MULTIPLE SOLUTIONS FOR A CRITICAL KIRCHHOFF SYSTEM

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Abstract. We consider the nonlocal system

$$\begin{cases} -m\left(\int_\Omega |\nabla u|^2 \mathrm{d}x\right)\Delta u &=& F_u(x,u,v)+\mu_1|u|^4u, \text{ in } \Omega,\\ -l\left(\int_\Omega |\nabla v|^2 \mathrm{d}x\right)\Delta v &=& F_v(x,u,v)+\mu_2|v|^4v, \text{ in } \Omega,\end{cases}$$

with positive potentials m and l. The nonlinearity F is subcritical and locally superlinear at infinty. By using the Symmetric Mountain Pass Theorem we obtain multiple solutions for small value of μ_1 and μ_2.

1. Introduction

We consider the nonlocal variational system

$$(S_\mu)\qquad \begin{cases} -m\left(\int_\Omega |\nabla u|^2 \mathrm{d}x\right)\Delta u &=& F_u(x,u,v)+\mu_1|u|^4u, \text{ in } \Omega,\\ -l\left(\int_\Omega |\nabla v|^2 \mathrm{d}x\right)\Delta v &=& F_v(x,u,v)+\mu_2|v|^4v, \text{ in } \Omega,\\ u,v\in W_0^{1,2}(\Omega), \end{cases}$$

where $\Omega\subset\mathbb{R}^3$ is a bounded smooth domain, F is locallly superlinear and μ_1, $\mu_2>0$ are parameters. The potentials m and l belongs to the set $\mathcal{A}$ of all continuous functions $g\in C([0,+\infty),\mathbb{R})$ which satisfy

($\mathcal{A}_1$) $g(t)\geq g_0>0$, for any $t\geq 0$;
($\mathcal{A}_2$) $2G(t):=2\int_0^t g(s)\mathrm{d}s\geq g(t)t$, for any $t\geq 0$.

Concerning the nonlinearity F we assume that

(F_0) $F\in C^1(\Omega\times\mathbb{R}^2,\mathbb{R})$ is even with respect to the second variable;
(F_1) there holds $\lim_{|z|\to\infty}\frac{|\nabla F(x,z)|}{|z|^5}=0$, uniformly in Ω;
(F_2) there exist σ_1, $\sigma_2\in[0,2)$, c_0, c_1, $c_2\in(0,+\infty)$ such that

$$\frac{1}{4}\nabla F(x,z)\cdot z-F(x,z)\geq -c_0-c_1|s|^{\sigma_1}-c_2|t|^{\sigma_2},$$

for any $x\in\Omega$, $z=(s,t)\in\mathbb{R}^2$, where $z_1\cdot z_2$ stands for the inner product of z_1, $z_2\in\mathbb{R}^2$. Furthermore, if $\sigma_1\neq 0$ or $\sigma_2\neq 0$, we also suppose that, for some $K>0$,

$$\begin{cases} \mu_2\leq\mu_1, & \text{if } \sigma_1\neq 0 \text{ and } \sigma_2=0;\\ \mu_2\leq\mu_1\leq K\mu_1, & \text{if } \sigma_1\neq 0 \text{ and } \sigma_2\neq 0;\\ \mu_1\leq K\mu_2, & \text{if } \sigma_1=0 \text{ and } \sigma_2\neq 0.\end{cases}$$

(F_3) there exist θ_1 $\theta_2\in(2,6)$ and $c_3,c_4,c_5\in(0,+\infty)$ such that

$$F(x,s,t)\leq c_3|s|^{\theta_1}+c_4|t|^{\theta_2}+c_5,\quad \forall\, x\in\Omega,\ (s,t)\in\mathbb{R}^2;$$

(F_4) there exists an open set $\Omega_0\subset\Omega$, with positive measure, such that $\lim_{|s|\to\infty}\frac{F(x,s,0)}{|s|^4}=+\infty$, uniformly in Ω_0.

We state below the main result of this paper.

Theorem 1.1. *Suppose that F satisfies $(F_0)-(F_4)$. Suppose also that m, $l\in\mathcal{A}$ and there exist a, $b>0$ such that*

$$m(t)\leq a+bt,\quad \forall\, t\geq 0. \tag{1.1}$$

Then, for any $k\in\mathbb{N}$, there exists $\mu_k^*>0$ such that the problem (S_μ) has at least k pairs of nonzero solutions for any μ_1, $\mu_2\in(0,\mu_k^*)$.

