POSITIVE AND NODAL SOLUTIONS FOR A NONLINEAR SCHRÖDINGER-POISSON SYSTEM WITH SIGN-CHANGING POTENTIALS

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Abstract. We use a variational approach to deal with the system

$$-\Delta u + V(x)u + K(x)\phi u = a(x)|u|^{p-1}u, \qquad -\Delta\phi = K(x)u^2, \qquad x \in \mathbb{R}^3,$$

whith $3 < p < 5$. Under some mild conditions on the sign-changing potentials V and a we obtain two nonzero solutions. One of them is nonnegative and the other one changes sign.

1. Introduction

In this paper we look for minimal solutions for the system

$$\begin{cases} -\Delta u + V(x)u + K(x)\phi u = a(x)|u|^{p-1}u, & \text{in } \mathbb{R}^3, \\ -\Delta\phi = K(x)u^2, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $3 < p < 5$ and the potentials V, K and a satisfy some mild conditions. As quoted by Benci and Fortunato in [4, 5], this system works as a model describing solitary waves for the nonlinear stationary Schrödinger equation interacting with the electrostatic field and also in semiconductor theory, nonlinear optics and plasma physics.

In the recent years, many researches focused on existence, nonexistence, multiplicity and concentration of solutions for the above problem. In [10], the authors obtained a radial solution for constant positive potentials and $3 < p < 5$. In [11], the same result was proved for $3 \le p < 5$. In [22], the authors presented existence and nonexistece results also for the radial case and a larger spectrum of the power p. The nonradial case was treated in [7] for $V \equiv 1$. In that paper it is allowed that a changes sign with an integrability condition for the functin $a(x) - a_\infty$ and some other technical conditions. In [27], the author considered $V \equiv 1$ but a being a sign-changing function with both the positive and negative part unbounded. We also quote the paper [9], where the author considered $V \equiv K \equiv 1$, a changing sign and add a term $\lambda g(x)u$ on the right-hand of the equation. With some conditions on the parameter λ they obtained the existence of two positive solutions. We also mention the paper [15], which seems to be the first one to consider the potential K not constant.

In the aforementioned works the authors obtained nonnegative solutions. There are only a few works concerning sign-changing solutions. In [17], the authors considered constant positive potentials and obtained, for each $k \in \mathbb{N}$, a nodal solutions

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having exactly $k-1$ nodal domains (see also [14] for a dynamical approach and a less restricve condition on the growth of the nonlinearity). Nonradial nodal solutions in the semiclassical limit were obtained in [16]. In [24], the authors considered V not constant and obtained a nodal solution. Recently, the authors in [2] (see also [1]) obtained a nodal solution with V not constant but positive and an asymptotic condition similar to (V_1). We emphasize that, in all these previous works, it was supposed that K and a are constant and positive.

As we pointed out, in many papers the potentials has been supposed constant, radial or positive. The main objective of this paper is to complement all these works. In our first result, we look for nontrivial solution in the case that V and a can be nonconstant, nonradial and indefinite in sign. More specifically, if we denote $V^-(x) := \max\{-V(x), 0\}$ and

$$S := \inf\left\{\|\nabla u\|^2_{L^2(\mathbb{R}^3)}; u \in \mathcal{D}^{1,2}(\mathbb{R}^3),\ \|u\|_{L^6(\mathbb{R}^3)} = 1\right\},$$

we assume the following conditions.

(V_0) $V^- \in L^{3/2}(\mathbb{R}^3)$ and $\int_{\mathbb{R}^3} |V^-(x)|^{3/2} dx < S^{3/2}$;

(V_1) there exist $\gamma > 0$ and $C_V > 0$ such that

$$V(x) \le V_\infty - C_V e^{-\gamma|x|}, \quad \text{for a.e. } x \in \mathbb{R}^3,$$

where

$$0 < V_\infty := \lim_{|x|\to+\infty} V(x);$$

(a_0) $a \in L^\infty(\mathbb{R}^3)$;

(a_1) there exist $\theta > 0$ and $C_a > 0$ such that

$$a(x) \ge a_\infty - C_a e^{-\theta|x|}, \quad \text{for a.e. } x \in \mathbb{R}^3,$$

where

$$a_\infty := \lim_{|x|\to+\infty} a(x) > 0.$$

Concerning the potential K, we have a basic assumption of regularity which enable us to deal with a scalar problem. Actually, if $K \in L^2(\mathbb{R}^3)$, for each $u \in W^{1,2}(\mathbb{R}^3)$, the equation

$$-\Delta\phi = K(x)u^2, \quad \text{in } \mathbb{R}^3,$$

has a unique weak solution $\phi = \phi_u \in \mathcal{D}^{1,2}(\mathbb{R}^3)$. We can prove that system (1.1) is equivalent to the scalar equation

$$(P) \qquad \begin{cases} -\Delta u + V(x)u + K(x)\phi_u(x)u = a(x)|u|^{p-1}u, & \text{in } \mathbb{R}^3, \\ u \in W^{1,2}(\mathbb{R}^3). \end{cases}$$

Hence, we shall suppose that

(K_0) $K \in L^2(\mathbb{R}^3)$;

(K_1) there exist $\alpha > 0$ and $C_K > 0$ such that

$$0 \le K(x) \le C_K e^{-\alpha|x|} \quad \text{for a.e. } x \in \mathbb{R}^3.$$

The equation in (P) is the Euler-Lagrange equation of the energy functional $I : W^{1,2}(\mathbb{R}^3) \to \mathbb{R}$ given by

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^3}\left(|\nabla u|^2 + V(x)u^2\right)dx + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_u(x)u^2 dx - \frac{1}{p+1}\int_{\mathbb{R}^3} a(x)|u|^{p+1} dx,$$

We say that a solution u is a ground state solution if it has minimal energy $I(u)$ among all the nonzero solutions. The first result of this paper can be stated as follows.

