{R}^3} u(x)\mathrm{d}x$. Throughout this section we suppose that the conditions $(V_0)-(V_2)$ and $(a_0)-(a_1)$ hold.

We first consider the limit problem

$$(P_\infty) \qquad -\left(1+\|\nabla u\|_2^2\right)\Delta u + V_\infty u = a_\infty |u|^{p-1}u, \quad x \in \mathbb{R}^3,$$

whose energy functional is $I_\infty : H^1(\mathbb{R}^3) \to \mathbb{R}$ given by

$$I_\infty(u) := \frac{1}{2}\|u\|_*^2 + \frac{1}{4}\|\nabla u\|_2^4 - \frac{1}{p+1}\int a_\infty |u|^{p+1},$$

with

$$\|u\|_*^2 := \int \left(|\nabla u|^2 + V_\infty u^2\right).$$

Its Nehari manifold is

$$\mathcal{N}_\infty := \left\{u \in H^1(\mathbb{R}^3)\backslash\{0\} \,:\, I'_\infty(u)u = 0\right\}$$

and we can define the ground state level by setting

$$m_\infty := \inf_{u \in \mathcal{N}_\infty} I(u).$$

Lemma 2.1. *The problem (P_∞) has a nonnegative ground state solution $\omega \in H^1(\mathbb{R}^3)$ verifying*

$$I_\infty(\omega) = m_\infty = \max_{t \ge 0} I_\infty(t\omega).$$

Moreover, if we set

$$h^* := \left(1 + \|\nabla \omega\|_2^2\right)^{1/2},$$

we can obtain, for any $0 < \delta < \sqrt{V_\infty}$, a constant $c := c(\delta) > 0$ such that

$$|\omega(x)| \le c e^{-\frac{\delta}{h^*}|x|}, \quad \forall\, x \in \mathbb{R}^3.$$

Proof. The existence of the solution $\omega \in H^1(\mathbb{R}^3)$ as well as the alternative characterization of m_∞ is proved in [11]. In order to obtain the decay rate, we set $v(x) := \omega(xh^*)$ and notice that

$$-\Delta v = \frac{(h^*)^2\Big(a_\infty|\omega(xh^*)|^{p-1}\omega(xh^*) - V_\infty\omega(xh^*)\Big)}{1+\|\nabla\omega\|_2^2}, \quad x \in \mathbb{R}^3,$$

and therefore, by the definition of h^*, we get

$$-\Delta v + V_\infty v = a_\infty |v|^{p-1}v, \quad x \in \mathbb{R}^3.$$

As proved in the paper [7], for any $0 < \delta < \sqrt{V_\infty}$, there exists $c := c(\delta) > 0$ such that $|v(x)| \le ce^{-\delta|x|}$, $x \in \mathbb{R}^3$, and therefore the result follows from the definition of v. □

We now come back to the problem (P). From now on, we shall assume that the relation (1.2) holds with the number h^* given by the previous lemma.

We denote by X the space $H^1(\mathbb{R}^3)$ endowed with the norm

$$\|u\| := \left(\|\nabla u\|_2^2 + \int V(x)u^2\right)^{1/2}, \quad u \in X.$$

In view of $(V_1) - (V_2)$, the above norm is equivalent to the usual norm of $H^1(\mathbb{R}^3)$ (see [10]). The energy functional associated to the problem (P) is

$$I(u) := \frac{1}{2}\|u\|^2 + \frac{1}{4}\|\nabla u\|_2^4 - \int a(x)|u|^{p+1}, \quad u \in X.$$

Lemma 2.2. *If a satisfies (a_1), then the set*

$$\mathcal{N} := \Big\{u \in X\backslash\{0\} \,:\, I'(u)u = 0\Big\}$$

is a C^1-manifold. Moreover,

$$m := \inf_{u\in\mathcal{N}} I(u) > 0.$$

Proof. For each $n \in \mathbb{N}$, consider $\omega_n(x) := \omega(x + x_n)$ where $\omega \in X$ is given by Lemma 2.1 and $x_n := (0, 0, n)$. By (a_1), we have that $\int a(x)|\omega_n|^{p+1} \to \int a_\infty|\omega|^{p+1} > 0$. Hence, for any $n \ge n_0$, we have that

$$\int a(x)|\omega_n|^{p+1} > 0.$$

For this values of n, the function $f_n(t) := I(t\omega_n)$, $t > 0$, is positive near the origin and goes to $-\infty$ as $t \to +\infty$. So, f_n achieves its maximum value at $t_n > 0$. Since $f_n'(t_n) = 0$ we conclude that $t_n\omega_n \in \mathcal{N}$ and therefore $\mathcal{N}$ is non empty.

