MULTIPLE SOLUTIONS FOR A KIRCHHOFF EQUATION WITH CRITICAL GROWTH

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ABSTRACT. We consider the problem

$$-m\left(\int_\Omega |\nabla u|^2 dx\right)\Delta u = \lambda f(x,u) + \mu |u|^{2^*-2}u, \quad x \in \Omega, \qquad u \in H_0^1(\Omega),$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 3$, is a bounded smooth domain, $2^* = 2N/(N-2)$, λ, $\mu > 0$ and m is an increasing positive function. The function f is odd in the second variable and has superlinear growth. In our first result we obtain, for each $k \in \mathbb{N}$, the existence of k pairs of nonzero solutions for all $\mu > 0$ fixed and λ large. Under weaker assumptions of f, we also obtain a similar result if $N = 3$, $\lambda > 0$ is fixed and μ is close to 0. In the proofs, we apply variational methods.

1. Introduction

Consider the problem

$$-m(\|u\|^2)\Delta u = g(x,u), \text{ in } \Omega, \qquad u = 0, \text{ on } \partial\Omega, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, the norm in $H_0^1(\Omega)$ is $\|u\|^2 = \int_\Omega |\nabla u|^2 dx$, m is a positive function and the nonlinear function g has polynomial growth. It is called nonlocal due to the presence of the term $m(\|u\|^2)$. The equation has its origin in the theory of nonlinear vibration. For instance, in the model case $m(t) = a + bt$, with a, $b > 0$, it comes from the following model for the modified d'Alembert wave equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = g(x,u),$$

for free vibrations of elastic strings. Here, L is the length of the string, h is the area of the cross-section, E is the Young modulus of the material, ρ is the mass density and P_0 is the initial tension. This kind of nonlocal equation was first proposed by Kirchhoff [16] and it was considered theoretically or experimentally by several physicists after that (see [27, 7, 26, 25]). Nonlocal problems also appear in other fields as, for example, biological systems where u describes a process which depends on the average of itself (for instance, population density). We refer the reader to [10, 20, 19], and references therein, for more examples on the physical motivation of this problem.

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We are interested here in the case that g is a small order perturbation of the critical power, namely the following problem:

$$(P) \qquad \begin{cases} -m(\|u\|^2)\Delta u = \lambda f(x,u) + \mu |u|^{2^*-2}u, & x \in \Omega, \\ u \in H_0^1(\Omega), \end{cases}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 3$, is a bounded smooth domain, $2^* := 2N/(N-2)$, λ, $\mu > 0$ are parameters and the functions m and f verify

(m_0) $m \in C([0,+\infty),(0,+\infty))$ is increasing;

(f_0) $f \in C(\Omega \times \mathbb{R}, \mathbb{R})$ is odd in the second variable;

(f_1) there exists $q \in (2, 2^*)$ such that

$$\lim_{|s|\to\infty} \frac{f(x,s)}{|s|^{q-1}} = 0, \quad \text{uniformly in } \Omega;$$

(f_2) there exists $\theta \in (2, 2^*)$ such that

$$0 < \theta F(x,s) := \theta \int_0^s f(x,t)dt \leq s f(x,s), \quad \forall\, x \in \Omega,\ s \neq 0;$$

(f_3) there holds

$$\lim_{s\to 0} \frac{f(x,s)}{s} = 0, \quad \text{uniformly in } \Omega.$$

Under the above conditions, it is well-known that the weak solutions of the problem are the critical points of the energy functional

$$I_\lambda(u) := \frac{1}{2}M(\|u\|^2) - \lambda \int_\Omega F(x,u)dx - \frac{\mu}{2^*}\int_\Omega |u|^{2^*}dx, \qquad u \in H_0^1(\Omega),$$

where M and F are primitives of the functions m and $f(x,\cdot)$, respectively. Since f is odd, the functional I is even and therefore we may expect that this symmetry provides multiple critical points. In the first result of this paper, we show that this true if the parameter λ is large.

Theorem 1.1. *Suppose that m and f satisfy (m_0) and $(f_0)-(f_3)$, respectively, and $\mu > 0$. Then, for any given $k \in \mathbb{N}$, there exists $\lambda_k^* > 0$ such that the problem (P) has at least k pairs of nonzero solutions for all $\lambda \geq \lambda_k^*$.*

In the proof, we apply a version of the Symmetric Mountain Pass Theorem. The noncompactness of the embedding $H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$ is overcome by the ideas of Brezis and Nirenberg [6] and the Concentration Compactness Principle of Lions [21]. Since for high dimensions $N \geq 4$ the critical power 2^* is smaller than or equal to 4, the integral $\int_\Omega |u|^{2^*}dx$ does not dominate the fourth-term $M(\|u\|^2)$. We deal with this difficult by using a truncation argument (see [1]) which consists in considering a truncated equation and, after solving this new problem, prove that its solutions have small norm and therefore solve the original problem. We also emphasize that the presence of a nonlocal term in the functional turns the proof of the geometric conditions more involved than that of [2] (see Proposition 3.3).

After J.L.Lions [20] presented an abstract functional analysis framework to deal with the evolution equation related with (P), this kind of problem has been extensively studied (see [1, 4, 3, 17, 5] and references therein). As far as we know, the first paper dealing with Kirchhoff type equation via variational methods was [1]. By assuming some technical conditions of the functions m and f, they obtained a solution for the problem (1.1). Since then, there is a vast literature concerning existence, nonexistence, multiplicity and concentration behavior of solutions for nonlocal problems. We just quote [14, 23, 15, 22, 29] for subcritical problems and [13, 12, 18] for critical growth problems. Our first result is closely related to that of [11], where the author considered $(f_1)-(f_3)$ and obtained a positive weak solution u_λ for $\lambda > 0$ large, and also proved that $\|u_\lambda\| \to 0$, as $\lambda \to +\infty$. We finally mention the work [24], where the author obtained, for $N = 3$ and $m(t) = a + tb$, the existence of one positive solution for any $\lambda > 0$. Our first theorem complements the aforementioned works since we consider multiple solutions for a critical equation under a very weak condition for the nonlocal term m.

In the second part of the paper, we suppose that $N = 3$ and consider the effect of the parameter μ on the number of solutions by assuming that m verifies:

$(\widehat{m_0})$ $m \in C([0, +\infty), (0, +\infty))$;
(m_1) $m(t) \geq \alpha_0 > 0$, for any $t \geq 0$;
(m_2) $2M(t) \geq m(t)t$, for any $t \geq 0$;
(m_3) there exist $a > 0$ and $b \geq 0$ such that

$$m(t) \leq a + bt, \quad t \geq 0.$$

A simple computation shows that the function $m(t) = \alpha_0 + bt^\delta$, with $\delta \in [0, 1]$, verifies all the above conditions, and therefore the model case of linear m can be considered. They also hold for the function $m(t) = \alpha_0(1 + \ln(1 + t))$. Obviously, all these functions satisfy the condition (m_0) of our first theorem.

