SEMILINEAR ELLIPTIC PROBLEMS INVOLVING EXPONENTIAL CRITICAL GROWTH IN THE HALF-SPACE

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Abstract. We perform an weighted Sobolev space approach to prove a Trudinger-Moser type inequality in the upper half-space. As applications, we derive some existence and multiplicity results for the problem

$$\begin{cases} -\Delta u + h(x)|u|^{q-2}u & = & a(x)f(u), & \text{in } \mathbb{R}^2_+, \\ \frac{\partial u}{\partial \nu} + u & = & 0, & \text{on } \partial\mathbb{R}^2_+, \end{cases}$$

under some technical condition on a, b and the the exponential nonlinearity f. The ideas can also be used to deal with Neumann boundary conditions.

1. Introduction and main results

In the paper [2], Alama-Tarantello considered the indefinite semilinear problem

$$\begin{cases} -\Delta u & = & \mu u + a(x)|u|^{p-2}u - b(x)|u|^{q-2}u, & \text{in } \Omega, \\ u & = & 0, & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $2 < p < q < 2^* := 2N/(N-2)$ and a, b are nonnegative weight functions in $L^1(\Omega)$. Under suitable conditions on the parameter μ, they proved existence, nonexistence and multiplicity of solutions according to the behavior of the competing terms $a|u|^{p-2}u$ and $b|u|^{q-2}u$ as determined by suitable integrability properties of the ratio a^q/b^p. After this, indefinite elliptic problem has been studied by many authors in bounded and unbounded domain. We specially quote the papers Rădulescu-Repovš [19] and Chabrowski [6], where it is considered the semilinear elliptic problem

$$\begin{cases} -\Delta u & = & \lambda a(x)|u|^{p-2}u - b(x)|u|^{q-2}u, & \text{in } \Omega, \\ u & = & 0, & \text{on } \partial\Omega, \end{cases}$$

where $1 < p < 2 < q < 2^*$ and $a, b \in L^\infty(\Omega)$ are such that $\operatorname{ess\,inf}_{x\in\Omega} a(x) > 0$ and $\operatorname{ess\,inf}_{x\in\Omega} b(x) > 0$. The authors proved results of existence, nonexistence and multiplicity of solutions according the range of the parameter λ.

In this paper, we address the following nonlinear elliptic problem

$$(P) \qquad \begin{cases} -\Delta u + h(x)|u|^{q-2}u & = & a(x)f(u), & \text{in } \mathbb{R}^2_+, \\ \frac{\partial u}{\partial \nu} + u & = & 0, & \text{on } \partial\mathbb{R}^2_+, \end{cases}$$

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where $2 \leq q < \infty$, $\mathbb{R}^2_+ := \{x = (x_1, x_2) \in \mathbb{R}^2, x_2 > 0\}$ denotes the upper half-space and ν is the unit outward normal vector on the boundary $\partial\mathbb{R}^2_+$. The basic assumptions on the functions appearing in our problem are

(a_0) $a \in C(\mathbb{R}^2_+, \mathbb{R}_+)$;
(h_0) $h \in C(\mathbb{R}^2_+, \mathbb{R}_+)$;
(f_0) $f \in C(\mathbb{R})$ is such that, for any $\alpha > 0$,

$$\lim_{|s|\to\infty} \frac{f(s)}{e^{\alpha|s|^2}} = 0;$$

(f_1) $\lim_{s\to 0} f(s)/s = 0$;
(f_2) there exists $\mu > q$ such that

$$0 < \mu F(s) \leq f(s)s, \quad \forall\, s \neq 0.$$

Roughly speaking, the growth condition (f_0) says that the problem has subcritical exponential growth with respect to some kind of Trudinger-Moser inequality. The main idea is looking for critical points of the associated energy functional

$$I(u) := \frac{1}{2}\int_{\mathbb{R}^2_+} |\nabla u|^2\, dx + \frac{1}{2}\int_{\partial\mathbb{R}^2_+} u^2\, dx_1 + \frac{1}{q}\int_{\mathbb{R}^2_+} h(x)|u|^q\, dx - \int_{\mathbb{R}^2_+} a(x)F(u)\, dx,$$

with $F(s) := \int_0^s f(t)\, dt$ and the domain of I is an appropriated Banach space E^q. Notice that the L^2-norm on the boundary does not appear in the definition of I. Actually, we shall prove that our working space E^q is such that the first two terms in the above expression is exactly a half of the square of the norm, and therefore the functional behaves like that of the bounded domain case.

In our first result we prove the following:

Existence

Theorem 1.1. *Suppose (a_0), (h_0), $(f_0) - (f_2)$ and $q \geq 2$. If*

(a_1) *there exist $c_1 > 0$ and $\beta > 2$ such that*

$$a(x) \leq \frac{c_1}{(1+|x|)^\beta}, \quad \textit{for all } x \in \mathbb{R}^2_+,$$

then problem (P) has a nonzero weak solution.

Condition (a_1) provides compactness for the Sobolev embedding of E^q into some weighted Lebesgue spaces. This is important to recover compactness for the functional I, since we are dealing with a problem in the unbounded domain $\mathbb{R}^2_+$. Another key point is the proof of an appropriated Trudinger-Moser inequality which guarantees that $u \mapsto \int_{\mathbb{R}^2_+} a(x)F(u)$ is a well defined C^1-functional. Actually, we present a general abstract framework to deal with problems with Robin (or Neumann) boundary condition in the upper half-space (see Section 2).

In our second result we suppose that a behaves like $(1+|x|)^{-2}$. In this new setting, we argue as in [2] and recover compactness by assuming an integrability condition of a special function involving both a and h. More specifically, we prove the following:

Existence2

Theorem 1.2. *Suppose (a_0), (h_0), $(f_0) - (f_2)$ and $q > 2$. If*

$(\widetilde{a_1})$ *there exists $c_1 > 0$ such that*

$$a(x) \leq \frac{c_1}{(1+|x|)^2}, \quad \textit{for all } x \in \mathbb{R}^2_+;$$

(h_1) *the following holds*

$$\int_{\mathbb{R}^2_+} \frac{(a(x))^{q/(q-2)}}{(h(x))^{2/(q-2)}}\, dx < +\infty,$$

then problem (P) *has a nonzero weak solution.*

As it is natural, if we have symmetry we can obtain more and more solutions. So, in our last result we obtain multiple solutions for the problem:

th2 **Theorem 1.3.** *Under the same conditions of Theorems 1.1 or 1.2, if* f *is odd and*

(f_3) *there exist* $c_{\mu_0} > 0$ *and* $\mu_0 > q$ *such that*

$$F(s) \geq c_{\mu_0}|s|^{\mu_0}, \quad \forall\, s \in \mathbb{R},$$

then problem (P) *has infinitely many weak solutions.*

The study of elliptic problems involving Robin boundary condition has a wide literature, see [6, 5, 20] and references therein. Such kind of boundary conditions arise on some important biological models, see for example [8] for this particular aspect. Concerning the mathematical point of view, our main motivation comes from the papers [10, 15], where the authors studied the quasilinear elliptic problem

$$\left\{ \begin{array}{rcll} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) & = & \lambda a(x)|u|^{p-2}u - b(x)|u|^{q-2}u, & \text{in } \Omega, \\ |\nabla u|^{p-2}(\nabla u \cdot \nu) + h(x)|u|^{p-2} & = & 0, & \text{on } \partial\Omega, \end{array} \right.$$

for an exterior domain Ω. Under certain conditions on the power p, q and the functions a, b, some results of existence, nonexistence and multiplicity are derived. It is worthwhile to mention that the Hardy type inequality (see [15, Lemma 1])

$$\int_\Omega \frac{|u|^p}{(1+|x|)^p} dx \leq C_0 \left(\int_\Omega |\nabla u|^p dx + \int_\Gamma \frac{\nu \cdot x}{(1+|x|)^p} |u|^p d\Gamma \right),$$

for $1 < p < N$ and some $C_0 > 0$ plays a fundamental role in their approaches. In order to prove this inequality it was fundamental to assume that there exist $C_0, C_1 > 0$ such that

$$\frac{C_0}{(1+|x|)^{p-1}} \leq h(x) \leq \frac{C_1}{(1+|x|)^{p-1}}.$$

Here, we present a new Hardy type inequality (see Lemma 2.4) which allows us to consider $h \equiv 1$.