By the Sobolev embedding and (a_0) we have $\|u\|^2 \le \int a(x)|u|^{p+1} \le c_1\|u\|^{p+1}$, for any $u \in \mathcal{N}$ and some some $c_1 > 0$. Since $(p+1) > 2$, we obtain $\rho > 0$ such that

$$\|u\|^2 \ge \rho, \quad \forall\, u \in \mathcal{N}. \tag{2.1}$$

If we define $J : X \to \mathbb{R}$ by $J(u) := I'(u)u$, we can use $u \in \mathcal{N}$, a simple computation and $(p+1) > 4$ to get

$$J'(u)u = \Big(2 - (p+1)\Big)\|u\|^2 + \Big(4 - (p+1)\Big)\|\nabla u\|_2^4 < 0. \tag{2.2}$$

This, (2.1) and the Implicit Function Theorem imply that $\mathcal{N}$ is a C^1-manifold.

Finally, if $u \in \mathcal{N}$, we have that

$$I(u) = I(u) - \frac{1}{p+1} I'(u)u \geq \left(\frac{1}{2} - \frac{1}{p+1}\right)\|u\|^2, \tag{2.3}$$

and it follows from (2.1) that $m > 0$. The lemma is proved. □

Proposition 2.3. *If V and a satisfy (V_2) and (a_1), respectively, then*

$$m < m_\infty.$$

Proof. Let ω_n and t_n as in the proof of Lemma 2.2. Since $t_n\omega_n \in \mathcal{N}$, we can use (V_2) and (a_1) to write

$$t_n^{-2}\left(\|\omega\|_*^2 + o_n(1)\right) + \|\nabla\omega\|_2^4 = t_n^{p-3}\left(\int a_\infty|\omega|^{p+1} + o_n(1)\right),$$

where $o_n(1)$ stands for a quantity approaching zero as $n \to +\infty$. By the above expression (t_n) is bounded. Hence, we may assume that $t_n \to t_0 \geq 0$. By (2.1),

$$0 < \rho \leq \|t_n\omega_n\|^2 = t_n^2\left(\|\omega\|_*^2 + o_n(1)\right),$$

and therefore $t_0 > 0$.

We now notice that

$$m \leq I(t_n\omega_n) = I_\infty(t_n\omega_n) + \frac{t_n^2}{2}V_n + \frac{t_n^{p+1}}{p+1}A_n,$$

with

$$V_n := \int \Big(V(x) - V_\infty\Big)\omega_n^2, \quad A_n := \int \Big(a_\infty - a(x)\Big)|\omega_n|^{p+1}.$$

We claim that, for some constants c_1, $c_2 > 0$, there hold

$$V_n \leq -c_1 e^{-\mu n}, \qquad A_n \leq c_2 e^{-\gamma n}, \tag{2.4}$$

If this is true, we have that

$$m \leq I_\infty(t_n\omega_n) + t_n^2 e^{-\mu n}\left(-\frac{1}{2}c_1 + \frac{t_n^{p-1}}{p+1}c_2 e^{(\mu-\gamma)n}\right).$$

Since $\mu < \gamma$ and $t_n \to t_0 > 0$, we obtain $n_0 \in \mathbb{N}$ such that $m < I_\infty(t_n\omega_n)$, for $n \geq n_0$. It follows from Lemma 2.1 that

$$m < I_\infty(t_n\omega_n) = I_\infty(t_n\omega) \leq \max_{t\geq 0} I_\infty(t\omega) = I_\infty(\omega) = m_\infty$$

and we have done.

