SOLITON SOLUTIONS FOR A GENERALIZED QUASILINEAR ELLIPTIC PROBLEM

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Abstract. We establish existence and multiplicity of solutions for the elliptic quasilinear Schrödinger equation

$$-\mathrm{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + V(x)u = h(x,u),\ x \in \mathbb{R}^N,$$

where g is a suitable function, V is a coercive like potential and the nonlinearity h is superlinear at infinity and at the origin. In the proofs, we apply minimization on the Nehari manifold and Ljusternick-Schnirelman theory.

1. Introduction

Let us consider the nonlinear Schrödinger equation

$$i\partial_t z = -\Delta z + W(x)z - \Delta l(|z|^2)l'(|z|^2)z - h(x,z), \quad x \in \mathbb{R}^N,\ t > 0,$$

where $W \in C(\mathbb{R}^N, \mathbb{R})$ is a potential, $l \in C(\mathbb{R}^+, \mathbb{R})$, $h \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ is a given nonlinearity and we look for solutions $z \in C(\mathbb{R} \times \mathbb{R}^N, \mathbb{C})$ with finite energy. This equation has been accepted as a model in physical phenomena depending on the function l. For instance, if $l(t) = 1$ we have the classical semilinear Schrödinger equation [16]. When $l(t) = t$, the equation arises from fluid mechanics, plasma physics and dissipative quantum mechanics, see [26, 19, 11, 14]. We also refer to [15, 4, 13] for further physical applications.

If we are interested in solitary wave solutions, namely solutions with the special form $z(t,x) = \exp(-i\mathcal{E}t)u(x)$, with $\mathcal{E} \in \mathbb{R}$ and u being a real valued function, we are lead to consider the equation

$$-\Delta u + V(x)u - \Delta[l(u^2)]l'(u^2)u = h(x,u), \quad x \in \mathbb{R}^N, \tag{1.1}$$

with $V(x) = W(x) + \mathcal{E}$. In the simplest case $l(t) = 1$, we have a semilinear equation and there exist a lot of papers concerning existence, non-existence, multiplicity and concentration behavior of solutions (see [28, 7, 5, 17] and its references). In the superfluid film case, namely $l(t) = t^{\alpha/2}$, for $\alpha > 0$, the problem also has been extensively studied during the last years, see [24, 25, 27, 31, 21, 33].

In order to present the object of study of this paper we notice that, if we set

$$g(t) = \sqrt{1 + 2(tl'(t^2))^2},$$

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then the problem (1.1) can be written as

$$(P)\qquad \begin{cases} -\mathrm{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + V(x)u = h(x,u), & x \in \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N). \end{cases}$$

It will be considered in a general framework, by assuming that the function g verifies

(g_0) $g \in C^1(\mathbb{R},\mathbb{R})$ is positive, even, non-decreasing in $(0,+\infty)$ and satisfies

$$g_\infty := \lim_{t\to\infty} \frac{g(t)}{t} \in (0,+\infty), \quad \beta := \sup_{t\in\mathbb{R}} \frac{tg'(t)}{g(t)} \le 1.$$

Notice that this includes the fluid mechanics, plasma physics and dissipative quantum mechanics case $g(t) = \sqrt{1+2t^2}$.

In the aforementioned works, different conditions are assumed on the potential V. We consider here a class which includes the coercive ones. More specifically, we suppose the following:

(V_0) $V \in C(\mathbb{R}^N,\mathbb{R})$;
(V_1) $\inf_{x\in\mathbb{R}^N} V(x) = V_0 > 0$;
(V_2) for all $M > 0$, there holds

$$\text{measure}\left(\{x \in \mathbb{R}^N : V(x) \le M\}\right) < +\infty.$$

The condition (V_2) is satisfied, for example, if $\lim_{|x|\to\infty} V(x) = +\infty$. Actually, ($V_2$) could be replaced by any other hypotheses which provides compactness of the embedding for the set $\{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)u^2dx < +\infty\}$ into the Lebesgue spaces $L^q(\mathbb{R}^N)$, for $2 \le q < 2^* := 2N/(N-2)$ (see [2] for some weaker conditions).

Formally, the Euler-Lagrange functional associated to (P) is

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} g(u)^2|\nabla u|^2dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)u^2dx - \int_{\mathbb{R}^N} H(x,u)dx,$$

with $H(x,\tau) := \int_0^\tau h(x,t)dt$. It is well known that I is not well defined in the whole space $H^1(\mathbb{R}^N)$, since there exist functions $u \in H^1(\mathbb{R}^N)$ such that $\int_{\mathbb{R}^N} g(u)^2|\nabla u|^2dx$ is not finite. In the case $g(t) = \sqrt{1+2t^2}$, this difficult was avoided in [6, 20] by considering a change of variable related with the solutions of the ODE

$$f'(t) = \frac{1}{\sqrt{1+2f^2(t)}}, \quad \text{in } (0,+\infty), \qquad f(0) = 0,$$

and working in an Orlicz-Sobolev framework. Since our function g is more general than those of [6, 20], we borrow an idea from [32], which consists in defining $G(t) := \int_0^t g(\tau)d\tau$, notice that $G \in C^1(\mathbb{R},\mathbb{R})$ is invertible and consider the functional

$$J(v) := \frac{1}{2}\int \left(|\nabla v|^2 + V(x)[G^{-1}(v)]^2\right) - \int H(x,G^{-1}(v)),$$

defined on the Orlicz space

$$E := \left\{v \in H^1(\mathbb{R}^N) : \int V(x)[G^{-1}(v)]^2 < \infty\right\}.$$

As we shall see in the next section this space has good properties. Moreover, if we assume the natural conditions

(h_0) $h \in C^1(\mathbb{R}^N \times \mathbb{R},\mathbb{R})$;
(h_1) there exists $C > 0$ and $p \in (2,2^*)$ such that

$$|h(x,t)| \le C(|t| + g(t)|G(t)|^{p-1}), \quad \text{for all } (x,t) \in \mathbb{R}^N \times \mathbb{R},$$

it can be proved that $J \in C^1(E, \mathbb{R})$ and that, if v is a sufficiently smooth critical point of J, then $u = G^{-1}(v)$ is a classical solution of the problem (P). We refer the reader to [9, 21] for more details concerning the change of variable used here.

We are interested in the case that the function h is superlinear at the origin and at the infinity. Hence, we define

$$\mathcal{H}_g(x,t) := \frac{h(x,t)}{g(t)G(t)}, \quad x \in \mathbb{R}^N,\, t \in \mathbb{R},$$

and suppose the following:

(h_2) $\lim_{t\to 0} \mathcal{H}_g(x,t) = 0$, uniformly in $x \in \mathbb{R}^N$;
(h_3) $\lim_{|t|\to+\infty} \mathcal{H}_g(x,t) = +\infty$, uniformly in $x \in \mathbb{R}^N$.

Under the monotonicity condition

(h_4) for any $x \in \mathbb{R}^N$, the function $\mathcal{H}_g(x,\cdot)$ is decreasing in $(-\infty, 0)$ and increasing in $(0, +\infty)$,

we prove the following existence result:

Theorem 1.1. *Suppose that g and V satisfy (g_0) and $(V_0) - (V_2)$, respectively. If h satisfies $(h_0) - (h_4)$, then the problem (P) has a ground state solution.*

For the classical semilinear case $g(t) = 1$, the function $\mathcal{H}_g(x,t)$ turns to be $h(x,t)/t$, and therefore the conditions $(h_2) - (h_3)$ are the usual one for superlinear problems, see [34]. Concerning (h_4), it is a version of the classical monotonicity condition on the ration $h(x,t)/t$ and is extensively used (in the semilinear case) to ensure unique projection properties on the Nehari manifold (see [29, 30] for related results). For quasilinear elliptic problems there exist some related results concerning on the existence of ground state solutions via the Nehari method. For example, assuming that $g(t) = \sqrt{1+t^2}$, the monotonicity condition given in hypothesis (h_4) is equivalent to assume that $h(x,t)/t^3$ is strictly increasing for $t > 0$ and strictly decreasing for $t < 0$. We refer the reader to the important works [9, 25] where the authors have used a minimization argument in the Nehari manifold to guarantee that the problem

$$-\Delta u - \Delta(u)^2 u + V(x)u = h(x,u), \quad x \in \mathbb{R}^N,$$

admits one ground state solution in $H^1(\mathbb{R}^N)$. In the present work, taking into account that g is a general function satisfying (g_0), we employ the Nehari method to obtain existence of ground states solutions for (P) when the nonlinear term h interacts with g. Hence, Theorem 1.1 complement and/or extends the aforementioned works. We emphasize that h is not a powerlike function and g behaves like t at infinity.

Under our assumptions, we prove that the Nehari manifold

$$\mathcal{N} = \{v \in E \setminus \{0\} : J'(v)v = 0\}$$

is a C^1-manifold (see Proposition 3.2). In [9], the authors pointed out that it was not known if $\mathcal{N}$ is regular when the involved functions are only continuous. Although this result is expected in our differentiable set, it is worthing to notice that we need to perform hard calculations due to the generality of the function g. Differently from [25], we deal here with an unbounded potential V, more general functions g and nonlinearities h which can be non-autonomous and non-homogeneous. Although in [8] they also considered nonlinearities without homogeneity, they assumed a condition which implies

$$(2+\delta)H(t) \le \frac{G(t)}{g(t)}h(t), \quad t>0,$$

for some $\delta>0$, which is an Ambrosetti-Rabinowitz type condition. We prove here that the natural conditions $(h_2)-(h_4)$ are sufficient to get the expected existence result for the superlinear case.