For the nonlinearity f, besides (f_0), we shall suppose that

$(\widehat{f_1})$ there holds

$$\lim_{|s|\to\infty} \frac{f(x, s)}{|s|^5} = 0, \quad \text{uniformly in } \Omega;$$

(f_4) there exist $\sigma \in [0, 2)$ and $c_1, c_2 \in (0, +\infty)$ such that

$$\frac{1}{4} f(x, s)s - F(x, s) \geq -c_1 - c_2|s|^\sigma, \quad x \in \Omega, \ s \in \mathbb{R},$$

where $F(x, s) := \int_0^s f(x, t)dt$;

(f_5) there exists an open set $\Omega_0 \subset \Omega$ with positive measure, such that

$$\liminf_{|s|\to\infty} \frac{F(x, s)}{s^4} = +\infty, \quad \text{uniformly in } \Omega_0.$$

Our result in the 3-dimensional case can be stated as follows:

Theorem 1.2. *Suppose that $N = 3$, m satisfies $(\widehat{m_0})$, $(m_1) - (m_3)$ and $\lambda > 0$. Suppose also that f satisfies (f_0), $(\widehat{f_1})$, (f_4), (f_5) and one of the conditions below:*

(f_6) *there exist $q \in (2, 6)$ and $c_3, c_4 \in (0, +\infty)$ such that*

$$F(x, s) \leq c_3|s|^q + c_4, \quad x \in \Omega, \ s \in \mathbb{R};$$

or

(f_7) *the function*

$$a(x) := \limsup_{s\to 0} \frac{F(x, s)}{s^2}$$

is such that $a^+(x) := \max\{a(x), 0\} \in L^\infty(\Omega)$.

Then, for any given $k \in \mathbb{N}$, there exists $\mu_k^* > 0$ such that the problem (P) has at least k pairs of nonzero solutions for all $\mu \in (0, \mu_k^*)$.

Obviously, $(\widehat{f_1})$ is weaker than (f_1). If we do not have a large parameter multiplying the term $f(x,u)$, the truncation argument used in the proof of Theorem 1.1 does not work, and therefore it will be natural to consider the modified Ambrosetti-Rabinowitz condition (f_2) with $\theta > 4$ (see [9, 24, 13, 15, 29]). In this case, we have that $F(x,s) \geq C_1|s|^\theta$ for any $x \in \Omega$ and $s \in \mathbb{R}$. So, the condition (f_4) is weaker than (f_2) with $\theta > 4$. Moreover, the superlinearity condition (f_5) holds only on a set of positive measure and therefore our conditions on f are weaker than those of Theorem 1.1. Unfortunately, the truncation argument does not work in this weak setting and we are not able to prove the second theorem if $N \geq 4$. The main point is that we do not know if the norm of the solutions given by Theorem 1.2 goes to zero as $\mu \to 0^+$.

It is worthwhile to mention that, although the local version of Theorem 1.2 was considered in [28], our result is new even in the local case. Actually, in this case we can prove our results with the quotient in (f_5) being $F(x,s)/s^2$ (see Remark 5.3), and therefore our condition (f_5) is more general than the hypothesis (f_6) of [28]. Moreover, our condition (f_7) is weaker than the condition (f_7) of [28]. Thus, our second theorem generalize Theorems A and C of [28] besides complement the aforementioned works.

An example of nonlinearity verifying all the hypothesys of Theorem 1.1 is $f(x,s) = a(x)|s|^{q-2}s$, with $a \in L^\infty(\Omega)$ positive and $q \in (2, 2^*)$, or even a finite sums of this kind of functions with different (and positive) $a_i \in L^\infty(\Omega)$ and $q_i \in (2, 2^*)$. For the second theorem, we pick $q \in (4, 6)$ and notice that, since the superlinearity condition is just local, we can allow the potential a to vanish in a proper set of positive measure of Ω. Actually, we may also consider examples where f is negative, for instance, $f(s) \sim s$ near the origin, $f(s) \sim s^{q-1}$ at infinity and f is negative and bounded in some intervals $(s_i^-, s_i^+) \subset (0, +\infty)$.

In the next section, we prove a local compactness property for the energy functional under the setting of Theorem 1.1 which is proved in Section 3. In Section 4 we prove compactness for $\mu > 0$ small and the final Section 5 is devoted to the proof of Theorem 1.2.

2. The local Palais Smale condition

Throughout the paper we write $\int_\Omega u$ instead of $\int_\Omega u(x)dx$. We are going to work on the space $H_0^1(\Omega)$ endowed with the norm

$$\|u\| := \left(\int_\Omega |\nabla u|^2 dx \right)^{1/2}.$$

We also denote by $\|u\|_p$ the $L^p(\Omega)$-norm of a function $u \in L^p(\Omega)$, for any $1 \leq p \leq \infty$. Throughout the two next sections we assume that (m_0) holds and that $\mu > 0$ is fixed.

Let $\theta > 2$ from (f_2) and $a > 0$ be a number in the range of m verifying

$$m(0) < a < \frac{\theta}{2} m(0). \tag{2.1}$$

Since m is increasing, there exists $s_0 > 0$ such that $m(s_0) = a$. We define $m_a \in C([0, +\infty), \mathbb{R}^+)$ by setting

$$m_a(s) := \begin{cases} m(s), & \text{if } 0 \leq s \leq s_0, \\ a, & \text{if } s \geq s_0, \end{cases}$$

and consider the truncated problem

$$(P_a) \qquad \begin{cases} -m_a(\|u\|^2)\Delta u = \lambda f(x,u) + \mu |u|^{2^*-2}u, & x \in \Omega, \\ u \in H_0^1(\Omega). \end{cases}$$

We say that $u \in H_0^1(\Omega)$ is a weak solution of (P_a) if

$$m_a(\|u\|^2) \int_\Omega (\nabla u \cdot \nabla \phi) = \lambda \int_\Omega f(x,u)\phi + \mu \int_\Omega |u|^{2^*-2}u\phi, \quad \forall\, \phi \in H_0^1(\Omega),$$

with an analogous definition for weak solutions of (P). By using $(f_0)-(f_1)$ and standard calculations we can prove that the energy function $I_{a,\lambda}$ given by

$$I_{a,\lambda}(u) := \frac{1}{2} M_a(\|u\|^2) - \lambda \int_\Omega F(x,u) - \frac{\mu}{2^*} \int_\Omega |u|^{2^*}, \quad u \in H_0^1(\Omega),$$

is well defined. In the above definition we are denoting $M_a(s) := \int_0^s m_a(t)dt$ and $F(x,s) := \int_0^s f(x,t)dt$. Moreover, $I_{a,\lambda}$ belongs to $C^1(H_0^1(\Omega),\mathbb{R})$ and the weak solutions of (P_a) are the critical points of $I_{a,\lambda}$.