In the aforementioned papers the authors deal only with the Sobolev case $N \geq 3$ (in the semilinear case i.e, $p = 2$). To the best of our knowledge, the borderline case $N = 2$ has not been considered, except when f is a pure power. Here, our main interest is to study problem (P) with the nonlinearity f having exponential growth. One of the main difficult in our task relies on the fact that we can not use Schwarz Symmetrization as in [4, 12, 1], since is not possible to perform a reflection argument because the weight function a is not well defined in the whole space via the usual reflection. In order to overcome these difficulties we combine ideas of Kufner-Opic [13] with some arguments of Yang-Zhu [18] and the recent paper by Do Ó-Sani-Zhang [9].

We finally mention that, as a byproduct of the abstract framework introduced here, we are also able to deal with Neumann boundary conditions. More specifically, if we assume $(a_0) - (a_1)$ and that the nonlinearity f satisfy the assumptions in

Theorems 1.1 and 1.3, with some minor modifications we can obtain the same kind of results for the problem

$$\begin{cases} -\Delta u + \dfrac{1}{(1+x_2)^2} u &= a(x)f(u), \quad \text{in } \mathbb{R}^2_+, \\ \dfrac{\partial u}{\partial \nu} &= 0, \qquad\qquad \text{on } \partial\mathbb{R}^2_+. \end{cases}$$

The paper is organized as follows: in the next section we prove a Hardy type inequality, we introduce the variational setting to deal with (P) and also prove a Trudinger-Moser type inequality. In Section 3, as an application of the previous results, we obtain existence and multiplicity of solutions for problem (P).

2. Variational framework

Throughout the paper we denote by $C_0^\infty(\mathbb{R}^2)$ the set all functions with derivatives of any order and with compact support. If $(X, \|\cdot\|_X)$ is any normed vector space, $R>0$ and $a \in X$, then $B_R(a) := \{y \in X : \|y-a\|_X < R\}$ and $B_R^c(a) := X \setminus B_R(a)$. When $a = 0$, we write only B_R and B_R^c, respectively. The points $x \in \mathbb{R}^2$ will be written as $x = (x_1, x_2)$, with $x_1,\ x_2 \in \mathbb{R}$ and $B_R^+ := B_R(0) \cap \mathbb{R}^2_+$. Finally, we denote by C, C_1, C_2, …, positive constants (possibly different).

Let $C_\delta^\infty(\mathbb{R}^2_+)$ be the space of $C_0^\infty(\mathbb{R}^2)-$functions restricted to $\mathbb{R}^2_+$ and define the weighted Sobolev space E as the completion of $C_\delta^\infty(\mathbb{R}^2_+)$ with respect to the norm

$$\|u\| := \left[\int_{\mathbb{R}^2_+} \left(|\nabla u|^2 + \frac{u^2}{(1+x_2)^2} dx\right)\right]^{1/2}.$$

Given a positive function $\omega \in L^1_{loc}(\mathbb{R}^2_+)$ and $p \geq 1$, we denote by L^p_ω the weighted Lebesgue space

$$L^p_\omega := L^p(\mathbb{R}^2_+, \omega) = \left\{ u : \mathbb{R}^2_+ \to \mathbb{R} \,:\, \|u\|_{L^p_\omega} := \left(\int_{\mathbb{R}^2_+} |u|^p \omega(x) dx\right)^{1/p} < +\infty \right\}.$$

In our first results we establish some embedding results from E into weighted Lebesgue spaces. We start with following:

SobolevLimiteCase **Proposition 2.1.** *The weighted Sobolev embedding $E \hookrightarrow L^p_{(1+x_2)^{-2}}$ is continuous for any $2 \leq p < \infty$. The same holds for the Sobolev trace embedding $E \hookrightarrow L^p(\partial\mathbb{R}^2_+)$.*

Proof. First notice that, for any $1 \leq m < N$, we can apply the Gagliardo-Nirenberg-Sobolev inequality and a suitable reflection argument to obtain a constant $C = C(m,N) > 0$ such that

$$\left(\int_{\mathbb{R}^N_+} |v|^{m^*} dx\right)^{m/m^*} \leq C \int_{\mathbb{R}^N_+} |\nabla v|^m dx, \quad \forall\, v \in C_0^\infty(\mathbb{R}^N),$$

where $m^* := Nm/(N-m)$. Picking $m = 1$ and $N = 2$, we obtain

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$$\left(\int_{\mathbb{R}^2_+} |v|^2 dx\right)^{1/2} \leq C \int_{\mathbb{R}^2_+} |\nabla v| dx, \quad \forall\, v \in C_0^\infty(\mathbb{R}^2). \tag{2.1}$$

Given $u \in C_0^\infty(\mathbb{R}^2)$, we can use the above inequality with $v = (1+x_2)^{-2}u^2$ to get

$$\left(\int_{\mathbb{R}^2_+} \frac{u^4}{(1+x_2)^2}dx\right)^{1/2} \leq 2C\int_{\mathbb{R}^2_+} \frac{u^2}{(1+x_2)^2}dx + 2C\int_{\mathbb{R}^2_+} \frac{|u||\nabla u|}{(1+x_2)}dx,$$

where we have used $(1+x_2)^{-1} \leq 1$, for $x_2 > 0$. Using Young's inequality in the last term above we obtain

$$\left(\int_{\mathbb{R}^2_+} \frac{u^4}{(1+x_2)^2}dx\right)^{1/2} \leq C_1\int_{\mathbb{R}^2_+} \left(|\nabla u|^2 + \frac{u^2}{(1+x_2)^2}\right)dx,$$

where $C_1 := 3C$. Hence, we conclude that $E \hookrightarrow L^4_{(1+x_2)^{-2}}$. If $2 < p_0 < 4$, we can use interpolation to obtain $0 < \theta < 1$ such that

$$\|u\|_{L^{p_0}_{(1+x_2)^{-2}}} \leq \|u\|^{\theta}_{L^2_{(1+x_2)^{-2}}}\|u\|^{1-\theta}_{L^4_{(1+x_2)^{-2}}} \leq C_2\|u\|.$$

In particular, since $2 < 3 < 4$, one has $E \hookrightarrow L^3_{(1+x_2)^{-2}}$. Thus, applying (2.1) with $v = (1+x_2)^{-2}|u|^3$ and repeating the argument, we get

$$\begin{aligned}\left(\int_{\mathbb{R}^2_+} \frac{|u|^6}{(1+x_2)^2}dx\right)^{1/2} &\leq 3C\int_{\mathbb{R}^2_+} \frac{|u|^3}{(1+x_2)^2}dx + 3C\int_{\mathbb{R}^2_+} \frac{u^2|\nabla u|}{(1+x_2)}dx \\ &\leq C_3\|u\|^3_{L^3_{(1+x_2)^{-2}}} + C_4\left(\|u\|^4_{L^4_{(1+x_2)^{-2}}} + \int_{\mathbb{R}^2_+}|\nabla u|^2dx\right),\end{aligned}$$

which implies that $E \hookrightarrow L^6_{(1+x_2)^{-2}}$. If $2 \leq p_0 \leq 6$, we can use interpolation again to write

$$\|u\|_{L^{p_0}_{(1+x_2)^{-2}}} \leq \|u\|^{\theta_1}_{L^2_{(1+x_2)^{-2}}}\|u\|^{1-\theta_1}_{L^6_{(1+x_2)^{-2}}} \leq C_5\|u\|.$$

Iterating this process with $k = 4, 5, \ldots$, one has $E \hookrightarrow L^{2k}_{(1+x_2)^{-2}}$. Hence, given $p \geq 2$, it is sufficient to choose $k \geq 2$ such that $2 < p < 2k$ and use interpolation as above to conclude that the embedding $E \hookrightarrow L^p_{(1+x_2)^{-2}}$ is continuous.