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We notice that condition (F_2) is weaker than the usual Ambrosetti-Ranbinowitz type condition since, in our setting, it is allowed that F takes negative values. Moreover, the superlinearity condition (F_4) holds only when $x \in \Omega_0$ and the second variable goes to infinity. Actually, we can prove an analogous theorem when the (local) superlinearity condition holds on the third variable (see Remark 3.4) and the function l verifies $l(t) \leq a + bt$.

Given a bounded smooth domain $\Omega \subset \mathbb{R}^N$ and a positive function w, the Kirchhoff equation

$$-w\left(\int_\Omega |\nabla u|^2 \mathrm{d}x\right)\Delta u = g(x,u), \text{ in } \Omega, \qquad u = 0, \text{ on } \partial\Omega,$$

has its origin in the theory of nonlinear vibration. For instance, in the model case $w(t) = a + bt$, with a, $b > 0$, it comes from the following model for the modified d'Alembert wave equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = g(x,u),$$

proposed by Kirchhoff in [9]. Here L is the length of the string, h is the area of cross-section, E is the Young modulus of the material, ρ is the mass density and P_0 is the initial tension. Nonlocal problems also appear in other fields as, for example, biological systems where u describes a process which depends on the average of itself (for instance, population density). We refer the reader to [3, 12] for more examples on the physical motivation of this problem. As far as we know, the first paper dealing with Kirchhoff type equation via variational methods was [1]. Since then, there is a vast literature concerning the existence, nonexistence, multiplicity and concentration behavior of solutions for scalar nonlocal problems with critical growth (see [4, 14, 11, 6] and references therein). Although the literature for the system is not so huge, we can cite the papers [2, 5, 8, 10, 16, 7] which contain some results which are related but not comparable with ours.

In the next section, we prove a local compactness result for the associated energy functional. In Section 3, we prove the main theorem.

2. The local Palais-Smale condition

In what follows we write $\int_\Omega u$ instead of $\int_\Omega u(x)dx$. We denote by $\|u\|_p$ the $L^p(\Omega)$-norm of a function $u \in L^p(\Omega)$, for any $1 \leq p \leq \infty$.

Let H be the Hilbert space $W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega)$ endowed with the norm $\|(u,v)\| := \left[\int_\Omega \left(|\nabla u|^2 + |\nabla v|^2\right)\right]^{1/2}$. For each component of the vector $(u,v) \in H$, we also denote $\|\cdot\| = \left(\int_\Omega |\nabla \cdot|^2\right)^{1/2}$. By using $(F_0) - (F_3)$ we can prove that the functional $I_{\mu_1,\mu_2} : H \to \mathbb{R}$ given by

$$I_{\mu_1,\mu_2}(u,v) := \frac{1}{2}M(\|u\|^2) + \frac{1}{2}L(\|v\|^2) - \frac{\mu_1}{6}\|u\|_6^6 - \frac{\mu_2}{6}\|v\|_6^6 - \int_\Omega F(x,u,v),$$

where $M(t) := \int_0^t m(s)\mathrm{d}s$ and $L(t) := \int_0^t l(s)\mathrm{d}s$, belongs to $C^1(H,\mathbb{R})$. Moreover, the critical points of I_{μ_1,μ_2} are precisely the weak solutions of (S_μ).

Given $c \in \mathbb{R}$, we say that I_{μ_1,μ_2} satisfies the Palais-Smale condition at level c ($(PS)_c$ for short) if any sequence $(z_n) \subset H$ such that

$$\lim_{n\to+\infty} I_{\mu_1,\mu_2}(z_n) = c, \qquad \lim_{n\to+\infty} I'_{\mu_1,\mu_2}(z_n) = 0,$$

has a convergent subsequence. The aim of this section is to prove the following local compactness result:

Proposition 2.1. *Given $C^* > 0$ there exits $\mu^* > 0$ such that I_{μ_1,μ_2} satisfies $(PS)_c$ for any $c < C^*$ and μ_1, $\mu_2 \in (0,\mu^*)$.*

The proof will be done in several steps. The first one is to verify that Palais-Smale sequences are bounded.

Lemma 2.2. *If $(z_n) = ((u_n,v_n)) \subset H$ is such that $I_{\mu_1,\mu_2}(z_n) \to c$ and $I'_{\mu_1,\mu_2}(z_n) \to 0$, then $(z_n) \subset H$ is bounded.*