It remains to prove (2.4). Since $|x - x_n| \leq |x| + n$, we infer from (V_2) that

$$\begin{aligned} V_n &\leq -c_V \int e^{-\mu|x|}\omega_n^2 = -c_V \int e^{-\mu|x-x_n|}\omega^2 \\ &\leq -c_V e^{-\mu n} \int e^{-\mu|x|}\omega^2 = -c_1 e^{-\mu n}. \end{aligned}$$

For the second inequality we pick $\delta \in \left(\frac{\gamma h_*}{p+1}, \sqrt{V_\infty}\right)$ and obtain, from Lemma 2.1, $c = c(\delta) > 0$ such that

$$|\omega(x)| \leq c e^{-\frac{\delta}{h^*}|x|}, \quad \forall\, x \in \mathbb{R}^3.$$

Hence, recalling that $n - |x| \leq |x - x_n|$, it follows from (a_1) and the former argument that

$$A_n \leq c_a \int e^{-\gamma|x|}|\omega_n|^{p+1} \leq c \cdot c_a e^{-\gamma n} \int e^{\left(\gamma - \frac{\delta(p+1)}{h^*}\right)|x|}.$$

Since $\gamma < \frac{\delta(p+1)}{h^*}$, the last integral above is finite and the proof is finished. □

Lemma 2.4. *There exists a bounded sequence* $(u_n) \subset \mathcal{N}$ *such that*

$$I(u_n) \to m, \qquad I'(u_n) \to 0.$$

Moreover, $u_n \rightharpoonup u_0$ *weakly in* X *with* $I'(u_0) = 0$.

Proof. The Ekeland Variational Principle provides $(u_n) \subset \mathcal{N}$ and $(\lambda_n) \subset \mathbb{R}$ such that

$$I(u_n) \to m, \qquad I'(u_n) + \lambda_n J'(u_n) \to 0,$$

with $J(u) = I'(u)u$. Using (2.2) and a standard argument we can show that $\lambda_n \to 0$, and therefore $I'(u_n) \to 0$. Moreover, by (2.3), we have that (u_n) is bounded and therefore we may assume that, for some $u_0 \in X$, there holds

$$u_n \rightharpoonup u_0, \text{ weakly in } X \text{ and } \mathcal{D}^{1,2}(\mathbb{R}^3), \qquad \|\nabla u_n\|_2^2 \to A^2, \tag{2.5}$$

with $A \in \mathbb{R}$. Hence, we can easily conclude that u_0 weakly satisfies

$$-\left(1 + A^2\right)\Delta u_0 + V(x)u_0 = a(x)|u_0|^{p-1}u_0, \quad x \in \mathbb{R}^3. \tag{2.6}$$

If $u_0 = 0$ the lemma is proved. So, we may assume that $u_0 \not\equiv 0$ and we shall prove that $A^2 = \|\nabla u_0\|_2^2$. First notice that $\|\nabla u_0\|_2^2 \leq \liminf_{n\to\infty} \|\nabla u_n\|_2^2 = A^2$. Suppose, by contradiction, that $\|\nabla u_0\|_2^2 < A^2$. Since u_0 satisfies (2.6) we have that

$$\int a(x)|u_0|^{p+1} = \|u_0\|^2 + \|\nabla u_0\|_2^2 A^2 > 0.$$

Hence, there exists $t_0 > 0$ such that $t_0 u_0 \in \mathcal{N}$. Thus, by using (2.6) and $\|\nabla u_0\|_2^2 < A^2$, we obtain

$$t_0^2\|u_0\|^2 + t_0^4\|\nabla u_0\|_2^4 = t_0^{p+1}\left(\|u_0\|^2 + A^2\|\nabla u_0\|_2^2\right) > t_0^{p+1}\left(\|u_0\|^2 + 2\|\nabla u_0\|_2^4\right),$$

and therefore

$$\left(t_0^2 - t_0^{p+1}\right)\|u_0\|^2 + \left(t_0^4 - t_0^{p+1}\right)\|\nabla u_0\|_2^4 > 0.$$

Since $(p+1) > 4$ we infer that $t_0 \in (0,1)$ and we can use (2.5) to get

$$\begin{aligned} m &\leq I(t_0u_0) - \frac{1}{p+1} I'(t_0u_0)t_0u_0 \\ &< \Big(\frac{1}{2} - \frac{1}{p+1}\Big)\|u_0\|^2 + \Big(\frac{1}{4} - \frac{1}{p+1}\Big)\|\nabla u_0\|_2^4 \\ &\leq \liminf_{n\to+\infty} \left[\Big(\frac{1}{2} - \frac{1}{p+1}\Big)\|u_n\|^2 + \Big(\frac{1}{4} - \frac{1}{p+1}\Big)\|\nabla u_n\|_2^4\right] \\ &= \liminf_{n\to+\infty} \left[I(u_n) - \frac{1}{p+1} I'(u_n)u_n\right] = m, \end{aligned} \tag{2.7}$$

which is a contradiction. Hence, $A^2 = \|\nabla u_0\|_2^2$ and it follows from (2.6) that $I'(u_0) = 0$. □

We are ready to prove our first theorem.

