TWO SOLUTIONS FOR A FOURTH ORDER NONLOCAL PROBLEM WITH INDEFINITE POTENTIALS

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Abstract. We study the nonlocal equation

$$\Delta^2 u - m\left(\int_\Omega |\nabla u|^2 dx\right)\Delta u = \lambda a(x)|u|^{q-2}u + b(x)|u|^{p-2}u, \text{ in } \Omega,$$

subject to the boundary condition $u = \Delta u = 0$ on $\partial\Omega$. For m continuous and positive we obtain a nonnegative solution if $1 < q < 2 < p \leq 2N/(N-4)$ and $\lambda > 0$ small. If the affine case $m(t) = \alpha + \beta t$, we obtain a second solution if $4 < p < 2N/(N-4)$ and $N \in \{5, 6, 7\}$. In the proofs we apply variational methods.

1. Introduction

Consider the semilinear problem

$$-\Delta u = \lambda a(x)|u|^{q-2}u + b(x)|u|^{p-2}u \text{ in } \Omega, \ \ u \in W_0^{1,2}(\Omega),$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $N \geq 3$, $\lambda > 0$ is a parameter, $1 < q < 2 < p \leq 2N/(N-2)$ and a, b are potentials defined in Ω. In a celebrated paper, Ambrosetti, Brezis and Cerami [2] supposed that $a \equiv 1$, $b \equiv 1$ and obtained two positive solutions if $\lambda > 0$ is small. In [10], de Figueiredo, Gossez and Ubilla generalized this result by considering nonconstant sign changing potentials. In this setting the Maximun Principle can fail and therefore the solutions are only nonnegative. Some other results for the Laplacian operator can be found in [1, 23, 20, 26] and references therein. We also quote [8, 4, 15] for the p-Laplacian, fractional Laplacian and Kirchhoff operator, respectively.

We consider here a nonlocal fourth-order version of the above problem, namely

$$(P_\lambda) \qquad \begin{cases} \Delta^2 u - m\left(\int_\Omega |\nabla u|^2 dx\right)\Delta u = \lambda a(x)|u|^{q-2}u + b(x)|u|^{p-2}u, \text{ in } \Omega \\ u = \Delta u = 0 \text{ on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $N \geq 5$ and $\Delta^2 u = \Delta(\Delta u)$ is the biharmonic operator. The equation in (P_λ) is related with the so called Berger plate model (see [5, 9])

$$u_{tt} + \Delta^2 u + \left(Q + \int_\Omega |\nabla u|^2 dx\right)\Delta u = f(x, u, u_t),$$

and it is a simplification of the von Karman plate equation that describes large deflection of plate. The parameter Q describes in-plane forces applied to the plate

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and the function f represents transverse loads which may depend on the displacement u and the velocity u_t. The equation is also related with some models which describe the bending equilibrium states of a beam subjected to a force $f(x,u)$ and other elastic force (see [25]), namely

$$u_{tt} + \frac{EI}{\rho}u_{xxxx} - \left(\frac{H}{\rho} + \frac{EA}{2\rho L}\int_0^L |u_x|^2 dx\right) u_{xx} = f(x,u).$$

More recent references with important details about the physical motivation can be found in [16, 3, 14, 27].

In [13], the authors supposed that m is increasing, $a \equiv 1$, $b \equiv 1$ and obtained infinitely many solutions, for $1 < q < 2$, $p = 2^* := 2N/(N-4)$ and $\lambda > 0$ small. This result was partially extended in [22] where they assumed that $b \equiv 1$, the (nonautonomous) concave term were of type $\lambda h(x,u)$, with $h(x,u) \geq 0$ if $u \geq 0$, and a technical assumption on the growth of the function m. Other results for positive potentials in unbounded domains can be found in [21, 17] and references there in.

Here we are going to consider sign-changing potentials under mild regularity conditions. More specifically, we suppose that

- (m_1) $m \in C([0,+\infty))$ is positive;
- (a_1) $a \in L^{\sigma_q}(\Omega)$, for some $\sigma_q > 2^*/(2^*-q)$;
- (a_2) if we set
$$\Omega_a^+ := \{x \in \Omega : a(x) > 0\},$$
then there exist $x_0 \in \Omega_a^+$ and $\delta > 0$ such that $B_\delta(x_0) \subset \Omega_a^+$;
- (b_1) $b \in L^\infty(\Omega)$.

In our first result we prove the following:

Theorem 1.1. *Suppose that $1 < q < 2 < p \leq 2^*$. If (m_1), $(a_1)-(a_2)$ and (b_1) hold, then there exists $\lambda^* > 0$ such that, for each $\lambda \in (0,\lambda^*)$, the problem (P_λ) has a nonnegative nonzero solution.*

In the proof we minimize the energy functional $I_\lambda : W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega) \to \mathbb{R}$ given by

$$I_\lambda(u) := \frac{1}{2}\|u\|^2 + \frac{1}{2}M\left(\int_\Omega |\nabla u|^2 dx\right) - \frac{\lambda}{q}\int a(x)(u^+)^q - \frac{1}{p}\int b(x)(u^+)^p,$$

where $M(t) := \int_0^t m(\tau)d\tau$ and $u^+(x) := \max\{u(x),0\}$. Although this is a standard process in concave-convex problems we have a serious difficult for proving that the minimum point is nonnegative. Actually, the usual procedure of using the negative part of the solution as a test function in I_λ does not work here since, differently of the second order problem, this function may not belong to $W^{2,2}(\Omega)$. It is worthwile to mention that some ideas for proving nonnegativity of solutions were presented in [24, 18] for different fourth-order problems. However, all of these authors assume that the right-hand side of the equation is nonnegative and therefore their techniques fail in our case. Actually, the argument used here (see Proposition 2.3) has interest in itself and we believe that it can be used in other kind of minimization problems which involve high-order operators with indefinite nonlinearities.

In our next theorem we look for a second solution in the case that m is a affine function. This occurs, for example, in the Berger plate model mentioned above.

Although a and b can change sign, for obtaining a second solution it is important to assure that, in some ball, both of them are positive. Hence, we suppose that

(m_2) for some $\alpha > 0$ and $\beta \geq 0$ there holds

$$m(t) = \alpha + \beta t, \quad t \geq 0;$$

(ab_1) if we set

$$\Omega_b^+ := \{x \in \Omega : b(x) > 0\},$$

then there exist $x_0 \in \Omega_a^+$ and $\delta > 0$ such that $B_\delta(x_0) \subset (\Omega_a^+ \cap \Omega_b^+)$;

(b_2) there exists $c_1 > 0$ and $\gamma \in (\frac{N-4}{2}, N)$ such that

$$\|b\|_\infty - b(x) \leq c_1 |x - x_0|^\gamma, \quad \text{for a.e. } x \in B_\delta(x_0),$$

where δ, x_0 come from (ab_1);

and prove the following multiplicity result:

Theorem 1.2. *Suppose that $N \in \{5, 6, 7\}$, $1 < q < 2$ and $4 < p \leq 2^*$. If (m_2), (a_1), (ab_1) and $(b_1) - (b_2)$ hold, then there exists $\lambda^* > 0$ such that the problem (P_λ) has at least two nonzero solution for each $\lambda \in (0, \lambda^*)$.*

The second solution will be obtained via the Mountain Pass Theorem. Due to the presence of the nonlocal term the classical calculations turn to be more involved. In the critical case $p = 2^*$ we also have an extra difficult since the embedding $W^{2,2}(\Omega) \hookrightarrow L^{2^*}(\Omega)$ is no longer compact. We follow the ideias introduced by Brezis and Nirenberg [7] (see also [6]) to get a local compactness result. Thanks to some fine estimates we can prove that the minimax level of the associated functional belongs to the correct range. At this point, it is very important to know that the first solution (given by Theorem 1.1) is nonnegative.