Proof. Given $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|s|^{\sigma_1} \leq \varepsilon|s|^6 + C_\varepsilon, \quad |t|^{\sigma_2} \leq \varepsilon|t|^6 + C_\varepsilon, \quad \forall\,(s,t) \in \mathbb{R}^2, \tag{2.1}$$

where σ_1, $\sigma_2 \in [0,2)$ come from (F_2). This, $(\mathcal{A}_2)$ and (F_2) provide

$$\begin{aligned} c + o_n(1) + o_n(1)\|z_n\| &\geq I_{\mu_1,\mu_2}(z_n) - \frac{1}{4}I'_{\mu_1,\mu_2}(z_n)z_n \\ &\geq \frac{\mu_1}{12}\|u_n\|_6^6 + \frac{\mu_2}{12}\|v_n\|_6^6 - c_0|\Omega| - c_1\|u_n\|_{\sigma_1}^{\sigma_1} - c_2\|v_n\|_{\sigma_2}^{\sigma_2} \\ &\geq \left(\frac{\mu_1}{12} - \varepsilon c_1\right)\|u_n\|_6^6 + \left(\frac{\mu_2}{12} - \varepsilon c_2\right)\|v_n\|_6^6 - (c_0 + 2C_\varepsilon)|\Omega|. \end{aligned}$$

Picking $\varepsilon > 0$ small, we obtain C_1, $C_2 > 0$ such that

$$\|z_n\|_6^6 \le C_1 + C_2\|z_n\|. \tag{2.2}$$

Since m, $l \in \mathcal{A}$, we have that $m(t) \ge m_0$, $l(t) \ge l_0$, for any $t \ge 0$. This, $I_{\mu_1,\mu_2}(z_n) = c + o_n(1)$, $(\mathcal{A}_1) - (\mathcal{A}_2)$ and (F_3) provide

$$\begin{aligned} \frac{l_0}{4}\|u_n\|^2 + \frac{m_0}{4}\|v_n\|^2 &\le \frac{1}{2}M(\|u_n\|^2) + \frac{1}{2}L(\|v_n\|^2) \\ &\le C\|z_n\|_6^6 + c_5|\Omega| + c_3\|u_n\|_{\theta_1}^{\theta_1} + c_4\|v_n\|_{\theta_2}^{\theta_2} + c + o_n(1). \end{aligned}$$

Since θ_1, $\theta_2 \in (2,6)$, there holds an inequality analogous to (2.1) with θ_i instead of σ_i. Hence, we infer from (2.2) that

$$\frac{\min\{m_0; l_0\}}{4}\|z_n\|^2 \le C_3\|z_n\|_6^6 + C_4 \le C_5\|z_n\| + C_6,$$

and therefore (z_n) is bounded in H. □

The next lemma is a version of [15, Lemma 3.1].

Lemma 2.3. *Suppose that $(z_n) = ((u_n, v_n)) \subset H$ is such that $z_n \rightharpoonup z = (u, v)$ weakly in H. Then, up to a subsequence,*

$$\lim_{n\to+\infty}\int_\Omega |F_u(x, z_n)u_n - F_u(x, z)u| = 0 = \lim_{n\to+\infty}\int_\Omega |F_v(x, z_n)v_n - F_u(x, z)v|.$$

Proof. We only prove the first statement. Notice that, up to a subsequence, $(u_n, v_n) \to (u, v)$ stronly in $L^2(\Omega)\times L^2(\Omega)$, $(u_n(x), v_n(x)) \to (u(x), v(x))$ for a.e. $x \in \Omega$ and

$$\max\{\|u\|_6^6, \|u_n\|_6^6, \|v\|_6^6, \|v_n\|_6^6\} \le C_1, \quad \forall\, n \in \mathbb{N}. \tag{2.3}$$

Moreover, for any $\varepsilon > 0$, we can use (F_1) and Young's inequality to obtain $C_\varepsilon > 0$ such that

$$|F_u(x, s, t)s| \le C_\varepsilon + \frac{\varepsilon}{2}(|s|^6 + |t|^5|s|) \le C_\varepsilon + \varepsilon(|s|^6 + |t|^6), \quad \forall\, x \in \Omega,\ (s, t) \in \mathbb{R}^2.$$

Now, given $\delta > 0$, we can pick $0 < \varepsilon < \delta/(8C_1)$ and apply Egorov's Theorem to obtain a measurable set $\widehat{\Omega} \subset \Omega$ such that $F_u(\cdot, z_n)u_n \to F_u(\cdot, z)u$ uniformly in $\widehat{\Omega}$ and $|\Omega\backslash\widehat{\Omega}| < \delta/(4C_\varepsilon)$.