Proof of Theorem 1.1. Let $(u_n) \subset \mathcal{N}$ be the sequence given by Lemma 2.4 and u_0 its weak limit. Suppose, by contradiction, that $u_0 \equiv 0$. Given $\varepsilon > 0$, we can use (V_2) to obtain $R > 0$ such that

$$\left| \int_{\mathbb{R}^3 \setminus B_R(0)} (V(x) - V_\infty) u_n^2 dx \right| \leq \varepsilon \|u_n\|_2^2. \tag{2.8}$$

Moreover, by using (V_0) and Hölder's inequality, we get

$$\int_{B_R(0)} |V(x) - V_\infty| u_n^2 \leq \|V - V_\infty\|_{L^t(B_R(0))} \|u_n\|^2_{L^{2t'}(B_R(0))}.$$

Since $t > 3/2$, we have that $2t' \in (2,6)$. Thus, we can use the compact embedding $H_0^1(B_R(0)) \hookrightarrow L^{2t'}(B_R(0))$ to conclude that the right-hand side above goes to zero. This, (2.8) and the boundedness of (u_n) imply that $\int V(x)u_n^2 = \int V_\infty u_n^2 + o_n(1)$. Since $a \in L^\infty(\mathbb{R}^3)$, a similar argument holds for $\int a(x)|u_n|^{p+1}$, and therefore

$$\|u_n\|^2 = \|u_n\|_*^2 + o_n(1), \quad \int a(x)|u_n|^{p+1} = \int a_\infty |u_n|^{p+1} + o_n(1). \tag{2.9}$$

Since $u_n \in \mathcal{N}$, it follows that

$$\|u_n\|_*^2 + \|\nabla u_n\|_2^4 = \int a_\infty |u_n|^{p+1} + o_n(1).$$

So, if we consider $t_n > 0$ such that $t_n u_n \in \mathcal{N}_\infty$, we can argue as in the proof of Lemma 2.4 to get

$$\left(t_n^2 - t_n^{p+1}\right)\|u_n\|_*^2 + \left(t_n^4 - t_n^{p+1}\right)\|\nabla u_n\|_2^4 = o_n(1).$$

The first equality in (2.9) together with (2.1) provide $\|u_n\|_* \geq \rho_1 > 0$, for n large. Hence, the above expression implies that $t_n \to 1$.

Now, using the boundedness of (u_n) and (2.9) again, we obtain

$$\begin{aligned} m_\infty &\leq I_\infty(t_n u_n) - \frac{1}{p+1} I'_\infty(t_n u_n)(t_n u_n) \\ &= \Big(\frac{1}{2} - \frac{1}{p+1}\Big) t_n^2 \|u_n\|_*^2 + \Big(\frac{1}{4} - \frac{1}{p+1}\Big) t_n^4 \|\nabla u_n\|_2^4 \\ &= \Big(\frac{1}{2} - \frac{1}{p+1}\Big) \|u_n\|^2 + \Big(\frac{1}{4} - \frac{1}{p+1}\Big) \|\nabla u_n\|_2^4 + o_n(1) \\ &= I(u_n) - \frac{1}{p+1} I'(u_n) u_n + o_n(1). \end{aligned}$$

Taking the limit we conclude that $m_\infty \leq m$, which contradicts Proposition 2.3. Hence, $u_0 \not\equiv 0$ is a non zero solution of the problem (P). The same trick used in (2.7) shows that $I(u_0) = m$, that is, u_0 is a ground state solution.