We make now some comments about the dimension restriction on the statement of our second theorem. Actually, an important feature of the techniques used i our proof is the interaction between the integral $\int_\Omega b(x)|u|^{2^*} dx$ and the fourth order term $\|u\|^4_{W_0^{1,2}(\Omega)}$ of the energy functional. Notice that, if $N = 8$, then $2^* = 4$ and therefore these terms has the same degree. The situation becomes more difficult for high dimensions $N > 8$ since is this case the nonlocal term dominates the critical one. It is interest to notice that the range for N covered by our result is exactly the so called noncritical dimensions for the biharmonic operator (see [24]). We learn in a recent paper of Naimen [19] that the same occurs for the second order problem, where the critical dimension is $N = 4$ and the spectrum of solution is very different from the 3-dimensional case. We finally notice that, although the first solution is nonnegative, the sign argument used in the proof of Theorem 1.1 does not apply for the second solution and therefore we have no information about its sign.

The paper contains two more sections. In Section 2 we prove Theorem 1.1 and in the last one we prove our multiplicity result.

2. The first solution

Let H be the Hilbert space $W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)$ endowed with inner product and associated norm given by

$$\langle u, v \rangle := \int_\Omega (\Delta u \Delta v) dx, \qquad \|u\| := \left(\int_\Omega (\Delta u)^2 dx \right)^{1/2},$$

for any u, $v \in H$. We also use the following notations:

$$\langle u, v\rangle_* := \int_\Omega (\nabla u \cdot \nabla v)dx, \qquad \|u\|_* := \left(\int_\Omega |\nabla u|^2 dx\right)^{1/2}.$$

We know that the embedding $H \hookrightarrow L^r(\Omega)$ is continuous for $1 \leq t \leq 2^*$ and compact if $1 \leq t < 2^*$. Hence, the following constant is well defined

$$S_t := \inf_{u \in H\setminus\{0\}} \frac{\int_\Omega (\Delta u)^2 dx}{\left(\int_\Omega |u|^r dx\right)^{2/r}}, \tag{2.1}$$

for any $1 \leq t \leq 2^*$.

From now on we denote by $\|u\|_t$ the $L^t(\Omega)$-norm of a function $u \in H$. Moreover, for shortness, we write $\int u$ instead of $\int_\Omega u(x)dx$.

Lemma 2.1. *The space H is compactly embedded into $W_0^{1,2}(\Omega)$.*

Proof. Let $(\varphi_k)_{k\in\mathbb{N}}$ be the eigenfunctions of $(-\Delta, W_0^{1,2}(\Omega))$ and $(\lambda_k)_{k\in\mathbb{N}}$ it associated eigenvalues. It is well known that they are orthogonal in $W_0^{1,2}(\Omega)$, $L^2(\Omega)$ and also in H. Hence, if $u = \sum_{k=1}^\infty a_k\varphi_k \in H$, we can compute

$$\|u\|^2 = \sum_{k=1}^\infty a_k^2\|\varphi_k\|^2 = \sum_{k=1}^\infty a_k^2\lambda_k\|\varphi_k\|_*^2 \geq \lambda_1 \sum_{k=1}^\infty a_k^2\|\varphi_k\|_*^2 = \lambda_1\|u\|_*^2.$$

It follows that the embedding $H \hookrightarrow W_0^{1,2}(\Omega)$ is continuous. Moreover, for any $u \in H$, there holds

$$\|u\|_*^2 = \int(\nabla u \cdot \nabla u) = -\int(u\Delta u) \leq \|u\|_2\|u\|,$$

and therefore the compactness in the statement is a consequence of the compactness of the embedding $W_0^{1,2}(\Omega) \hookrightarrow L^2(\Omega)$. □

Since we intending to apply variational methods, we introduce the energy functional

$$I_\lambda(u) := \frac{1}{2}\|u\|^2 + \frac{1}{2}M(\|u\|_*^2) - \frac{\lambda}{q}\int a(x)(u^+)^q - \frac{1}{p}\int b(x)(u^+)^p, \qquad u \in H,$$

where $M(t) := \int_0^t m(\tau)d\tau$ and $u^+(x) := \max\{u(x), 0\}$. Under the setting of Theorem 1.1, we have that $I_\lambda \in C^1(H, \mathbb{R})$. Moreover, the solutions of (P_λ) are precisely the critical points of I_λ.

Lemma 2.2. *There exist $\rho, \delta > 0$ such that $I_\lambda(u) \geq \delta > 0$, for any $u \in H$ such that $\|u\| = \rho$, provided $\lambda > 0$ is small enough.*

Proof. Hölder's inequality implies that $\int a(x)(u^+)^q \leq \|a\|_{\sigma_q}\|u\|_{q\sigma_q'}^q$, for any $u \in H$. Since an analogous computation holds for the term $\int b(x)(u^+)^p$ and M is nonnegative, we can use (2.1) to obtain

$$\begin{aligned} I_\lambda(u) &\geq \frac{1}{2}\|u\|^2 - \frac{\lambda}{q}\|a\|_{\sigma_q}\|u\|_{q\sigma_q'}^q - \frac{1}{p}\|b\|_\infty\|u\|_p^p \\ &\geq \frac{\|u\|^q}{2}\left\{\|u\|^{2-q} - \frac{2}{p}\|b\|_\infty S_p^{-p/2}\|u\|^{p-q} - 2\frac{\lambda}{q}\|a\|_{\sigma_q} S_{q\sigma_q'}^{-q/2}\right\}. \end{aligned}$$

For $B := (2/p)\|b\|_\infty S_p^{-p/2}$, the function

$$f(t) := t^{2-q} - Bt^{p-q}, \qquad t > 0,$$

achieves its maximum value at

$$\rho := \left[\frac{(2-q)}{B(p-q)}\right]^{1/(p-2)}.$$

Let $\rho_0 := f(\rho)$ and notice that, for any $\|u\| = \rho$, there holds

$$I_\lambda(u) \geq \frac{\rho^q}{2}\left\{\rho_0 - 2\frac{\lambda}{q}\|a\|_{\sigma_q} S_{q\sigma_q'}^{-q/2}\right\} \geq \frac{\rho^q}{2}\frac{\rho_0}{2} = \delta > 0,$$

whenever

$$\lambda \leq \frac{\rho_0}{4}\frac{q}{\|a\|_{\sigma_q}} S_{q\sigma_q'}^{q/2}. \tag{2.2}$$

The lemma is proved. □

Proposition 2.3. *Let $\rho > 0$ be as in the above lemma. We have that*

$$-\infty < c_0 := \inf_{u \in \overline{B_\rho(0)}} I_\lambda(u) < 0.$$

Moreover, if $u_0 \in B_\rho(0)$ is such that $I_\lambda(u_0) = c_0$, then $u_0 \geq 0$ in Ω is a nonzero solution of (P_λ).