Theorem 1.1. *Suppose that* $3 < p < 5$ *and the potentials satisfy* $(V_0) - (V_1)$, $(a_0) - (a_1)$ *and* $(K_0) - (K_1)$. *If*

$$\gamma < \min\{\theta, \alpha\}, \quad \theta < (p+1)\sqrt{V_\infty} \quad \text{and} \quad \alpha < 2\sqrt{V_\infty},$$

then the problem (P) *has a nonnegative ground state solution.*

In the proof we apply variational methods. Although this is rather standard we need to overcome the lack of compactness of the embedding of $W^{1,2}(\mathbb{R}^3)$ into the Lebesgue spaces. This is done by using some comparison arguments of the minimax level of the energy functional and that of the limit problem. In the calculations, the restrictions on the size of the parameters play an important role. Another difficult relies on the fact the potentials V and a can change sign. In this way, some known arguments of unique projection in the Nehari manifold do not hold here. Theorem 1 complements the results of [10, 11, 22, 7, 27] and improves that of [13], since we have here weaker conditions on the parameters γ, θ and α.

In our second result we obtain the existence of nodal solution with a stronger condition on the parameter α. More specifically, we shall prove the following theorem.

Theorem 1.2. *Suppose that* $3 < p < 5$ *and the potentials satisfy* $(V_0) - (V_1)$, $(a_0) - (a_1)$ *and* $(K_0) - (K_1)$. *If*

$$\gamma < \min\{\theta, \alpha\}, \quad \theta < (p+1)\sqrt{V_\infty} \quad \text{and} \quad \alpha < \sqrt{V_\infty},$$

then the problem (P) *has a nodal minimal solution.*

Although the proof follows the same lines of those presented in the first theorem, the calculations here are more involved. Actually, in some results on nodal solutions, the authors tried to minimize the energy functional restricted to some set which contains all the nodal solutions. Recalling that the map $u \mapsto u^+$ is not differentiable, we conclude that the same holds for this set. Since for any nodal solution of (P) we have that $I'(u^+)u^+ < 0$, the usual set $\{u \in W^{1,2}(\mathbb{R}^3) : I'(u^\pm)u^\pm = 0\}$ is not a good choice for minimization set. Hence, we follow [1], by using $\{u \in W^{1,2}(\mathbb{R}^3) : I'(u)u^+ = 0 = I'(u)u^-\}$. Again, we need to make some trick calculations since the potentials V and a can change sign. The (stronger) condition on the exponent α is of technical nature and appears when we try to correct localize the minimax level of the energy functional in order to get compactness. Theorem 2 complements the results of [17, 14, 23, 24, 2].

We finaly mention the papers [20, 8, 26] where variants of our problem are considered with some of the potentials changing sign. In all those results, there is no information about the sign of the solution. However, our results are in some sense related (an not comparable) with those.

The paper is organized in the following way: in the next section, after presenting the abstract framework, we prove the existence of the nonnegative solution. In Section 3 we obtained nodal solutions for an auxiliary problem and in the final Section 4 we prove Theorem 1.2.

2. The positive solution

We first establish the variational framework to deal with the problem (P). For any $2 \leq q \leq \infty$, we denote by $\|u\|_q$ the L^q-norm of a function $u \in L^q(\mathbb{R}^3)$. For saving notation, we write only $\int u$ to denote $\int_{\mathbb{R}^3} u(x)\mathrm{d}x$. We also denote $u^+(x) := \max\{u(x), 0\}$ and $u^-(x) := \max\{-u(x), 0\}$.

If we set

$$X := \left\{ u \in W^{1,2}(\mathbb{R}^3) : \int V(x)u^2 < \infty \right\},$$

according to [12, Lemma 2.1], the quadratic form $u \mapsto \int (|\nabla u|^2 + V^+(x)u^2)$ defines a norm in X, which is equivalent to the usual norm of $W^{1,2}(\mathbb{R}^3)$. Moreover, we can use Hölder's inequality to obtain

$$\int V^-(x)u^2 \leq \|V^-\|_{3/2}\|u\|_6^2 \leq S^{-1}\|V^-\|_{3/2} \int |\nabla u|^2, \tag{2.1}$$

and therefore we conclude that the norm

$$\|u\| := \left(\int (|\nabla u|^2 + V(x)u^2) \right)^{\frac{1}{2}}.$$

is well defined and is equivalent to $\|\cdot\|_{W^{1,2}(\mathbb{R}^3)}$.

Since $K \in L^2(\mathbb{R}^3)$, a straightforward application of the Lax-Milgram Theorem implies that, for any given $u \in W^{1,2}(\mathbb{R}^3)$, there exists a unique $\phi = \phi_u \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ which weakly solves $-\Delta\phi = K(x)u^2$ in $\mathbb{R}^3$. Recalling the expression of the Green function of the Laplacian we obtain

$$\phi_u(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{K(y)u^2(y)}{|x-y|} dy. \tag{2.2}$$

We can construct the application $\phi : W^{1,2}(\mathbb{R}^3) \to \mathcal{D}^{1,2}(\mathbb{R}^3)$ which associates, for each $u \in W^{1,2}(\mathbb{R}^3)$, the function ϕ_u defined above. We collect below some properties of the map ϕ (see [7, Lemma 2.1] and [13, Lemma 2.2]).

Lemma 2.1. *For any $u \in X$ there holds*

1. $\phi_u(x) \geq 0$ *for a.e.* $x \in \mathbb{R}^3$;
2. $\phi_{tu} = t^2\phi_u$, *for any* $t > 0$;
3. *there exists* $c_1 > 0$ *such that* $\int |\nabla\phi_u|^2 \leq c_1\|u\|_6^2$;
4. *If* $u_n \rightharpoonup u$ *weakly in* X, *then*

$$\lim_{n\to\infty} \int K(x)\phi_{u_n}(x)u_n^2 = \int K(x)\phi_u(x)u^2$$

and

$$\lim_{n\to\infty} \int K(x)\phi_{u_n}(x)u_n\varphi = \int K(x)\phi_u(x)u\varphi, \qquad \forall\, \varphi \in X.$$

The main interest in the map ϕ comes from the fact that it enables us dealing with system (S) as a single equation. Actually, it can be proved that $(u, \phi_u) \in W^{1,2}(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$ is a solution of (S) if, and only if, it is a weak solution of the nonlocal problem (P) quoted in the introduction. Its equation has a variational structure and its energy functional is $I : X \to \mathbb{R}$ given by

$$I(u) := \frac{1}{2} \int \left(|\nabla u|^2 + V(x)|u|^2\right) + \frac{1}{4} \int K(x)\phi_u(x)u^2 - \frac{1}{p+1} \int a(x)|u|^{p+1}.$$

In view of (2.1), we can prove that $I \in C^1(X, \mathbb{R})$ and

$$I'(u)\varphi = \langle u, \varphi\rangle_X + \int K(x)\phi_u u\varphi - \int a(x)|u|^{p-1}u\varphi, \qquad \forall\, \varphi \in X.$$

Hence, the critical points of I are precisely the weak solutions of (P).