Now we will prove the trace embedding. In order to do that we compute, for $p \geq 2$ fixed,

$$\begin{aligned}|u(x_1,0)|^p &= -\int_0^{+\infty} \frac{\partial}{\partial x_2}\left(\frac{|u|^p}{(1+x_2)^2}\right)dx_2 \\ &\leq p\int_0^{+\infty} \frac{|u|^{p-1}|\nabla u|}{(1+x_2)^2}dx_2 + 2\int_0^{+\infty}\frac{|u|^p}{(1+x_2)^3}dx_2.\end{aligned}$$

Integrating, using Hölder's inequality together and $(1+x_2)^{-1} < 1$, if $x_2 \geq 0$, we obtain

$$\int_{\partial\mathbb{R}^2_+} |u(x_1,0)|^p dx_1 \leq p\|u\|^{p-1}_{L^{2(p-1)}_{(1+x_2)^{-2}}}\left(\int_{\mathbb{R}^2_+}|\nabla u|^2dx\right)^{1/2} + 2\|u\|^p_{L^p_{(1+x_2)^{-2}}}.$$

Since $2(p-1) \geq 2$, we obtain from the first part of the proof that

$$\|u\|^p_{L^p(\partial\mathbb{R}^2_+)} \leq C_6\|u\|^{p-1}\|u\| + C_7\|u\|^p \leq C_8\|u\|^p,$$

which completes the proof of Proposition 2.1. □

Remark 2.2. *The above embeddings fail if* $p=\infty$. *In fact, considering the function* $u(x_1,x_2) := (1+x_2)^2\ln(1-\ln|x|)$ *if* $(x_1,x_2)\in B_1^+$ *and zero otherwise, where* $B_1^+ := \{x=(x_1,x_2)\in\mathbb{R}^2_+ : |x|<1\}$, *one can see that* $u\in E$ *but* $u\notin L^\infty\left(\mathbb{R}^2_+,(1+x_2)^{-2}\right)$.

In what follows we show that $(1+x_2)^{-2}$ can de replaced by a large class of weights. Essentially, it is important the decay rate of the weight at infinity. If it is suitable, we are also able to obtain compactness properties. This will be useful for our applications in the next section.

compact **Lemma 2.3.** *Let* $\omega\in C(\mathbb{R}^2_+,\mathbb{R})$. *Suppose there exists* $c_1>0$ *and* $\beta\geq 2$ *such that*

$$0<\omega(x)\leq\frac{c_1}{(1+|x|)^\beta},\qquad \forall\, x\in\mathbb{R}^2_+.$$

Then the Sobolev embedding $E\hookrightarrow L^p_\omega$ *is continuous for any* $2\leq p<\infty$. *Moreover, the embedding is compact if* $\beta>2$.

Proof. The first statement is a direct consequence of the inequality $(1+|x|)^{-\beta}\leq(1+x_2)^{-2}$ and the embedding $E\hookrightarrow L^p_{(1+x_2)^{-2}}$. So, we prove only the second statement.

Suppose that $\beta>2$, $(u_k)\subset E$ is such that $u_k\rightharpoonup 0$ weakly in E and consider $\varepsilon>0$. From the Sobolev embedding we get $\|u_k\|_{L^p_{(1+x_2)^{-2}}}\leq C_1$. Since $\beta>2$, we can choose $R>0$ such that $(1+x_2)^2/(1+|x|)^\beta\leq\varepsilon/(2C_1)$ for any $|x|\geq R$. Hence,

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$$\int_{\mathbb{R}^2_+\setminus B_R^+}\frac{|u_k|^p}{(1+|x|)^\beta}\,dx=\int_{\mathbb{R}^2_+\setminus B_R^+}\frac{|u_k|^p}{(1+x_2)^2}\frac{(1+x_2)^2}{(1+|x|)^\beta}\,dx<\frac{\varepsilon}{2},\tag{2.2}$$

for any $k\in\mathbb{N}$. On the other hand, since the restriction operator $u\mapsto u_{|_{B_R^+}}$ is continuous from E into $E(B_R^+):=\left\{v_{|_{B_R^+}} : v\in E\right\}$ and the embedding $E(B_R^+)\hookrightarrow L^p(B_R^+,(1+|x|)^{-\beta})$ is compact, along a subsequence we have that

$$\lim_{k\to+\infty}\int_{B_R^+}\frac{|u_k|^p}{(1+|x|)^\beta}\,dx=0.$$

Since $\varepsilon>0$ is arbitrary, the above expression and (2.2) imply that, along a subsequence, $u_k\to 0$ strongly in $L^p_{(1+|x|)^{-\beta}}$. It follows from (a_1) that $u_k\to 0$ strongly in L^p_ω and the lemma is proved. □

A fundamental result in the context of this paper regards on a weighted Hardy-type inequality. This is the subject of next lemma (see [14] for a similar result in dimension $N\geq 3$).

Hardy **Lemma 2.4.** *The following inequality holds*

$$\int_{\mathbb{R}^2_+}\frac{u^2}{(1+x_2)^2}dx\leq 4\left(\int_{\mathbb{R}^2_+}|\nabla u|^2dx+\int_{\partial\mathbb{R}^2_+}u^2dx_1\right),\quad \forall\, u\in C_0^\infty(\mathbb{R}^2).$$

Proof. If $u\in C_0^\infty(\mathbb{R}^2)$ and $t\in\mathbb{R}$, we have

embedding2
$$\int_{\mathbb{R}^2_+}\left(\frac{u}{(1+x_2)}+tu_{x_2}\right)^2dx=\int_{\mathbb{R}^2_+}\left[\frac{u^2}{(1+x_2)^2}+t^2u_{x_2}^2+2t\frac{uu_{x_2}}{(1+x_2)}\right]dx.\tag{2.3}$$

Since the normal unit vector pointing out of $\partial\mathbb{R}^2_+$ is $(0,-1)$, we can use the Divergence Theorem to get

$$\int_{\mathbb{R}^2_+} \frac{u u_{x_2}}{(1+x_2)} dx = -\int_{\partial\mathbb{R}^2_+} u^2\, dx_1 - \int_{\mathbb{R}^2_+} \frac{u u_{x_2}}{(1+x_2)} dx + \int_{\mathbb{R}^2_+} \frac{u^2}{(1+x_2)^2} dx,$$

which implies that

$$2\int_{\mathbb{R}^2_+} \frac{u u_{x_2}}{(1+x_2)} dx = \int_{\mathbb{R}^2_+} \frac{u^2}{(1+x_2)^2} dx - \int_{\partial\mathbb{R}^2_+} u^2 dx_1.$$

Combining the above expression and (2.3), we obtain

embedding4

$$\int_{\mathbb{R}^2_+} \left(\frac{u}{(1+x_2)} + t u_{x_2} \right)^2 dx = At^2 + Bt + C \geq 0, \quad \forall\, t \in \mathbb{R}, \tag{2.4}$$

where

$$A := \int_{\mathbb{R}^2_+} u_{x_2}^2 dx, \qquad B := \int_{\mathbb{R}^2_+} \frac{u^2}{(1+x_2)^2} dx - \int_{\partial\mathbb{R}^2_+} u^2\, dx_1, \quad C := \int_{\mathbb{R}^2_+} \frac{u^2}{(1+x_2)^2} dx.$$

It follows from (2.4) that $B^2 - 4AC \leq 0$, and therefore

$$B^2 \leq 4\left(\int_{\mathbb{R}^2_+} |\nabla u|^2 dx \right) \left(\int_{\mathbb{R}^2_+} \frac{u^2}{(1+x_2)^2} dx \right).$$

The above inequality and

$$B^2 \geq \left(\int_{\mathbb{R}^2_+} \frac{u^2}{(1+x_2)^2} dx \right)^2 - 2\left(\int_{\mathbb{R}^2_+} \frac{u^2}{(1+x_2)^2} dx \right) \left(\int_{\partial\mathbb{R}^2_+} u^2\, dx_1 \right)$$

imply that

$$\int_{\mathbb{R}^2_+} \frac{u^2}{(1+x_2)^2} dx \leq 4\int_{\mathbb{R}^2_+} |\nabla u|^2 dx + 2\int_{\partial\mathbb{R}^2_+} u^2\, dx_1$$

and the result follows. □

As a consequence of Lemma 2.1 and Lemma 2.4 we have the following result.