Proof. Since I_λ maps bounded sets in bounded sets, we have that $c_0 > -\infty$. Let $B_\delta(x_0) \subset \Omega_a^+$ be as in the hypothesis (a_2) and take a nonnegative function $\varphi \in C_0^\infty(B_\delta(x_0))$ such that $\int a(x)\varphi^q > 0$. If $0 < t < 1$, we get

$$I_\lambda(t\varphi) \leq \frac{t^2}{2}\|\varphi\|^2 + \frac{c_1}{2}t^2\|\varphi\|_*^2 - \lambda\frac{t^q}{q}\int a(x)\varphi^q - \frac{t^p}{p}\int b(x)\varphi^p,$$

with $c_1 := \max_{s \in [0, \|\varphi\|_*^2]} m(s)$. Hence

$$\limsup_{t \to 0^+} \frac{I_\lambda(t\varphi)}{t^q} \leq -\frac{\lambda}{q}\int a(x)\varphi^q < 0$$

and therefore $I_\lambda(t\phi) < 0$, for any $t > 0$ small. This proves that $c_0 < 0$.

We have that

$$I_\lambda(t\varphi) \leq \frac{t^2}{2}\|\varphi\|^2 - \lambda\frac{t^q}{q}\int a(x)\varphi^q - \frac{t^p}{p}\int b(x)\varphi^p,$$

with $c_1 := \max_{s \in [0, \|\phi\|_*^2]} m(s)$. Hence

$$\limsup_{t \to 0^+} \frac{I_\lambda(t\varphi)}{t^q} \leq -\frac{\lambda}{q}\int a(x)\varphi^q < 0$$

and therefore $I_\lambda(t\phi) < 0$, for any $t > 0$ small. This proves that $c_0 < 0$.

If $u_0 \in B_\rho(0)$ is such that $I_\lambda(u_0) = c_0 < 0$ then $u_0 \neq 0$ and $I_\lambda'(u_0) = 0$. Arguing as in [6] we can prove that u_0 verifies the boundary conditions of (P_λ) in the trace sense. Hence, u_0 is a solution of (P_λ). In order to prove that $u_0 \geq 0$ in Ω, we first notice that, by elliptic regularity, $u_0 \in C^2(\Omega) \cap C(\overline{\Omega})$. Suppose, by contradiction, that the open set $\Omega_{u_0}^- := \{x \in \Omega : u_0(x) < 0\}$ is nonempty. Since $u_0 \equiv 0$ on $\partial\Omega_{u_0}^-$, if $\Delta u_0 \leq 0$ in $\Omega_{u_0}^-$ we could infer from the Maximum Principle that $u_0 \equiv 0$ in $\Omega_{u_0}^-$, which does not make sense. Hence,

$$A_1 := \Delta u(x_1) > 0, \quad A_2 := -u(x_1) > 0, \tag{2.3}$$

for some $x_1 \in \Omega_{u_0}^-$. For simplicity, we shall assume that $x_1 = 0$ and $B_1(0) \subset \Omega_{u_0}^-$.

Let $\psi \in C^\infty(\mathbb{R}^N)$ be such that $\psi \equiv 0$ in $\mathbb{R}^N \setminus B_1(0)$, $\|\nabla\psi\|_\infty \le c_2$ and $\int_{B_1(0)} \Delta\psi(y)dy > 0$. We pick $\delta > 0$ in such way that the function $\phi(x) := 1+\delta\psi(x)$ verifies

$$\phi \equiv 1 \text{ in } \mathbb{R}^N \setminus B_1(0), \qquad \phi(x) \ge 1/2, \quad \forall\, x \in \mathbb{R}^N.$$

Since A_1, $A_2 > 0$, by taking δ smaller if necessary, we may assume that

$$0 < \Gamma_1 := \int_{B_1(0)} \left[A_1 - A_2\Delta\phi(y)\right]^2 dy < \Gamma_2 := \int_{B_1(0)} A_1^2 dy.$$

For any $0 < \varepsilon < 1$, we set

$$u_\varepsilon(x) = \left[1 + \varepsilon^2\left(\phi\left(\frac{x}{\varepsilon}\right) - 1\right)\right] u_0(x), \quad x \in \mathbb{R}^N.$$

and call ϕ_ε the function into the brackets above.

Clearly $u_\varepsilon \in H$ and, since ϕ_ε is nonnegative, we also have that $u_\varepsilon^+ = u_0^+$. We claim that, for $\varepsilon > 0$ small, there holds

$$\int(\Delta u_\varepsilon)^2 < \int(\Delta u_0)^2, \quad M(\|u_\varepsilon\|_*^2) + \int(\Delta u_\varepsilon)^2 < M(\|u_0\|_*^2) + \int(\Delta u_0)^2. \tag{2.4}$$

If this is true, we conclude that $u_\varepsilon \in B_\rho(0)$ verifies $u_\varepsilon \ne u_0$ and $I_\lambda(u_\varepsilon) < I_\lambda(u_0) = c_0$, which contradicts the definition of c_0.

It remains to prove (2.4). We first notice that, by using a simple computation, we can prove that $\|u_\varepsilon\|_*^2 \to \|u_0\|_*^2$, as $\varepsilon \to 0^+$, and therefore the second inequality in (2.4) is a consequence of the first one and the continuity of M. For the proof of the former we set

$$\Gamma_{1,\varepsilon} := \int_{\{|x|<\varepsilon\}} (\phi_\varepsilon\Delta u + 2\nabla\phi_\varepsilon \cdot \nabla u_0 + u_0\Delta\phi_\varepsilon)^2 dx.$$

and

$$\Gamma_{2,\varepsilon} := \int_{\{|x|<\varepsilon\}} (\Delta u_0)^2 dx.$$

The change of variables $y := x/\varepsilon$ provides

$$\Gamma_{1,\varepsilon} = \varepsilon^N\Gamma_1 + o(\varepsilon^N), \quad \Gamma_{2,\varepsilon} = \varepsilon^N\Gamma_2 + o(\varepsilon^N),$$

as $\varepsilon \to 0^+$. Since $\Gamma_1 < \Gamma_2$, we conclude that $\Gamma_{1,\varepsilon} < \Gamma_{2,\varepsilon}$, for $\varepsilon > 0$ small. Hence, for this values of ε, we obtain

$$\int_\Omega (\Delta u_\varepsilon)^2 dx = \Gamma_{1,\varepsilon} + \int_{\{|x|\ge\varepsilon\}} (\Delta u_0)^2 dx < \Gamma_{2,\varepsilon} + \int_{\{|x|\ge\varepsilon\}} (\Delta u_0)^2 dx = \int_\Omega (\Delta u_0)^2 dx,$$

and we have done.