Since our potentials can be nonradial, it is necessary to overcome the lack of compactness of the embedding of X into the Lebesgue spaces. Thus, we are going to consider the limit problem

$$(P_\infty) \qquad -\Delta u + V_\infty u = a_\infty |u|^{p-1}u, \quad x \in \mathbb{R}^3,$$

with associated energy functional $I_\infty : X \to \mathbb{R}$ given by

$$I_\infty(u) := \frac{1}{2}\int (|\nabla u|^2 + V_\infty u^2) - \frac{1}{p+1}\int a_\infty |u|^{p+1}.$$

Its Nehari manifold is

$$\mathcal{N}_\infty := \Big\{ u \in X\backslash\{0\}; I'_\infty(u)u = 0 \Big\}$$

and the following result can be found in [6].

Proposition 2.2. *The problem (P_∞) has a positive radial solution $\omega \in W^{1,2}(\mathbb{R}^3)$ such that*

$$I_\infty(\omega) = m_\infty = \inf_{u \in \mathcal{N}_\infty} I_\infty(u).$$

Moreover, for any $0 < \delta < \sqrt{V_\infty}$, there exists $C = C(\delta) > 0$ such that

$$\omega(x) \le C e^{-\delta|x|}, \quad \forall\, x \in \mathbb{R}^3.$$

In order to obtain our first solution, we are going to minimize the functional I restricted to its Nehari manifold

$$\mathcal{N} := \Big\{ u \in X\backslash\{0\} : I'(u)u = 0 \Big\}.$$

We present in the sequel some important properties of this set. They are standard when the potential a is positive. In our case, even the potential being indefinite, we take advantage of the positivy of the aymptotic limit a_∞.

Lemma 2.3. *The following hold*

1. *$\mathcal{N}$ is non-empty and it is a C^1-manifold;*
2. *for any $u \in \mathcal{N}$, we have that $I(u) = \max_{t \ge 0} I(tu)$;*
3. *we have that*

$$m := \inf_{u \in \mathcal{N}} I(u) > 0.$$

Proof. Let ω be the solution given by Proposition 2.2, $x_n := (0, 0, n) \in \mathbb{R}^3$ and

$$\omega_n(x) := \omega(x + x_n), \qquad \forall\, x \in \mathbb{R}^3. \tag{2.3}$$

If follows from (a_1), the Lebesgue Theorem and the change of variables $x \mapsto (x + x_n)$ that

$$\lim_{n\to\infty} \int a(x)|\omega_n|^{p+1} = \int a_\infty |\omega|^{p+1} dx > 0. \tag{2.4}$$

Hence, for n large, $\int a(x)|\omega_n|^{p+1} > 0$. Since $3 < p < 5$, a simple calculation shows that the function $t \mapsto I(t\omega_n)$ achieves its global maximum in some $t_n > 0$ such that $I'(t_n\omega_n)(t_n\omega_n) = 0$, that is, $t_n\omega_n \in \mathcal{N} \neq \emptyset$. Since $\int K(x)\phi_u(x)u^2 \geq 0$ and $(p+1) > 2$, it is easy to obtain $c_1 > 0$ such that

$$\|u\| \geq c_1 > 0, \qquad \forall\, u \in \mathcal{N}. \tag{2.5}$$

Hence, if define $J(u) := I'(u)u$, we can use item 1 of Lemma 2.1 and $3 < p < 5$ to obtain, for any $u \in \mathcal{N}$,

$$J'(u)u \leq (2 - (p+1))\|u\|^2 < 0, \tag{2.6}$$

and therefore we can use the Implicit Function Theorem to guarantee the regularity of $\mathcal{N}$. In the last inequality above, we have used K, $\phi_u \geq 0$ and $3 < p < 5$.

If $u \in \mathcal{N}$ we have that $\|u\|^2 + \int K(x)\phi_u(x)u^2 = \int a(x)|u|^{p+1} > 0$ and therefore the second statement can be proved in a standard way. For the last one, it is sufficient to notice that

$$I(u) = I(u) - \frac{1}{p+1}I'(u)u \geq \Big(\frac{1}{2} - \frac{1}{p+1}\Big)\|u\|^2, \tag{2.7}$$

and use (2.5). The lemma is proved. □

We present below a key estimate for the proof of Theorem 1.1.

Proposition 2.4. *We have that $m < m_\infty$.*

Proof. For ω_n defined in (2.3), we can use (2.4) to conclude that the number t_n appearing in the proof of Lemma 2.3 is such that $t_n\omega_n \in \mathcal{N}$ for any $n \geq n_0$. Since $\|\omega_n\|_6 = \|\omega\|_6$, it follows from item 3 of Lemma 2.1 that

$$\int K(x)\phi_{\omega_n}(x)\omega_n^2 \leq \|\phi_{\omega_n}\|_6\|K\|_2\|\omega_n\|_6^2 \leq c_2\|K\|_2\|\omega\|_6^4,$$

for some $c_2 > 0$. This and $I'(t_n\omega_n)t_n\omega_n = 0$ imply that

$$0 \leq t_n^{-2}\|w_n\|^2 + c_2\|K\|_2\|\omega\|_6^4 - t_n^{p-3}\int a(x)|\omega_n|^{p+1},$$

from which it follows that (t_n) is bounded. Hence, we may assume that $t_n \to t_0 \geq 0$. By (2.5) and (V_1), we have that

$$0 < c_1 \leq t_n^2\|\omega_n\|^2 = t_n^2(\|\omega\|^2 + o_n(1)),$$

and therefore $t_0 > 0$.