Lemma 2.5. *The quantity*

$$\|u\|_E := \left(\int_{\mathbb{R}^2_+} |\nabla u|^2 dx + \int_{\partial\mathbb{R}^2_+} u^2 dx_1 \right)^{1/2}$$

defines on E a norm which is equivalent to $\|\cdot\|$.

Proof. If $u \in C_0^\infty(\mathbb{R}^2)$, it follows from Lemma 2.4 that $\|u\|^2 \leq 5\|u\|_E^2$. On the other hand, since $(1+x_2)^{-1} < 1$, if $x_2 \geq 0$, we have that

$$\begin{aligned} |u(x_1,0)|^2 &= -\int_0^{+\infty} \frac{\partial}{\partial x_2} \left(\frac{u^2}{(1+x_2)^2} \right) dx_2 \\ &\leq \int_0^{+\infty} \frac{2|u||\nabla u|}{(1+x_2)^2}\, dx_2 + 2\int_0^{+\infty} \frac{u^2}{(1+x_2)^3}\, dx_2 \\ &\leq \int_0^{+\infty} |\nabla u|^2\, dx_2 + 3\int_0^{+\infty} \frac{u^2}{(1+x_2)^2}\, dx_2. \end{aligned}$$

Integrating this inequality we obtain

$$\int_{\partial\mathbb{R}^2_+} |u(x_1,0)|^2 dx_1 \le \int_{\mathbb{R}^2_+} |\nabla u|^2 dx + 3\int_{\mathbb{R}^2_+} \frac{u^2}{(1+x_2)^2} dx.$$

Therefore,

$$\|u\|_E^2 \le 2\int_{\mathbb{R}^2_+} |\nabla u|^2 dx + 3\int_{\mathbb{R}^2_+} \frac{u^2}{(1+x_2)^2} dx \le 3\|u\|^2,$$

and the proof is complete. □

Since E is embedded into Lebesgue spaces of arbitrary order, it is natural to ask if we can obtain embeddings from E into Orlicz space. In order to give a positive answer to this question we consider, from now on, the following weight

$$b(x) := \frac{1}{(1+|x|)^2}.$$

According to Lemma 2.3, the embedding $E \hookrightarrow L_b^p$ is continuous for any $2 \le p < \infty$. Moreover, we have the following Trudinger-Moser type result:

Trudinger-Moser

Proposition 2.6. *For any $\alpha > 0$ we have that $(e^{\alpha u^2} - 1) \in L_b^1$. Moreover, there exists $\alpha_0 > 0$ such that*

$$L(\alpha, b) := \sup_{\{u\in E\,:\,\|u\|_E \le 1\}} \int_{\mathbb{R}^2_+} b(x)(e^{\alpha u^2} - 1)dx < +\infty,$$

for any $0 < \alpha \le \alpha_0$.

To prove our main proposition we will combine the ideas of Kufner-Opic [13] and Yang-Zhu [18]. First we recall a local estimate concerning the Trudinger-Moser inequality.

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Lemma 2.7. *For any $R > 0$, there exists a constant $A_0 > 0$ such that for any $y \in \mathbb{R}^2$ and $v \in W_0^{1,2}(B_R(y))$ with $\|\nabla v\|_{L^2(B_R(y))} \le 1$ we have*

$$\int_{B_R(y)} (e^{4\pi v^2} - 1)\, dx \le A_0 R^2 \int_{B_R(y)} |\nabla v|^2\, dx.$$

Proof. For the proof, we refer the reader to [17, Lemma 4.1] or [18, Lemma 1]. □

Our strategy for proving Proposition 2.6 consists in consider, for any $u \in E$, its extension to the whole $\mathbb{R}^2$ defined by

$$\bar{u}(x_1,x_2) := \begin{cases} u(x_1,x_2), & x_2 > 0 \\ u(x_1,-x_2), & x_2 < 0. \end{cases}$$

The following holds:

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Lemma 2.8. *Given $R > 1$, there exists $\alpha_1 > 0$ and $A_1 = A_1(R) > 0$ such that*

$$\sup_{\{u\in E\,:\,\|u\|_E \le 1\}} \int_{B_R} b(x)(e^{\alpha \bar{u}^2} - 1)\, dx \le A_1,$$

for any $0 < \alpha \le \alpha_1$.

Proof. Consider a cut-off function $\varphi \in C_0^\infty(B_{2R})$ such that $0 \le \varphi \le 1$, $\varphi \equiv 1$ in B_R and $\|\nabla\varphi\|_{L^\infty(B_{2R})} \le C_1/R$ for some $C_1 > 0$. Note that $\varphi\bar{u} \in H_0^1(B_{2R})$ and

$$\begin{aligned}\int_{B_{2R}} |\nabla(\varphi\bar{u})|^2 dx &\le 2\left(\int_{B_{2R}} \varphi^2|\nabla\bar{u}|^2 dx + \int_{B_{2R}} |\nabla\varphi|^2\bar{u}^2 dx\right)\\ &\le 2\left(\int_{B_{2R}} |\nabla\bar{u}|^2 dx + \frac{C_1^2}{R^2}\int_{B_{2R}} \bar{u}^2 dx\right)\\ &\le 2\left(\int_{B_{2R}} |\nabla\bar{u}|^2 dx + C_1^2\frac{(1+2R)^2}{R^2}\int_{B_{2R}} b(x)\bar{u}^2 dx\right),\end{aligned}$$

and therefore

$$\int_{B_{2R}} |\nabla(\varphi\bar{u})|^2 dx \le C_2 \int_{B_{2R}} \left(|\nabla\bar{u}|^2 + b(x)\bar{u}^2\right) dx,$$

with $C_2 := 2\max\{1, (3C_1)^2\} > 0$. Hence, $v := \varphi\bar{u}/\sqrt{10C_2} \in H_0^1(B_{2R})$ verifies

$$\begin{aligned}\|\nabla v\|^2_{L^2(B_{2R})} &\le \frac{1}{10}\int_{\mathbb{R}^2} \left(|\nabla\bar{u}|^2 + b(x)\bar{u}^2\right) dx\\ &\le \frac{1}{5}\int_{\mathbb{R}^2_+} \left(|\nabla u|^2 + \frac{u^2}{(1+x_2)^2}\right) dx \le \|u\|_E^2 \le 1,\end{aligned}$$

where we have used $\|u\|^2 \le 5\|u\|_E^2$ and $b(x) \le (1+x_2)^{-2}$, if $x_2 \ge 0$. Since $b \le 1$ and $\varphi \equiv 1$ in B_R, we obtain

$$\int_{B_R} b(x)(e^{\alpha\bar{u}^2} - 1)\, dx \le \int_{B_R} (e^{\alpha(\varphi\bar{u})^2} - 1)\, dx \le \int_{B_{2R}} (e^{10C_2\alpha v^2} - 1)\, dx$$

and we can use Lemma 2.7 to conclude that

$$\int_{B_R} b(x)(e^{\alpha\bar{u}^2} - 1) dx \le A_0(2R)^2,$$

if $0 < \alpha \le \alpha_1 := 4\pi/(10C_2)$. The lemma is proved. □

We now take the control of the integral outside a ball.

duro **Lemma 2.9.** *There exist $\alpha_2 > 0$ and $A_2 > 0$ such that*

$$\sup_{\{u\in E\,:\,\|u\|_E\le 1\}} \int_{B_{3r}^c} b(x)(e^{\alpha\bar{u}^2} - 1)\, dx \le A_2,$$

for any $r > 1$ and $0 < \alpha \le \alpha_2$.