Since we prove regularity for the Nehari manifold, we are able to set our problem in a Ljusternik-Schnirelmann framework, see [5, 34]. This is a useful tool in order to find multiplicity of solutions for quasilinear elliptic problems. If Y is a closed subset of a topological space X, we denote by $\mathrm{cat}_X(Y)$ the Ljusternik-Schnirelmann category of Y in X, namely the least number of closed and contractible sets in X which cover Y.

In our second result we consider a singularly perturbed version of the problem (P), namely

$$(P_\varepsilon) \qquad \begin{cases} -\varepsilon^2 \mathrm{div}(g^2(u)\nabla u) + \varepsilon^2 g(u)g'(u)|\nabla u|^2 + V(x)u = h(u), & x\in\mathbb{R}^N,\\ u\in H^1(\mathbb{R}^N), \end{cases}$$

with $\varepsilon>0$. As in the first result, we assume that the function h verifies the autonomous version of the conditions $(h_0)-(h_4)$. Our aim is to establish, for small values of $\varepsilon>0$, a relation between the number of solutions of the problem and the topology of the set

$$M := \{x\in\mathbb{R}^N : V(x)=V_0\}.$$

For our multiplicity result we add two technical conditions:

(V_3) there holds

$$V_0 < V_\infty := \liminf_{|x|\to\infty} V(x);$$

(h_5) there exist $2<p_1<p_2<2^*$ such that

$$\lim_{t\to 0}\frac{h'(t)}{g(t)|G(t)|^{p_1-2}} = 0, \qquad \lim_{|t|\to+\infty}\frac{h'(t)}{g(t)|G(t)|^{p_2-2}} < +\infty.$$

It is important to emphasize that the coercive case $V_\infty=+\infty$ is allowed. Moreover, under (V_3), the set M is compact. The growth condition (h_5) has already appeared in the semilinear case $g(t)=1$ and it provides some kind of splitting result for Palais-Smale sequences (see Lemma 4.5).

For $\delta>0$, we define the set

$$M_\delta := \{x\in\mathbb{R}^N : \mathrm{dist}(x,M)\le\delta\}$$

and prove the following multiplicity result:

Theorem 1.2. *Suppose that g and V satisfy (g_0) and $(V_0)-(V_3)$, respectively. If h satisfies $(h_0)-(h_5)$ then, for any $\delta>0$, there exists $\varepsilon_\delta>0$ such that, for any $\varepsilon\in(0,\varepsilon_\delta)$, the problem (P_ε) has at least $\mathrm{cat}_{M_\delta}(M)$ solutions.*

The key ingredient in the proof is obtaining a relationship between the topology of the set M and that of subsets level of the energy functional. In order to do this, we need to consider an autonomous version of the problem, namely

$$-\mathrm{div}(g^2(v)\nabla u) + g(v)g'(v)|\nabla u|^2 + \mu u = h(v), \quad x\in\mathbb{R}^N,$$

for $\mu > 0$ fixed. We prove (see Theorem 4.2) that, under $(h_0) - (h_5)$, this problem has a ground stated solution. Although this is an auxiliary step for the proof of Theorem 1.2, this result has interest in yourself and complement some of the aforementioned works.

It is worthwhile to mention the paper [22], where the authors considered a very general equation which does not permit a change of variables approach. They used a q-Laplacian regularization method to obtain infinitely many solutions by assuming technical conditions on the gradient of the C^1-potential V, that h is odd and verifies an Ambrosetti-Rabinowitz type condition (see also [23] for a bounded domain case). Despite the generality of the equation in their paper, we have no hypotheses on the gradient of V, the function h here is not necessarily odd and we consider just the natural superlinear condition (h_3).

As far we know, there are no multiplicity results for general quasilinear equations via Ljusternik-Schnirelmann theory. Actually, since we have proved that $(h_0)-(h_4)$ are the correct assumptions to guarantee regularity for $\mathcal{N}$, we believe that some calculations performed here can be useful to extend many results of the semilinear case for this general setting. The main contribution of the second part of the paper is providing multiplicity of solutions for a huge class of quasilinear Schrödinger equations taking into account the fact that g can be general.

We finish this introduction by presenting some examples of functions which satisfy our hypotheses. First we notice that, in some settings, Theorem 1.2 provides an arbitrarily large number of solutions. Actually, suppose $M = \{x_n : n \geq 1\} \cup \{x\}$, where $x_n \to x$ and $x_n \neq x$ for infinitely many indices. Then, for any fixed $k \in \mathbb{N}$, it can be proved that $\mathrm{cat}_{M_\delta}(M) \geq k$, if $\delta > 0$ is small. Hence, for $\varepsilon > 0$ small, we can find at least k solutions for the problem (P_ε) (see [5] for more details). Concerning examples for the function h, we first quote $h(t) = g(t)|G(t)|^{p-2}G(t)$. A simples computation shows that it satisfies $(h_0) - (h_4)$ for any $p \in (2, 2^*)$. Furthermore, it also verifies (h_5) for each $2 < p_1 < p_2 < 2^*$ such that $p_1 < p - 1$ and $p_2 > p - 1/2$. Actually, this can be infered from the limits $\lim_{|t|\to\infty} G(t)/t^2 < +\infty$ and $\lim_{t\to 0} G(t)/t = g(0) > 0$ which are consequence of L'Hospital's rule. Another example is $h(t) = g(t)G(t)\ln(1 + |t|)$, with the extra restriction $2 < p_1 < 5/2 < p_2 < 2^*$ in (h_5). This last assumption makes sense only for $N < 10$. More generally, we can consider $h(t) = g(t)|G(t)|^{p-2}G(t)\ln(1 + |t|)$ with $p \in (2, 2^*)$, $p_1 < p$ and $p_2 > p + 1/2$.

The paper is organized as follows: in the next section, we present the variational framework to deal with the problem as well as the main properties of the function g. In Section 3, we consider the Nehari approach in order to get our main results. Section 4 is devoted to the autonomous version of (P). In the final section, we prove our multiplicity result.

2. The variational framework

Hereafter we write $\int u$ instead of $\int_{\mathbb{R}^N} u(x)\mathrm{d}x$ and denote by $\|\cdot\|_{L^p}$ the $L^p(\mathbb{R}^N)$-norm, for $p \geq 1$.

As quoted in the introduction, the problem (P) is formally the Euler-Lagrange equation associated with the functional

$$I(u) = \frac{1}{2}\int g(u)^2\,|\nabla u|^2 + \frac{1}{2}\int V(x)u^2 - \int H(x,u). \tag{2.1}$$

Since it is not well defined in $H^1(\mathbb{R}^N)$, we shall follow [32] and use the change of variables $v := G(u)$, where the function G is defined as $G(t) := \int_0^t g(\tau)\mathrm{d}\tau$. We list in the sequel the main properties of the function G^{-1} whose proof can be found in [10].

Lemma 2.1. *Suppose that g satisfies (g_0). Then, the function $G^{-1} \in C^2(\mathbb{R},\mathbb{R})$ satisfies the following properties:*

(g_1) *G^{-1} is increasing;*

(g_2) $\left|G^{-1}(t)\right| \le \frac{|t|}{g(0)}$, *for all $t \in \mathbb{R}$;*

(g_3) $\lim\limits_{t\to\pm\infty} \frac{G^{-1}(t)}{g(G^{-1}(t))} = \pm\frac{1}{g_\infty}$;

(g_4) $1 \le \frac{tg(t)}{G(t)} \le 2$ *and* $1 \le \frac{G^{-1}(t)\, g\left(G^{-1}(t)\right)}{t} \le 2$, *for all $t \neq 0$;*

(g_5) $\left|G^{-1}(t)\right| \ge \begin{cases} G^{-1}(1)\,|t|, & \text{for all } |t| \le 1, \\ G^{-1}(1)\sqrt{|t|}, & \text{for all } |t| \ge 1\,; \end{cases}$

(g_6) *G^{-1} is concave and, for all $s \ge 1$, $t \in \mathbb{R}$, there holds*
$$s[G^{-1}(t)]^2 \le [G^{-1}(st)]^2 \le s^2[G^{-1}(t)]^2;$$

(g_7) *$[G^{-1}]^2$ is convex and, for all $s \in [0,1]$, $t \in \mathbb{R}$, there holds*
$$s^2[G^{-1}(t)]^2 \le [G^{-1}(st)]^2 \le s[G^{-1}(t)]^2.$$

Let X be the Hilbert space
$$X := \left\{ u \in H^1(\mathbb{R}^N) : \int V(x)u^2 < \infty \right\},$$
endowed with the inner product
$$\langle u, v\rangle := \int (\nabla u \cdot \nabla v + V(x)uv), \quad \text{for all } u, v \in X.$$
It is well known that the embedding $X \hookrightarrow L^q(\mathbb{R}^N)$ is continuous for $q \in [2, 2^*]$ and compact for $q \in [2, 2^*)$ (see [12]).