Notice that, by the definition of m_a, if $u \in H_0^1(\Omega)$ is a weak solution of (P_a) such that $\|u\| < s_0$, then $m_a(\|u\|^2) = m(\|u\|^2)$ and therefore u is also a weak solution of the original problem (P). Hence, we are going to look for multiple critical points of $I_{a,\lambda}$ with small norm.

Lemma 2.1. *Suppose that f satisfies $(f_0)-(f_2)$. If $(u_n) \subset H_0^1(\Omega)$ is such that $I_{a,\lambda}(u_n) \to c$ and $I'_{a,\lambda}(u_n) \to 0$, then (u_n) is bounded in $H_0^1(\Omega)$.*

Proof. Condition (m_0) and the definition of m_a imply that $M_a(s) \geq m(0)s$ and $m_a(s) \leq a$, for any $s \in \mathbb{R}$. Hence, we can use (f_2) to get

$$c + o_n(1) + o_n(1)\|u_n\| = I_{a,\lambda}(u_n) - \frac{1}{\theta} I'_{a,\lambda}(u_n)u_n \geq \left(\frac{m(0)}{2} - \frac{a}{\theta} \right) \|u_n\|^2,$$

where $o_n(1)$ stands for a quantity approaching zero as $n \to +\infty$. By (2.1) the term into the parenthesis above is positive and we have done. □

If we set

$$S := \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\|u\|^2}{(\int_\Omega |u|^{2^*})^{2/2^*}},$$

we can state the following well-known result due to Lions [21]:

Lemma 2.2. *Suppose that $(u_n) \subset H_0^1(\Omega)$ is such that $u_n \rightharpoonup u$ weakly in $H_0^1(\Omega)$ and $|u_n|^{2^*} \rightharpoonup \nu$, $|\nabla u_n|^2 \rightharpoonup \zeta$ weakly in the sense of measures, where ν and ζ are non-negative and bounded measures on $\overline{\Omega}$. Then there exist a countable index set J, which can be empty, and a family $\{x_j\}_{j \in J} \subset \overline{\Omega}$ such that*

$$\nu = |u|^{2^*}\mathrm{d}x + \sum_{j \in J} \nu_j \delta_{x_j}, \qquad \zeta \geq |\nabla u|^2 \mathrm{d}x + \sum_{j \in J} \zeta_j \delta_{x_j}, \tag{2.2}$$

with ν_j, $\zeta_j > 0$ satisfying $S\nu_j^{2/2^*} \leq \zeta_j$, for all $j \in J$.

In what follows we prove that, for some special sequences, the set J must be finite.

Lemma 2.3. *Let $(u_n) \subset H_0^1(\Omega)$ be as in the statement of Lemma 2.2. If $I'_{a,\lambda}(u_n) \to 0$, then the set J is empty or finite. Moreover,*

$$\nu_j \geq \left(\frac{m(0)S}{\mu}\right)^{N/2}, \quad \forall\, j \in J. \tag{2.3}$$

Proof. Let $\phi \in C_0^\infty(\mathbb{R}^N, [0,1])$ be such that $\phi \equiv 1$ in $B_{1/2}(0)$ and $\phi \equiv 0$ in $\mathbb{R}^N \setminus B_1(0)$. Suppose that $J \neq \varnothing$, fix $j \in J$ and define $\phi_\varepsilon(x) := \phi(\frac{x-x_j}{\varepsilon})$. Since $I'(u_n)(\phi_\varepsilon u_n) = o_n(1)$, we have that

$$m(\|u_n\|^2)\left(A_{n,\varepsilon} + \int_\Omega |\nabla u_n|^2\phi_\varepsilon\right) = o_n(1) + \mu\int_\Omega |u_n|^{2^*}\phi_\varepsilon + \lambda\int_\Omega f(x,u_n)u_n\phi_\varepsilon, \tag{2.4}$$

with $A_{n,\varepsilon} := \int_\Omega u_n(\nabla u_n \cdot \nabla\phi_\varepsilon)$. By using (f_3) and the sub-critical growth condition (f_1), we can prove that $\int_\Omega f(x,u_n)u_n\phi_\varepsilon \to \int_\Omega f(x,u)u\phi_\varepsilon$, as $n \to +\infty$. Thus, since (m_0) implies that $m(t) \geq \alpha_0 > 0$, for any $t \geq 0$, we infer from (2.4) and Lemma 2.2 that

$$\alpha_0\left(\limsup_{n\to+\infty} A_{n,\varepsilon} + \int_{\overline{\Omega}} \phi_\varepsilon d\zeta\right) \leq \mu\int_{\overline{\Omega}} \phi_\varepsilon d\nu + \lambda\int_\Omega f(x,u)u\phi_\varepsilon.$$

We claim that

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty} A_{n,\varepsilon} = 0. \tag{2.5}$$

If this is true can take $\varepsilon \to 0$ and use the Lebesgue Theorem to get $\alpha_0\zeta_j \leq \mu\nu_j$. Recalling that $S\nu_j^{2/2^*} \leq \zeta_j$, we obtain

$$m(0)S\nu_j^{2/2^*} \leq m(0)\zeta_j \leq \alpha_0\zeta_j \leq \mu\nu_j,$$

and therefore $\nu_j \geq (m(0)S/\mu)^{N/2}$. Hence,

$$\nu(\overline{\Omega}) \geq \sum_{j\in J}\nu_j \geq \sum_{j\in J}\left(\frac{m(0)S}{\mu}\right)^{N/2}.$$

Since $\nu(\overline{\Omega}) < +\infty$, we conclude that set J is finite.

In order to prove (2.5), we compute

$$\begin{aligned}
|A_{n,\varepsilon}| &\leq \frac{\||\nabla\phi|\|_\infty}{\varepsilon}\left(\int_{B_\varepsilon(x_j)} |\nabla u_n|^2 dx\right)^{1/2}\left(\int_{B_\varepsilon(x_j)} |u_n|^2 dx\right)^{1/2} \\
&\leq \frac{\||\nabla\phi|\|_\infty}{\varepsilon}\|u_n\|\left(\int_{B_\varepsilon(x_j)} |u_n|^2 dx\right)^{1/2} \\
&\leq \frac{d_1}{\varepsilon}\left(\int_{B_\varepsilon(x_j)} |u|^2 dx + o_n(1)\right)^{1/2},
\end{aligned}$$

with $d_1 > 0$. Since $\int_{B_\varepsilon(x_j)} |u|^2 dx = O(\varepsilon^3)$, as $\varepsilon \to 0$, equation (2.5) follows from the above inequality. □

If E is a real Banach space and $I \in C^1(E,\mathbb{R})$, we say that I satisfies the Palais-Smale condition at level $c \in \mathbb{R}$, $(PS)_c$ for short, if every sequence $(u_n) \subset E$ such that $I(u_n) \to c$ and $I'(u_n) \to 0$ possesses a convergent subsequence.