Proof. Given $r \ge 1$ and $\sigma > r$ we define the annuli

$$A_r^\sigma := \{x \in B_r^c : |x| < \sigma\} = \{x \in \mathbb{R}^2 : r < |x| < \sigma\}.$$

A trick adaption of Besicovitch covering lemma [11] (see [9, estimate (4.8)]) shows that there exist a sequence of points $\{x_k\}_k \in A_1^\sigma$ and a universal constant $\theta > 0$ such that

cov1 (2.5)
$$A_1^\sigma \subseteq \bigcup_k U_k^{1/2}, \qquad \sum_k \chi_{U_k}(x) \le \theta, \quad \forall\, x \in \mathbb{R}^2,$$

where $U_k^{1/2} := B_{|x_k|/6}(x_k)$ and χ_{U_k} denotes the function characteristic of $U_k := B_{|x_k|/3}(x_k)$.

In order to estimate the integral of $\bar{u}$ in A^{σ}_{3r}, we fix $1<r<\sigma$ and follows as in [13] we introduce the set of indices

$$K_{r,\sigma} := \left\{ k \in \mathbb{N} : \ U_k^{1/2} \cap B_{3r}^c \neq \varnothing \right\}.$$

It is easy to see that, if $U_k \cap B_{3r}^c \neq \varnothing$, then $U_k \subset B_r^c$. Moreover, since $1<r<3r$, we have that $A^{\sigma}_{3r} \subset A^{\sigma}_1$. Thus

inclusion (2.6)
$$A^{\sigma}_{3r} \subseteq \bigcup_{k \in K_{r,\sigma}} U_k^{1/2} \subseteq \bigcup_{k \in K_{r,\sigma}} U_k \subseteq B_r^c \subseteq B_1^c$$

and we obtain

bound3 (2.7)
$$\int_{A^{\sigma}_{3r}} b(x)(e^{\alpha \bar{u}^2}-1)dx \leq \sum_{k \in K_{r,\sigma}} \int_{U_k^{1/2}} b(x)(e^{\alpha \bar{u}^2}-1)\,dx.$$

Now notice that

$$\frac{2}{3}|x_k| \leq |x| \leq \frac{4}{3}|x_k|, \quad \forall\, x \in U_k,$$

and therefore

covering2 (2.8)
$$\frac{1}{(1+(4/3)|x_k|)^2} \leq b(x) \leq \frac{1}{(1+(2/3)|x_k|)^2}, \quad \forall\, x \in U_k.$$

For any $k \in K_{r,\sigma}$ fixed, in view of (2.8) we get

bound4 (2.9)
$$\int_{U_k^{1/2}} b(x)(e^{\alpha \bar{u}^2}-1)dx \leq \frac{1}{(1+(2/3)|x_k|)^2} \int_{U_k^{1/2}} (e^{\alpha \bar{u}^2}-1)dx.$$

Now, consider a cut-off function $\varphi_k \in C_0^{\infty}(U_k)$ such that $0 \leq \varphi_k \leq 1$ in U_k, $\varphi_k \equiv 1$ in $U_k^{1/2}$ and $\|\nabla \varphi_k\|_{L^{\infty}(U_k)} \leq C/|x_k|$ for some constant $C>0$. Note that $\varphi_k \bar{u} \in H_0^1(U_k)$ and, by (2.8),

$$\begin{aligned}
\int_{U_k} |\nabla(\varphi_k \bar{u})|^2 dx &\leq 2\left(\int_{U_k} |\varphi_k|^2 |\nabla \bar{u}|^2 dx + \int_{U_k} |\nabla \varphi_k|^2 \bar{u}^2 dx \right) \\
&\leq 2\left(\int_{U_k} |\nabla \bar{u}|^2 dx + \frac{C^2}{|x_k|^2} \int_{U_k} \bar{u}^2 dx \right) \\
&\leq 2\left(\int_{U_k} |\nabla \bar{u}|^2 dx + C^2 \frac{(1+(4/3)|x_k|)^2}{|x_k|^2} \int_{U_k} b(x)\bar{u}^2 dx \right).
\end{aligned}$$

Since $x_k \in A^{\sigma}_r$, we have that $|x_k|>r>1$. This and the above estimate imply that

$$\int_{U_k} |\nabla(\varphi_k \bar{u})|^2 dx \leq C_3 \int_{U_k} \left(|\nabla \bar{u}|^2 + b(x)\bar{u}^2 \right) dx,$$

where $C_3 := 2\max\{1,(7C/3)^2\}$. Therefore, the function $v_k := \varphi_k \bar{u}/(\sqrt{10 C_3}) \in H_0^1(U_k)$ and

$$\begin{aligned}
\|\nabla v_k\|^2_{L^2(U_k)} &\leq \frac{1}{10} \int_{U_k} \left(|\nabla \bar{u}|^2 + b(x)\bar{u}^2 \right) dx \leq \frac{1}{10} \int_{\mathbb{R}^2} \left(|\nabla \bar{u}|^2 + b(x)\bar{u}^2 \right) dx \\
&\leq \frac{1}{5} \int_{\mathbb{R}^2_+} \left(|\nabla u|^2 + \frac{u^2}{(1+x_2)^2} \right) dx \leq \|u\|_E^2 \leq 1.
\end{aligned}$$

Thus, applying Lemma 2.7 with $B_R(y) = U_k$ and $v = v_k$, we get

$$\int_{U_k^{1/2}} (e^{\alpha(\varphi_k \bar{u})^2}-1)dx \leq \int_{U_k} (e^{10 C_3 \alpha v_k^2}-1)dx \leq C_0 3^{-2} |x_k|^2 \int_{U_k} |\nabla v_k|^2 dx,$$

for any $0 < \alpha \leq \alpha_2 := 4\pi/(10C_3)$ and hence

$$\int_{U_k^{1/2}} (e^{\alpha \bar{u}^2} - 1)dx \leq \frac{C_0|x_k|^2}{10 \cdot 3^2} \int_{U_k} \left(|\nabla \bar{u}|^2 + b(x)\bar{u}^2\right) dx.$$

This, together with (2.7), (2.9) and the fact that $s^2/(1+cs)^2 \leq 1/c^2$ for any $c, s > 0$ imply that

$$\begin{aligned}\int_{A_{3r}^\sigma} b(x)(e^{\alpha \bar{u}^2} - 1)dx &\leq \frac{C_0}{90} \sum_{k \in K_{r,\sigma}} \frac{|x_k|^2}{(1 + (2/3)|x_k|)^2} \int_{U_k} \left(|\nabla \bar{u}|^2 + b(x)\bar{u}^2\right) dx \\ &\leq \frac{C_0}{40} \sum_{k \in K_{r,\sigma}} \int_{B_r^c} \left(|\nabla \bar{u}|^2 + b(x)\bar{u}^2\right) \chi_{U_k} dx,\end{aligned}$$

where we have used (2.6) in the last inequality. It follows from (2.5) that

$$\int_{A_{3r}^\sigma} b(x)(e^{\alpha \bar{u}^2} - 1)dx \leq \frac{\theta C_0}{40} \int_{B_r^c} \left(|\nabla \bar{u}|^2 + b(x)\bar{u}^2\right) dx.$$

Taking the limit as $\sigma \to +\infty$ and arguing as before we get

$$\int_{B_{3r}^c} b(x)(e^{\alpha \bar{u}^2} - 1)dx \leq 2\frac{\theta C_0}{40} \int_{B_r^c \cap \mathbb{R}_+^2} \left(|\nabla \bar{u}|^2 + b(x)\bar{u}^2\right) dx \leq \frac{\theta C_0}{4},$$

for any $0 < \alpha \leq \alpha_2 := 4\pi/(10C_3)$. The lemma is proved. □

We are now ready to prove the main result of this subsection.

Proof of Proposition 2.6. For any $u \in E$, we have that

$$\int_{\mathbb{R}_+^2} b(x)(e^{\alpha u^2} - 1)dx = \frac{1}{2} \int_{\mathbb{R}^2} b(x)(e^{\alpha \bar{u}^2} - 1)\, dx.$$

Picking $r > 1$ and setting $R := 3r$, after splitting the last integral above into B_R and B_R^c, we can use Lemmas 2.8 and 2.9 to get

$$\int_{\mathbb{R}_+^2} b(x)(e^{\alpha u^2} - 1)dx \leq \frac{A_1 + A_2}{2},$$

whenever $\|u\|_E \leq 1$ and $0 < \alpha < \alpha_0 := \min\{\alpha_1, \alpha_2\}$. □

We finish this section by proving, in the trace sense, a Trudinger-Moser type inequality. It has interest in itself since it can be used ia large class of semilinear elliptic problems with nonlinear boundary conditions.