The definition of I and I_∞ imply that

$$m \leq I(t_n\omega_n) = I_\infty(t_n\omega_n) + \frac{t_n^2}{2}V_n + \frac{t_n^4}{4}K_n + \frac{t_n^{p+1}}{p+1}A_n, \tag{2.8}$$

for

$$V_n := \int (V(x) - V_\infty)\omega_n^2, \quad K_n := \int K(x)\phi_{\omega_n}(x)\omega_n^2$$

and

$$A_n := \int (a_\infty - a(x))|\omega_n|^{p+1}.$$

Claim: There exists $C > 0$ such that

$$V_n \leq -Ce^{-\gamma n}, \quad K_n \leq Ce^{-\alpha n}, \quad A_n \leq Ce^{-\theta n}. \tag{2.9}$$

If this is true, since $I_\infty(t_n\omega_n) = I_\infty(t_n\omega) \le I_\infty(\omega) = m_\infty$, the above inequalities and (2.8) provide

$$m \le m_\infty + Ct_n^2 e^{-\gamma n}\left[-\frac{1}{2} + \frac{t_n^2}{4}e^{(\gamma-\alpha)n} + \frac{t_n^{p-1}}{p+1}e^{(\gamma-\theta)n}\right].$$

Recalling that $t_n \to t_0 > 0$ and $\gamma < \min\{\alpha, \theta\}$, we see that the expression into the brackets above is negative for n large.

It remains to verify the claim. Since $n - |x| \le |x - x_n|$, we can use (K_1) and Holder's inequality to get

$$K_n \le \int K(x)\phi_{\omega_n}(x)\omega_n^2 \le C_K e^{-\alpha n}\|\phi_{\omega_n}\|_6 \left(\int e^{\alpha 6/5|x|}\omega^{12/5}\right)^{5/6}. \tag{2.10}$$

Since $\alpha < 2\sqrt{V_\infty}$, we can use Proposition 2.2 to guarantee that the last integral above is finite. Moreover, by the embedding $\mathcal{D}^{1,2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$ and item 3 of Lemma 2.1, we have that (ϕ_{ω_n}) is bounded in $L^6(\mathbb{R}^3)$. The inequality for K_n follows from the above considerations.

An analogous argument and $\theta < (p+1)\sqrt{V_\infty}$ provides

$$A_n \le C_a \int e^{-\theta|x-x_n|}\omega^{p+1} \le C_a e^{-\theta n}\int e^{\theta|x|}\omega^{p+1} = Ce^{-\theta n}.$$

Since $|x + x_n| \le |x| + n$, we have that

$$V_n \le -C_V \int e^{-\gamma|x-x_n|}\omega^2 \le -C_V e^{-\gamma n}\int e^{-\gamma|x|}\omega^2$$

and the proof is finished. □

We are ready to prove our first result.

Proof of Theorem 1.1. The Ekeland Variational Principle provides $(u_n) \subset \mathcal{N}$ such that $I(u_n) \to m$ and $\|I'(u_n)_{|\mathcal{N}}\| \to 0$. Using (2.6), we can prove that $I'(u_n) \to 0$, in such way that (u_n) is a Palais-Smale sequence of I. A standard procedure proves that this sequence is bounded. Hence, along a subsequence, we have that $u_n \rightharpoonup u$ weakly in X. Morevoer, it follows from item 4 of Lemma 2.1 that $I'(u) = 0$.

Suppose, by contradiction, that $u \equiv 0$. Then, $u_n \to 0$ in $L^s_{loc}(\mathbb{R}^N)$ for any $s \in [2, 2^*)$. Thus, we can use (V_1), (a_1), (K_1) and the same calculation performed in (2.10) to obtain

$$\int a(x)|u_n|^{p+1} = \int a_\infty|u_n|^{p+1} + o_n(1), \qquad \int K(x)\phi_{u_n}u_n^2 = o_n(1), \tag{2.11}$$

$$\|u_n\|_*^2 := \left(\int |\nabla u_n|^2 + V_\infty u_n^2\right) = \|u_n\|^2 + o_n(1).$$

Thus, recalling that $I'(u_n)u_n = 0$, we get

$$\|u_n\|_*^2 = \int a_\infty|u_n|^{p+1} + o_n(1). \tag{2.12}$$

Let $t_n > 0$ be such that $t_n u_n \in \mathcal{N}_\infty$. By using that $u_n \in \mathcal{N}$, (2.5) and the first equality in (2.11), we conclude that $\int a(x)|u_n|^{p+1} \ge c_0 > 0$. Hence, the above equality implies that $t_n \to 1$ as $n \to +\infty$. It follows that

$$\begin{aligned} I_\infty(t_n u_n) &= I_\infty(t_n u_n) - \frac{1}{p+1} I'_\infty(t_n u_n)(t_n u_n) \\ &= \Big(\frac{1}{2} - \frac{1}{p+1}\Big) t_n^2 \|u_n\|_*^2 = \Big(\frac{1}{2} - \frac{1}{p+1}\Big) \|u_n\|^2 + o_n(1) \\ &= I(u_n) - \frac{1}{p+1} I'(u_n)u_n + o_n(1) = I(u_n) + o_n(1), \end{aligned}$$

and therefore

$$m_\infty \leq I_\infty(t_n u_n) = I(u_n) + o_n(1) = m + o_n(1).$$

Taking the limit into the above expression we obtain $m_\infty \leq m$, which contradicts Proposition 2.4. Thus, we have that $u \not\equiv 0$.

We have proved that $u \in \mathcal{N}$ is a nonzero critical point. Hence, we can use item 4 of Lemma 2.1 to obtain

$$\begin{aligned} m &\leq I(u) - \frac{1}{p+1} I'(u)u = \left(\frac{1}{2} - \frac{1}{p+1}\right) \|u\|^2 - \left(\frac{1}{4} - \frac{1}{p+1}\right) \int K(x)\phi_u u^2 \\ &\leq \liminf_{n\to+\infty} \left(I(u_n) - \frac{1}{p+1} I'(u_n)u_n \right) = m, \end{aligned}$$

from which it follows that $I(u) = m$.