Proposition 2.4. *Suppose that f satisfies $(f_0)-(f_2)$ and define*

$$c^* := \min\left\{\mu\left(\frac{1}{\theta}-\frac{1}{2^*}\right)\left(\frac{m(0)S}{\mu}\right)^{N/2}, \left(\frac{m(0)}{2}-\frac{a}{\theta}\right)s_0^2\right\}.$$

Then the functional $I_{a,\lambda}$ satisfies the (PS_c) condition at any level $c<c^*$.

Proof. Let $(u_n)\subset H_0^1(\Omega)$ be such that $I'_{a,\lambda}(u_n)\to 0$ and $I_{a,\lambda}(u_n)\to c<c^*$. We start by proving that the set J given by Lemma 2.2 is empty. Indeed, suppose by contradiction that there exists some $j\in J$. If we consider ϕ_ε as in the proof of Lemma 2.3, we can use Lemma 2.1, (f_2) and (2.1) to get

$$\begin{aligned} c+o_n(1) &= I_{a,\lambda}(u_n)-\frac{1}{\theta}I'_{a,\lambda}(u_n)u_n \\ &\geq \mu\left(\frac{1}{\theta}-\frac{1}{2^*}\right)\int_\Omega |u_n|^{2^*} \geq \mu\left(\frac{1}{\theta}-\frac{1}{2^*}\right)\int_\Omega |u_n|^{2^*}\phi_\varepsilon. \end{aligned}$$

Taking the limit and using (2.3), we conclude that

$$c\geq \mu\left(\frac{1}{\theta}-\frac{1}{2^*}\right)\nu_j \geq \mu\left(\frac{1}{\theta}-\frac{1}{2^*}\right)\left(\frac{m(0)S}{\mu}\right)^{N/2},$$

which contradicts $c<c^*$.

Since the set J is empty, we can use (2.2) and the boundedness of Ω to conclude that

$$\lim_{n\to+\infty}\int_\Omega |u_n|^{2^*} = \int_\Omega |u|^{2^*},$$

where $u\in H_0^1(\Omega)$ is the $H_0^1(\Omega)$-weak limit of (u_n). Recalling that $I'_{a,\lambda}(u_n)u_n = o_n(1)$, we can use (f_1) to get

$$\lim_{n\to\infty} m_a(\|u_n\|^2)\|u_n\|^2 = \lambda\int_\Omega f(x,u)u+\mu\int_\Omega |u|^{2^*}. \tag{2.6}$$

On the other hand, if we set $\alpha_0 := \lim_{n\to\infty}\|u_n\|^2$, we can use $I'_{a,\lambda}(u_n)\to 0$ to obtain

$$m_a(\alpha_0^2)\int_\Omega(\nabla u\cdot\nabla\phi) = \lambda\int_\Omega f(x,u)\phi+\mu\int_\Omega |u|^{2^*-2}u\phi, \quad \forall\,\phi\in H_0^1(\Omega).$$

By picking $\phi=u$, we infer from (2.6) that

$$\lim_{n\to\infty} m_a(\|u_n\|^2)\|u_n\|^2 = m_a(\alpha_0^2)\|u\|^2.$$

Since m_a is continuous and positive, we have that $\alpha_0=\|u\|$. Hence, the weak convergence implies that $u_n\to u$ strongly in $H_0^1(\Omega)$ and the proposition is proved. □

3. Proof of Theorem 1.1

In this section, we present the proof of our first theorem. We are going to use the following version of Symmetric Mountain Pass Theorem (see [2, 28]):

Theorem 3.1. *Let $E=V\oplus W$ be a real Banach space with* $\dim V<\infty$. *Suppose $I\in C^1(E,\mathbb{R})$ is an even functional satisfying $I(0)=0$ and*

(I_1) *there exist $\rho,\ \alpha>0$ such that*

$$\inf_{u\in\partial B_\rho(0)\cap W} I(u)\geq\alpha;$$

(I_2) *there exist a subspace* $\widehat{V} \subset E$ *such that* $\dim V < \dim \widehat{V} < \infty$ *and*

$$\max_{u \in \widehat{V}} I(u) \leq M,$$

for some $M > 0$*;*

(I_3) I *satisfies* $(PS)_c$ *for any* $c \in (0, M)$*, with* M *as in* (I_2)*.*

Then I *possesses at least* $(\dim \widehat{V} - \dim V)$ *pairs of non-trivial critical points.*

In what follows we verify that the functional $I_{a,\lambda}$ satisfies the conditions (I_1) and (I_2).

Lemma 3.2. *Suppose that* f *satisfies* (f_0)*,* (f_1) *and* (f_3)*. Then, for each* $\lambda > 0$*, there exists* ρ_λ*,* $\alpha_\lambda > 0$ *such that*

$$\inf_{u \in \partial B_{\rho_\lambda}(0)} I_{a,\lambda}(u) \geq \alpha_\lambda.$$

Proof. Given $\varepsilon > 0$, we can use (f_1) and (f_3) to obtain $C_\varepsilon > 0$ such that

$$F(x,s) \leq \frac{\varepsilon}{2}|s|^2 + C_\varepsilon |s|^q, \quad \forall\, (x,s) \in \Omega \times \mathbb{R}.$$

Hence,

$$\begin{aligned} I_{a,\lambda}(u) &\geq \frac{m(0)}{2}\|u\|^2 - \frac{\varepsilon}{2}\int_\Omega |u|^2 - \frac{C_\varepsilon}{q}\lambda \int_\Omega |u|^q - \frac{\mu}{2^*}\int_\Omega u^{2^*} \\ &\geq \frac{1}{2}\left(m(0) - \frac{\varepsilon\lambda}{\lambda_1(\Omega)}\right)\|u\|^2 - d_1\lambda\|u\|^q - d_2\|u\|^{2^*}, \end{aligned}$$

where $\lambda_1(\Omega) > 0$ is the first eigenvalue of $(-\Delta, H_0^1(\Omega))$ and d_1, $d_2 > 0$ are constants independent of λ. The result easily follows if we choose $\varepsilon > 0$ small and use the inequality $2 < q < 2^*$. □