By setting $v := |u_0|$, we can easily conclude that $I(v) = I(u_0) = m$ and $v \in \mathcal{N}$. Hence $I'(v) = \lambda J'(v)$, for some $\lambda \in \mathbb{R}$. Since $I'(v)v = 0$ and $J'(v)v < 0$, it follows that $\lambda = 0$. Thus, $v \in X$ is a non negative ground state solution of (P). □

3. The nodal solution

In this section we obtain a sign changing solutions for the problem (P). We shall assume (V_1), (a_0) and denote $u^+(x) := \max\{u(x), 0\}$ and $u^-(x) := \max\{-u(x), 0\}$. Differently from the previous section we are going to work in a space of radial functions. So, we set

$$X_{rad} := X \cap \{u \in H^1(\mathbb{R}^3) \,:\, u(x) = u(|x|) \text{ for a.e. } x \in \mathbb{R}^3\}$$

and denote by I the same functional of the last section but now constrained to X_{rad}. We also define the following set

$$\mathcal{N}^\pm := \Big\{u \in X_{rad} \,:\, u^+, u^- \not\equiv 0,\, I'(u)u^+ = 0 = I'(u)u^-\Big\}.$$

Although it is not a differentiable manifold, we can obtain a nodal solution along a minimizing process. We first prove a version of Lemma 2.2 for this new set.

Lemma 3.1. *Suppose that a satisfies $(\widetilde{a_1})$. Then, the set $\mathcal{N}^\pm$ is non empty and*

$$m^\pm := \inf_{\mathcal{N}^\pm} I(u) > 0.$$

Proof. By $(\widetilde{a_1})$, we can obtain $R > 0$ such that $a(x) \geq a_\infty/2 > 0$, for a.e. $x \in \mathbb{R}^3/B_R(0)$. Hence, picking $u \in X_{rad}$ with support contained in $\mathbb{R}^3/B_R(0)$ and $u^\pm \not\equiv 0$, we have that

$$\int a(x)|u^\pm|^{p+1} > 0.$$

Let $h_u : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ be defined as $h_u(t,s) := I(tu^+ - su^-)$, that is,

$$\begin{aligned} h_u(t,s) = {} & \frac{t^2}{2}\|u^+\|^2 + \frac{s^2}{2}\|u^-\|^2 + \frac{1}{4}\Big(t^2\|\nabla u^+\|^2 + s^2\|\nabla u^-\|^2\Big)^2 \\ & - \frac{t^{p+1}}{p+1}\int a(x)|u^+|^{p+1} - \frac{s^{p+1}}{p+1}\int a(x)|u^-|^{p+1}. \end{aligned} \tag{3.1}$$

Since $(p+1) > 4$, we have that

$$\lim_{|(t,s)|\to\infty} h_u(t,s) = -\infty.$$

Hence, h_u attains its maximum value at some point $(t_u, s_u) \in \mathbb{R}^2$.

We claim that $t_u,\ s_u > 0$. Indeed, suppose by contradiction that $s_u = 0$. Hence, $(t_u, 0)$ is the maximum point of h_u. Picking $s > 0$ small in such way that $I(su^-) > 0$, we obtain from (3.1) that

$$h_u(t_u,0) = I(t_u u^+) < I(t_u u^+) + I(su^-) + \frac{t_u^2 s^2}{2}\|\nabla u^+\|_2^2\|\nabla u^-\|_2^2 = h_u(t_u,s),$$

which is a contradiction. The proof that $s_u > 0$ is analogous.

Since $t_u,\ s_u > 0$ we have that

$$\frac{\partial h_u}{\partial t}(t_u,s_u) = I'(t_u u^+ - s_u u^-)u^+ = 0$$

and

$$\frac{\partial h_u}{\partial s}(t_u,s_u) = -I'(t_u u^+ - s_u u^-)u^- = 0,$$

which imply $(t_u u^+ - s_u u^-) \in \mathcal{N}^{\pm}$, proving the first statement of the lemma. The second one is an easy consequence of $\mathcal{N}^{\pm} \subset \mathcal{N}$ and (2.1). □

Lemma 3.2. *Suppose that a satisfies* $(\widetilde{a_1})$. *Then, for any* $u \in \mathcal{N}^{\pm}$, *the function* $h_u : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ *defined in* (3.1) *attains its (strict) global maximum value at* $(1,1)$.

Proof. Since $I'(u)u^{\pm} = 0$, we have that $\int a(x)|u^{\pm}|^{p+1} > 0$. Thus, arguing as in the proof of Lemma 3.1, we conclude that h_u has a maximum point (t_u, s_u) such that $t_u,\ s_u > 0$.