We also define the Orlicz-Sobolev space
$$E := \left\{ v \in H^1(\mathbb{R}^N) : \int V(x)[G^{-1}(v)]^2 < \infty \right\}.$$
Since $[G^{-1}]^2$ is a convex function, we can argue as in [25, 24] to conclude that E is a Banach space when endowed with the norm
$$\|v\| := \||\nabla v|\|_{L^2} + [\![v]\!], \quad \text{for all } v \in E,$$
where
$$[\![v]\!] := \inf_{\xi>0} \frac{1}{\xi}\left\{ 1 + \int V(x)[G^{-1}(\xi v)]^2 \right\}.$$

By a weak solution of (P) we mean a function $u \in H^1(\mathbb{R}^N) \cap L^\infty_{loc}(\mathbb{R}^N)$ such that, for all $\varphi \in C_0^\infty(\mathbb{R}^N)$, there holds
$$\int [g^2(u)\nabla u \nabla\varphi + g(u)g'(u)|\nabla u|^2\varphi + V(x)u\varphi] = \int h(x,u)\varphi.$$
After the change of variables $u = G^{-1}(v)$ in the map given in (2.1), we obtain the following functional
$$J(v) := \frac{1}{2}\int \left(|\nabla v|^2 + V(x)[G^{-1}(v)]^2\right) - \int H(x, G^{-1}(v)), \quad v \in E.$$

Under the growth conditions (g_0) and (h_1), we have that $J \in C^1(E, \mathbb{R})$ and its critical points are weak solutions of the problem

$$-\Delta v + V(x)\frac{G^{-1}(v)}{g\left(G^{-1}(v)\right)} = \frac{h(x, G^{-1}(v))}{g\left(G^{-1}(v)\right)}, \quad x \in \mathbb{R}^N.$$

Moreover, if $v \in E \cap C^2(\mathbb{R}^N)$ is a critical point of J, then the function $u = G^{-1}(v)$ is a classical solution of (P) (see [6]).

We list below the main properties of the space E.

Proposition 2.2. *Suppose that V satisfies $(V_0) - (V_2)$. Then the space E has the following properties:*

(1) *if $(v_n) \subset E$ is such that $v_n(x) \to v(x)$ a.e. in $\mathbb{R}^N$ and*

$$\lim_{n\to+\infty} \int V(x)[G^{-1}(v_n)]^2 = \int V(x)[G^{-1}(v)]^2,$$

then

$$\lim_{n\to+\infty} [\![v_n - v]\!] = 0;$$

(2) *the embeddings $E \hookrightarrow \mathcal{D}^{1,2}(\mathbb{R}^N)$, $E \hookrightarrow H^1(\mathbb{R}^N)$ and $X \hookrightarrow E$ are continuous;*

(3) *the map $v \mapsto G^{-1}(v)$ from E to $L^q(\mathbb{R}^N)$ is continuous for $q \in [2, 2\cdot 2^*]$ and compact for $q \in [2, 2 \cdot 2^*)$;*

(4) *if $v \in E$ and $u = G^{-1}(v)$, then*

$$\|ug(u)\| \leq 4\|v\|;$$

(5) *If $v_n \rightharpoonup 0$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ and $\left(\int V(x)[G^{-1}(v_n)]^2\right)$ is bounded then, up to a subsequence, $G^{-1}(v_n) \to 0$ strongly in $L^q(\mathbb{R}^N)$ for any $2 \leq q < 2 \cdot 2^*$;*

(6) *if $v \in E$, then*

$$[\![v]\!] \leq 2 \max\left\{ \int V(x)[G^{-1}(v)]^2, \left(\int V(x)[G^{-1}(v)]^2 \right)^{1/2} \right\};$$

(7) *if $v \in E$, then*

$$[\![v]\!] \geq \frac{1}{4} \min\left\{ \int V(x)[G^{-1}(v)]^2, \left(\int V(x)[G^{-1}(v)]^2 \right)^{1/2} \right\};$$

(8) *if $v \in E$, then*

$$\frac{1}{16}\min\left\{||v||, ||v||^2\right\} \leq Q(v) \leq 16 \max\left\{||v||, ||v||^2\right\},$$

where

$$Q(v) := \int (|\nabla v|^2 + V(x)[G^{-1}(v)]^2).$$

Proof. The proof of items $(1) - (7)$ can be found in [10]. For the last item, we take $v \in E$ and consider two cases. If $[\![v]\!] \leq \||\nabla v|\|_{L^2}$, it follows from the definition of $\|\cdot\|$ that

$$\min\{\|v\|, \|v\|^2\} \leq \|v\|^2 \leq 4\|\nabla v\|_2^2 \leq 4Q(v). \tag{2.2}$$

If $[\![v]\!] > \||\nabla v|\|_{L^2}$, we can use (6) to get

$$\|v\| \leq 2[\![v]\!] \leq 4 \max\left\{ \int V(x)[G^{-1}(v)]^2, \left(\int V(x)[G^{-1}(v)]^2 \right)^{1/2} \right\}.$$

Hence, $\|v\| \le 4\max\{Q(v), \sqrt{Q(v)}\}$. This and (2.2) imply that, for any $v \in E$, there holds

$$Q(v) \ge \frac{1}{16}\min\left\{\|v\|, \|v\|^2\right\}.$$

The proof of the second inequality in (8) can be proved with the same argument. We omit the details. □

We finish this section with two technical results.

Lemma 2.3. *Suppose that h satisfies $(h_0), (h_3)$ and (h_4). Then, for each $x \in \mathbb{R}^N$, the function*

$$\mathcal{L}(x,t) := \frac{h(x,t)G(t)}{g(t)} - 2H(x,t), \quad t \in \mathbb{R}, \tag{2.3}$$

is non-increasing in $(-\infty, 0)$ and non-decreasing in $(0, +\infty)$. Moreover, for each $x \in \mathbb{R}^N$, there holds

$$\lim_{|t|\to\infty} \mathcal{L}(x,t) = +\infty. \tag{2.4}$$

Proof. Since

$$\frac{\partial}{\partial t}\mathcal{L}(x,t) = G^2(t)\frac{\partial}{\partial t}\left\{\frac{h(x,t)}{g(t)G(t)}\right\}, \tag{2.5}$$

the first statement is a direct consequence of (h_4). For the second one, we notice that

$$\frac{\partial}{\partial s}\frac{H(x,s)}{G(s)^2} = \frac{h(x,s)G(s) - 2H(x,s)g(s)}{G(s)^3} = \frac{g(s)}{G(s)^3}\mathcal{L}(x,s).$$

By fixing $0 < t_0 < t$, we can integrate the last identity over $[t_0, t]$ to get

$$\frac{H(x,t)}{G(t)^2} - \frac{H(x,t_0)}{G(t_0)^2} = \int_{t_0}^{t} \frac{g(s)}{G(s)^3}\mathcal{L}(x,s)ds$$

Since $\mathcal{L}$ is non-decreasing in $[t_0, t]$, we deduce that

$$\frac{H(x,t)}{G(t)^2} - \frac{H(x,t_0)}{G(t_0)^2} \le \mathcal{L}(x,t)\int_{t_0}^{t} \frac{g(s)}{G(s)^3}ds = \mathcal{L}(x,t)\left(-\frac{1}{2G(t)^2} + \frac{1}{2G(t_0)^2}\right),$$

from which it follows that

$$\frac{H(x,t)}{G(t)^2} \le \frac{H(x,t_0)}{G(t_0)^2} + \mathcal{L}(x,t)\frac{1}{2G(t_0)^2}.$$

By using (h_3) and L'Hospital rule we conclude that the left-hand side above goes to infinity as $t \to +\infty$. Hence, $\lim_{t\to\infty}\mathcal{L}(x,t) = +\infty$. Using a similar argument in the interval $[t, t_0] \subset (-\infty, 0)$ we conclude that the same occurs when $t \to -\infty$. □

Lemma 2.4. *Suppose that h satisfies (h_0) and (h_1). Then, if $(v_n) \subset E$ is such that $v_n \rightharpoonup 0$ weakly in E, we have that*

$$\lim_{n\to+\infty} \max\left\{\left|\int \frac{h(x, G^{-1}(v_n))}{g(G^{-1}(v_n))}v_n\right|, \left|\int H(x, G^{-1}(v_n))\right|\right\} = 0.$$

Proof. By (h_1) and (h_2) , for any $\varepsilon > 0$ there exists $C_1 = C_1(\varepsilon) > 0$ such that

$$|h(x,t)| \leq \varepsilon g(t)|G(t)| + C_1 g(t)|G(t)|^{p-1}, \quad \text{for all } (x,t) \in (\mathbb{R}^N, \mathbb{R}). \tag{2.6}$$

Hence, we can use the embedding $E \hookrightarrow H^1(\mathbb{R}^N)$, (g_2) and (g_5) to obtain

$$\begin{aligned} \left| \int \frac{h(x, G^{-1}(v_n))}{g(G^{-1}(v_n))} v_n \right| &\leq \varepsilon \int |v_n|^2 + C_1 \int |v_n|^p \\ &\leq \varepsilon C_2 + C_3 \int \left(|G^{-1}(v_n)|^p + |G^{-1}(v_n)|^{2p} \right). \end{aligned}$$