□

We are ready to prove our first main result.

Proof of Theorem 1.1. Let $\rho > 0$ and c_0 as in the last proposition and $(u_n) \subset \overline{B_\rho(0)}$ be a minimizing sequence for c_0. Since $I \ge \delta > 0$ on $\partial B_\rho(0)$ and $c_0 < 0$, we have that $u_n \in B_\rho(0)$ for all $n \ge n_0$. Thus, by the Ekeland's Variational Principle [12], we may assume that

$$I_\lambda(u_n) \to c_0, \qquad I_\lambda'(u_n) \to 0.$$

The boundedness of (u_n) and Lemma 2.1 provides $u_0 \in H$ such that, up to a subsequence,

$$\left\{\begin{array}{l} u_n \rightharpoonup u_0 \text{ weakly in } X, \\ u_n \to u_0 \text{ strongly in } W_0^{1,2}(\Omega) \text{ and } L^r(\Omega), \text{ for } 1 \le r < 2^*, \\ u_n^+(x) \to u_0^+(x), \text{ for a.e. } x \in \Omega. \end{array}\right. \tag{2.5}$$

If $\sigma := (2^*-1)/(q-1)$, then $\sigma' = (2^*-1)/(2^*-q)$ and we can use Young's inequality to obtain, for a.e. $x \in \Omega$,

$$|a(x)(u_n^+)^{q-1}|^{2^*/(2^*-1)} \le \frac{1}{\sigma}|u_n|^{2^*} + \frac{1}{\sigma'}|a(x)|^{2^*/(2^*-q)}.$$

Since $\sigma_q > 2^*/(2^*-q)$, we infer from the above inequality that the sequence $(a(\cdot)(u_n^+)^{q-1})$ is bounded in $L^{2^*/(2^*-1)}(\Omega)$. So, recalling that $(2^*)' = 2^*/(2^*-1)$, we conclude that

$$\lim_{n\to+\infty} \int a(x)(u_n^+)^{q-1}\phi = \int a(x)(u_0^+)^{q-1}\phi, \quad \forall\, \phi \in H \subset L^{2^*}(\Omega).$$

Since $b \in L^\infty(\mathbb{R}^N)$, a simpler argument shows that

$$\lim_{n\to+\infty} \int b(x)(u_n^+)^{p-1}\phi = \int b(x)(u_0^+)^{p-1}\phi, \quad \forall\, \phi \in H.$$

The strong convergence in $W_0^{1,2}(\Omega)$ implies that

$$\lim_{n\to+\infty} m(\|u_n\|_*^2) \int (\nabla u_n \cdot \nabla \phi) = m(\|u_0\|_*^2) \int (\nabla u_0 \cdot \nabla \phi).$$

All together, the above equalities provide

$$0 = \lim_{n\to+\infty} I'_\lambda(u_n)\phi = I'_\lambda(u_0)\phi, \qquad \forall\, \phi \in H,$$

and therefore $I'_\lambda(u_0) = 0$.

Since $(2^*/q)' = 2^*/(2^*-q)$, it follows from (a_1) that $q\sigma'_q < 2^*$. This and (2.5) show that $|u_n(x)| \le \psi(x)$ a.e. in Ω, for some $\psi \in L^{q\sigma'_q}(\Omega)$. By using Young's inequality again we obtain, for a.e. $x \in \Omega$,

$$|a(x)(u_n^+)^q| \le |a(x)||\psi(x)|^q \le \frac{1}{\sigma_q}|a(x)|^{\sigma_q} + \frac{1}{\sigma'_q}\psi(x)^{q\sigma'_q}.$$

The right-hand side above belongs to $L^1(\Omega)$, and therefore we can use (2.5) and the Lebesgue Theorem to obtain

$$\lim_{n\to+\infty} \int a(x)(u_n^+)^q = \int a(x)(u_0^+)^q.$$

Moreover, if we set

$$A_n := \frac{1}{2}M(\|u_n\|_*^2) - \frac{1}{p}m(\|u_n\|_*^2)\|u_n\|_*^2,$$

the strong convergence in $W_0^{1,2}(\Omega)$ provides

$$\lim_{n\to+\infty} A_n = A_0 := \frac{1}{2}M(\|u_0\|_*^2) - \frac{1}{p}m(\|u_0\|_*^2)\|u_0\|_*^2.$$

Thus, from the weak convergence of (u_n) in H, we get

$$\begin{aligned} c_0 &= \liminf_{n\to+\infty}\left(I_\lambda(u_n)-\frac{1}{p}I'_\lambda(u_n)u_n\right)\\ &= \liminf_{n\to+\infty}\left\{\left(\frac{1}{2}-\frac{1}{p}\right)\|u_n\|^2+A_n+\lambda\left(\frac{1}{p}-\frac{1}{q}\right)\int a(x)(u_n^+)^q\right\}\\ &\geq \left(\frac{1}{2}-\frac{1}{p}\right)\|u_0\|^2+A_0+\lambda\left(\frac{1}{p}-\frac{1}{q}\right)\int a(x)(u_0^+)^q\\ &= I_\lambda(u_0)-\frac{1}{p}I'_\lambda(u_0)u_0=I_\lambda(u_0). \end{aligned}$$

Since $u_0\in\overline{B_\rho(0)}$, we conclude that $I_\lambda(u_0)=c_0$. Lemma 2.2 implies that $u_0\in B_\rho(0)$ and therefore we infer from Proposition 2.3 that $u_0\geq 0$ is a nonzero solution of (P_λ). □

3. The second solution

We devote this section to the proof of Theorem 1.2. We start by noticing that, since (ab_1) is stronger than (a_2), under the setting of our second theorem we can apply Theorem 1.1 to obtain a first solution which is non negative and has negative energy. This solution will be denoted by u_0 from now on. We also assume that $m(t)=\alpha+\beta t$, in such way that

$$M(t)=\alpha t+\frac{\beta}{2}t^2,\quad t\geq 0. \tag{3.1}$$

Since we are intending to apply the Mountain Pass Theorem we first prove the following:

Lemma 3.1. *Suppose that there exists $x_1\in\mathbb{R}^N$ and $\eta>0$ such that $B_\eta(x_1)\subset\Omega_b^+$. If $\phi\in C_0^\infty(B_\delta(x_1))\setminus\{0\}$ is nonnegative, then*

$$\lim_{t\to+\infty}I_\lambda(u_0+t\phi)=-\infty,$$

whenever $4<p\leq 2^*$.