We claim that $t_u, s_u \leq 1$. For the proof, we first consider the case $s_u \leq t_u$. Recalling that $\frac{\partial h_u}{\partial t}(t_u,s_u) = 0$, we get

$$\begin{aligned} t_u^{p+1}\int a(x)|u^+|^{p+1} &= t_u^2\|u^+\|^2 + t_u^4\|\nabla u^+\|_2^4 + t_u^2 s_u^2\|\nabla u^+\|_2^2\|\nabla u^-\|_2^2 \\ &\leq t_u^2\|u^+\|^2 + t_u^4\|\nabla u^+\|_2^4 + t_u^4\|\nabla u^+\|_2^2\|\nabla u^-\|_2^2. \end{aligned}$$

On the other hand, since $u \in \mathcal{N}^{\pm}$, we have that

$$\int a(x)|u^+|^{p+1} = \|u^+\|^2 + \|\nabla u^+\|_2^4 + \|\nabla u^+\|_2^2\|\nabla u^-\|_2^2.$$

and therefore

$$\Big(t_u^2 - t_u^{p+1}\Big)\|u^+\|^2 + \Big(t_u^4 - t_u^{p+1}\Big)\|\nabla u^+\|_2^4 + \Big(t_u^4 - t_u^{p+1}\Big)\|\nabla u^+\|_2^2\|\nabla u^-\|_2^2 \geq 0.$$

Since $(p+1) > 4$ we infer from the above expression that $t_u \leq 1$. The case $t_u \leq s_u$ is analogous. Moreover, by using $\frac{\partial h_u}{\partial s}(t_u, s_u) = 0$ and the same argument we can prove that $s_u \leq 1$.

Now, since

$$h_u(t_u, s_u) = I(t_u u^+ - s_u u^-) - \frac{1}{p+1} I'(t_u u^+ - s_u u^-)(t_u u^+ - s_u u^-),$$

we have that

$$h_u(t_u, s_u) = \left(\frac{1}{2} - \frac{1}{p+1}\right)\left(t_u^2\|u^+\|^2 + s_u^2\|u^-\|^2\right) +$$

$$\left(\frac{1}{4} - \frac{1}{p+1}\right)\left\{t_u^4\|\nabla u^+\|_2^4 + s_u^4\|\nabla u^-\|_2^4 + 2s_u^2 t_u^2\|\nabla u^+\|_2^2\|\nabla u^-\|_2^2\right\}.$$

If $\min\{t_u, s_u\} < 1$, then $h_u(t_u, s_u) < h_u(1,1)$, which does not make sense. Hence, $t_u = s_u = 1$. Finally, the same calculation performed above shows that this maximum point is strict. □

Lemma 3.3. *There exists $u_1 \in \mathcal{N}^{\pm}$ such that $I(u_1) = m^{\pm}$.*

Proof. Let $(u_n) \subset \mathcal{N}^{\pm}$ be such that $I(u_n) \to m^{\pm}$. By (2.3), (u_n) is bounded and we may assume that

$$u_n^{\pm} \rightharpoonup u^{\pm}, \quad \text{weakly in } X_{rad}. \tag{3.2}$$

Since $I'(u_n)u_n^{\pm} = 0$, we have that

$$\|u_n^{\pm}\|^2 + \left(\|\nabla u_n^+\|^2 + \|\nabla u_n^-\|^2\right)\|\nabla u_n^{\pm}\|^2 = \int a(x)|u_n^{\pm}|^{p+1}.$$

As in the proof of (2.1) we can obtain $\rho_1 > 0$ such that $\|u_n^{\pm}\|^2 \geq \rho_1$. Thus, the above equality implies that $\int a(x)|u_n^{\pm}|^{p+1} \geq \rho_1$. Taking the limit and recalling that X_r is compactly embedded in $L^{p+1}(\mathbb{R}^3)$ we get

$$\int a(x)|u^{\pm}|^{p+1} \geq \rho_1 > 0.$$

The above inequality and the same argument used in the proof of Lemma 3.1 provides $(t_u, s_u) \in \mathbb{R}^2$ such that $u_1 := (t_u u^+ - s_u u^-) \in \mathcal{N}^{\pm}$. By using Lemma 3.2, the weak convergence in (3.2) and the compact embedding again we obtain

$$\begin{aligned} m^{\pm} &\leq I(u_1) \leq \liminf_{n\to\infty} I(t_u u_n^+ - s_u u_n^-) \\ &= \liminf_{n\to\infty} h_{u_n}(t_u, s_u) \\ &\leq \liminf_{n\to\infty} h_{u_n}(1,1) = \liminf_{n\to\infty} I(u_n) = m^{\pm}. \end{aligned}$$

The lemma is proved. □

We present now the proof of our second theorem.