By item (3) of Proposition 2.2 we have that $G^{-1}(v_n) \to 0$ strongly in $L^q(\mathbb{R}^N)$, for any $q \in [2, 2 \cdot 2^*)$. Hence,

$$\limsup_{n \to +\infty} \left| \int \frac{h(x, G^{-1}(v_n))}{g(G^{-1}(v_n))} v_n \right| \leq \varepsilon C_2,$$

and we conclude tat $\int \frac{h(x,G^{-1}(v_n))}{g(G^{-1}(v_n))} v_n \to 0$. A similar argument holds for $\int H(x, G^{-1}(v_n))$. □

3. Existence of solution via the Nehari approach

Throughout this section, for any given $v \in E$, we write $u := G^{-1}(v)$. We are going to prove Theorem 1.1 by using minimization over the Nehari manifold. So, we first define the set

$$\mathcal{N} := \{v \in E \backslash \{0\} : J'(v)v = 0\}.$$

Alternatively, if $\mathcal{J} : E \to \mathbb{R}$ is given by $\mathcal{J}(v) := J'(v)v$, the set $\mathcal{N}$ can be equivalently written as $\mathcal{J}^{-1}(0) \setminus \{0\}$. Hence, for any $v \in \mathcal{N}$, there holds

$$\mathcal{J}(v) = \int |\nabla v|^2 + \int V(x) \frac{G^{-1}(v)}{g(G^{-1}(v))} v - \int \frac{h(x, G^{-1}(v))}{g(G^{-1}(v))} v = 0 \tag{3.1}$$

We shall prove that this set is a C^1-manifold which has some useful properties. For this purpose, it is important to consider, for each fixed $v \in E \setminus \{0\}$, the fibering map

$$\gamma_v(t) := J(tv), \quad t > 0, v \in E \backslash \{0\}.$$

It satisfies the following.

Lemma 3.1. *Suppose that h satisfies $(h_0) - (h_3)$. Then*

$$\lim_{t \to 0} \frac{\gamma_v'(t)}{t} > 0, \qquad \lim_{t \to \infty} \frac{\gamma_v(t)}{t^2} = -\infty, \qquad \lim_{t \to \infty} \frac{\gamma_v'(t)}{t} = -\infty. \tag{3.2}$$

Proof. Let $v \in E \setminus \{0\}$ be fixed and notice that

$$\frac{\gamma_v'(t)}{t} = \int |\nabla v|^2 + \int V(x) \frac{G^{-1}(tv)}{tg(G^{-1}(tv))} v - \int \frac{h(x, G^{-1}(tv))}{tg(G^{-1}(tv))}. \tag{3.3}$$

By using (g_4), we get

$$\frac{G^{-1}(ts)}{tg(G^{-1}(ts))} s = \frac{ts}{G^{-1}(ts) g(G^{-1}(ts))} \left[\frac{G^{-1}(ts)}{t} \right]^2 \geq \frac{1}{2} \left[\frac{G^{-1}(ts)}{t} \right]^2 \geq 0,$$

for any t, $s \neq 0$. Hence, we infer from (3.3) that

$$\frac{\gamma_v'(t)}{t} \geq \int |\nabla v|^2 - \int \frac{h(x, G^{-1}(tv))}{tg(G^{-1}(tv))}.$$

By using (h_2) and the Lebesgue Theorem, we obtain

$$\lim_{t\to 0}\int \frac{h(x,G^{-1}(tv))}{tg(G^{-1}(tv))}v=\lim_{t\to 0}\int \frac{h(x,G^{-1}(tv))}{g(G^{-1}(tv))G(G^{-1}(tv))}v^2=0. \tag{3.4}$$

Thus, the first limit in (3.2) holds true. We now use (g_6) to compute

$$\frac{\gamma_v(t)}{t^2}\le \frac{1}{2}Q(v)-\int \frac{H(x,G^{-1}(tv))}{t^2}, \tag{3.5}$$

for any $t\ge 1$. Using (h_3) and L'Hospital rule we conclude that $\lim_{|s|\to+\infty}\frac{H(x,s)}{G(s)^2}=+\infty$. By (g_3), we have that $G^{-1}(tv)\to+\infty$, as $t\to+\infty$, in the set $\Omega_v^+=\{x\in\mathbb{R}^N : v(x)>0\}$. The same function goes to $-\infty$ in the set $\Omega_v^-=\{x\in\mathbb{R}^N : v(x)<0\}$. Since $v\ne 0$, all these considerations and the Lebesgue Theorem imply that

$$\lim_{t\to\infty}\int \frac{H(x,G^{-1}(tv))}{t^2}=\lim_{t\to\infty}\int \frac{H(x,G^{-1}(tv))}{G(G^{-1}(tv))^2}v^2=\infty$$

and therefore the second limit in (3.2) is a consequence of (3.5). For the last one we notice that, by (g_3),

$$\lim_{t\to+\infty}V(x)\frac{G^{-1}(tv(x))}{g(G^{-1}(tv(x)))}\frac{1}{t}v(x)=0,$$

a.e. in Ω_v^+. By using (g_3) again, we can prove that the same occurs on the set $\mathbb{R}^N\setminus\Omega_v^+$. It is sufficient now to take the limit as $t\to+\infty$ in (3.3) and use the Lebesgue Theorem. □

We present below the main properties of the set $\mathcal{N}$. They are well known in the semilinear case $g(t)=1$ (see [34] for instance).

Proposition 3.2. *Suppose that h satisfies $(h_0)-(h_4)$. Then*

- (N_1) *for any $v\in E\setminus\{0\}$, there exists a unique $t_v>0$ such that $t_vv\in\mathcal{N}$. In particular, $\mathcal{N}$ is non-empty;*
- (N_2) *there exists $\rho>0$ such that*
$$\|v\|\ge\rho,\quad \text{for all } v\in\mathcal{N};$$
- (N_3) *the set $\mathcal{N}$ is a C^1-manifold;*
- (N_4) *if $v\in\mathcal{N}$ is a critical point of J constrained to $\mathcal{N}$, then $J'(v)=0$;*
- (N_5) *if $v\in\mathcal{N}$, then*
$$\max_{t\ge 0}J(tv)=J(v);$$
- (N_6) *if $(v_n)\subset\mathcal{N}$ is such that $\|v_n\|\to+\infty$, then $J(v_n)\to+\infty$.*

Proof. Let $v\in E\setminus\{0\}$ be fixed. A simple calculation shows that $tv\in\mathcal{N}$ if, and only if, $\gamma_v'(t)=0$. Taking into account the first and third limit in (3.2) and the continuity of γ_v', we obtain $t_v>0$ such that $t_vv\in\mathcal{N}$.

In order to prove the uniqueness of t_v, we first notice that $\gamma_v'(t)=0$ is equivalent to

$$k_v(t):=\int\left(\phi(x,t)-V(x)\psi(x,t)\right)v^2=\int|\nabla v|^2,$$

with

$$\phi(x,t):=\frac{h(x,G^{-1}(tv))}{g(G^{-1}(tv))G(G^{-1}(tv))},\qquad \psi(x,t):=\frac{G^{-1}(tv)}{g(G^{-1}(tv))G(G^{-1}(tv))}.$$

So, it is sufficiency to prove that the above equation has at most one solution in $(0,+\infty)$.

For each $x \in \Omega_v^{\pm} := \{x \in \mathbb{R}^N : v(x) \neq 0\}$ we set $s := G^{-1}(tv)$. Since s has the same sign of $v(x)$, we can use (h_4) to conclude that

$$\begin{aligned} \frac{d}{dt}\phi(x,t) &= \left(\frac{d}{ds}\frac{h(x,s)}{g(s)G(s)}\right)\frac{ds}{dt} \\ &= \left(\frac{d}{ds}\frac{h(x,s)}{g(s)G(s)}\right)\frac{v(x)}{g(G^{-1}(tv(x)))} > 0, \end{aligned}$$

for any $t \in (0,+\infty)$ and for a.e. $x \in \Omega_v^{\pm}$. Analogously, we can use (g_4) to obtain

$$\begin{aligned} \frac{d}{dt}\psi(x,t) &= \left(\frac{d}{ds}\frac{s}{G(s)g(s)}\right)\frac{v(x)}{g(G^{-1}(tv(x)))} \\ &= \frac{1}{g(s)G(s)}\left\{1 - \frac{sg(s)}{G(s)} - \frac{sg'(s)}{g(s)}\right\}\frac{v(x)}{g(G^{-1}(tv(x)))} \\ &\leq -\left(\frac{sg'(s)}{g(s)^2G(s)}\right)\frac{v(x)}{g(G^{-1}(tv(x)))} < 0, \end{aligned}$$

for any $t \in (0,+\infty)$ and for a.e. $x \in \Omega_v^{\pm}$. Since $v \neq 0$, the set $\Omega_v^{\pm}$ has positive measure. Hence, the above estimates imply that the function $k_v(t)$ is increasing in $(0,+\infty)$. So, the equation $k_v(t) = \int |\nabla v|^2$ has at most one solution in this set and the proof of (N_1) is concluded.