Considering $\overline{u} = |u|$, we can use (2.2) to see that $\phi_u = \phi_{\overline{u}}$. Hence, an easy computation shows that $\overline{u} \in \mathcal{N}$ and $I(\overline{u}) = I(u) = m$. It follows from the Lagrange Multiplier Theorem that $I'(\overline{u}) = \lambda J'(\overline{u})$, for some $\lambda \in \mathbb{R}$. This implies that $0 = I'(\overline{u})\overline{u} = \lambda J'(\overline{u})\overline{u}$. By (2.6), $J'(\overline{u})\overline{u} < 0$, and therefore $\lambda = 0$. We conclude that $\overline{u} \geq 0$ is a non negative solution of the problem (P). Since $I(\overline{u}) = m$, it is a ground state solution. The theorem is proved. □

Remark 2.5. *If $\bar{u}$ is the non negative solution given by Theorem 1.1, we can use (K_1) to conclude that*

$$-\Delta \bar{u} + V(x)\bar{u} \leq a(x)|\bar{u}|^p, \qquad \forall\, x \in \mathbb{R}^3,$$

in the weak sense. Arguing as in [18, Theorem 1.11], *we can prove that $\bar{u}(x) \to 0$ as $|x| \to +\infty$. Hence, as in the proof of* [19, Theorem 3.1], *we can verify that, for any $\delta > 0$, there exists $C = C(\delta) > 0$, such that*

$$\bar{u}(x) \leq C e^{-\delta |x|}, \qquad \forall\, x \in \mathbb{R}^3. \tag{2.13}$$

3. The auxiliary problem

In this section we prove an auxiliary result which will be useful in the proof of the second theorem. We start by noticing that the nodal solutions of the problem (P) belongs to the set

$$\mathcal{N}^\pm := \Big\{ u \in \mathcal{N} : u^\pm \not\equiv 0,\ I'(u)u^+ = 0 = I'(u)u^- \Big\}.$$

For $R > 0$, we consider

$$\mathcal{N}_R^\pm := \mathcal{N}^\pm \cap H_0^1(B_R(0)).$$

We state in the sequel the main result of this section.

Proposition 3.1. *For any $R > 0$, the infimum*

$$m_R^{\pm} := \inf_{u \in \mathcal{N}_R^{\pm}} I(u)$$

is achieved at some $u_R \in \mathcal{N}_R^{\pm}$ such that $I'(u_R)\varphi =$, for any $\varphi \in H_0^1(B_R(0))$.

In order to prove the above result we shall adapt some arguments from [3, 1].

Lemma 3.2. *Let $u \in \mathcal{N}_R^{\pm}$ and define*

$$h^u(t,s) := I(tu^+ + su^-), \qquad \forall\, t,\, s \geq 0.$$

Then h^u attains its maximum at the point $(1,1) \in \mathbb{R}^2$.

Proof. Since $I'(u)u^{\pm} = 0$, we have that

$$\|u^{\pm}\|^2 + \int K(x)\phi_u(x)(u^{\pm})^2 = \int a(x)|u^{\pm}|^{p+1} > 0,$$

and therefore $\lim_{|(t,s)| \to +\infty} h^u(t,s) = -\infty$, from which it follows that the maximum is attained at some point $(t_0, s_0) \in [0,+\infty) \times [0,+\infty)$.

<u>Claim 1</u>: we have that $s_0,\, t_0 > 0$

Indeed, suppose by contradiction that $s_0 = 0$. In this case, since $h^u(0,0) = 0$ and $p > 3$, we have that $t_0 > 0$. Moreover, since $\int a(x)(u^-)^{p+1} > 0$, we obtain $I(su^-) > 0$, for small $s > 0$. By using $\int \phi_{u^+}(x)(u^-)^2 = \int \phi_{u^-}(x)(u^+)^2$, we get

$$h^u(t_0,0) = I(t_0u^+) < I(t_0u^+) + I(su^-) + \frac{s^2t_0^2}{2}\int K(x)\phi_{u^-}(x)(u^+)^2 = h^u(t_0,s),$$

which does not make sense. A similar argument shows that $t_0 > 0$.

<u>Claim 2</u>: we have that $s_0,\, t_0 \in (0,1]$

We just prove that $t_0 \leq 1$, since the argument for the other inequality is analogous. Recalling that the partial derivatives of $I(tu^+ + su^-)$ vanishes at (t_0, s_0), we obtain

$$t_0^2\|u^+\|^2 + t_0^4\int K(x)\phi_{u^+}(x)(u^+)^2 +$$
$$+ s_0^2t_0^2\int K(x)\phi_{u^-}(x)(u^+)^2 = t_0^{p+1}\int a(x)(u^+)^{p+1}.$$

If we suppose that $s_0 \leq t_0$, the equality above and $\phi_u = \phi_{u^+} + \phi_{u^-}$ imply that

$$t_0^{-2}\|u^+\|^2 + \int K(x)\phi_u(x)(u^+)^2 \geq t_0^{p-3}\int a(x)(u^+)^{p+1}.$$

From this and $I'(u)u^+ = 0$, we obtain

$$(t_0^{-2} - 1)\|u^+\|^2 \geq (t_0^{p-3} - 1)\int a(x)(u^+)^{p+1},$$

and therefore $t_0 \leq 1$. For the case $t_0 \leq s_0$ it is sufficient to use

$$s_0^2\|u^-\|^2 + s_0^4\int K(x)\phi_{u^-}(x)(u^-)^2 +$$
$$+ s_0^2t_0^2\int K(x)\phi_{u^+}(x)(u^-)^2 = s_0^{p+1}\int a(x)(u^-)^{p+1}$$

and argue as above.

Claim 3: h^u does not attain its maximum in $(0,1]^2 \setminus \{(1,1)\}$

Since $h^u(t_0,s_0) = I(t_0u^+ + s_0u^-) - \frac{1}{p+1}I'(t_0u^+ + s_0u^-)(t_0u^+ + s_0u^-)$, we have that

$$h^u(t_0,s_0) = \left(\frac{1}{2} - \frac{1}{p+1}\right)(t_0^2\|u^+\|^2 + s_0^2\|u^-\|^2)+$$

$$\left(\frac{1}{4} - \frac{1}{p+1}\right)\left\{t_0^4\int \phi_{u^+}(x)(u^+)^2 + s_0^4\int \phi_{u^-}(x)(u^-)^2 + 2s_0^2t_0^2\int \phi_{u^-}(x)(u^+)^2\right\},$$

where we also have used $\phi_{u^-}(x)(u^+)^2 = \phi_{u^+}(x)(u^-)^2$. If $s_0 < 1$ or $t_0 < 1$, it follows from the above equality that

$$h^u(t_0,s_0) < I(u^+ + u^-) - \frac{1}{p+1}I'(u^+ + u^-)(u^+ + u^-) = h^u(1,1),$$

which is absurd. □

Lemma 3.3. *For any $R > 0$, there exists $u_R \in \mathcal{N}_R^{\pm}$ such that $I(u_R) = m_R^{\pm}$.*