Proof. From (3.1) we have that $M(\|u_0+t\phi\|_*^2)=O(t^4)$, as $t\to+\infty$. Hence, recalling that $\phi\equiv 0$ outside $B_\eta(x_1)\subset\Omega_b^+$, a straightforward computation provides

$$\begin{aligned} I_\lambda(u_0+t\phi) &\leq O(t^2)+O(t^4)-O(t^q)-\frac{1}{p}\int_{\Omega_b^+}b(x)(u_0+t\phi)^p\,\mathrm{d}x\\ &\leq O(t^4)-c_1\int_{B_\delta(x_1)}b(x)(u_0+t\phi)^p\\ &\leq O(t^4)+O(1)-c_2t^p\int_{B_\delta(x_1)}b(x)\phi^p, \end{aligned}$$

as $t\to+\infty$. The result follows from $p>4$. □

Lemma 3.2. *If $p=2^*$ and u_0 is the unique nonzero critical points of I_λ, then I_λ satisfies $(PS)_c$ condition for every*

$$c<\bar{c}:=I_\lambda(u_0)+\frac{2}{N}\frac{1}{\|b\|_\infty^{(N-4)/4}}S^{N/4}.$$

Proof. Let $(u_n) \subset H$ be such that $I_\lambda(u_n) \to c$ and $I'_\lambda(u_n) \to 0$. From (3.1), we get

$$\begin{aligned} O(1)\|u_n\| &= I_\lambda(u_n) - \frac{1}{p} I'_\lambda(u_n)u_n \\ &\geq \left(\frac{1}{2} - \frac{1}{p}\right)\|u_n\|^2 - \lambda\left(\frac{1}{q} - \frac{1}{p}\right) S_{q\sigma'_q}^{-q/2} \|a\|_{\sigma_q} \|u_n\|^q, \end{aligned}$$

as $n \to +\infty$. It follows from $1 < q < 2$ that the sequence (u_n) is bounded in H. So, along a subsequence we have that

$$\begin{cases} u_n \rightharpoonup u \text{ weakly in } H, \\ u_n \to u_0 \text{ strongly in } W_0^{1,2}(\Omega) \text{ and } L^{q\sigma'_q}(\Omega), \\ u_n^+(x) \to u_0^+(x), \text{ for a.e. } x \in \Omega, \end{cases} \tag{3.2}$$

where we have used Lemma 2.1 and (a_1). Hence $\int a(x)(u_n^+)^q \to \int a(x)(u^+)^q$ and, if we set $v_n := u_n - u$, we can use the Brezies-Lieb lemma to get

$$\begin{aligned} 0 = I'_\lambda(u_n)u_n &= \|u_n\|^2 + m(\|u_n\|_*^2)\|u_n\|_*^2 - \lambda \int a(x)(u_n^+)^q - \int b(x)(u_n^+)^{2^*} \\ &= \|u\|^2 + \|v_n\|^2 + m(\|u\|_*^2)\|u\|_*^2 - \lambda \int a(x)(u^+)^q + o(1) \\ &\quad - \int b(x)(u^+)^{2^*} - \int b(x)(v_n^+)^{2^*} \\ &= I'_\lambda(u)u + \|v_n\|^2 - \int b(x)(v_n^+)^{2^*}. \end{aligned}$$

As in the proof of Theorem 1.1 we have that $I'_\lambda(u) = 0$, and therefore there exists $l \geq 0$ such that

$$\lim_{n\to+\infty} \|v_n\|^2 = l = \lim_{n\to+\infty} \int b(x)(v_n^+)^{2^*}.$$

We claim that $l = 0$. If this is true, it follows that $\|u_n - u\| \to 0$ and we have done.

In order to prove that $l = 0$, we first notice that

$$\int b(x)(v_n^+)^{2^*} \leq \|b\|_\infty S^{-2^*/2} \left(\int (\Delta v_n)^2\right)^{2^*/2}.$$

Taking the limit we obtain $l \leq \|b\|_\infty S^{-2^*/2} l^{2^*/2}$. If $l > 0$, we infer from this last inequality that

$$l \geq \frac{1}{\|b\|_\infty^{(N-4)/4}} S^{N/4}. \tag{3.3}$$

On the other hand, the same argument of the beginning of the proof provides

$$c + o(1) = I_\lambda(u_n) = I_\lambda(u) + \frac{1}{2}\|v_n\|^2 - \frac{1}{2^*}\int b(x)(v_n^+)^{2^*} + o(1),$$

as $n \to +\infty$. Taking the limit and using (3.3) we get

$$c = I_\lambda(u) + \left(\frac{1}{2} - \frac{1}{2^*}\right) l = I_\lambda(u) + \frac{2}{N} l \geq I_\lambda(u) + \frac{2}{N}\frac{1}{\|b\|_\infty^{(N-4)/4}} S^{N/4}.$$

But $I'_\lambda(u) = 0$ and therefore, by hypothesis, either $u = 0$ or $u = u_0$. Since $\max\{I_\lambda(0), I_\lambda(u_0)\} \leq 0$, the above inequality contradicts $c < \bar{c}$. □

At this point we recall that, for $t = 2^*$ in (2.1) , the best constant $S := S_{2^*}$ is achieved by the family of functions

$$U_{\varepsilon,y}(x) := C_N \left(\frac{\varepsilon}{\varepsilon^2 + |x-y|^2} \right)^{(N-4)/2},$$

with $\varepsilon > 0$, $y \in \mathbb{R}^N$ and $c_N := [N(N-4)(N^2-4)]^{(N-4)/8}$. This function is a classical solution of the equation $\Delta^2 u = u^{2^*-1}$ in $\mathbb{R}^N$, with $N > 4$.

In order to correct localize the minimax level of the energy functional we take $B_\delta(x_0) \subset (\Omega_a^+ \cap \Omega_b^+)$ as in the condition (ab_1), pick $r > 0$ small in such way that $B_{2r}(x_0) \subset B_\delta(x_0)$ and fix a smooth function satisfying $\phi \equiv 1$ in $B_r(x_0)$ and $\phi \equiv 0$ outside $B_{2r}(x_0)$. We also define

$$u_\varepsilon(x) := \phi(x) U_{\varepsilon,x_0}(x),$$

and consider the L^{2^*}-normalized function

$$v_\varepsilon(x) := \frac{u_\varepsilon(x)}{\|u_\varepsilon\|_{2^*}}. \tag{3.4}$$

In what follows we assume, without loss of generality, that $x_0 = 0$.