Proposition 2.10. *Let $\alpha_0 > 0$ be given by Proposition 2.6. Then, for any $\alpha > 0$ and $u \in E$ we have that $(e^{\alpha u^2(\cdot,0)} - 1) \in L^1(\partial\mathbb{R}_+^2, b(\cdot,0))$. Moreover,*

$$L^t(\alpha, b) := \sup_{\{u \in E:\ \|u\|_E \leq 1\}} \int_{\partial\mathbb{R}_+^2} \left[b(x_1,0)(e^{\alpha u^2(x_1,0)} - 1)\right] dx_1 < +\infty,$$

for any $0 < \alpha < \alpha_0/2$.

Proof. Let $u \in C_0^\infty(\mathbb{R}^2)$ and notice that

$$b(x_1,0)(e^{\alpha u^2(x_1,0)} - \alpha u^2(x_1,0) - 1) = -\int_0^\infty \left(b(x)(e^{\alpha u^2} - \alpha u^2 - 1)\right)_{x_2} dx_2.$$

A straightforward computation shows that

$$\begin{aligned}\left(b(x)(e^{\alpha u^2}-\alpha u^2-1)\right)_{x_2} &= -\frac{2x_2}{(1+|x|)^3|x|}(e^{\alpha u^2}-\alpha u^2-1)\\ &\quad +2b(x)\alpha u\left(e^{\alpha u^2}-1\right)u_{x_2}.\end{aligned}$$

Since $(e^{\alpha s^2}-\alpha s^2-1)\leq(e^{\alpha s^2}-1)$, for any $s\in\mathbb{R}$, we obtain

$$\left|\frac{2x_2}{(1+|x|)^3|x|}(e^{\alpha s^2}-\alpha s^2-1)\right|\leq 2b(x)(e^{\alpha s^2}-1).$$

Let p, $m>2$ be such that $(1/p)+(1/m)+(1/2)=1$. Using Young's inequality, $b(x)\leq 1$ and the elementary inequality $(e^s-1)^m\leq(e^{ms}-1)$, for any $s\geq 0$, we get

$$\left|b(x)u(e^{\alpha u^2}-1)u_{x_2}\right|\leq C_1\left[b(x)|u|^p+b(x)(e^{m\alpha u^2}-1)+b(x)|\nabla u|^2\right],$$

for a.e. $x\in\mathbb{R}^2$ and some constant $C_1=C_1(m,p)>0$. From the estimates above we conclude that

$$\begin{aligned}\int_{\partial\mathbb{R}^2_+} b(x_1,0)(e^{\alpha u^2(x_1,0)}-\alpha u^2(x_1,0)-1)dx_1 &\leq C_2\int_{\mathbb{R}^2_+}\left(b(x)|u|^p+|\nabla u|^2\right)dx\\ &\quad +C_2\int_{\mathbb{R}^2_+} b(x)(e^{\alpha m u^2}-1)dx,\end{aligned}$$

with $C_2:=2\alpha C_1+2$. This and $b(x)\leq 1$ imply that

$$\int_{\partial\mathbb{R}^2_+} b(x_1,0)(e^{\alpha u^2(x_1,0)}-1)dx_1\leq C_3\|u\|_E^2+C_2\int_{\mathbb{R}^2_+}\left(b(x)|u|^p+b(x)(e^{\alpha m u^2}-1)\right)dx,$$

with $C_3:=\max\{\alpha,C_2\}$. Recalling that $\alpha<\alpha_0/2$, we can pick $m>2$ such that $\alpha m<\alpha_0$. If follows from the above expression, the embedding $E\hookrightarrow L^p_b$ and Proposition 2.6 that

$$\int_{\partial\mathbb{R}^2_+} b(x_1,0)(e^{\alpha u^2(x_1,0)}-1)\,dx_1\leq C_3\|u\|_E^2+C_4\|u\|_E^p+L(\alpha m,b),$$

whenever $u\in C_0^\infty(\mathbb{R}^2)$ is such that $\|u\|_E\leq 1$. A density argument shows that $L^t(\alpha,b)\leq C_3+C_4+L(\alpha m,b)$. $\square$

3. Applications

In this section we apply our embedding theorems. We consider the problem (P) and notice that, since h does not belong to any Lebesgue space, we need to introduce a suitable subspace of E in order to use variational methods. So, we consider

$$E^q:=\left\{u\in E:\int_{\mathbb{R}^2_+} h(x)|u|^q dx<\infty\right\},$$

equipped with the norm

$$\|u\|_{E^q}:=\left[\|u\|_E^2+\left(\int_{\mathbb{R}^2_+} h(x)|u|^q dx\right)^{2/q}\right]^{1/2}.$$

Since one of the terms of the energy functional associated to (P) has the form $\int a(x)F(u)$ and the nonlinearity f can behave like e^s at infinity, we need the following Trudinger-Moser type inequality:

T-M **Lemma 3.1.** *Suppose* $(\widetilde{a_1})$. *Then, for any* $u \in E$ *and* $\alpha > 0$, *we have that* $(e^{\alpha u^2} - 1) \in L^1_a$. *Moreover, there exists* $\alpha_0 > 0$ *such that*

$$L(\alpha, a) = \sup_{\{u \in E:\ \|u\|_E \leq 1\}} \int_{\mathbb{R}^2_+} a(x)(e^{\alpha u^2} - 1)dx < +\infty,$$

for any $0 < \alpha \leq \alpha_0$.

Proof. The result is a direct consequence of Proposition 2.6 and the inequality $a(x) \leq c_1(1 + |x|)^2$, where $c_1 > 0$ comes from (a_1). □

Using the above lemma, $(f_0) - (f_1)$ and Proposition 2.1, we can prove that the functional $I : E^q \to \mathbb{R}$ defined by

$$I(u) := \frac{1}{2}\|u\|_E^2 + \frac{1}{q}\int_{\mathbb{R}^2_+} h(x)|u|^q dx - \int_{\mathbb{R}^2_+} a(x)F(u)dx,$$

is well defined, $I \in C^1(E^q, \mathbb{R})$ and

$$\langle I'(u), \varphi\rangle = \int_{\mathbb{R}^2_+} (\nabla u \cdot \nabla \varphi)dx + \int_{\partial\mathbb{R}^2_+} u\varphi\, dx_1 + \int_{\mathbb{R}^2_+} h(x)|u|^{q-2}u\varphi dx - \int_{\mathbb{R}^2_+} a(x)f(u)\varphi dx,$$

for any $\varphi \in E^q$. Thus, weak solutions of (P) are exactly the critical points of I.

We shall prove our existence result as an application of the Mountain Pass Theorem. So, the first step is to prove the following:

geome **Proposition 3.2.** *Suppose* (f_1), (f_2), (a_1), (h_1) *and* $q \geq 2$. *Then*

(i) *there exist* $\rho, C > 0$ *such that* $I_{|\partial B_\rho(0)} \geq C$;
(ii) *there exists* $e \in E^q$ *with* $\|e\|_{E^q} > \rho$ *such that* $I(e) < 0$.