Proof. Let $(u_n) \subset \mathcal{N}_R^{\pm}$ be such that $I(u_n) \to m_R^{\pm}$. By (2.7), (u_n) is bounded, and therefore, up to a subsequence, there exists $u \in H_0^1(B_R(0))$ such that $u_n \rightharpoonup u$ weakly in $H_0^1(B_R(0))$, $u_n \to u$ strongly in $L^{p+1}(B_R(0))$ and $u_n(x) \to (x)$ a.e. in $B_R(0)$. Arguing as in the proof of (2.5) we obtain $\rho > 0$ such that

$$\int_{B_R(0)} a(x)|u_n^{\pm}|^{p+1}dx = \|u_n^{\pm}\|^2 + \int_{B_R(0)} K(x)\phi_{u_n}(x)(u_n^{\pm})^2dx \geq \|u_n^{\pm}\|^2 \geq \rho.$$

Hence, taking the limit, we conclude that $\int_{B_R(0)} a(x)|u^{\pm}|^{p+1}dx \geq \rho > 0$. Following the same lines of Claim 1 on the proof of Lemma 3.2, we can verify that h^u attains its maximum at some point $(t_u, s_u) \in \mathbb{R}^2$ such that t_u, $s_u > 0$. Thus

$$\frac{\partial}{\partial t}h^u(t_u,s_u) = 0 = \frac{\partial}{\partial s}h^u(t_u,s_u),$$

which is equivalent to $u_R := (t_uu^+ + s_uu^-) \in \mathcal{N}_R^{\pm}$. Hence, we can use the above convergences, item 4 of Lemma 2.1 and Lemma 3.2 to conclude that

$$\begin{aligned} m_R^{\pm} \leq I(t_uu^+ + s_uu^-) &\leq \liminf_{n\to\infty} I(t_uu_n^+ + s_uu_n^-) \\ &\leq \liminf_{n\to\infty} h^{u_n}(t_u,s_u) \\ &\leq \liminf_{n\to\infty} h^{u_n}(1,1) = \liminf_{n\to\infty} I(u_n) = m_R^{\pm}. \end{aligned}$$

The lemma is proved. □

We are ready to prove the main result of this section.

Proof of Proposition 3.1. For $R > 0$, let u_R be given by the last lemma. It suffices to check tat $I'(u_R)\varphi = 0$ for any $\varphi \in H_0^1(B_R(0))$. Arguing by contradiction, we suppose that this is not true. Then there exists δ, $\lambda > 0$ such that $\|I'(v)\| > \lambda$ whenever $\|v - u_R\| < 3\delta$. If we consider

$$g(t,s) := tu_R^+ + su_R^-,$$

for s, $t > 0$, it follows from Lemma 3.2 that, for some open set $D \subset \mathbb{R}^2$ containing $(1,1)$, there holds

$$\rho := \max_{(t,s)\in\partial D} I(g(t,s)) < m_R^{\pm}. \tag{3.1}$$

For $\varepsilon < \min\{(m_R^\pm - \rho)/2, \delta\lambda/8\}$ and $S = B_\delta(u_R)$, [25, Lemma 2.3] provides a deformation $\eta \in C([0,1] \times H_0^1(B_R(0)), H_0^1(B_R(0)))$ such that

(a) $\eta(1,u) = u$, if $u \notin I^{-1}[m_R^\pm - 2\varepsilon, m_R^\pm + 2\varepsilon]$,

(b) $\eta(1, I^{m_R^\pm+\varepsilon} \cap S) \subset I^{m_R^\pm - \varepsilon}$,

(c) $I(\eta(1,u)) \le I(u)$, for any $u \in H_0^1(B_R(0))$.

If we define $h(t,s) := \eta(1, g(t,s))$, it follows from Lemma 3.2, (c) and (b) that

$$\max_{(t,s)\in D} I(h(t,s)) < m_R^\pm. \tag{3.2}$$

Consider the maps

$$\psi(t,s) = \left(I'(g(t,s))u_R^+, I'(g(t,s))u_R^-\right),$$
$$\Psi(t,s) = \left(t^{-1}I'(h(t,s))h(t,s)^+, s^{-1}I'(h(t,s))h(t,s)^-\right),$$

and notice that, since $u_R \in \mathcal{N}_R^\pm$, we have that $\psi(t,s) = (0,0)$ if, and only if, $(t,s) = (1,1)$. Hence, a straightforward computation provides $\deg(\psi, D, (0,0)) = \operatorname{sgn}\det\psi'(1,1) = 1$. On the other hand, since $\rho < (m_R^\pm - 2\varepsilon)$, it follows from (3.1) and (a) that $h \equiv \mathrm{Id}$ on ∂D, in such way that $\deg(\Psi, D, (0,0)) = 1$. Thus, there exists $(t,s) \in D$ such that $h(t,s) \in \mathcal{N}_R^\pm$, which contradicts (3.2) and finishes the proof. □

4. The nodal solution

In this section we prove our second theorem by showing that the

$$m^\pm := \inf_{u\in\mathcal{N}^\pm} I(u)$$

is achieved. This will be done as a limit process on the solutions obtained in the last section.