It follows from (3.1) and (g_4) that

$$\frac{1}{2}Q(v) = \frac{1}{2}\int(|\nabla v|^2 + V(x)u^2) \leq \int \frac{1}{g(u)}h(x,u)v. \tag{3.6}$$

Given $\varepsilon > 0$, it follows from (h_1) and (h_2) that, for some $C_1 = C_1(\varepsilon) > 0$, there holds

$$|h(x,t)| \leq \varepsilon g(t)|G(t)| + C_1 g(t)|G(t)|^{2^*-1}, \quad \text{for all } (x,t) \in \mathbb{R}^N \times \mathbb{R}.$$

Hence, picking $\varepsilon = V_0/4$, we obtain

$$\int \frac{1}{g(u)}h(x,u)v \leq \frac{V_0}{4}\int |G(u)||v| + C_1\int |G(u)|^{2^*-1}|v|.$$

Since $v = G(u)$, we can use $V(x) \geq V_0$, the embedding $E \hookrightarrow \mathcal{D}^{1,2}(\mathbb{R}^N)$ and (3.6), to write

$$\frac{1}{2}Q(v) \leq \frac{1}{4}Q(v) + C_2\|v\|^{2^*},$$

for some $C_2 > 0$. This and item (8) of Proposition 2.2 provides

$$\frac{1}{4\cdot 16}\min\{\|v\|, \|v\|^2\} \leq \frac{1}{4}Q(v) \leq C_2\|v\|^{2^*}$$

If follows that statement (N_2) holds for

$$\rho := \min\left\{1, \left(\frac{1}{64C_2}\right)^{1/(2-2^*)}\right\} > 0.$$

For proving (N_3) we take $v \in E$ and notice that, by (3.1), we have that

$$\mathcal{J}'(v)\phi = J''(v)(v,\phi) + J'(v)\phi, \quad \text{for all } \phi \in E.$$

Picking $\phi = v$, we obtain

$$\mathcal{J}'(v)v = J''(v)(v,v), \tag{3.7}$$

A direct computation gives

$$\begin{aligned} J''(v)(\phi,\psi) &= \int \nabla\phi\cdot\nabla\psi + \int V(x)\left(\frac{1}{g(u)^2} - \frac{ug'(u)}{g(u)^3}\right)\phi\psi \\ &\quad + \int\left(-\frac{1}{g(u)^2}h'(x,u) + \frac{g'(u)}{g(u)^3}h(x,u)\right)\phi\psi, \end{aligned}$$

for any ϕ, $\psi \in E$. Picking $\phi = \psi = v$ and using (3.1) and (3.7), we get

$$\mathcal{J}'(v)v = \int V(x)\Gamma_1(v) + \int \Gamma_2(v) \tag{3.8}$$

with

$$\Gamma_1(v) := \frac{1}{g(u)^2}v^2 - \frac{u}{g(u)}v - \frac{ug'(u)}{g(u)^3}v^2$$

and

$$\Gamma_2(v) := -\frac{1}{g(u)^2}h'(x,u)v^2 + \frac{1}{g(u)}h(x,u)v + \frac{g'(u)}{g(u)^3}h(x,u)v^2.$$

We claim that $\max\{\Gamma_1(v),\Gamma_2(v)\} < 0$. If this is true, we infer from (3.8) that $\mathcal{J}'(v)v < 0$, for all $v \in \mathcal{N}$. Recalling that $\mathcal{N} = \mathcal{J}^{-1}(\{0\}) \setminus \{0\}$ and the elements of $\mathcal{N}$ are far way the origin, we can use the Implicit Function Theorem to conclude that $\mathcal{N}$ is a C^1-manifold, which is exactly the statement (N_3).

Before presenting the proof of the above claim let us suppose that $v \in \mathcal{N}$ is a critical point of J constrained to $\mathcal{N}$. Then it follows that $J'(v) = \lambda\mathcal{J}'(v)$, for some Lagrange Multiplier $\lambda \in \mathbb{R}$. Thus, we have that

$$\mathcal{J}(v) = J'(v)v = \lambda\mathcal{J}'(v)v.$$

Since $\mathcal{J}(v) = 0$ and $\mathcal{J}'(v)v < 0$, we have that $\lambda = 0$, that is, $J'(v) = 0$ as stated in (N_4).

In what follows we prove that $\max\{\Gamma_1(v),\Gamma_2(v)\} < 0$. By using (g_4), we get

$$\frac{1}{g(u)^2}v^2 \le \frac{1}{g(u)^2}vg(G^{-1}(v))G^{-1}(v) = \frac{u}{g(u)}v.$$

Since $g'(t)t \ge 0$ for any $t \in \mathbb{R}$, the above expression provides $\Gamma_1(v) \le 0$. In order to estimate $\Gamma_2(v)$ we notice that hypothesis (h_4) implies that

$$h'(x,t) > \left(\frac{g'(t)}{g(t)} + \frac{g(t)}{G(t)}\right)h(x,t),$$

for all $x \in \mathbb{R}^N$ and $t \neq 0$. Thus, recalling that $G(u) = v$, we obtain

$$\begin{aligned} \Gamma_2(v) &< -\left(\frac{g'(u)}{g(u)} + \frac{g(u)}{G(u)}\right)\frac{h(x,u)}{g(u)^2}v^2 + \frac{h(x,u)}{g(u)}v + \frac{g'(u)h(x,u)}{g(u)^3}v^2. \\ &= -\frac{h(x,u)}{g(u)G(u)}v^2 + \frac{h(x,u)}{g(u)}v = 0. \end{aligned}$$

This establishes (N_4).

For proving (N_5) we notice that, by the first limit in (3.2), the function $\gamma_v(t)$ is increasing near the origin. Since $\gamma_v(0) = 0$ and the second limit in (3.2) imply that $\gamma_v(t) \to -\infty$, as $t \to +\infty$, we conclude that the function $\gamma_v(t)$ attains its maximum value at some $t_v > 0$. Since $\gamma_v'(t_v) = 0$, we have that $t_v v \in \mathcal{N}$. The uniqueness of projection given by (N_1) implies that $t_v = 1$ and therefore (N_5) holds.

Now we shall prove that J is coercive on the Nehari manifold $\mathcal{N}$. Suppose, by contradiction, that there exist $(v_n) \in \mathcal{N}$ such that $\|v_n\| \to +\infty$, but $J(v_n) \le C_1$, for some $C_1 > 0$. If we take $K > 0$ free for now and define $w_n := \frac{v_n}{\|v_n\|}$, we can use the above inequality and (N_5) to get

$$\frac{1}{2}\int \left(|\nabla K w_n|^2 + V(x)[G^{-1}(Kw_n)]^2\right) - \int H(x, G^{-1}(Kw_n)) \le C_1. \tag{3.9}$$

Since (w_n) is bounded in E and $E \hookrightarrow \mathcal{D}^{1,2}(\mathbb{R}^N)$ we have that, up to a subsequence, $w_n \rightharpoonup w$ weakly in $\mathcal{D}^{1,2}(\mathbb{R}^N)$.

If $w \equiv 0$, it follows from the property (8) of Proposition 2.2 that

$$\int V(x)[G^{-1}(Kw_n)]^2 \le Q(Kw_n) \le 16 \max\{\|Kw_n\|, \|Kw_n\|^2)\} = 16K.$$

Hence, by (5) of Proposition 2.2, we conclude that $G^{-1}(w_n) \to 0$ strongly in $L^q(\mathbb{R}^N)$ for any $q \in [2, 2 \cdot 2^*)$. In particular, there exists $h_q \in L^q(\mathbb{R}^N)$ such that, for a.e. $x \in \mathbb{R}^N$, $|G^{-1}(w_n(x))| \le h_q(x)$ and $G^{-1}(w_n(x)) \to 0$. It follows from (h_1) and the Lebesgue Theorem that $\int H(x, G^{-1}(Kw_n)) \to 0$. Hence, we infer from (3.9), that

$$Q(w_n) = \int \left(|\nabla K w_n|^2 + V(x)[G^{-1}(Kw_n)]^2\right) \le C_2,$$

for some $C_2 > 0$, independent of n and K. Using (8) of Proposition 2.2 again we deduce that

$$\frac{K}{16} = \frac{1}{16}\min\{\|Kw_n\|, \|Kw_n\|^2\} \le Q(w_n) \le C_2,$$

which does not make sense, since $K > 0$ is arbitrary.