Lemma 3.3. *As $\varepsilon \to 0^+$, there hold*

$$\|v_\varepsilon\|^2 = S + O(\varepsilon^{N-4}), \qquad \|v_\varepsilon\|_*^2 = \begin{cases} O(\varepsilon^2), & \text{if } N = 7, \\ O(\varepsilon^2 |\ln \varepsilon|), & \text{if } N = 6, \\ O(\varepsilon), & \text{if } N = 5. \end{cases}$$

Proof. It is proved in [6, Eqns. (7.8) and (7.9)] that

$$\|u_\varepsilon\|^2 = S^{N/4} + O(\varepsilon^{N-4}), \qquad \|u_\varepsilon\|_{2^*}^{2^*} = S^{N/4} + O(\varepsilon^N).$$

Hence,

$$\|v_\varepsilon\|^2 = \frac{S^{N/4} + O(\varepsilon^{N-4})}{(S^{N/4} + O(\varepsilon^N))^{2/2^*}} = \frac{S^{N/4} + O(\varepsilon^{N-4})}{S^{(N-4)/4} + O(\varepsilon^N)} = S^{N/4} + O(\varepsilon^{N-4}),$$

and the first equality is proved. For the second one, we notice that

$$\nabla U_\varepsilon = c_N (4-N) \frac{\varepsilon^{(N-4)/2}}{(\varepsilon^2 + |x|^2)^{(N-2)/2}} x.$$

By using the definition of u_ε and a straightforward computation we obtain

$$\begin{aligned} \int |\nabla u_\varepsilon|^2 &= \int |\nabla \phi|^2 U_\varepsilon^2 + 2 \int \phi U_\varepsilon (\nabla \phi \cdot \nabla U_\varepsilon) + \int_{B_{2r}(0)} \phi^2 |\nabla U_\varepsilon|^2 dx \\ &= O(\varepsilon^{N-4}) + c_N^2 (4-N)^2 \varepsilon^{N-4} \int_{B_{2r}(0)} \frac{|x|^2}{(\varepsilon^2 + |x|^2)^{N-2}} dx. \end{aligned}$$

Hence, the change of variables $y := x/\varepsilon$ provides

$$\int |\nabla u_\varepsilon|^2 = O(\varepsilon^{N-4}) + c_N^2 (4-N)^2 \varepsilon^2 \Gamma_\varepsilon,$$

with

$$\Gamma_\varepsilon := \int_{B_{2r/\varepsilon}(0)} \frac{|y|^2}{(\varepsilon^2 + |y|^2)^{N-2}} dy.$$

If $N = 7$ then

$$\Gamma_\varepsilon \le \int_{\mathbb{R}^7} \frac{|y|^2}{(1 + |y|^2)^5} dy = C_1 < +\infty,$$

and therefore

$$\|u_\varepsilon\|_*^2 = O(\varepsilon^{N-4}) + O(\varepsilon^2) = O(\varepsilon^2), \quad \text{if } N = 7. \tag{3.5}$$

For $N \in \{5,6\}$ we have that $\Gamma_\varepsilon \to +\infty$, as $\varepsilon \to 0^+$, and we need a more precise estimate. We consider first the case $N=6$ and compute, for $\varepsilon < 2r$,

$$\begin{aligned}\Gamma_\varepsilon &= \int_{B_1(0)} \frac{|y|^2}{(\varepsilon^2+|y|^2)^4}dy + \int_{\{1\le|y|\le(2r/\varepsilon)\}} \frac{|y|^2}{(|y|^2)^4}dy \\ &= O(1) + C_2\int_1^{2r/\varepsilon} \frac{1}{s}ds = O(|\ln\varepsilon|),\end{aligned}$$

from which it follows that

$$\|u_\varepsilon\|_*^2 = O(\varepsilon^{N-4}) + O(\varepsilon^2|\ln\varepsilon|) = O(\varepsilon^2|\ln\varepsilon|), \quad \text{if } N = 6. \tag{3.6}$$

Finally, if $N=5$, we can proceed as above to obtain

$$\Gamma_\varepsilon = O(1) + C_3\int_1^{2r/\varepsilon} 1 ds = O(\varepsilon^{-1}),$$

and therefore

$$\|u_\varepsilon\|_*^2 = O(\varepsilon^{N-4}) + O(\varepsilon) = O(\varepsilon), \quad \text{if } N = 5.$$

This, (3.5) and (3.6) imply that

$$\|u_\varepsilon\|_*^2 = \begin{cases} O(\varepsilon^2), & \text{if } N=7,\\ O(\varepsilon^2|\ln\varepsilon|), & \text{if } N=6,\\ O(\varepsilon), & \text{if } N=5.\end{cases}$$

Since $\|u_\varepsilon\|_{2^*}^2 \to S^{(N-4)/4} > 0$, as $\varepsilon \to 0^+$, the same equality holds for $\|v_\varepsilon\|_*^2$ and the lemma is proved. □

Proposition 3.4. *Suppose that a, b verify (a_1), (b_1), and (ab_1). Then, for any $\varepsilon > 0$ small, the function v_ε defined in* (3.4) *is such that*

$$\max_{t>0} I_\lambda(u_0 + tv_\varepsilon) < \bar{c} := I_\lambda(u_0) + \frac{1}{N}\frac{1}{\|b\|_\infty^{(N-2)/2}}S^{N/2}.$$

Proof. For any $\varepsilon > 0$, it follows from Lemma 3.1 that the function $t \mapsto I(u_0 + tv_\varepsilon)$ achieves its maximum at a point $t_\varepsilon > 0$. Moreover, arguing as in [11, Lemma 4.1], we can prove that $(t_\varepsilon)_{\varepsilon>0}$ is bounded for $\varepsilon \in (0,1]$.

Since $I'_\lambda(u_0)v_\varepsilon = 0$, we have that

$$\langle u_0, v_\varepsilon\rangle + \langle u_0, v_\varepsilon\rangle_* m(\|u_0\|_*^2) = \lambda\int a(x)u_0^{q-1}v_\varepsilon + \int b(x)u_0^{p-1}v_\varepsilon.$$

This, (3.1) and a straightforward computation provide

$$m_\varepsilon := I_\lambda(u_0 + t_\varepsilon v_\varepsilon) = I_\lambda(u_0) + \frac{t_\varepsilon^2}{2}\|v_\varepsilon\|^2 - \frac{\lambda}{q}\Gamma_{1,\varepsilon} - \frac{1}{2^*}\Gamma_{2,\varepsilon} + t_\varepsilon^2\Gamma_{3,\varepsilon}, \tag{3.7}$$

with

$$\Gamma_{1,\varepsilon} := \int a(x)\left[(u_0+t_\varepsilon v_\varepsilon)^q - u_0^q - qt_\varepsilon u_0^{q-1}v_\varepsilon\right],$$

$$\Gamma_{2,\varepsilon} := \int b(x)\left[(u_0+t_\varepsilon v_\varepsilon)^{2^*} - u_0^{2^*} - 2^*t_\varepsilon u_0^{2^*-1}v_\varepsilon\right],$$

$$\Gamma_{3,\varepsilon} := \|v_\varepsilon\|_*^2\left(\frac{\alpha}{2} + \frac{\beta t_\varepsilon^2}{4}\|v_\varepsilon\|_*^2 + \frac{\beta}{2}\|u_0\|_*^2 + \beta t_\varepsilon\langle u_0, v_\varepsilon\rangle_*\right) + \beta\langle u_0, v_\varepsilon\rangle_*^2.$$

It follows from the Mean Value Theorem that, for a.e. $x \in \Omega$,

$$(u_0 + t_\varepsilon v_\varepsilon)^q = u_0^q + qt(u_0 + t_\varepsilon \theta v_\varepsilon)^{q-1} v_\varepsilon,$$

for some $\theta(x) \in [0,1]$. Since $v_\varepsilon \equiv 0$ outside $B_{2r}(0) \subset \mathrm{supp}(a)$, we get

$$a(x)(u_0 + t_\varepsilon v_\varepsilon)^q = a(x)u_0^q + qt_\varepsilon a(x)(u_0 + t_\varepsilon \theta(x) v_\varepsilon)^{q-1} v_\varepsilon \geq a(x)(u_0^q + qt_\varepsilon u_0^{q-1} v_\varepsilon),$$

for a.e. $x \in \Omega$. Thus, $\Gamma_{1,\varepsilon} \geq 0$.