Lemma 4.1. *We have that*

$$\lim_{R\to+\infty} m_R^\pm = m^\pm.$$

Proof. Given $\varepsilon > 0$ we consider $u_\varepsilon \in \mathcal{N}^\pm$ such that $I(u_\varepsilon) \le m^\pm + \varepsilon$. We have that, for some $R_0 > 0$ large enough, $u_\varepsilon^\pm \not\equiv 0$ in $B_{R_0}(0)$. We now take $R > 4R_0$, consider a cutoff function $\xi_R \in C_0^\infty(\mathbb{R}^3)$ such that $\xi_R \equiv 1$ in $B_{R/4}(0)$ and $\xi_R \equiv 0$ in $\mathbb{R}^3 \setminus B_{3R/4}(0)$, and define the function

$$u_{\varepsilon,R}(x) := \xi_R(x)u_\varepsilon(x).$$

Since

$$\lim_{R\to+\infty}\int_{B_R(0)} a(x)|u_{\varepsilon,R}^\pm|^{p+1}dx = \int a(x)|u_\varepsilon^\pm|^{p+1} > 0,$$

we have that $\int_{B_R(0)} a(x)|u_{\varepsilon,R}^\pm|^{p+1}dx > 0$, for $R > 0$ large enough. As in the proof of Lemma 3.2, we can obtain t_R, $s_R \in (0,+\infty)$ such that $(t_Ru_{\varepsilon,R}^+ + s_Ru_{\varepsilon,R}^-) \in \mathcal{N}_R^\pm$. Thus, since $u_\varepsilon \in \mathcal{N}^\pm$, we get

$$\begin{aligned} m_R^\pm \le I(t_Ru_{\varepsilon,R}^+ + s_Ru_{\varepsilon,R}^-) &= I(t_Ru_\varepsilon^+ + s_Ru_\varepsilon^-) + o_R(1) \\ &\le I(u_\varepsilon) + o_R(1) \le m^\pm + \varepsilon + o_R(1), \end{aligned}$$

where $o_R(1)$ denotes a quantity approaching zero as $R \to +\infty$. Since $\varepsilon > 0$ is arbitrary, we conclude that

$$\limsup_{R\to+\infty} m_R^\pm \le m^\pm.$$

Since $\mathcal{N}_R^{\pm} \subset \mathcal{N}^{\pm}$, the result follows from the above inequality. □

The key estimate for the proof of Theorem 1.2 is the following.

Proposition 4.2. *We have that*

$$m^{\pm} < m + m_{\infty}.$$

Proof. Let $\bar{u}$ the solution given by Theorem 1.1. For any $n \in \mathbb{N}$ and $(t,s) \in [\frac{1}{2}, 2]^2$, we define, for $x \in \mathbb{R}^3$,

$$\psi_n(x) := t\bar{u}(x) - s\omega_n(x),$$

where ω_n was defined in (2.3). We claim that there exists $n_0 \in \mathbb{N}$ such that, for any $n \geq n_0$ and $(s,t) \in [\frac{1}{2}, 2]^2$,

$$I(\psi_n) < m + m_{\infty}.$$

In order to prove the claim we first notice that, for some $C > 0$, there holds

$$\int K(x)\phi_{\psi_n}(x)\psi_n^2 \leq t^4 \int K(x)\phi_u(x)u^2 + Ce^{-\alpha n}, \tag{4.1}$$

for any $n \geq n_0$ and $(t,s) \in [\frac{1}{2}, 2]^2$. Indeed, since $\bar{u}$, $\omega \in X$ are non negative, we have that

$$\int K(x)\phi_{\psi_n}(x)\psi_n^2 \leq t^2 \int K(x)\phi_{\psi_n}(x)u^2 + s^2 \int K(x)\phi_{\psi_n}(x)\omega_n^2.$$

By (2.9),

$$K_n = \int K(x)\phi_{\omega_n}(x)\omega_n^2 \leq c_1 e^{-\alpha n},$$

for some $c_1 > 0$. Fubini's Theorem and the same argument used to get the above inequality provide

$$\int K(x)\phi_{\omega_n}(x)u^2 = \int K(x)\phi_u(x)\omega_n^2 \leq c_2 e^{-\alpha n}.$$

The estimate (4.1) is a consequence of the above inequalities and the equality $\phi_{\omega_n} \leq \phi_{t\bar{u}} + \phi_{s\omega_n}$.

We infer from (4.1) that

$$I(\psi_n) \leq I(t\bar{u}) + I_{\infty}(s\omega_n) + \frac{s^2}{2}V_n + \frac{s^{p+1}}{p+1}A_n - D_n - E_n + Ce^{-\alpha n}, \tag{4.2}$$

with

$$D_n := ts \int (\nabla \bar{u} \cdot \nabla \omega_n + V(x)\bar{u}\omega_n),$$

$$E_n := \frac{1}{p+1} \int a(x)\Big(|\psi_n|^{p+1} - |t\bar{u}|^{p+1} - |s\omega_n|^{p+1}\Big).$$

and A_n and V_n, defined in the proof of Proposition 2.4, verifying

$$V_n \leq -c_1 e^{-\gamma n}, \qquad A_n \leq c_1 e^{-\theta n}.$$

By Remark 2.5, we have that $|\bar{u}(x)| \leq Ce^{-\mu|x|}$, for some $\gamma < \mu < \sqrt{V_{\infty}}$. Hence, we can use [13, Proposition 3.2] to get

$$|E_n| \leq c_2 \int \Big(|t\bar{u}|^p s\omega_n + |s\omega_n|^p t\bar{u}\Big) \leq c_2 e^{-\mu n},$$

for n large. Moreover, using that $\bar{u}$ is a solution of (P) and condition (K_1), we obtain

$$|D_n| = ts\Big|\int a(x)|\bar{u}|^{p-1}\bar{u}\omega_n - \int K(x)\phi_{\bar{u}}(x)\bar{u}\omega_n\Big| \le c_3e^{-\mu n} + c_4e^{-\alpha n}.$$

Since

$$I(t\bar{u}) \le I(\bar{u}) = m, \qquad I_\infty(s\omega_n) = I(s\omega) \le I_\infty(\omega) = m_\infty,$$

all the above inequalities can be replaced in (4.2) to provide

$$I(\psi_n) \le m + m_\infty + e^{-\gamma n}\left[-c_3 + c_2e^{(\gamma-\theta)n} + c_3e^{(\gamma-\mu)n} + c_4e^{(\gamma-\alpha)n}\right]$$

Taking n large, the claim follows from $\gamma < \min\{\alpha, \theta, \mu\}$.