We shall prove that $w \not\equiv 0$ also provides a contradiction. Since $|v_n(x)| \to +\infty$ a.e. for $x \in \Omega_w^{\pm} := \{x \in \mathbb{R}^N : w(x) \neq 0\}$, it follows from Fatou's Lemma, (g_3) and Lemma 2.3 that

$$\liminf_{n\to+\infty} \int_{\Omega_w^{\pm}} \mathcal{L}(x, G^{-1}(v_n)) \ge \int_{\Omega_w^{\pm}} \liminf_{n\to\infty} \mathcal{L}(x, G^{-1}(v_n)) = \infty. \tag{3.10}$$

On the other hand, by (g_4), we have that

$$\frac{[G^{-1}(t)]^2}{2} \le \frac{tG^{-1}(t)}{g(G^{-1}(t))} \le [G^{-1}(t)]^2, \quad \text{for all } t \neq 0. \tag{3.11}$$

Thus, recalling that $J(v_n) \le C_1$, using (3.1) and $\mathcal{L}(x,t) \ge 0$, we obtain

$$\begin{aligned} C_1 &\ge \frac{1}{2}\int \left(|\nabla v_n|^2 + V(x)\frac{G^{-1}(v_n)}{g(G^{-1}(v_n))}v_n\right) - \int H(x, G^{-1}(v_n)) \\ &= \frac{1}{2}\int \left(\frac{h(x, G^{-1}(v_n))G(G^{-1}(v_n))}{g(G^{-1}(v_n))} - 2H(x, G^{-1}(v_n))\right) \\ &= \frac{1}{2}\int \mathcal{L}(x, G^{-1}(v_n)) \ge \frac{1}{2}\int_{\Omega_w^{\pm}} \mathcal{L}(x, G^{-1}(v_n)). \end{aligned}$$

The above expression contradicts (3.10) and therefore the proof is finished. □

We devote the rest of this section to the proof of our existence theorem. The main idea is to consider the minimization problem

$$c_0 := \inf_{v\in\mathcal{N}} J(v). \tag{3.12}$$

We notice that, if $v \in \mathcal{N}$, it follows from (3.11) and Lemma 2.3 that

$$J(v) = J(v) - \frac{1}{2}J'(v)v \ge \frac{1}{2}\int \mathcal{L}(x, G^{-1}(v)) > 0.$$

Hence, the number c_0 is well defined. We shall prove that it is attained at $\mathcal{N}$ by a nonzero solution of (P).

Proof of Theorem 1.1. Let $(v_n) \subset \mathcal{N}$ be such that $J(v_n) \to c_0$. By (N_6), we may assume that, up to a subsequence, $v_n \rightharpoonup v$ weakly in E. Since $v_n \in \mathcal{N}$ we can use (g_4) to obtain

$$\frac{1}{2}Q(v_n) \leq \int |\nabla v_n|^2 + \int V(x)\frac{G^{-1}(v_n)}{g(G^{-1}(v_n))}v_n = \int \frac{h(x,G^{-1}(v_n))}{g(G^{-1}(v_n))}v_n.$$

If $v = 0$, we infer from Lemma 2.4 that $Q(v_n) \to 0$ and therefore, by item (8) of Proposition 2.2, we would have $\|v_n\| \to 0$, contradicting (N_2). This shows $v \neq 0$ and therefore there exists $t_v > 0$ such that $v_0 := t_v v \in \mathcal{N}$. By using (g_7) we conclude that $Q : E \to \mathbb{R}$ is convex. Thus, the functional J is weakly lower semicontinuous and it follows that

$$c_0 \leq J(v_0) = J(t_v v) \leq \liminf_{n\to\infty} J(t_v v_n) \leq \liminf_{n\to\infty} J(v_n) = c_0.$$

Hence, $J(v_0) = c_0$. Arguing as in the proof of (N_4) we can conclude that v_0 is a critical point of I constrained to $\mathcal{N}$. Thus, $v_0 \in \mathcal{N}$ is a solution of (P). The theorem is now proved.

4. The autonomous problem

For each $\mu > 0$, we consider in this section the problem

$$(AP_\mu) \qquad \begin{cases} -\mathrm{div}(g^2(v)\nabla u) + g(v)g'(v)|\nabla u|^2 + \mu u = h(v), \ x \in \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases}$$

where the autonomous function h verifies $(h_0) - (h_5)$.

We denote by H_μ the space $H^1(\mathbb{R}^N)$ endowed with the norm

$$\|v\|_{H_\mu} := \left(\int |\nabla v|^2 + \mu |v|^2\right)^{1/2}, \quad v \in H_\mu.$$

It can be proved that, for the constant potential $V(x) = \mu$, the Orlicz space defined in Section 2 is equal to $H^1(\mathbb{R}^N)$. Moreover, if we define the Orlicz norm by

$$\|v\|_{\mathcal{O}_\mu} := \||\nabla v|\|_{L^2} + \inf_{\xi>0} \frac{1}{\xi}\left\{1 + \mu\int [G^{-1}(\xi v)]^2\right\}, \quad v \in H_\mu,$$

convergence in the norms $\|\cdot\|_{H_\mu}$ and $\|\cdot\|_{\mathcal{O}_\mu}$ are equivalent. More precisely, we have the following:

Proposition 4.1. *If*

$$Q_\mu(v) := \int \left(|\nabla v|^2 + \mu[G^{-1}(v)]^2\right), \quad v \in H_\mu,$$

then

$$\frac{1}{16}\min\{\|v\|_{\mathcal{O}_\mu}, \|v\|^2_{\mathcal{O}_\mu}\} \leq Q_\mu(v) \leq 16\max\{\|v\|_{\mathcal{O}_\mu}, \|v\|^2_{\mathcal{O}_\mu}\} \tag{4.1}$$

and

$$c_1\min\{\|v\|_{\mathcal{O}_\mu}, \|v\|^2_{\mathcal{O}_\mu}\} \leq \|v\|^2_{H_\mu} \leq c_2\max\{\|v\|_{\mathcal{O}_\mu}, \|v\|^{2^*}_{\mathcal{O}_\mu}\},$$

for some constants $c_1, c_2 > 0$.

Proof. We just prove the second statement, since the first one follows as in Proposition 2.2. For $v \in H^1(\mathbb{R}^N)$, it follows from (g_5) and the Gagliardo-Nirenberg inequality that, for constants $C_i > 0$, there holds

$$\begin{aligned}\int v^2 &\le \frac{1}{G^{-1}(1)^2}\int_{|v|\le 1}[G^{-1}(v)]^2 + C_1\int_{|v|\ge 1}[G^{-1}(v)^2]^{2^*}\\ &\le C_2\int [G^{-1}(v)]^2 + C_1\left(\int\left|\frac{2G^{-1}(v)}{g(G^{-1}(v))}\nabla v\right|^2\right)^{2^*/2}\\ &\le C_2\int [G^{-1}(v)]^2 + C_3\||\nabla v|\|_{L^2}^{2^*}.\end{aligned}$$

Thus,

$$\|v\|_{H_\mu}^2 \le \||\nabla v|\|_{L^2}^2 + \mu C_2\int [G^{-1}(v)]^2 + \mu C_3\||\nabla v|\|_{L^2}^{2^*} \le C_4\max\{Q_\mu(v), Q_\mu(v)^{2^*/2}\}.$$

On the other hand, using property (g_2) we obtain

$$\|v\|_{H_\mu}^2 \ge \||\nabla v|\|_{L^2}^2 + \mu g(0)^2\int [G^{-1}(v)]^2 \ge \min\{1, g(0)^2\}Q_\mu(v).$$

All together, the above inequalities and (4.1) prove the result. □

We consider the functional $I_\mu : H_\mu \to \mathbb{R}$ given by

$$I_\mu(v) := \frac{1}{2}\int |\nabla v|^2 + \frac{\mu}{2}\int [G^{-1}(v)]^2 - \int H(G^{-1}(v)).$$

As before, if $v \in H_\mu \cap C^2(\mathbb{R}^N)$ is a critical point of I_μ, then the function $u = G^{-1}(v)$ is a classical solution of (AP_μ).

The main result of this section if the following.

Theorem 4.2. *Suppose that g satisfies (g_0) and (h) satisfies $(h_0) - (h_5)$. Then, for any $\mu > 0$, the autonomous problem (AP_μ) has a ground state solution.*

In order to prove this result we define

$$\mathcal{M}_\mu := \left\{v \in H_\mu \backslash \{0\} : I'_\mu(v)v = 0\right\}, \tag{4.2}$$

Arguing as in the Section 3 we can check that $\mathcal{M}_\mu$ is a C^1-manifold verifying properties analogous to $(N_1) - (N_5)$. Hence, it is well defined the number

$$m_\mu := \inf_{u \in \mathcal{M}_\mu} I_\mu(v) > 0.$$

The property (N_6) also holds, but we need a different proof since we lost the compact embeddings.

Proposition 4.3. *The functional I_μ is coercive in $\mathcal{M}_\mu$.*

Proof. Let $(v_n) \subset \mathcal{M}_\mu$ be such that $\|v_n\|_\mu \to \infty$ and suppose, by contradiction, that $I_\mu(v_n) \le C$, for some $C > 0$. We set $w_n := v_n/\|v_n\|_{H_\mu}$ and claim that, for some sequence $(y_n) \subset \mathbb{R}^N$ and constants r, $\eta > 0$, there holds

$$\liminf_{n\to+\infty}\int_{B_r(y_n)} |w_n|^2 \ge \eta. \tag{4.3}$$

Indeed, if this is not true, it follows from a result due to Lions [18, Lemma I.1] that $w_n \to 0$ in $L^p(\mathbb{R}^N)$, where $p \in (2, 2^*)$ comes from the condition (h_1). Since $\mathcal{M}_\mu$ satisfies a property analogous to (N_5), we have that $I_\mu(v_n) \ge I_\mu(tv_n)$, for all $t > 0$.