Proof. If $\alpha_0 > 0$ is given by Lemma 3.1, we can use (f_0) to get

$$\lim_{|s| \to +\infty} \frac{f(s)}{|s|^q(e^{\alpha_0 s^2} - 1)} = 0,$$

Given $\varepsilon > 0$, the above expression and (f_1) provide $C_1 > 0$ such that

calor1
$$|F(s)| \leq \frac{\varepsilon}{2}s^2 - C_1|s|^{q+1}(e^{\alpha_0 s^2} - 1), \quad \forall\, s \in \mathbb{R}. \tag{3.1}$$

It follows from Hölder's inequality that

calor2
$$\int_{\mathbb{R}^2_+} a(x)|u|^{q+1}(e^{\alpha_0 u^2} - 1)dx \leq \|u\|^{q+1}_{L_a^{(q+1)2}} \left(\int_{\mathbb{R}^2_+} a(x)(e^{\alpha_0 u^2} - 1)^2 dx\right)^{1/2}. \tag{3.2}$$

Since $(e^{\alpha_0 s^2} - 1)^2 \leq (e^{2\alpha_0 s^2} - 1)$ and $\|u\|_E \leq \|u\|_{E^q}$, we obtain from Lemma 3.1 a constant $C_2 > 0$ such that

$$\int_{\mathbb{R}^2_+} a(x)(e^{\alpha_0 u^2} - 1)^2 dx \leq \int_{\mathbb{R}^2_+} a(x)\left(e^{2\alpha_0\|u\|_E^2\left(\frac{u}{\|u\|_E}\right)^2} - 1\right)dx \leq C_2,$$

if $\|u\|^2_{E^q} \leq (1/2)$. This, (3.1), (3.2) and Lemma 2.3 imply that

$$\int_{\mathbb{R}^2_+} a(x)F(u)dx \leq C_3\varepsilon\|u\|_E^2 + C_4\|u\|^{q+1}_{E^q},$$

for some constants C_3, $C_4 > 0$. Picking $\varepsilon = 1/(4C_3)$, we obtain

$$I(u) \geq \frac{1}{4}\|u\|_E^2 + \frac{1}{q}\|u\|_{L_h^q}^q - C_4\|u\|_{E^q}^{q+1},$$

if $\|u\|_{E^q}^2 \leq (1/2)$. For such u we have that $\|u\|_E^q \leq \|u\|_E^2$. Moreover, since $q \geq 2$, we also have $\|u\|_{E^q}^q \leq C_5 \left(\|u\|_E^q + \|u\|_{L_h^q}^q\right)$, for some $C_5 > 0$. Thus, we conclude that

$$I(u) \geq C_6\|u\|_{E^q}^q - C_4\|u\|_{E^q}^{q+1}, \quad \forall u \in E^q \cap B_{\sqrt{1/2}}(0),$$

with $C_6 := C_5^{-1} \min\left\{4^{-1}, q^{-1}\right\}$. The first statement of the lemma is an easy consequence of the above inequality.

In order to prove (ii) we notice that, by (f_2), there exists C_7, $C_8 > 0$ such that

$$F(s) \geq C_7|s|^\mu - C_8, \quad \forall\, s \in \mathbb{R}.$$

So, given a nonzero function $\varphi \in E^q$ with support in the compact set Ω, we obtain

$$I(t\varphi) \leq \frac{t^2}{2}\|\varphi\|_E^2 + \frac{t^q}{q}\int_\Omega h(x)|\varphi|^q dx - C_7 t^\mu \int_\Omega a|\varphi|^\mu dx + C_8|\Omega|\|a\|_{L^\infty(\Omega)}.$$

Hence $I(t\varphi) \to -\infty$, as $t \to +\infty$, and it is sufficient to set $e := t_0\varphi$, with $t_0 > 0$ large enough. This finishes the proof of the lemma. □

In the next steps we shall prove a compactness property for the functional I. First, a compact embedding result for the space E^q:

compact4 **Lemma 3.3.** *Suppose $(\widetilde{a_1})$ and (h_1). Then the weighted Sobolev embedding $E^q \hookrightarrow L_a^2$ is compact.*

Proof. Let $(u_k) \subset E^q$ be such that $u_k \rightharpoonup 0$ weakly in E^q. The Sobolev embedding provides $C > 0$ be such that $\|u_k\|_{L_h^q}^2 \leq C$. Hence, using Hölder's inequality with exponents $q/2$ and $q/(q-2)$ we obtain, for any $R > 0$,

$$\int_{\mathbb{R}_+^2 \setminus B_R^+} a(x)|u_k|^2\, dx \leq C\left(\int_{\mathbb{R}_+^2 \setminus B_R^+} \frac{a(x)^{q/(q-2)}}{h(x)^{2/(q-2)}}\, dx\right)^{(q-2)/q}.$$

Given $\varepsilon > 0$, we can use the above expression together with the integrability condition of (h_1) to choose $R > 0$ large in such way that

$$\int_{\mathbb{R}_+^2 \setminus B_R^+} a(x)|u_k|^2\, dx \leq \frac{\varepsilon}{2},$$

for any $k \in \mathbb{N}$. On the other hand, as in the proof of Lemma 2.3,

$$\lim_{k\to+\infty} \int_{B_R^+} a(x)|u_k|^2\, dx = 0.$$

Since $\varepsilon > 0$ is arbitrary, the result is proved. □

Lemma 3.4. *If $(u_k) \subset E^q$ is such that $u_k \rightharpoonup u$ weakly in E^q and $I'(u_k) \to 0$, then $I'(u) = 0$.*

Proof. Let $\varphi \in C_\delta^\infty(\mathbb{R}_+^2)$ and $\Omega \subset \overline{\mathbb{R}_+^2}$ its compact support. Since $\|u_k\|_{W^{1,2}(\Omega)} \leq \|u_k\| \leq C$, we may assume that, for any $p \geq 1$, $u_n \to u$ strongly in $L^p(\Omega)$ and $u_n(x) \to u(x)$ for a.e. $x \in \Omega$. Using (f_0) and (f_1) we can check that $f(u_k) \in L^1(\Omega)$

for any $k \in \mathbb{N}$. Moreover, since $\langle I'(u_k), u_k \to 0$, as $k \to +\infty$, we obtain $C_1 > 0$ such that

$$\int_{\mathbb{R}^2_+} f(u_k)u_k dx = o_k(1) + \|u_k\|_E^2 + \|u_k\|_{L_h^q}^q \le C_1,$$

where $o_k(1)$ stands for a quantity approaching zero as $k \to +\infty$. It follows from [7, Lemma 2.1] that $f(u_k) \to f(u)$, as $k \to +\infty$, strongly in $L^1(\Omega)$, and therefore

filme1 (3.3)
$$\lim_{k\to+\infty} \int_{\mathbb{R}^2_+} f(u_k)\varphi\, dx = \int_{\mathbb{R}^2_+} f(u)\varphi\, dx.$$

The strong convergence in $L^{q-1}(\Omega)$ provides $\psi \in L^{q-1}(\Omega)$ such that $|u_k(x)| \le \psi(x)$ for a.e. $x \in \Omega$. Thus,

$$|h(x)|u_k(x)|^{q-1}\varphi(x)| \le \|h\|_{L^\infty(\Omega)}\|\varphi\|_{L^\infty(\Omega)}|\psi(x)|^{q-1}.$$

The right-hand side above belongs to $L^1(\Omega)$ and therefore we infer from the Lebesgue Theorem that

$$\lim_{k\to+\infty} \int_{\mathbb{R}^2_+} h(x)|u_k|^{q-2}u_k\varphi\, dx = \int_{\mathbb{R}^2_+} h(x)|u|^{q-2}u\varphi\, dx.$$

This, (3.3) and the weak convergence of (u_k) imply that

$$0 = \lim_{k\to+\infty} \langle I'(u_k), \varphi\rangle = \langle I'(u), \varphi\rangle, \qquad \forall \varphi \in C_\delta^\infty(\mathbb{R}^2_+).$$

By density we conclude that $I'(u) = 0$. $\square$

We recall that I satisfies the Palais-Smale condition at level $c \in \mathbb{R}$ if any sequence $(u_k) \in E^q$ such that $I(u_k) \to c$ and $I'(u_k) \to 0$ has a convergent subsequence. In the next result we show that this is true in our setting.

PS1 **Proposition 3.5.** *Suppose* (a_1) *and* $q \ge 2$. *Then* I *satisfies the Palais-Smale condition at any level* $c \in \mathbb{R}$. *The same holds if we drop* (a_1) *by both the conditions* $(\widetilde{a_1})$ *and* (h_1).