Proposition 3.3. *Suppose that* f *satisfies* $(f_0) - (f_2)$*. Then, for any given* $k \in \mathbb{N}$ *and* $M^* > 0$*, there exists* $\lambda_k^* > 0$ *with the following property: for any* $\lambda \geq \lambda_k^*$ *we can find a* k*-dimensional subspace* $V_k^\lambda \subset H_0^1(\Omega)$ *such that*

$$\max_{u \in V_k^\lambda} I_{a,\lambda}(u) < M^*.$$

Proof. Let $\varphi \in C_0^\infty(B_1(0))$, choose $\{x_1, \ldots, x_k\} \subset \Omega$ and $\delta > 0$ such that, for i, $j \in I := \{1, \ldots, k\}$, $B_\delta(x_i) \subset \Omega$ and $B_\delta(x_i) \cap B_\delta(x_j) = \varnothing$, if $i \neq j$. For each $i \in I$, we set $\varphi_i^\delta(x) := \varphi(\frac{x - x_i}{\delta})$ and notice that

$$A_\delta := \frac{\|\varphi_i^\delta\|^2}{\|\varphi_i^\delta\|_\theta^2} = \delta^{\left(N-2-\frac{2N}{\theta}\right)}\frac{\|\varphi\|^2}{\|\varphi\|_\theta^2}. \tag{3.1}$$

Since $\mathbb{R}^k$ is finite dimensional, there exists $d_1 = d_1(k,\theta)$ such that

$$\sum_{i=1}^k |y_i|^\theta \geq d_1 \left(\sum_{i=1}^k |y_i|^2\right)^{\theta/2}, \quad \forall\, (y_1, \ldots, y_k) \in \mathbb{R}^k. \tag{3.2}$$

Hence, if we set

$$V_{k,\delta} := \operatorname{span}\{\varphi_1^\delta, \ldots, \varphi_k^\delta\},$$

we have that, for any $u = \sum_{i=1}^{k} \alpha_i \varphi_i^\delta \in V_{k,\delta}$, there holds

$$
\begin{aligned}
\int_\Omega |u|^\theta dx &= \int_{B_\delta(x_1)\cup\cdots\cup B_\delta(x_k)} \Big|\sum_{i=1}^{k} \alpha_i \varphi_i^\delta\Big|^\theta dx \\
&= \sum_{i=1}^{k} \|\alpha_i \varphi_i^\delta\|_\theta^\theta \geq d_1 \left(\sum_{i=1}^{k} \|\alpha_i \varphi_i^\delta\|_\theta^2\right)^{\theta/2} \qquad (3.3)\\
&= d_1 \left(\sum_{i=1}^{k} A_\delta^{-1} \|\alpha_i \varphi_i^\delta\|^2\right)^{\theta/2} = d_2 \delta^{-(N-2-\frac{2N}{\theta})\frac{\theta}{2}} \|u\|^\theta,
\end{aligned}
$$

where $d_2 = d_1\|\varphi\|^2\|\varphi\|_\theta^{-2}$ and we have used (3.2), (3.1) and the fact that the support of the functions φ_i^δ are disjoint.

By using (f_3), we obtain d_3, $d_4 > 0$ such that

$$
F(x,s) \geq d_3|s|^\theta - d_2, \quad \forall\, (x,s) \in \Omega \times \mathbb{R}.
$$

This and (3.3) provide

$$
\begin{aligned}
I_{a,\lambda}(u) &\leq \frac{a}{2}\|u\|^2 - \lambda \sum_{i=1}^{k} \int_{B_\delta(x_i)} F(x,u) \\
&\leq \frac{a}{2}\|u\|^2 - \lambda d_2 d_3 \delta^{-(N-2-\frac{2N}{\theta})\frac{\theta}{2}} \|u\|^\theta + \lambda d_2 k \delta^N \omega_N,
\end{aligned}
$$

where ω_N is the volume of the unitary ball $B_1(0) \subset \mathbb{R}^N$. Thus, for positive constants $d_5 = d_5(k,\theta)$, $d_6 = d_6(k,N)$ and

$$
\gamma := -\left(N - 2 - \frac{2\theta}{N}\right)\frac{\theta}{2} > 0,
$$

there holds

$$
I_{a,\lambda}(u) \leq \frac{a}{2}\|u\|^2 - \lambda d_5 \delta^\gamma \|u\|^\theta + \lambda d_6 \delta^N, \quad \forall\, u \in V_{k,\delta}. \qquad (3.4)
$$

Since $\theta < 2^*$, we have that $\gamma < N$ and therefore we can pick $\gamma_0 \in (\gamma, N)$ and consider the function

$$
h_\delta(t) := \frac{a}{2}t^2 - d_5\delta^{-\gamma_0+\gamma} t^\theta + d_6 \delta^{-\gamma_0+N}, \quad t > 0.
$$

It attains its maximum value at $t_\delta = \left[a(d_5\theta)^{-1}\delta^{\gamma-\gamma_0}\right]^{1/(\theta-2)}$. This and $\gamma_0 \in (\gamma, N)$ imply that $h_\delta(t_\delta) \to 0$ as $\delta \to 0^+$. Thus, there exists $\delta^* = \delta^*(l,\theta,N,a) > 0$ such that

$$
\max_{t\geq 0} h_\delta(t) \leq \frac{M^*}{2}, \quad \forall\, \delta \in (0, \delta^*].
$$

We now set $\lambda_k^* := (\delta^*)^{-\gamma_0}$. Let $\lambda \geq \lambda_k^*$ and define the k-dimensional subspace $V_k^\lambda := V_{k,\delta}$ for $\delta = \lambda^{-1/\gamma_0}$. Since $\delta^{-\gamma_0} = \lambda \geq \lambda_k^* = (\delta^*)^{-\gamma_0}$, we obtain $\delta \leq \delta^*$. Thus, for any $u \in V_k^\lambda$, we can use (3.4) and the above inequality to get

$$
I_{a,\lambda}(u) \leq \frac{a}{2}\|u\|^2 - \delta^{-\gamma_0} d_5 \delta^\gamma \|u\|^\theta + \delta^{-\gamma_0} d_6 \delta^N \leq \max_{t\geq 0} h_\delta(t) \leq \frac{M^*}{2},
$$

and we have done. □

We are ready to prove our first theorem.

Proof of Theorem 1.1. Let $k \in \mathbb{N}$ be given. We are going to apply Theorem 3.1 with $W = H_0^1(\Omega)$. Condition (I_1) is a direct consequence of Lemma 3.2. In order to verify the other conditions, we consider $M^* < c^*$, with c^* as in Proposition 2.4. Condition (I_3) follows from Proposition 2.4. It remains to obtain a k-dimensional subspace such that (I_2) holds. But that condition always hold for the subsapce V_k^λ given by Proposition 3.3, if we take $\lambda \geq \lambda_k^*$.