It follows from the above inequality for F_u and (2.3) that

$$\int_{\Omega\backslash\widehat{\Omega}} |F_u(x, z_n)u_n - F_u(x, z)u| \le 2C_\varepsilon|\Omega \setminus \widehat{\Omega}| + 4C_1\varepsilon \le \delta.$$

This and the uniform convergence in $\widehat{\Omega}$ provide

$$\limsup_{n\to+\infty}\int_\Omega |F_u(x, z_n)u_n - F_u(x, z)u| \le \limsup_{n\to+\infty}\left(\int_{\widehat{\Omega}} + \int_{\Omega\backslash\widehat{\Omega}}\right) \le \delta.$$

Since $\delta > 0$ is arbitrary, the result follows from the above inequality. □

Let S the best constant of the Sobolev embedding $W_0^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$, namely

$$S := \inf_{u\in W_0^{1,2}(\Omega)\backslash\{0\}} \frac{\|u\|^2}{(\int_\Omega |u|^6)^{2/6}}.$$

We denote by $C(\overline{\Omega})$ the set of all continuos functions $u : \overline{\Omega} \to \mathbb{R}$ endowed with the norm $\|u\|_{C(\overline{\Omega})} := \max_{x\in\overline{\Omega}}|u(x)|$ and by $\mathcal{M}(\overline{\Omega})$ its dual, namely the set of Radon measures. Arguing along the same lines of the proof of the concentration-compactness lemma of Lions [13, Lemma 2.1], we can prove the following version of that result for our system:

Lemma 2.4. *Suppose that $(z_n) = ((u_n, v_n)) \subset H$ verifies*

$$\begin{cases} u_n \rightharpoonup u,\ v_n \rightharpoonup v, & \text{weakly in } W_0^{1,2}(\Omega), \\ |\nabla u_n|^2 \rightharpoonup \zeta,\ |\nabla v_n|^2 \rightharpoonup \overline{\zeta}, & \text{in the weak}^\star \text{ topology } \sigma(\mathcal{M}(\overline{\Omega}), C(\overline{\Omega})), \\ |u_n|^6 \rightharpoonup \nu,\ |v_n|^6 \rightharpoonup \overline{\nu}, & \text{in the weak}^\star \text{ topology } \sigma(\mathcal{M}(\overline{\Omega}), C(\overline{\Omega})), \end{cases}$$

where ζ, $\overline{\zeta}$, ν, $\overline{\nu} \in \mathcal{M}(\overline{\Omega})$ are nonnegative bounded measures in $\overline{\Omega}$. Then there exist enumerable sets J_1 and J_2, which can be empty, and two families $\{x_j\ ,\ j \in J_1\}$ and $\{y_j\ ,\ j \in J_2\}$ of points in $\overline{\Omega}$ such that

(a) $\nu = |u|^6dx + \sum_{j\in J_1} \nu_j\delta_{x_j}; \quad \overline{\nu} = |v|^6dx + \sum_{j\in J_2} \overline{\nu}_j\delta_{y_j};$

(b) $\zeta \ge |\nabla u|^2dx + \sum_{j\in J_1} \zeta_j\delta_{x_j}; \quad \overline{\zeta} \ge |\nabla v|^2dx + \sum_{j\in J_2} \zeta_j\delta_{y_j},$

where ν_j, $\overline{\nu}_j, \zeta_j, \overline{\zeta}_j > 0$. *Moreover,* $S\nu_j^{1/3} \leq \zeta_j$, *for any* $j \in J_1$, *and* $\overline{\nu}_j^{1/3} \leq \overline{\zeta}_j$, *for any* $j \in J_2$.

We prove in the sequel that, for some special sequences, the sets J_1 and J_2 are finite.

Lemma 2.5. *Let* $(z_n) \subset H$ *be as in Lemma 2.4 and suppose that* $I'_{\mu_1,\mu_2}(z_n) \to 0$. *Then* J_1 *and* J_2 *are finite. Moreover,*

$$\nu_j \geq (m_0 S/\mu_1)^{3/2}, \quad \forall\, j \in J_1; \qquad \overline{\nu}_j \geq (l_0 S/\mu_2)^{3/2}, \quad \forall\, j \in J_2. \tag{2.4}$$

Proof. We only pove that J_1 is finite, since the argument for J_2 is analogous. Let $\phi \in C_0^\infty(\mathbb{R}^N, [0,1])$ be such that $\phi \equiv 1$ in $B_{1/2}(0)$ and $\phi \equiv 0$ in $\mathbb{R}^N \setminus B_1(0)$. Suppose that $J_1 \neq \varnothing$, fix $j \in J_1$ and define $\phi_\varepsilon(x) := \phi((x - x_j)/\varepsilon)$, for $\varepsilon > 0$. Since $(\phi_\varepsilon z_n) \subset H$ is bounded we have that $I'_{\mu_1,\mu_2}(z_n)(\phi_\varepsilon u_n, 0) = o_n(1)$, and therefore

$$m(\|u_n\|^2)\left(A_{n,\varepsilon} + \int_\Omega |\nabla u_n|^2 \phi_\varepsilon\right) = o_n(1) + \mu_1 \int_\Omega |u_n|^6 \phi_\varepsilon + \int_\Omega F_u(x, u_n, v_n) u_n \phi_\varepsilon,$$