Proof. Let $(u_k) \subset E^q$ be such that $I(u_k) \to c$ and $I'(u_k) \to 0$. We claim that (u_k) has a bounded subsequence. Indeed, computing $I(u_k) - (1/\mu)\langle I'(u_k), u_k\rangle$ and using (f_2), we get

$$\left(\frac{1}{2} - \frac{1}{\mu}\right)\|u_k\|_E^2 + \left(\frac{1}{q} - \frac{1}{\mu}\right)\|u_k\|_{L_h^q}^q \le c + o_k(1) + o_k(1)\|u_k\|_{E^q}.$$

Suppose, by contradiction, that $\|u_k\|_{E^q} \to \infty$ as $k \to \infty$. Since $2 \le q < \mu$, the above expression implies that

PS7 (3.4)
$$\lim_{k\to+\infty} \frac{\|u_k\|_E^2}{\|u_k\|_{E^q}} = 0, \quad \lim_{k\to+\infty} \frac{\|u_k\|_{L_h^q}^q}{\|u_k\|_{E^q}} = 0.$$

The first equality above combined with the fact that

$$\frac{\|u_k\|_E^2}{\|u_k\|_{E^q}} + \frac{\|u_k\|_{L_h^q}^2}{\|u_k\|_{E^q}} = \|u_k\|_{E^q} \to \infty, \quad \text{as} \quad k \to \infty,$$

shows that

$$\lim_{k\to+\infty} \frac{\|u_k\|_{L_h^q}^2}{\|u_k\|_{E^q}} = +\infty.$$

Consequently, $\|u_k\|^2_{L^q_h} \to +\infty$ and we can use $q \geq 2$ and (3.4) to conclude that

$$\lim_{k\to+\infty} \frac{\|u_k\|^2_{L^q_h}}{\|u_k\|_{E^q}} = \lim_{k\to+\infty} \|u_k\|^{2-q}_{L^q_h} \frac{\|u_k\|^q_{L^q_h}}{\|u_k\|_{E^q}} = 0,$$

which does not make sense. Hence, (u_k) has a bounded subsequence. For simplicity, we write only (u_k) to denote this subsequence.

Suppose first that (a_1) holds. Then, in view of Lemma 2.3, we may assume that, for some $u \in E^q$,

$$\begin{cases} u_k \rightharpoonup u & \text{weakly in } E^q, \\ u_k(x) \to u(x) & \text{for a.e. } x \in \mathbb{R}^2_+, \\ u_k \to u & \text{strongly in } L^2_a. \end{cases}$$

By the last lemma we have that $I'(u) = 0$, and therefore

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$$o_k(1) = \langle I'(u_k) - I'(u), u_k - u\rangle = \|u_k - u\|^2_E + A(k) - B(k), \tag{3.5}$$

where

$$A(k) := \int_{\mathbb{R}^2_+} h(x)\left(|u_k|^{q-2}u_k - |u|^{q-2}u\right)(u_k - u)\, dx$$

and

$$B(k) := \int_{\mathbb{R}^2_+} a(x)\left(f(x,u_k) - f(x,u)\right)(u_k - u)\, dx.$$

We claim that $B(k) = o_k(1)$. In fact, we have

$$|B(k)| \leq B_1(k) + B_2(k),$$

with

$$B_1(k) := \int_{\mathbb{R}^2_+} a(x)|f(x,u_k)||u_k - u| dx, \quad B_2(k) := \int_{\mathbb{R}^2_+} a(x)|f(x,u)||u_k - u| dx.$$

Let $C_1 > 0$ be such that $\|u_k\|_E \leq C_1$, for all $k \in \mathbb{N}$ and choose $0 < \alpha < \alpha_0/(2C_1)$, where α_0 comes from Lemma 3.1. Using $(f_0)-(f_1)$ we obtain C_2, $C_3 > 0$ such that

$$|f(x,s)| \leq C_2|s| + C_3(e^{\alpha s^2} - 1), \quad \forall\, x \in \mathbb{R}^2_+,\ s \in \mathbb{R}.$$

Holder's inequality and the same argument used in the proof of Lemma 3.2 imply that

$$\begin{aligned} B_1(k) \quad &\leq \quad C_2\|u_k\|_{L^2_a}\|u_k - u\|_{L^2_a} \\ &\quad + C_3\|u_k - u\|_{L^2_a}\left[\int_{\mathbb{R}^2_+} a(x)\left(e^{2\alpha\|u_k\|^2_E\left(\frac{u_k}{\|u_k\|_E}\right)^2} - 1\right) dx\right]^{1/2}. \end{aligned}$$

Since $2\alpha\|u_k\|^2_E \leq \alpha_0$ and $u_k \to u$ strongly in L^2_a (see Lemma 2.3), the above expression implies that $B_1(k) \to 0$. The same arguments shows that $B_2(k) = o_k(1)$.

From the above remarks and (3.5) we infer that

$$o_k(1) = \|u_k - u\|^2_E + A(k).$$

We now recall that $(|s_1|^{q-2}s_1 - |s_2|^{q-2}s_2)(s_1 - s_2) \geq C_4|s_1 - s_2|^q$, for all s_1, $s_2 \in \mathbb{R}$ and some $C_4 = C_4(q)$ (see [16, inequality (2.2)]). This and the above equality imply that

$$\|u_k - u\|^2_E + C_4\|u_k - u\|^q_{L^q_h} \leq o_k(1)$$

and therefore $u_k \to u$ strongly in E^q.

For the case that $(\widetilde{a_1})$ and (h_1) hold we can argue along the same lines but using Lemma 3.3 instead of Lemma 2.3. $\square$

We are ready to prove our existence results.

Proof of Theorems 1.1 and 1.2. By the last proposition the functional I satisfies the Palais-Smale condition. Hence, it follows from Proposition 3.2 and the classical Mountain Pass Theorem that there exists $u_0 \in E^q \setminus \{0\}$ such that $I'(u_0) = 0$. This function is a nonzero weak solution of (P). $\square$

In order to prove our multiplicity result we shall use the following version of the symmetric Mountain Pass Theorem.

sym **Theorem 3.6.** *Let $\mathcal{E}$ be a real infinite-dimensional Banach space and $\mathcal{I} \in C^1(\mathcal{E}, R)$ an even functional satisfying the Palais-Smale condition at any level and the following hypotheses:*

$(\mathcal{I}_1)$ *$\mathcal{I}(0) = 0$ and there are constants ρ, $C > 0$ such that $\mathcal{I}_{|_{\partial B_\rho(0)}} \geq C$;*

$(\mathcal{I}_2)$ *for any finite dimensional $\tilde{X} \subset X$, $\tilde{X} \cap \{u \in X : \mathcal{I}(u) \geq 0\}$ is bounded.*

Then $\mathcal{I}$ has an unbounded sequence of critical values.

We finish the paper proving Theorem 1.3.

Proof of Theorem 1.3. Since f is odd we have that I is even and $I(0) = 0$. By Proposition 3.2(i) the functional verifies $(\mathcal{I}_1)$. Moreover, by Proposition 3.5, it also satisfies Palais-Smale.

Let $\tilde{E} \subset E^q$ be a finite dimensional subspace and $(u_k) \subset \tilde{E}$ be such that $I(u_k) \geq 0$. Using (f_3) and the equivalence of norms in $\tilde{E}$, we obtain

$$\begin{aligned} 0 < I(u_k) &\leq \frac{1}{2}\|u_k\|_E^2 + \frac{1}{q}\|u_k\|_{L_h^q}^q - c_{\mu_0}\|u_k\|_{L_a^{\mu_0}}^{\mu_0} \\ &\leq C_1\|u_k\|_{E^q}^2 + C_2\|u_k\|_{E^q}^q - C_3\|u_k\|_{E^q}^{\mu_0}, \end{aligned}$$

for some constants C_1, C_2, $C_3 > 0$. Since $\mu_0 > q \geq 2$, the above expression implies that (u_k) is bounded. Hence, $(\mathcal{I}_2)$ holds and Theorem 3.6 provides infinitely many critical points for I. $\square$

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