Proof of Theorem 1.2. By the previous lemma, there exists $u_1 \in \mathcal{N}^{\pm}$ such that $I(u_1) = m^{\pm} > 0$. Since u_1 clearly changes sign, it is sufficient to prove that $I'(u_1) = 0$. Suppose, by contradiction, that this is not the case. Then, there exist $\delta, \lambda > 0$ such that $\|I'(v)\| > \lambda$ whenever $\|v - u_1\| < 3\delta$. Setting $g(t,s) := tu_1^+ - su_1^-$, we can use Lemma 3.2 to obtain $D \subset \mathbb{R}^2$ such that $(1,1) \in D$ and

$$\alpha := \max_{(t,s)\in\partial D} I(g(t,s)) = \max_{(t,s)\in\partial D} h_{u_1}(t,s) < m^{\pm}. \tag{3.3}$$

For $\varepsilon < \min\{(m^{\pm} - \alpha)/2, \lambda\delta/8\}$ and $S := B_\delta(u_1)$, it follows from [15, Lemma 2.3] that there exists $\eta \in C([0,1] \times X_{rad}, X_{rad})$ verifying

(i) $\eta(1,u) = u$, if $u \notin I^{-1}([m^{\pm} - 2\varepsilon, m^{\pm} + 2\varepsilon])$;
(ii) $\eta(1, I^{m^{\pm}+\varepsilon} \cap S) \subset I^{m^{\pm}-\varepsilon}$;
(iii) $I(\eta(1,u)) \le I(u)$, for any $u \in X_{rad}$.

By Lemma 3.2, (ii) and (iii) it follows that

$$\max_{(t,s)\in D} I(\eta(1, g(t,s))) < m^{\pm}. \tag{3.4}$$

We define

$$\psi(t,s) := (I'(g(t,s))u_1^+, I'(g(t,s))u_1^-) \tag{3.5}$$

and

$$\Psi(t,s) := (t^{-1}I'(f(t,s))f(t,s)^+, s^{-1}I'(f(t,s))f(t,s)^-), \tag{3.6}$$

where $f(t,s) := \eta(1, g(t,s))$. Since $u_1 \in \mathcal{N}^{\pm}$, then $\psi(t,s) = 0$ if, and only if, $(t,s) = (1,1) \in D$. Thus, it follows from the definition of the Brouwer degree that

$$\deg(\psi, D, 0) = \operatorname{sgn} \det \psi'(1,1).$$

A direct computation shows that $\det \psi'(1,1) = (a_{i,j})$ with

$$a_{1,1} = \|u^+\|^2 + 3\|\nabla u^+\|_2^4 + \|\nabla u^-\|_2^2\|\nabla u^+\|_2^2 - p\int a(x)|u^+|^{p+1}$$

$$a_{2,2} = -\|u^-\|^2 - 3\|\nabla u^-\|_2^4 - \|\nabla u^-\|_2^2\|\nabla u^+\|_2^2 + p\int a(x)|u^-|^{p+1}$$

and

$$a_{1,2} = 2\|\nabla u^-\|_2^2\|\nabla u^+\|_2^2, \quad a_{2,1} = -2\|\nabla u^-\|_2^2\|\nabla u^+\|_2^2.$$

Since $u \in \mathcal{N}^{\pm}$, we have that

$$\begin{aligned} a_{1,1} &= 2\|\nabla u^+\|_2^4 - (p-1)\int a(x)|u^+|^{p+1} \\ &\le 2\left(\|\nabla u^+\|_2^4 - \int a(x)|u^+|^{p+1}\right) \\ &\le 2\left(-\|u^+\|^2 - \|\nabla u^-\|_2^2\|\nabla u^+\|_2^2\right) \\ &\le a_{2,1}. \end{aligned}$$