where $A_{n,\varepsilon} := \int u_n(\nabla u_n \cdot \nabla \phi_\varepsilon)$. Since $m(t) \geq m_0$, for any $t \geq 0$, Lemma 2.4 implies that

$$m_0 \left(\limsup_{n\to+\infty} A_{n,\varepsilon} + \int_{\overline{\Omega}} \phi_\varepsilon d\zeta\right) \leq \mu_1 \int_{\overline{\Omega}} \phi_\varepsilon d\nu + \int_\Omega f(x,u) u \phi_\varepsilon. \tag{2.5}$$

We claim that $\lim\limits_{\varepsilon\to 0} \limsup\limits_{n\to\infty} A_{n,\varepsilon} = 0$. If this is true, we can take the limit as $\varepsilon \to 0$ in the above equality and use the Lebesgue Theorem to get $m_0 \zeta_j \leq \mu_1 \nu_j$. Recalling that $S\nu_j^{1/3} \leq \zeta_j$, we obtain $m_0 S \nu_j^{1/3} \leq m_0 \zeta_j \leq \mu_1 \nu_j$, and therefore $\nu_j \geq (m_0 S/\mu_1)^{3/2}$. Thus, $\nu(\overline{\Omega}) \geq \sum_{j\in J} \nu_j \geq \sum_{j\in J} (m_0 S/\mu_1)^{3/2}$ and we conclude that J_1 is finite.

In order to prove the claim, we first use Hölder's inequality to compute

$$|A_{n,\varepsilon}| \leq \int_\Omega |u_n||\nabla u_n||\nabla \phi_\varepsilon| \leq C \left(\int_\Omega |u_n|^2 |\nabla \phi_\varepsilon|^2\right)^{1/2}.$$

Thus, the change of variables $y := (x - x_j)/\varepsilon$ and the Sobolev embedding provide

$$\limsup_{n\to\infty} |A_{n,\varepsilon}| \leq C \left(\int_\Omega |u|^2 |\nabla \phi_\varepsilon|^2 dx\right)^{1/2} = \varepsilon^{(N-2)/2} C \left(\int_{\{|y|\leq\varepsilon\}} |u(y\varepsilon + x_j)|^2 |\nabla \phi(y)|^2 dy\right)^{1/2},$$

and the result follows from $N \geq 3$. □

We are ready to prove our local compactness result.

Proof of Proposition 2.1. We first assume that σ_1, $\sigma_2 \in [0,2)$ given in (F_2) are both nonzero. Given $C^* > 0$, we fix μ_1, $\mu_2 \in (0, \widetilde{\mu})$, with $\widetilde{\mu} > 0$ to be choosed later. Let $(z_n) = ((u_n, v_n)) \subset H$ be such that $I_{\mu_1,\mu_2}(z_n) \to c \leq C^*$ and $I'_{\mu_1,\mu_2}(z_n) \to 0$, as $n \to +\infty$. According to Lemma 2.2, we may assume that all the hypotheses of Lemma 2.4 holds. By using the same notation of that lemma we define

$$Q_1 := \sum_{j\in J_1} \nu_j, \quad Q_2 := \sum_{j\in J_2} \overline{\nu}_j,$$

in such way that

$$\lim_{n\to+\infty} \|u_n\|_6^6 = \|u\|_6^6 + Q_1, \qquad \lim_{n\to+\infty} \|v_n\|_6^6 = \|v\|_6^6 + Q_2. \tag{2.6}$$

<u>*Claim*</u>*:* If $\widetilde{\mu} > 0$ is small, then $Q_1 < (m_0 S/\mu_1)^{3/2}$ and $Q_2 < (l_0 S/\mu_2)^{3/2}$.
Indeed, by (F_2) we have that

$$I_{\mu_1,\mu_2}(z_n) - \frac{1}{4} I'_{\mu_1,\mu_2}(z_n) z_n \geq \frac{\mu_1}{12}\|u_n\|_6^6 + \frac{\mu_2}{12}\|v_n\|_6^6 - c_0|\Omega| - c_1\|u_n\|_{\sigma_1}^{\sigma_1} - c_2\|v_n\|_{\sigma_2}^{\sigma_2}.$$

By taking the limit, using (2.6) and the embeddings $W_0^{1,2}(\Omega) \hookrightarrow L^{\sigma_i}(\Omega)$ and $L^{\sigma_1}(\Omega) \hookrightarrow L^6(\Omega)$, we get

$$\frac{\mu_1}{12} Q_1 + \frac{\mu_2}{12} Q_2 \leq M + c_0|\Omega| + c_1 a_1 \|u\|_6^{\sigma_1} - \frac{\mu_1}{12}\|u\|_6^6 + c_2 a_2 \|u\|_6^{\sigma_2} - \frac{\mu_2}{12}\|v\|_6^6. \tag{2.7}$$