In view of the claim, to obtain the inequality $m^\pm < m + m_\infty$, it is sufficient to obtain $(t_0, s_0) \in [\frac{1}{2}, 2]^2$ such that $t_0\bar{u} - s_0\omega_n \in \mathcal{N}^\pm$. With this purpose, we define

$$h^\pm(t, s, n) := I'(t\bar{u} - s\omega_n)(t\bar{u} - s\omega_n)^\pm.$$

By using $3 < p < 5$, $I'(\bar{u})u^+ = 0$ and a straightforward calculation, we get

$$h^+(1/2, 0, n) > 0, \qquad h^+(2, 0, n) < 0.$$

Moreover, since $w_n \rightharpoonup 0$ weakly in X and ω is a solution of the limit problem, we can use (a_1), (V_1) and item 4 of Lemma 2.1 to conclude that

$$\begin{aligned} h^-(0, 1/2, n) &= (1/2)^2\int(|\nabla\omega_n|^2 + V(x)\omega_n^2) - (1/2)^{p+1}\int a(x)\omega_n^{p+1} \\ &= ((1/2)^2 - (1/2)^{p+1})\int(|\nabla\omega|^2 + V_\infty\omega^2) + o_n(1), \end{aligned}$$

and therefore, for n large, we have that $h^-(0, 1/2, n) > 0$. The same argument provides $h^-(2, 0, n) < 0$. It follows from Miranda Theorem [21] that, for some $(t_0, s_0) \in [\frac{1}{2}, 2]^2$, there holds $h^\pm(t_0, s_0, n) = 0$, for n large. But this is equivalent to $(t_0\bar{u} - s_0\omega_n) \in \mathcal{N}^\pm$. □

Inspired by [2], we can use all the above results to prove our second theorem as follows.

Proof of Theorem 1.2. For each $n \in \mathbb{N}$, let $u_n \in \mathcal{N}_n^\pm$ be the function given by Proposition 3.1 with $R = n$. Since $\mathcal{N}_n^\pm \subset \mathcal{N}$ we can check that (u_n) is bounded. Hence, we may suppose that $u_n \rightharpoonup u$ weakly in X. In view of Proposition 3.1, we have that u is a critical point of I. We shall prove that $u^\pm \neq 0$.

Suppose, by contradiction, that $u^+ = 0$. Let a, $b \in \mathbb{R}$ defined as

$$a := \lim_{n\to+\infty}\left\{I(u_n^+) + \frac{1}{4}\int K(x)\phi_{u_n^-}(x)(u_n^+)^2\right\},$$

$$b := \lim_{n\to+\infty}\left\{I(u_n^-) + \frac{1}{4}\int K(x)\phi_{u_n^+}(x)(u_n^-)^2\right\}.$$

Since $I(u_n) = m_n^\pm$, a simple calculation, Lemma 4.1 and Proposition 4.2 provide

$$a + b = \lim_{n\to+\infty} I(u_n) = m^\pm < m + m_\infty. \tag{4.3}$$

Let $t_n > 0$ be such that $t_nu_n^+ \in \mathcal{N}_\infty$, that is,

$$\|u_n^+\|_*^2 = t_n^{p-1}\int a_\infty(u_n^+)^{p+1}. \tag{4.4}$$

We claim that $t_n \to 1$. Indeed, arguing as in the proof of Lemma 3.3, we obtain

$$\int a(x)(u_n^+)^{p+1} \geq \|u_n^+\|^2 \geq \rho, \tag{4.5}$$

for some $\rho > 0$. Since $I'(u_n)u_n^+ = 0$, we have that

$$\|u_n^+\|^2 + \int K(x)\phi_{u_n}(u_n^+)^2 = \int a(x)(u_n^+)^{p+1}.$$

Since (u_n^+) weakly converges to zero, we can use (K_1) and Holder's inequality to conclude that $\int K(x)\phi_{u_n}(x)(u_n^+)^2 \to 0$. Thus, it follows from (V_1), (a_1) and (4.5) that

$$\|u_n^+\|_*^2 = \int a_\infty (u_n^+)^{p+1} + o_n(1) \geq \rho + o_n(1).$$

This and (4.4) imply that $t_n \to 1$ as $n \to +\infty$. Thus, as in the proof of Theorem 1.1, we can verify that $I_\infty(t_n u_n^+) = I(u_n^+) + o_n(1)$. Hence,

$$m_\infty \leq I_\infty(t_n u_n^+) = I(u_n^+) + o_n(1) \leq I(u_n^+) + \frac{1}{4}\int K(x)\phi_{u_n^-}(u_n^+)^2 + o_n(1) = a.$$

and it follows from (4.3) that $b < m$.

On the other hand, if we consider $s_n > 0$ such that $s_n u_n^- \in \mathcal{N}$, we have that

$$0 = I'(u_n)u_n^- = I'(u_n^-)u_n^- + \int K(x)\phi_{u_n^-}(x)(u_n^+)^2. \tag{4.6}$$

This implies that $I'(u_n^-)u_n^- < 0$. Thus, $s_n \leq 1$ and we get

$$\begin{aligned} m &\leq I(s_n u_n^-) = I(s_n u_n^-) - \frac{1}{p+1} I'(s_n u_n^-)(s_n u_n^-) \\ &= \Big(\frac{1}{2} - \frac{1}{p+1}\Big) s_n^2 \|u_n^-\|^2 + \Big(\frac{1}{4} - \frac{1}{p+1}\Big) s_n^4 \int K(x)\phi_{u_n^-}(x)(u_n^-)^2 \\ &\leq I(u_n^-) - \frac{1}{p+1} I'(u_n^-)(u_n^-). \end{aligned}$$

This and (4.6) provide

$$\begin{aligned} m &\leq I(u_n^-) + \frac{1}{p+1}\int K(x)\phi_{u_n^+}(x)(u_n^-)^2 \\ &\leq I(u_n^-) + \frac{1}{4}\int K(x)\phi_{u_n^+}(x)(u_n^-)^2 = b + o_n(1), \end{aligned}$$

and we conclude that $m \leq b$. This contradiction shows that $u^+ \neq 0$. Similarly, $u^- \neq 0$.

We have obtained a nodal solution u of the problem. It verifies,

$$\begin{aligned} m^\pm &\leq I(u) = I(u) - \frac{1}{p+1} I'(u)u \\ &= \Big(\frac{1}{2} - \frac{1}{p+1}\Big)\|u_1\|^2 + \Big(\frac{1}{4} - \frac{1}{p+1}\Big)\int K(x)\phi_{u_1}(x)u_1^2 \\ &\leq \liminf_{n\to\infty}\Big\{I(u_n) - \frac{1}{p+1} I'(u_n)u_n\Big\} = m^\pm, \end{aligned}$$

and therefore $I(u) = m^\pm$. The theorem is proved. □

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