An analogous argument shows that $a_{2,2} \ge a_{1,2}$. Hence, $\det \psi'(1,1) = a_{1,1}a_{2,2} - a_{1,2}a_{2,1} < 0$ and we conclude that

$$\deg(\psi, D, 0) = -1.$$

The choice of $\varepsilon > 0$, (3.3) and (i) imply that $g = f$ on ∂D. It follows from (3.5) and (3.6) that $\psi = \Psi$ on ∂D and

$$\deg(\Psi, D, 0) = \deg(\psi, D, 0) = -1.$$

Hence, there exists $(t, s) \in D$ such that $f(t, s) \in \mathcal{N}^{\pm}$, which contradicts (3.4). Thus, $I'(u_1) = 0$ and the theorem is proved. □

Acknowledgment

The authors thank the referee for pointing out a small gap in the first version of the proof Theorem 1.1.

References

[1] C.O. Alves, F.J.S.A. Correia and T.F. Ma, *Positive solutions for a quasilinear elliptic equation of Kirchhoff type,* Comput. Math. Appl. **49** (2005), 85-93.

[2] C.O. Alves and M.A. Souto, *Existence of least energy nodal solution for a Schrödinger-Poisson system in bounded domains*, Z. Angew. Math. Phys. **65** (2014), 1153-1166.

[3] A. Azzollini, *The elliptic Kirchhoff equation in $\mathbb{R}^N$ perturbed by a local nonlinearity*, Differential Integral Equations **25** (2012), 543-554.

[4] A. Azzollini, *A note on the elliptic Kirchhoff equation in $\mathbb{R}^N$ perturbed by a local nonlinearity*, Commun. Contemp. Math., **17** (2015), Art. ID 1450039, 5 pp.

[5] A. Azzollini, P. d'Avenia and A. Pomponio, *Multiple critical points for a class of nonlinear functionals*, Ann. Mat. Pura Appl. **190** (2011), 507-523.

[6] A.M. Batista and M.F. Furtado *Positive and nodal solutions for a nonlinear Schrödinger-Piosson system with sign-changing potentials,* Nonlinear Anal. Real World Appl. **39** (2018), 142-156.

[7] H. Berestycki and P.L. Lions, emphNonlinear scalar field equations. I. Existence of a ground state, Arch. Rational Mech. Anal. **82**, (1983), 313-345.

[8] H. Chen and H. Liu, *Multiple solutions for an indefinite Kirchoff-type equation with sign-changing potential,* Electronic J. Diff. Equations **274** (2015), 1-9.

[9] G.M. Figueiredo and J.R. Santos Junior, *Existence of a least energy nodal solution for a Schrödinger-Kirchhoff equation with potential vanishing at infinity,* J. Math. Phys. **56** (2015), no. 5.

[10] M.F. Furtado, L.A. Maia and E.S. Medeiros, *Positive and nodal solutions for a nonlinear Schrödinger equation with indefinite potential,* Adv. Nonlinear Stud. **8** (2008), 353-373.

[11] X. He and W. Zou, *Existence and concentration behavior of positive solutions for a Kirchhoff equation in $\mathbb{R}^3$,* J. Differential Equations **252** (2012), no. 2, 1813-1834.

[12] G. Kirchhoff, Mechanik, Teubner,Leipzig, 1883.

[13] G. Li and H. Ye, *Existence of positive ground state solutions for the nonlinear Kirchhoff type equations in $\mathbb{R}^3$,* J. Differential Equations **257** (2014), 566-600.

[14] J.L. Lions, *On some questions in boundary value problems of mathematical physics.* International Symposium on Continuum, Mechanics and Partial Differential Equations, Rio de Janeiro(1977), Mathematics Studies, North- Holland, Amsterdam, **30** (1978), 284-346.

[15] M. Willem, *Minimax Theorems*, Birkhauser, Boston, Basel, Berlim, (1996).

[16] Y. Wu and Y. Huang, *Sign-changing solutions for Schrödinger equations with indefinite supperlinear nonlinearities,* J. Math. Anal. Appl. **401** (2013), 850-860.

Alex M. Batista
Universidade de Brasília, Departamento de Matemática, 70910-900, Brasília-DF, Brazil
e-mail: `alexdemourabatista@hotmail.com`

Marcelo F. Furtado
Universidade de Brasília, Departamento de Matemática, 70910-900, Brasília-DF, Brazil
e-mail: `mfurtado@unb.br`