We now consider the inequality

$$(z+y)^s \geq z^s + y^s + sz^{s-1}y + szy^{s-1} - C_\mu z^{s-\mu} y^\mu,$$

for all z, $y \geq 0$, $s > 2$ and $1 < \mu < s-1$. By picking $s = 2^*$ and recalling that $b(x) \geq 0$ in the support of v_ε, we get

$$\Gamma_{2,\varepsilon} \geq \int b(x) \left[t_\varepsilon^{2^*} v_\varepsilon^{2^*} + 2^* t_\varepsilon^{2^*-1} u_0 v_\varepsilon^{2^*-1} - C_\mu t_\varepsilon^\mu u_0^{2^*-\mu} v_\varepsilon^\mu \right].$$

This, $\Gamma_{1,\varepsilon} \geq 0$ and (3.7) provide

$$\begin{aligned} m_\varepsilon \quad \leq \quad & I_\lambda(u_0) + \left(\frac{t_\varepsilon^2}{2} \|v_\varepsilon\|^2 - \frac{t_\varepsilon^{2^*}}{2^*} \|b\|_\infty \right) + \frac{t_\varepsilon^{2^*}}{2^*} \int (\|b\|_\infty - b(x)) v_\varepsilon^{2^*} \\ & + t_\varepsilon^2 \Gamma_{3,\varepsilon} - t_\varepsilon^{2^*-1} \int b(x) u_0 v_\varepsilon^{2^*-1} + \frac{1}{2^*} C_\mu t_\varepsilon^\mu \int b(x) u_0^{2^*-\mu} v_\varepsilon^\mu, \end{aligned} \tag{3.8}$$

where we have used

$$-\frac{t_\varepsilon^{2^*}}{2^*} \int b(x) v_\varepsilon^{2^*} = \frac{t_\varepsilon^{2^*}}{2^*} \int (\|b\|_\infty - b(x)) v_\varepsilon^{2^*} - \frac{t_\varepsilon^{2^*}}{2^*} \|b\|_\infty.$$

A simple computation provides

$$\begin{aligned} \max_{t \geq 0} \left(\frac{t^2}{2} \|v_\varepsilon\|^2 - \frac{t^{2^*}}{2^*} \|b\|_\infty \right) \quad & = \quad \frac{2}{N} \frac{1}{\|b\|_\infty^{(N-4)/4}} \left(\|v_\varepsilon\|^2 \right)^{N/4} \\ & = \quad \frac{2}{N} \frac{1}{\|b\|_\infty^{(N-4)/4}} S^{N/4} + O(\varepsilon^{N-4}). \end{aligned} \tag{3.9}$$

Moreover, it follows from (ab_1) and Lemma 3.3 that

$$\begin{aligned} \int (\|b\|_\infty - b(x)) v_\varepsilon^{2^*} \quad & \leq \quad \frac{c_1}{\|u_\varepsilon\|_{2^*}^{2^*}} \int_{B_{2r}(0)} \left[\frac{\varepsilon^{(N-4)/2}}{(\varepsilon^2 + |x|^2)^{(N-2)/2}} \right]^{2N/(N-4)} |x|^\gamma dx \\ & \leq \quad c_2 \varepsilon^N \int_{B_{2r}(0)} \frac{|x|^\gamma}{(\varepsilon^2 + |x|^2)^N} dx \\ & = \quad c_2 \varepsilon^\gamma \int_{B_{(2r)/\varepsilon}(0)} \frac{|y|^\gamma}{(1 + |y|^2)^N} dy. \end{aligned}$$

Since $\gamma < N$, we conclude that

$$\int (\|b\|_\infty - b(x)) v_\varepsilon^{2^*} = O(\varepsilon^\gamma). \tag{3.10}$$

According to [6, Eqn. (7.10)], for some $A_0 > 0$ we have that,

$$\int v_\varepsilon^s = A_0 \varepsilon^{N-(N-4)s/2} + o(\varepsilon^{N-(N-4)s/2}), \quad \text{if } s > N/(N-4).$$

Thus, since $\mu < 2^* - 1 = (N+4)/(N-4) > N/(N-4)$ and $b \in L^\infty(\Omega)$, we obtain

$$\int b(x) u_0 v_\varepsilon^{2^*-1} = A_0 O(\varepsilon^{(N-4)/2}), \tag{3.11}$$

and

$$\int b(x) u_0^{2^*-\mu} v_\varepsilon^{\mu} = A_0 O(\varepsilon^{N-(N-4)\mu/2}) = O(\varepsilon^{(N-4)/2+\bar{\mu}}), \tag{3.12}$$

with $\bar{\mu} = \bar{\mu}(N) > 0$. In this last equality we have used that $N - (N-4)\mu/2 > (N-4)/2$.

Since $\|v_\varepsilon\|_* \to 0$, as $\varepsilon \to 0^+$, we can use the Cauchy-Schwarz inequality and the boundedness of $(t_\varepsilon)_{\varepsilon>0}$ to obtain $C_2 > 0$ such that

$$|\Gamma_{3,\varepsilon}| \le C_2 \|v_\varepsilon\|_*^2,$$

for $\varepsilon > 0$ small. Thus, replacing (3.9)$-$(3.12) in (3.8), we get

$$m_\varepsilon \le \bar{c} + \varepsilon^{(N-4)/2}\left[O(\varepsilon^{\gamma-(N-4)/2}) - A_0 + O(\varepsilon^{\bar{\mu}})\right] + C_2\|v_\varepsilon\|_*^2.$$

This and Lemma 3.3 provide

$$m_\varepsilon \le \bar{c} + \begin{cases} \varepsilon^{3/2}\left[O(\varepsilon^{\gamma-(N-4)/2}) - A_0 + O(\varepsilon^{\bar{\mu}}) + O(\varepsilon^{1/2})\right], & \text{if } N = 7, \\ \varepsilon\left[O(\varepsilon^{\gamma-(N-4)/2}) - A_0 + O(\varepsilon^{\bar{\mu}}) + O(\varepsilon|\ln \varepsilon|)\right], & \text{if } N = 6, \\ \varepsilon^{1/2}\left[O(\varepsilon^{\gamma-(N-4)/2}) - A_0 + O(\varepsilon^{\bar{\mu}}) + O(\varepsilon^{1/2})\right], & \text{if } N = 5. \end{cases}$$

Since $\gamma > (N-4)/2$, $A_0 > 0$ and $\bar{\mu} > 0$, we conclude that $m_\varepsilon < \bar{c}$, for any $\varepsilon > 0$ small. □

We are ready to proof our second main theorem.