Hence, if we consider $M > 1$, we get $C_1 \geq I_\mu(v_n) \geq I_\mu(M\|v_n\|^{-1}v_n)$ and therefore we can use (g_6), the definition of Q_μ, $\|w_n\|_{H_\mu} = 1$ and (4.1) to obtain

$$
\begin{aligned}
C_1 &\geq \frac{1}{2}\int \left(|\nabla M w_n|^2 + \mu G^{-1}(Mw_n)^2\right) - \int H(G^{-1}(Mw_n)) \\
&\geq \frac{M}{2}\int \left(|\nabla w_n|^2 + \mu G^{-1}(w_n)^2\right) - \int H(G^{-1}(Mw_n)) \qquad (4.4)\\
&\geq \frac{M}{32} - \int H(G^{-1}(Mw_n)).
\end{aligned}
$$

The strong convergence of (w_n) in $L^p(\mathbb{R}^N)$ and the same argument of the proof of Lemma 2.4 provide $\int H(G^{-1}(Mw_n)) \to 0$. Taking the limit in (4.4), we obtain $M \leq 32C_1$, which is absurd since $M > 1$ is arbitrary.

By (4.3), if we set $\tilde{w}_n(x) := w_n(x + y_n)$, there exists $\tilde{w} \in H_\mu$ such that $\tilde{w}_n \rightharpoonup \tilde{w}$ weakly in H_μ, $\tilde{w}_n \to \tilde{w}$ in $L^2_{loc}(\mathbb{R}^N)$ and $\tilde{w}_n(x) \to \tilde{w}(x)$ a.e. in $\mathbb{R}^N$. Moreover, by (4.3) and the local convergence in $L^2(\mathbb{R}^N)$, we have that $\tilde{w} \neq 0$. It follows from (g_4) that

$$C_1 \geq I_\mu(v_n) - \frac{1}{2}I'_\mu(v_n)v_n \geq \frac{1}{2}\int \mathcal{L}(G^{-1}(v_n)) = \frac{1}{2}\int \mathcal{L}(G^{-1}(\widetilde{v}_n)).$$

where $\widetilde{v}_n(x) := v_n(x + y_n) = \tilde{w}_n(x)\|v_n\|_{H_\mu}$. We have that $|\widetilde{v}_n(x)| \to \infty$ for a.e. $x \in \Omega^{\pm}_{\tilde{w}} := \{x \in \mathbb{R}^N : |\tilde{w}(x)| \neq 0\}$. Since this set has positive Lebesgue measure, we can use Fatou's Lemma and Lemma 2.3 to conclude that

$$C_1 \geq \frac{1}{2}\int \liminf_{n\to\infty} \mathcal{L}(G^{-1}(\widetilde{v}_n)) = +\infty,$$

which does not make sense. This contradiction finishes the proof. □

The following compactness result is a the keystone for the proof of Theorem 4.2.

Proposition 4.4. *Let $(w_n) \in \mathcal{M}_\mu$ be such that $I_\mu(w_n) \to m_\mu$ and $w_n \rightharpoonup w$ weakly in H_μ. Then there exists $(y_n) \subset \mathbb{R}^N$ such that $\widetilde{w}_n := w_n(\cdot + y_n) \to \widetilde{w} \in \mathcal{M}_\mu$ with $I_\mu(\widetilde{w}) = m_\mu$. Moreover, if $w \neq 0$, then (y_n) can be taken identically zero and therefore $w_n \to w$ in H_μ.*

Before proving this result we shall show how it can produce a solution for the autonomous problem.

Proof of Theorem 4.2. Let $(w_n) \subset \mathcal{M}_\mu$ be such that $I_\mu(w_n) \to m_\mu$. According to the last proposition, up to translations, this sequence converges to $w_\mu \in \mathcal{M}_\mu$ such that $I_\mu(\omega_\mu) = m_\mu$. It follows from (N_4) that this function is a ground state solution of (AP_μ). □

We devote the rest of this section for the proof of Proposition 4.4. We need two technical results. The fist one is a sort of the classical Lions result and the second one is (weak) version of the well known Splitting Lemma.

Lemma 4.5. *Suppose that $(v_n) \subset H_\mu$ is such that $I'_\mu(v_n)v_n \to 0$ and $v_n \rightharpoonup 0$ weakly in H_μ. If $v_n \not\to 0$ strongly in H_μ, then there exist a sequence $(y_n) \subset \mathbb{R}^N$ and constants $r, \eta > 0$ such that*

$$\liminf_{n\to\infty} \int_{B_r(y_n)} |v_n|^2 \geq \eta.$$

Proof. Indeed, if the conclusion does not hold, we have that $v_n \to 0$ in $L^p(\mathbb{R}^N)$ and therefore we can argue as in the proof of Lemma 2.4 to get $\int \frac{h(G^{-1}(v_n))}{g(G^{-1}(v_n))} v_n \to 0$. Hence, it follows from (3.11) that

$$\begin{aligned} o_n(1) + \int \frac{h(G^{-1}(v_n))}{g(G^{-1}(v_n))} v_n &= \int |\nabla v_n|^2 + \mu \int \frac{G^{-1}(v_n)}{g(G^{-1}(v_n))} v_n \\ &\geq \int |\nabla v_n|^2 + \frac{\mu}{2} \int [G^{-1}(v_n)]^2, \end{aligned}$$

where $o_n(1)$ stands for a quantity approaching zero as $n \to +\infty$. Thus, $Q_\mu(v_n) \to 0$ and we infer from Proposition 4.1 that $\|v_n\|_{H_\mu} \to 0$, contrary to the hypothesis. □

Lemma 4.6. *Suppose that $(w_n) \subset H_\mu$ is a Palais-Smale sequence of I_μ such that $w_n \rightharpoonup w$ weakly in H_μ. Then, for $z_n := w_n - w$, there holds*

$$I_\mu(z_n) = I_\mu(w_n) - I_\mu(w) + o_n(1)$$

and

$$I'_\mu(z_n) z_n = I'_\mu(w_n) w_n - I'_\mu(w) w + o_n(1).$$

Proof. Given $\delta > 0$, it follows from (h_5) that, for some $C_\delta > 0$ there holds

$$|h'(t)| \leq \delta g(t) |G(t)|^{p_1 - 2} + C_\delta g(t) |G(t)|^{p_2 - 2}, \tag{4.5}$$

and

$$|h(t)| \leq \delta p_1 |G(t)|^{p_1 - 1} + C_\delta p_2 |G(t)|^{p_2 - 1}, \tag{4.6}$$

for all $t \in \mathbb{R}$. For $t > 0$, we can use (g_0) to get

$$\begin{aligned} |H(t)| &\leq \delta \int_0^t \frac{p_1 g(\tau) G(\tau)^{p_1 - 1}}{g(\tau)} \mathrm{d}\tau + C_\delta \int_0^t \frac{p_2 g(\tau) G(\tau)^{p_2 - 1}}{g(\tau)} \mathrm{d}\tau \\ &\leq \frac{\delta}{g(0)} |G(t)|^{p_1} + \frac{C_\delta}{g(0)} |G(t)|^{p_2}. \end{aligned}$$

Since G is odd, we have the same inequality for $t < 0$. Hence, we obtain

$$|H(G^{-1}(t))| \leq \frac{\delta}{g(0)} |t|^{p_1} + \frac{C_\delta}{g(0)} |t|^{p_2}, \quad \text{for all } t \in \mathbb{R}.$$

Thus, arguing as in the proof of [1, Lemma 3.1], we can check that

$$\lim_{n \to +\infty} \int \left[H(G^{-1}(z_n)) - H(G^{-1}(w_n)) + H(G^{-1}(w)) \right] = 0.$$

Since $[G^{-1}]^2$ is convex, the same holds for the function Q_μ defined in Proposition 4.1. We can use the above equality to obtain

$$I_\mu(z_n) = I_\mu(w_n) - I_\mu(w) + o_n(1).$$

For the second statement we need to estimate the derivative of the function

$$f(t) := \frac{h(G^{-1}(t))}{g(G^{-1}(t))} t, \quad t \in \mathbb{R}.$$

If we set $s := G^{-1}(t)$, a straightforward computation provides

$$f'(t) = \frac{h'(s)}{g(s)^2} t + \frac{h(s)}{g(s)} - \frac{h(s) g'(s)}{g(s)^3} t,$$

and therefore we can use (g_0) and (g_4) to obtain

$$\begin{aligned}|f'(t)| &\leq |s|\left|\frac{t}{g(s)s}\right|\left|\frac{h'(s)}{g(s)}\right| + \frac{1}{g(s)}|h(s)|\left(1+\left|\frac{t}{g(s)s}\right|\left|\frac{sg'(s)}{g(s)}\right|\right)\\ &\leq |G^{-1}(t)|\left|\frac{h'(s)}{g(s)}\right| + \frac{2}{g(s)}|h(s)|\\ &\leq \frac{|t|}{g(0)}\left|\frac{h'(G^{-1}(t))}{g(G^{-1}(t))}\right| + \frac{2}{g(0)}|h(G^{-1}(t))|.\end{aligned}$$

This and (4.5)-(4.6) provides

$$|f'(t)| \leq c_1\delta|t|^{p_1-1} + c_2C_\delta|t|^{p_2-1}, \quad \text{for all } t \in \mathbb{R},$$

with $c_1 := (1+2p_1)g(0)^{-1}$ and $c_2 := (1+2p_2)g(0)^{-1}$. Using this estimate and the argument of [1, Lemma 3.1], we get

$$\lim_{n\to+\infty}\int\left[\frac{h(G^{-1}(z_n))}{g(G^{-1}(z_n))}z_n - \frac{h(G^{-1}(w_n))}{g(G^{-1}(w_n))}w_n + \frac{h(G^{-1}(w))}{g(G^{-1}(w))}w\right] = 0. \tag{4.7}$$

Since

$$0 \leq Q_\mu'(v)v \leq Q_\mu(v), \quad \text{for all } v \in H_\mu$$

and Q_μ is convex, we can use (4.7) to conclude that

$$I_\mu'(z_n)z_n = I_\mu'(w_n)w_n - I_\mu'(w)w + o_n(1).$$

This ends the proof. □

Proof of Proposition 4.4. Thanks to the Ekeland's Variational Principle we may assume that $I_\mu'(w_n) \to 0$. Since $w_n \to w$ in $L^q_{loc}(\mathbb{R}^N)$ for any $2 \leq q < 2^*$, we can easily conclude that $I_\mu'(w) = 0$ and $w_n(x) \to w(x)$ for a.e. $x \in \mathbb{R}^N$.