Since $I_{a,\lambda}(0) = 0$ and this functional is even, the above considerations and Theorem 3.1 provide, for each $\lambda \geq \lambda_k^*$, the existence of at least k pairs of nonzero solutions of the modified problem (P_a). Let $u \in H_0^1(\Omega)$ be one of these solutions. Since $I_{a,\lambda}(u) \leq M^* < c^*$, we can use the definition of c^* and (f_2) to obtain

$$\left(\frac{m(0)}{2} - \frac{a}{\theta}\right) s_0^2 \geq c^* > M^* = I_{a,\lambda}(u) - \frac{1}{\theta} I'_{a,\lambda}(u)u \geq \left(\frac{m(0)}{2} - \frac{a}{\theta}\right) \|u\|^2.$$

Hence, $\|u\| < s_0$ and it follows from the definition of m_a that $m_a(\|u\|^2) = m(\|u\|)$, that is, the function $u \in H_0^1(\Omega)$ weakly solves (P). The theorem is proved. □

4. The 3-dimensional case

From now on we focus on Theorem 1.2. We assume hereafter that the conditions $(\widehat{m_0})$, $(m_1) - (m_3)$ hold and that $\lambda > 0$ is fixed. For each $\mu > 0$, we can use conditions $(f_0) - (\widehat{f_1})$ to guarantee that the energy functional $I_\mu : H_0^1(\Omega) \to \mathbb{R}$ given by

$$I_\mu(u) := \frac{1}{2} M(\|u\|^2) - \lambda \int_\Omega F(x,u) - \frac{\mu}{6} \int_\Omega |u|^6, \quad u \in H_0^1(\Omega),$$

is well defined. Moreover, $I_\mu \in C^1(H_0^1(\Omega), \mathbb{R})$ and the critical points of I_μ are weak solutions of (P).

The proof of the Palais-Smale condition is completely different from that done for the functional $I_{a,\lambda}$. In this case, we have the following:

Proposition 4.1. *Suppose that f satisfies $(\widehat{f_1})$, (f_4) and one of the conditions (f_6) or (f_7). Then, given $M > 0$, there exist $\mu^* = \mu^*(\Omega, M, a, c_3, c_4, \sigma, \lambda) > 0$ such that I_μ satisfies the $(PS)_c$ condition for any $c < M$ and $\mu \in (0, \mu^*)$.*

The proof will be done in several steps. The first one is to prove that Palais-Smale sequences are bounded.

Lemma 4.2. *Suppose that f satisfies $(\widehat{f_1})$, (f_4) and one of the conditions (f_6) or (f_7). If $(u_n) \subset H_0^1(\Omega)$ is such that $I_\mu(u_n) \to c$ and $I'_\mu(u_n) \to 0$, then (u_n) is bounded in $H_0^1(\Omega)$.*

Proof. Let $(u_n) \subset H_0^1(\Omega)$ be such that $I_\mu(u_n) \to c$, $I'_\mu(u_n) \to 0$ and consider $\sigma \in [0, 2)$ given by (f_4). For any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that,

$$|s|^\sigma \leq \varepsilon |s|^6 + C_\varepsilon, \quad \forall\, s \in \mathbb{R}. \tag{4.1}$$

This, (m_2) and (f_4) show that, for n large, there hold

$$\begin{aligned} c+o_n(1)+o_n(1)\|u_n\| &\geq I_\mu(u_n)-\frac{1}{4}I'_\mu(u_n)u_n \\ &\geq \frac{\mu}{12}\|u_n\|_6^6+\lambda\int_\Omega\left(\frac{1}{4}f(x,u_n)u_n-F(x,u_n)\right) \\ &\geq \frac{\mu}{12}\|u_n\|_6^6-c_1\lambda|\Omega|-c_2\lambda\int_\Omega|u_n|^\sigma \\ &\geq \left(\frac{\mu}{12}-\varepsilon c_2\lambda\right)\|u_n\|_6^6-(c_1\lambda+C_\varepsilon)|\Omega|. \end{aligned}$$

By picking $\varepsilon>0$ small, we obtain d_1, $d_2>0$ such that

$$\|u_n\|_6^6\leq d_1+d_2\|u_n\|. \tag{4.2}$$

On the other hand, since $I_\mu(u_n)=c+o_n(1)$, it follows from (m_2), (m_1) and (f_6) that

$$\frac{\alpha_0}{4}\|u_n\|^2\leq\frac{1}{2}M(\|u_n\|^2)\leq\frac{\mu}{6}\|u_n\|_6^6+c_3\lambda|\Omega|+c_4\lambda\|u_n\|_q^q+c+o_n(1).$$

Since $2<q<6$, we have an inequality analogous to (4.1) with σ replaced by q, and therefore it follows from (4.2) that

$$\frac{\alpha_0}{4}\|u_n\|^2\leq d_3\|u_n\|_6^6+d_4\leq d_5\|u_n\|+d_6.$$

Hence, (u_n) is bounded in $H_0^1(\Omega)$.

Suppose now that f satisfies (f_7) instead of (f_6). Given $\varepsilon>0$, we can use $(\widehat{f_1})$ to obtain $d_7>0$ such that

$$|F(x,s)|\leq d_7+\varepsilon|s|^6.$$

Letting $n\to+\infty$ and recalling that $\varepsilon>0$ is arbitrary, we conclude that

$$\limsup_{s\to+\infty}\frac{|F(x,s)|}{s^6}=0.$$

This and (f_7) provides $C_\varepsilon>0$ such that

$$F(x,s)\leq C_\varepsilon|s|^6+\left(\|a^+\|_\infty+\varepsilon\right)s^2. \tag{4.3}$$

The proof now follows as in the first case. □

We prove now a version of Lemma 2.3 in this new setting.