Proof of Theorem 1.2. As quoted in the beginning of the section, we already have a solution u_0 with negative energy whenever (2.2) holds. The second solution will be obtained via the Mountain Pass Theorem.

We first consider the critical case $p = 2^*$. We take $\rho > 0$ given by Lemma 2.2 and consider $\phi \in H$ as in the statement of Lemma 3.1. We can obtain $t_0 > 0$ large in such way that $e := u_0 + t\phi$ satisfies $I_\lambda(e) \le I_\lambda(u_0)$. If we define

$$c_M := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_\lambda(\gamma(t)) \tag{3.13}$$

where $\Gamma := \{\gamma \in C([0,1], X) : \gamma(0) = u_0, \gamma(1) = e\}$, Lemma 2.2 and an usual intersection argument show that $c_M \ge \alpha > 0$. The Mountain Pass Theorem provides a sequence $(u_n) \subset H$ such that $I_\lambda(u_n) \to c_M$ and $I'_\lambda(u_n) \to 0$. According to Proposition 3.4, for $\varepsilon > 0$ small enough we have that $c_M < \bar{c}$, where $\bar{c}$ comes from Lemma 3.2. Hence, we have compactness on the level c_M and therefore, up to a subsequence, $u_n \to u_1$ strongly in H. By the regularity of I_λ we have that $I_\lambda(u_1) \ge \delta > 0$ and $I'_\lambda(u_1) = 0$. Hence, we have obtained a second solution.

The proof for the subcritical case $4 < p < 2^*$ is analogous (and easier) since, in this setting, a standard argument shows that the Palais-Smale condition holds at any level. □

References

[1] S. Alama and G. Tarantello, *On semilinear elliptic equations with indefinite nonlinearities*, Calculus of Variations in PDE **1** (1993), 439-475.

[2] A. Ambrosetti, H. Brezis H and G. Cerami, emphCombined effects of concave and convex nonlinearities in some elliptic problems, J. Funct. Anal. **122** (1994), 519-543.

[3] A.B. Ashfaque, F.M. Mahomed and F.D. Zaman, *Invariant boundary value problems for a fourth-order dynamic Euler-Bernoulli beam equation*, J. Math. Phys. **53**, 043703 (2012).

[4] M. Bhakta and D. Mukherjee, *Multiplicity results and sign changing solutions of non-local equations with concave-convex nonlinearities*, Diff. Int. Equ. **30** (2017), 387-422.

[5] M. Berger, *A new approach to the large deflection of plate*, J. Appl. Mech. **22** (1955), 465-472.

[6] F. Bernis, J. García Azorero and I. Peral, *Existence and multiplicity of nontrivial solutions in semilinear critical problems of fourth order.* Adv. Differential Equations **1** (1996), 219–240.

[7] H. Brezis, and L. Nirenberg, *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents.* Comm. Pure Appl. Math. **36** (1983), 437–477.

[8] L. Boccardo, M. Escobedo and I. Peral, *A Dirichlet problem involving critical exponents*, Nonlinear Anal. **24** (1995), 1639-1648.

[9] I. Chueshov and I. Lasiecka, *Long-time behavior of second order evolution equations witn nonlinear damping*, Mem. Amer. Math. Soc., voo. 165, Amer. Math. Soc., Providence, RI, 2008.

[10] D.G. de Figueiredo, J. P. Gossez and P. Ubilla, *Local superlinearity and sublinearity for indefinite semilinear elliptic problems*, J. Funct. Anal. **199** (2003), 452-467.

[11] P. Drábek and Y.X. Huang, *Multiplicity of positive solutions for some quasilinear elliptic equation in $\mathbb{R}^N$ with critical Sobolev exponent*, Journal Dif. Eq. **140** (1997), 106-132.

[12] I. Ekeland, *On the variational principle*, J. Math. Anal. Appl. **47** (1974), 324-353.

[13] G.M. Figueiredo and R.G. Nascimento, *Multiplicity of solutions for equations involving a nonlocal term and the biharmonic operator*, Electron. J. Differential Equations 2016, Paper No. 217, 15 pp.

[14] J.R. Kang, *Energy decay of solutions for an extensible beam equation with a weak nonlinear dissipation*, Math. Meth. Appl. Sci. **35** (2012), 1587-1593.

[15] J-F. Liao, Y. Pu, X-F. Ke and C-L. Tang, *Multiple positive solutions for Kirchhoff type problems involving concave-convex nonlinearities*, Comm. Pure Applied Anal. **16** (2017), 2157-2175.

[16] T-F. Ma and V. Narciso, *Global attractor for a model of extensible beam with nonlinear damping and source terms*, Nonlinear Analysis **73** (2010), 3402-3412.

[17] A. Mao and W. Wang, *Nontrivial solutions of nonlocal fourth order elliptic equation of Kirchhoff type in $\mathbb{R}^3$*, J. Math. Anal. Appl. **459** (2018), 556-563.

[18] J.L.F. Melo and E.M. dos Santos, *Positive solutions to a fourth-order elliptic problem by the Lusternik–Schnirelmann category.* J. Math. Anal. Applications **420** (2014), 532–550.

[19] D. Naimen, *The critical problem of Kirchhoff type elliptic equations in dimension four*, Journal Dif. Eq. **257** (2014), 1168-1193

[20] J.C.N. Pádua, E.A.B. Silva and S.H.M. Soares, *Positive solutions of critical semilinear problems involving a sublinear term on the origin*, Indiana Univ. Math. J. **55** (2006), 1091-1111.

[21] H. Song and C. Chen, *Infinitely Many Solutions for Schrödinger–Kirchhoff-Type Fourth-Order Elliptic Equations*, Proceedings of the Edinburgh Mathematical Society **60** (2017), 1003-1020.

[22] Y. Song and S. Shi, *Multiplicity of Solutions for Fourth-Order Elliptic Equations of Kirchhoff Type with Critical Exponent*, Journal of Dynamical and Control Systems **23** (2017), 375-386.

[23] E. Tokens, *A semilinear elliptic equation with convex and concave nonlinearities*, Top. Meth. Nonlinear Anal. **13** (1999), 251-271.

[24] R.C.A.M. Van der Vorst, *Fourth-order elliptic equations with critical growth*, C.R. Math. Acas. Sci. Paris **320** (1995), 295-299.

[25] S. Woinowsky-Krieger; *The effect of axial force on the vibration of hinged bars*, J. Appl. Mech. **17** (1950), 35-36.

[26] T.F. Wu, *Three positive solutions for Dirichlet problems involving critical Sobolev exponent and sign-changing weight*, Journal Dif. Eq. **249** (2010), 1549-1578.

[27] Z.-J. Yang; *On an extensible beam equation with nonlinear damping and source terms*, J. Differential Equations **254** (2013), 3903-3927.

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