Choosing $C_1 > 0$ such that $\frac{\sigma_1}{6C_1} = \frac{\mu_1}{12 a_1 c_1}$ and applying Young's inequality with exponents $r = 6/\sigma_1$ and $r' = 6/(6-\sigma_1)$ we obtain $\|u\|_6^{\sigma_1} \leq \frac{\sigma_1}{6C_1}\|u\|_6^6 + \frac{C_1^{r'-1}}{r'} = \frac{\mu_1}{12a_1c_1}\|u\|_6^6 + \frac{C_2}{(\mu_1)^{\sigma_1/(6-\sigma_1)}}$. Analogously, $\|v\|_6^{\sigma_2} \leq \frac{\mu_2}{12a_2c_2}\|v\|_6^6 + \frac{C_3}{(\mu_2)^{\sigma_2/(6-\sigma_2)}}$. So, it follows from (2.7) that

$$\frac{\mu_1}{12} Q_1 + \frac{\mu_2}{12} Q_2 \leq M + c_0|\Omega| + c_1 a_1 \frac{C_2}{(\mu_1)^{\sigma_1/(6-\sigma_1)}} + c_2 a_2 \frac{C_3}{(\mu_2)^{\sigma_2/(6-\sigma_2)}}.$$

Since $\mu_2 \le \mu_1 \le K\mu_2$, we obtain $A_i > 0$, $i = 1, \dots, 6$, independent of μ_1 and μ_2, such that

$$Q_1 \le \frac{A_1}{\mu_1} + \frac{A_2}{(\mu_1)^{6/(6-\sigma_1)}} + \frac{A_3}{(\mu_1)^{6/(6-\sigma_2)}}, \qquad Q_2 \le \frac{A_4}{\mu_2} + \frac{A_5}{(\mu_2)^{6/(6-\sigma_2)}} + \frac{A_6}{(\mu_2)^{6/(6-\sigma_2)}}.$$

Recalling that $\sigma_i \in [0, 2)$, we obtain $\max\{1; 6/(6-\sigma_1); 6/(6-\sigma_2)\} < 3/2$. Hence, picking $\widetilde{\mu} > 0$ such that $A_1\sqrt{\widetilde{\mu}} \le (m_0 S)^{3/2}/3$, there holds

$$\frac{A_1}{\mu_1} \le \frac{1}{3}\frac{(m_0 S)^{3/2}}{\mu_1\sqrt{\widetilde{\mu}}} \le \frac{1}{3}\left(\frac{m_0 S}{\mu_1}\right)^{3/2}.$$

By using the same argument (with a small $\widetilde{\mu}$ if necessary), we obtain $Q_1 < (m_0 S/\mu_1)^{3/2}$. The inequality for Q_2 can be proved in the same way.

By using the claim and (2.4) we conclude that the sets J_1 and J_2 of Lemma 2.4 are empty. Hence

$$\lim_{n\to+\infty}\int_\Omega |u_n|^6 = \int_\Omega |u|^6, \qquad \lim_{n\to+\infty}\int_\Omega |v_n|^6 = \int_\Omega |v|^6,$$

and we can use $I'_{\mu_1,\mu_2}(z_n)(u_n, 0) = o_n(1)$, $I'_{\mu_1,\mu_2}(z_n)(u, 0) = o_n(1)$ and Lemma 2.3 to get

$$o_n(1) = I_{\mu_1,\mu_2}(z_n)(u_n, 0) - I'_{\mu_1,\mu_2}(z_n)(u, 0) = m(\|u_n\|^2)\left(\|u_n\|^2 - \|u\|^2\right) + o_n(1).$$

Since m is continuous and positive we get $\|u\| \to \|u\|$, and therefore $u_n \to u$ strongly in $W_0^{1,2}(\Omega)$. The same argument shows that $v_n \to v$ strongly in $W_0^{1,2}(\Omega)$. Thus, the proposition holds if $\sigma_1 \neq 0$ and $\sigma_2 \neq 0$.