Lemma 4.3. *Suppose that f satisfies $(\widehat{f_1})$ and let $(u_n)\subset H_0^1(\Omega)$ be as in the statement of Lemma 2.2. If $I'_\mu(u_n)\to0$, then the set J is empty or finite. Moreover,*

$$\nu_j\geq\left(\frac{\alpha_0 S}{\mu}\right)^{\frac{3}{2}},\quad\forall\, j\in J. \tag{4.4}$$

Proof. Let $\phi\in C_0^\infty(\mathbb{R}^N,[0,1])$ as in the proof of Lemma 2.3. Since $I'_\mu(u_n)(\phi_\varepsilon u_n)=o_n(1)$, we have that

$$m(\|u_n\|^2)\left(A_{n,\varepsilon}+\int_\Omega|\nabla u_n|^2\phi_\varepsilon\right)=o_n(1)+\mu\int_\Omega|u_n|^6\phi_\varepsilon+\lambda\int_\Omega f(x,u_n)u_n\phi_\varepsilon,$$

with $A_{n,\varepsilon}:=\int_\Omega u_n(\nabla u_n\cdot\nabla\phi_\varepsilon)$. By using $(\widehat{f_1})$, the compactness embedding of $H_0^1(\Omega)$ into the Lebesgue spaces and a version of a compactness lemma due to Strauss (see [8, Lemma 1.2]) we can check that $\int f(u_n)u_n\phi_\varepsilon\to\int f(u)u\phi_\varepsilon$, as

$n \to +\infty$. Arguing as in as in the proof of Lemma 2.3, using (2.5) and (m_1), we obtain $\nu_j \geq (\alpha_0 S/\mu)^{3/2}$. Hence,

$$\nu(\overline{\Omega}) \geq \sum_{j\in J} \nu_j \geq \sum_{j \in J} \left(\frac{\alpha_0 S}{\mu}\right)^{3/2}. \tag{4.5}$$

and we conclude that set J is finite. □

We are ready to present the proof of our compactness result.

Proof of Proposition 4.1. Let $(u_n) \subset H_0^1(\Omega)$ be such that $I'_\mu(u_n) \to 0$ and $I_\mu(u_n) \to c < M$. By Lemma 4.2, this sequence is bounded in $H_0^1(\Omega)$ and therefore there exist $u \in H_0^1(\Omega)$ and two bounded measures ν, ζ satisfying all the hypotheses of Lemma 2.2.

Arguing as in the proof of Lemma 4.2 and using Hölder's inequality we get, for n large,

$$\begin{aligned} M &> I_\mu(u_n) - \frac{1}{4} I'_\mu(u_n)u_n \\ &\geq \frac{\mu}{12}\int_\Omega |u_n|^6 - c_1\lambda|\Omega| - c_2\lambda\int_\Omega |u_n|^\sigma \\ &\geq \mu d_1 \int_\Omega |u_n|^6 - d_2 - d_3\left(\int_\Omega |u_n|^6\right)^{\sigma/6} \end{aligned}$$

with $d_1 := 1/12$, $d_2 := c_1\lambda|\Omega|$ and $d_3 := c_2\lambda|\Omega|^{(6-\sigma)/6}$. Letting $n \to +\infty$ and recalling that $|u_n|^6 \rightharpoonup \nu$ weakly in the sense of measures, we obtain

$$\mu d_1 \nu(\overline{\Omega}) \leq M + d_2 + d_3 \nu(\overline{\Omega})^{\sigma/6}.$$

If $\nu(\overline{\Omega}) > 1$, we can use the above estimate to obtain

$$\nu(\overline{\Omega}) \leq \nu(\overline{\Omega})^{\sigma/6}\left(\frac{M + d_2 + d_3}{\mu d_1}\right).$$

Since $0 \leq \sigma < 2$, there exists $\widetilde{\mu} > 0$ such that

$$\nu(\overline{\Omega}) \leq \left(\frac{M + d_2 + d_3}{\mu d_1}\right)^{6/(6-\sigma)} \leq \left(\frac{\alpha_0 S}{\mu}\right)^{3/2}, \quad \forall\, \mu \in (0, \widetilde{\mu}). \tag{4.6}$$

If $\nu(\overline{\Omega}) \leq 1$, we can choose $\widehat{\mu} < \alpha_0 S$ and use a simple computation to obtain $\nu(\overline{\Omega}) < (\alpha_0 S \mu^{-1})^{3/2}$, for any $\mu \in (0, \widehat{\mu})$. Hence, if we set $\mu^* := \min\{\widetilde{\mu}, \widehat{\mu}\}$, we get

$$\nu(\overline{\Omega}) < \left(\frac{\alpha_0 S}{\mu}\right)^{3/2}, \quad \forall\, \mu \in (0, \mu^*),$$

and therefore it follows from (4.5) that the set J given by Lemma 2.2 is empty. Thus, we can use (2.2) and the boundedness of Ω to get

$$\int_\Omega |u_n|^6 \to \int_\Omega |u|^6.$$

Since (u_n) is bounded in $H_0^1(\Omega)$ we easily conclude that $\int_\Omega |u_n|^4 u_n u \to \int_\Omega |u|^6$. As before, we also have that $\int_\Omega f(x, u_n)(u - u_n) \to 0$. Thus,

$$o_n(1) = I'_\mu(u_n)u_n - I'_\mu(u_n)u = m(\|u_n\|^2)\left(\|u_n\|^2 - \|u\|^2\right) + o_n(1).$$

It follows from (m_1) that $\|u_n\| \to \|u\|$. This and the weak convergence of (u_n) finishes the proof. $\square$

5. Proof of Theorem 1.2

In order to present the proof of Theorems 1.2 we consider $(\varphi_j)_{j\in\mathbb{N}}$ the normalized eigenfunctions of $(-\Delta, H_0^1(\Omega))$. For any $m \in \mathbb{N}$, we set

$$V_m := \operatorname{span}\{\varphi_1, \ldots, \varphi_m\}$$

and notice that $H_0^1(\Omega) = V_m \oplus V_m^{\perp}$. Moreover, as proved in [28, Lemma 4.1], for any given $2 \le r < 6$ and $\delta > 0$, there is $m_0 \in \mathbb{N}$ such that, for all $m \ge m_0$,

$$\|u\|_r^r \le \delta \|u\|^r, \quad \forall\, u \in V_m^{\perp}. \tag{5.1}$$

We first prove that I_μ verifies the geometric condition (I_1).