If $\sigma_1 \neq 0$ and $\sigma_2 = 0$, equation (2.7) becomes

$$\frac{\mu_1}{12}Q_1 + \frac{\mu_2}{12}Q_2 \le M + c_0|\Omega| + c_1 a_1\|u\|_6^{\sigma_1} - \frac{\mu_1}{12}\|u\|_6^6,$$

and the result follows with the same argument used above. Finnally, if $\sigma_1 = \sigma_2 = 0$, it is suficient to consider $\mu_1 < \left(\frac{m_0 S^{3/2}}{3M}\right)^2$ and $\mu_2 < \left(\frac{l_0 S^{3/2}}{3M}\right)^2$. The proposition is proved. □

3. Proof of Theorems 1.1

The main theorems of this paper will be proved as an application of the following version of the Symmetric Mountain Pass Theorem.

Theorem 3.1. *Let $E = V \oplus W$ be a real Banach space with $\dim V < \infty$. Suppose that $I \in C^1(E, \mathbb{R})$ is an even functional satisfying $I(0) = 0$ and*

- (I_1) *there exist ρ, $\alpha > 0$ such that* $\inf_{u\in\partial B_\rho(0)\cap W} I(u) \ge \alpha$;
- (I_2) *there exist a subspace $\widehat{V} \subset E$ with $\dim V < \dim \widehat{V} < \infty$ such that, for some $M > 0$, there holds* $\max_{u\in\widehat{V}} I(u) \le M$;
- (I_3) *I satisfies the $(PS)_c$ for any $c \in (0, M)$.*

Then I possesses at least $(\dim \widehat{V} - \dim V)$ pairs of nonzero critical points.

In what follows we denote by $(\varphi_j)_{j\in\mathbb{N}}$ the normalized eigenfunctions of $\sigma(-\Delta, W_0^{1,2}(\Omega))$. For each $m \in \mathbb{N}$, we set $V_m := \mathrm{span}\{(\varphi_1, \varphi_1), \dots, (\varphi_m, \varphi_m)\}$ and notice that $H = V_m \oplus V_m^\perp$. It was proved in [15, Lemma 3.1] that, given $2 \le r < 6$ and $\delta > 0$, there exists $m_0 \in \mathbb{N}$ such that, for any $m \ge m_0$,

$$\int_\Omega |u|^r \le \delta\|u\|^r, \quad \forall\, u \in \mathrm{span}\{\varphi_1, \dots, \varphi_m\}^\perp \tag{3.1}$$

We prove in the sequel that the energy functional satisfies (I_1).

Lemma 3.2. *There exist $\overline{\mu} > 0$, $m \in \mathbb{N}$ and ρ, $\alpha > 0$ such that, for any μ_1, $\mu_2 \in (0, \overline{\mu})$, there holds*

$$I_{\mu_1,\mu_2}(z) \ge \alpha, \quad \forall\, z \in \partial B_\rho(0) \cap V_m^\perp.$$

Proof. By using $(\mathcal{A}_1)$, (F_3), the Sobolev embeddings and inequality (3.1) with $r = \sigma_i$ and $\delta > 0$ to be choosed later, we get

$$\begin{aligned} I(z) &\ge \frac{1}{4}m(\|u\|^2)\|u\|^2 + \frac{1}{4}l(\|v\|^2)\|v\|^2 - \int F(x, z) - \frac{\mu_1}{6}\|u\|_6^6 - \frac{\mu_2}{6}\|v\|_6^6 \\ &\ge \frac{m_0}{4}\|u\|^2 + \frac{l_0}{4}\|v\|^2 - c_3|\Omega| - c_4\|u\|_{\theta_1}^{\theta_1} - c_5\|v\|_{\theta_2}^{\theta_2} - \mu_1 b_1\|u\|^6 - \mu_2 b_2\|v\|^6 \\ &\ge \frac{m_0}{4}\|u\|^2 + \frac{l_0}{4}\|v\|^2 - c_3|\Omega| - c_4\delta\|u\|^{\theta_1} - c_5\delta\|v\|^{\theta_2} - \mu_1 b_1\|u\|^6 - \mu_2 b_2\|v\|^6, \end{aligned}$$

for any $z \in V_m^{\perp}$. Hence, for $c := (1/4)\min\{m_0; l_0\}$, there holds

$$I(z) \geq \|z\|^2(c - c_4\delta\|z\|^{\theta_1-2} - c_5\delta\|z\|^{\theta_2-2}) - c_3|\Omega| - \mu_1 b_1\|z\|^6 - \mu_2 b_2\|z\|^6,$$

If $\rho = \rho(\delta) > 0$ verifies $c_4\delta\rho^{\theta_1-2} + c_5\delta\rho^{\theta_2-2} = c/2$, then

$$I(z) \geq \frac{c}{2}\rho^2 - c_3|\Omega| - \mu_1 b_1\rho^6 - \mu_2 b_2\rho^6, \qquad \forall z \in \partial B_\rho(0) \cap V_m^{\perp}.$$

Notice that $\rho(\delta) \to +\infty$ as $\delta \to 0$, and therefore we can pick $\delta > 0$ such that $(c/2)\rho^2 - c_3|\Omega| > (c/4)\rho^2$. So, there exists $\overline{\mu} > 0$ small such that

$$I(z) \geq \rho^2\left(\frac{c}{4} - \mu_1 b_1\rho^4 - \mu_2 b_2\rho^4\right) =: \alpha > 0, \qquad \forall z \in \partial B_\rho(0) \cap V_m^{\perp}.$$

for any $\mu_1, \mu_2 \in (0, \overline{\mu})$. The lemma is proved. □

The local superlinearity condition (F_4) is used only in the next result.

Lemma 3.3. *Suppose that F and m satisfy (F_4) and (1.1). Then, for any $j \in \mathbb{N}$, there exist a j-dimensional subspace $\widehat{V}_j \subset H$ and $M > 0$ such that $\sup_{u\in\widehat{V}_j} I(z) \leq M$, for any $\mu_1, \mu_2 > 0$.*

Proof. Let $\Omega_0 \subset \Omega$ given by (F_4) and consider $(\varphi_j)_{j\in\mathbb{N}}$ the normalized eigenfunctions of $\sigma(-\Delta, W_0^{1,2}(\Omega_0))$. We define the subspace $\widehat{V}_j := \text{span}\{(\varphi_j, 0), \ldots, (\varphi_j, 0)\}$. Since $\widehat{V_j}$ is finite dimensional, there exists $C_1 = C_1(\widehat{V_j}) > 0$ such that

$$C_1\|u\|^4 \leq \|u\|_4^4, \quad \forall\, u \in \widehat{V_j}. \tag{3.2}$$

Let $b > 0$ be given in (1.1) and $\varepsilon > b/(4C_1)$. It follows from (F_4) and (F_0) that, $F(x, s, 0) \geq \varepsilon|s|^4 - C_2$, for any $x \in \Omega_0$, $s \in \mathbb{R}$, and some constant $C_2 = C_2(C_1, b) > 0$. This, (1.1) and (3.2) imply that, for any $z \in \widehat{V}_j$, there holds

$$I_{\mu_1,\mu_2}(z) \leq \frac{a}{2}\|u\|^2\left(\varepsilon C_1 - \frac{b}{4}\right)\|u\|^4 + C_2|\Omega| \leq \sup_{t>0}\left\{\frac{a}{2}t^2 + \varepsilon_0 t^4 + C_2|\Omega|\right\},$$

with $\varepsilon_0 = (\varepsilon C_1 - b/4) > 0$. The result follows if we call M the right-hand side above.

□

We are ready to prove our main result.

Proof of Theorem 1.1. Let $k \in \mathbb{N}$ be fixed. By Lemma 3.2, we can find $m \in \mathbb{N}$ large in such way that, for the decomposition $H = V \oplus W$, with $V := \langle(\varphi_1, 0)\ldots, (\varphi_m, 0)\rangle$, $W := V^{\perp}$, the functional I_{μ_1,μ_2} satisfies (I_1) for any μ_1, $\mu_2 \in (0, \overline{\mu})$. Moreover, by using Lemma 3.3, we obtain a subspace $\widehat{V}_{k+m} \subset H$ and $M > 0$ such that

$$\dim \widehat{V}_{k+m} = (k + m), \qquad \sup_{z\in\widehat{V}_{k+m}} I \leq M, \quad \forall\, \mu_1,\, \mu_2 > 0.$$

Hence, I_{μ_1,μ_2} satisfies (I_2). By considering M as above, we obtain from Proposition 2.1 a number $\mu^* > 0$ such that I_{μ_1,μ_2} satisfies (I_3), for any μ_1, $\mu_2 \in (0, \mu^*)$. Since $I_{\mu_1,\mu_2}(0) = 0$ and I_{μ_1,μ_2} is even, we can set $\mu_k^* := \min\{\overline{\mu}; \mu^*\}$ and use Theorem 3.1 to conclude that, for any $\mu \in (0, \mu_k^*)$, the functional I_{μ_1,μ_2} has at lesat $(k + m - m) = k$ pairs of nonzero critical points. □

Remark 3.4. *A simple inspecion of the proof of Lemma 3.3 shows that it also holds if we replace the bound condition in m by $l(t) \leq a + bt$, for any $t \geq 0$, and the superlinearity condition (F_4) by*

$(\widehat{F_4})$ *there exists an open set $\Omega_0 \subset \Omega$, with positive measure, such that $\lim_{|t|\to\infty} \frac{F(x,0,t)}{|t|4} = +\infty$, uniformly in Ω_0.*

Hence, in this new setting, we also get multiple solutions for the problem (S_μ).

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