Lemma 5.1. *Suppose that f satisfies (f_6) or (f_7). Then there exists*

$$\overline{\mu} = \overline{\mu}(a, b, \Omega, c_3, c_4, \lambda) > 0,$$

$m \in \mathbb{N}$ and ρ, $\alpha > 0$ such that, for any $\mu \in (0, \bar{\mu})$, there holds

$$I_\mu(u) \ge \alpha, \quad \forall u \in \partial B_\rho(0) \cap V_m^{\perp}.$$

Proof. Suppose first that (f_6) holds. If we use the inequality (5.1) with $r = q$ and $\delta > 0$ to be chosen later, we obtain

$$I_\mu(u) \ge \|u\|^2 \left(\frac{\alpha_0}{4} - \delta c_3 \lambda \|u\|^{q-2}\right) - c_4 \lambda |\Omega| - \frac{\mu}{6S^3}\|u\|^6, \quad \forall\, u \in V_m^{\perp},$$

where we have used (m_2), (m_1) and (f_6). If $\rho = \rho(\delta) > 0$ is such that $\delta c_3 \lambda \rho^{q-2} = \alpha_0/8$, we obtain

$$I_\mu(u) \ge \frac{\alpha_0}{8}\rho^2 - c_4\lambda|\Omega| - \frac{\mu}{6S^3}\rho^6, \quad \forall\, u \in \partial B_\rho(0) \cap V_m^{\perp}.$$

Since $\rho(\delta) \to +\infty$, as $\delta \to 0^+$, we can take $\delta > 0$ small in such way that $(\alpha_0/8)\rho^2 - c_4\lambda|\Omega| > (\alpha_0/16)\rho^2$, and therefore we can obtain $\bar{\mu} > 0$ such that,

$$I_\mu(u) \ge \frac{\alpha_0}{16}\rho^2 - \frac{\bar{\mu}}{6S^3}\rho^6 \ge \alpha > 0, \quad \forall\, u \in \partial B_\rho(0) \cap V_m^{\perp}.$$

The conclusion easily follows from the above inequality.

If (f_7) holds, we consider $\varepsilon > 0$ and use (m_2), (m_1) and (4.3) to get

$$I_\mu(u) \ge \frac{\alpha_0}{4}\|u\|^2 - \frac{(\mu + 6\lambda C_\varepsilon)}{6S^3}\|u\|^6 - \lambda(\|a^+\|_\infty + \varepsilon)\|u\|_2^2,$$

for any $u \in H_0^1(\Omega)$. Choosing $r = 2$ and $\delta = \frac{\alpha_0}{8\lambda(\|a^+\|_\infty + \varepsilon)}$ in (5.1), we obtain $m \in \mathbb{N}$ such that

$$I_\mu(u) \ge \frac{\alpha_0}{8}\rho^2 - \frac{(\mu + 6\lambda C_\varepsilon)}{6S^3}\rho^6, \quad \forall\, u \in \partial B_\rho(0) \cap V_m^{\perp}.$$

The lemma follows from the above inequality and the same argument used in the first case. $\square$

The local superlinearity condition (f_5) provides (I_2) as we can see from the next lemma.

Lemma 5.2. *Suppose that f satisfies (f_5). Then, for any given $l \in \mathbb{N}$, there is a l-dimensional subspace $\widehat{V} \subset H_0^1(\Omega)$ and a constant $M > 0$ such that*

$$\sup_{u \in \widehat{V}} I_\mu(u) \le M, \quad \forall\, \mu > 0.$$

Proof. Let $\Omega_0 \subset \Omega$ be given by condition (f_5), consider $(\phi_j)_{j\in\mathbb{N}}$ the normalized eigenfunctions of $(-\Delta, H_0^1(\Omega_0))$ and define the l-dimensional subspace

$$\widehat{V}_l := \operatorname{span}\{\phi_1, \dots, \phi_l\}.$$

Since $\widehat{V}_l$ has finite dimension there exists $d_1 = d(\widehat{V}) > 0$ such that

$$d_1\|u\|^4 \leq \|u\|_4^4, \quad \forall\, u \in \widehat{V}. \tag{5.2}$$

Given $\varepsilon > b/(4d_1)$, it follows from (f_5) and the continuity of F that, for some $d_2 = d_2(d_1, b)$, there holds

$$F(x,s) \geq \varepsilon|s|^4 - d_2, \quad \forall\, x \in \Omega_0,\, s \in \mathbb{R}.$$

This, (m_3) and (5.2) imply that, for any $u \in \widehat{V}_l$, we have that

$$I_\mu(u) \leq \frac{a}{2}\|u\|^2 - \left(\varepsilon\lambda d_1 - \frac{b}{4}\right)\|u\|^4 + d_2\lambda|\Omega| \leq \sup_{t>0}\left\{\frac{a}{2}t^2 - \varepsilon_0 t^4 + d_2\lambda|\Omega|\right\},$$

with $\varepsilon_0 := (\varepsilon\lambda d_1 - b/4) > 0$. If we denote by M the supremum of the right-hand side above, we can use $a > 0$ to conclude that $0 < M < +\infty$ and we have done. □

Remark 5.3. *In the local case $m \equiv 1$, the same conclusion of the last lemma holds if we drop (f_5) by the weaker condition*

$(\widehat{f_5})$ *there exists an open set $\Omega_0 \subset \Omega$ with positive measure, such that*

$$\lim_{|s|\to\infty} \frac{F(x,s)}{s^2} = +\infty, \quad \textit{uniformly in } \Omega_0.$$

Actually, if $\widehat{d_1} > 0$ is such that $\widehat{d_1}\|u\|^2 \leq \|u\|_2^2$, for all $u \in \widehat{V}$, the same argument provides

$$I_\mu(u) \leq \left(\frac{a}{2} - \varepsilon\lambda\widehat{d_1}\right)\|u\|^2 + d_2\lambda|\Omega| \leq \sup_{t>0}\left\{-\varepsilon_0 t^2 + d_2\lambda|\Omega|\right\},$$

with $\varepsilon_0 := \varepsilon\lambda\widehat{d_1} - (a/2) > 0$. Hence, the lemma holds with $M = d_2\lambda|\Omega|$.

We are ready to prove our last result.

Proof of Theorem 1.2. Let $k \in \mathbb{N}$ be fixed. Since all the previous results hold with conditions (f_6) or (f_7), we present the proof in a unified way.

By Lemma 5.1 we can find $m \in \mathbb{N}$ large such that, for the decomposition $H = V \oplus W$ with

$$V := \langle \varphi_1, \cdots, \varphi_m \rangle, \qquad W := \langle \varphi_1, \cdots, \varphi_m \rangle^\perp,$$

the functional I_μ verifies (I_1) for any $\mu \in (0, \bar{\mu})$. Moreover, by Lemma 5.2, we obtain a subspace $\widehat{V} \subset H_0^1(\Omega)$ and $M > 0$ such that

$$\dim \widehat{V} = (k+m), \qquad \sup_{u\in\widehat{V}} I_\mu(u) \leq M, \quad \forall\, \mu > 0.$$

Hence, I_μ satisfies (I_2). For the above choice of M we obtain, from Proposition 4.1, a number μ^* such that I_μ satisfies (I_3) for any $\mu \in (0, \mu^*)$. Since $I_\mu(0) = 0$ and I_μ is even, if we set $\mu_k^* := \min\{\bar{\mu}, \mu^*\}$, we can invoke Theorem 3.1 to conclude that, for all $\mu \in (0, \mu_k^*)$, the functional I_μ has at least $(k+m-m) = k$ pairs of nonzero critical points. The theorems are proved. □

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