We shall prove that, if $w \neq 0$, the proposition holds for the null sequence $y_n = 0$. First notice that $w \in \mathcal{M}_\mu$ and

$$I_\mu(w) = I_\mu(w) - \frac{1}{2}I_\mu'(w)w = \frac{1}{2}\int\left[\mu\left([G^{-1}(w)]^2 - \frac{G^{-1}(w)}{g(G^{-1}(w))}w\right) + \mathcal{L}(G^{-1}(w))\right],$$

where $\mathcal{L}$ was defined in (2.3). It follows from (g_4) and Lemma 2.3 that the term into the brackets above is non-negative. Hence, we can use Fatou's lemma to get

$$\begin{aligned}I_\mu(w) &\leq \liminf_{n\to\infty}\frac{1}{2}\int\left[\mu\left(G^{-1}(w_n)^2 - \frac{G^{-1}(w_n)}{g(G^{-1}(w_n))}w_n\right) + \mathcal{L}(G^{-1}(w_n))\right]\\ &= \liminf_{n\to\infty}\left(I_\mu(w_n) - \frac{1}{2}I_\mu'(w_n)w_n\right) = \liminf_{n\to\infty} I_\mu(w_n) = m_\mu,\end{aligned}$$

and therefore we conclude that $I_\mu(w) = m_\mu$.

In order to get the strong converge we suppose, by contradiction, that $(z_n) := (w_n - w)$ is such that $z_n \not\to 0$. We obtain from Lemma 4.5 a sequence $(y_n) \subset \mathbb{R}^N$ and constants r, $\eta > 0$ such that

$$\liminf_{n\to\infty}\int_{B_r(y_n)}|z_n|^2 \geq \eta.$$

Setting $\widetilde{z}_n := z_n(\cdot + y_n)$, we have that $\widetilde{z}_n \rightharpoonup \widetilde{z}$ weakly in H_μ. The above inequality and the local convergence of $(\widetilde{z}_n)$ in $L^2_{loc}(\mathbb{R}^N)$ implies that $\int_{B_r(0)}|\widetilde{z}(x)|^2dx > 0$. On the other hand, from Lemma 4.6 we conclude that $I_\mu(z_n) \to 0$ and $I'_\mu(z_n)z_n \to 0$. Thus,

$$o_n(1) = I_\mu(z_n) - \frac{1}{2}I'_\mu(z_n)z_n \geq \frac{1}{2}\int_{B_r(0)} \mathcal{L}(G^{-1}(z_n(x+y_n)))dx.$$

Taking the limit we obtain $\int_{B_r(0)} \mathcal{L}(G^{-1}(\widetilde{z}))dx \leq 0$ which is absurd, since $\mathcal{L}(t) > 0$ for any $t \neq 0$, G^{-1} is increasing and the set $\{x \in B_r(0) : \widetilde{z}(x) \neq 0\}$ has positive Lebesgue measure. This contradiction shows that $w_n \to w$ in H_μ.

It remains to consider the case $w = 0$. If $w_n \not\to 0$, by using Lemma 4.5 again, we obtain $(y_n) \subset \mathbb{R}^N$ and constants r, $\eta > 0$ such that

$$\liminf_{n\to+\infty} \int_{B_r(y_n)} |w_n|^2 \geq \eta.$$

If we set $\widetilde{w}_n := w_n(\cdot + y_n)$, we have that the $I_\mu(\widetilde{w}_n) \to m_\mu$ and $\widetilde{w}_n \rightharpoonup \widetilde{w}$ weakly in H_μ, with $\int_{B_r(0)} |\widetilde{w}|^2 dx \geq \eta > 0$. Since $\widetilde{w} \neq 0$, we can argue as in the first part of the proof to conclude that $\widetilde{w}_n \to \widetilde{w}$ strongly in H_μ. This finishes the proof. □

5. Multiplicity of solutions

In this section we prove our multiplicity result. We first notice that the problem (P_ε) is equivalent to

$$(\widetilde{P}_\varepsilon) \qquad \begin{cases} -\mathrm{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + V(\varepsilon x)u = h(u), \ x \in \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N) \end{cases}$$

We devote the rest of this section for the proof of the following:

Theorem 5.1. *Suppose that V and g satisfy $(V_0) - (V_3)$ and (g_0), respectively. Suppose also that h satisfies $(h_0) - (h_5)$. Then there exists $\varepsilon^* > 0$ such that, for any $\varepsilon \in (0, \varepsilon^*)$, the problem $(\widetilde{P}_\varepsilon)$ has at least cat(M) nonzero solutions*

The variational framework to deal with this problem is analogous to that used for the problem (P). Actually, we consider

$$E_\varepsilon := \left\{ v \in H^1(\mathbb{R}^N) : \int V(\varepsilon x)[G^{-1}(v)]^2 < \infty \right\}$$

endowed with the norm

$$\|v\|_\varepsilon := \||\nabla v|\|_{L^2} + \|v\|_\varepsilon$$

where

$$\|v\|_\varepsilon := \inf_{\xi>0} \frac{1}{\xi}\left[1 + \int V(\varepsilon x)[G^{-1}(\xi v)]^2\right].$$

The associated functional

$$J_\varepsilon(v) := \frac{1}{2}\int |\nabla u|^2 - \frac{1}{2}\int V(\varepsilon x)[G^{-1}(v)]^2 - \int H(G^{-1}(v)), \quad v \in E_\varepsilon,$$

belongs to $C^1(E_\varepsilon, \mathbb{R})$ and its critical points provide solutions to (P_ε). If we define

$$\mathcal{N}_\varepsilon := \{v \in E_\varepsilon : J'_\varepsilon(v)v = 0\},$$

we can use the former arguments to show that $\mathcal{N}_\varepsilon$ has properties analogous to $(N_1) - (N_6)$.

In the proof of Theorem 5.1 we shall apply the following abstract result for C^1-manifolds (see [34, Theorem 5.19]).

Theorem 5.2. *Let ψ be a C^1-functional defined on a C^1-manifold $\mathcal{V}$. If ψ is bounded from below and satisfies the Palais-Smale condition, then ψ has at least* $\mathrm{cat}(\mathcal{V})$ *distinct critical points.*

It is easy to prove that J_ε is bounded from below on $\mathcal{N}_\varepsilon$. Moreover, thanks to condition (V_2), we can also check that this functional satisfies the Palais-Smale condition. The difficult part is to relate the category of the set M with that of $\mathcal{N}_\varepsilon$. The following result, whose proof is similar to that present in [3, Lemma 4.3], will be used.

Lemma 5.3. *Let A, B^+, B^- be closed sets with $B^- \subset B^+$. Let $\beta : A \to B^+$, $\Phi : B^- \to A$ be two continuous maps such that $\beta \circ \Phi$ is homotopical equivalent to the embedding $\iota : B^- \to B^+$. Then* $\mathrm{cat}(A) \geq \mathrm{cat}_{B^+}(B^-)$.

In what follows we construct the maps Φ and β.

5.1. **The map Φ_ε.** Let $\omega \in \mathcal{M}_{V_0}$ be a ground state solution of the problem (AP_{V_0}) given by Theorem 4.2 and take $\xi > 0$ such that $\int_{B_{\xi/2}(0)} \omega(x)^2 dx > 0$. We consider a smooth cut-off function $\eta \in C^\infty(\mathbb{R}^+, [0,1])$ such that $\eta(s) = 1$ if $0 \leq s \leq \xi/2$ and $\eta(s) = 0$ if $s \geq \xi$. For each $\varepsilon > 0$ and $y \in M$ we define

$$\Psi_{\varepsilon,y}(x) := \eta(|\varepsilon x - y|)\omega\left(\frac{\varepsilon x - y}{\varepsilon}\right).$$

We recall that $M = \{x \in \mathbb{R}^N : V(x) = V_0\}$ and define $\Phi_\varepsilon : M \to \mathcal{N}_\varepsilon$ by setting

$$\Phi_\varepsilon(y) := t_\varepsilon \Psi_{\varepsilon,y},$$

where $t_\varepsilon > 0$ is the unique number such that $t_\varepsilon \Psi_{\varepsilon,y} \in \mathcal{N}_\varepsilon$. This function is well defined due to (N_1).

Lemma 5.4. *We have that*

$$\lim_{\varepsilon \to 0^+} J_\varepsilon(\Phi_\varepsilon(y)) = m_{V_0},$$

uniformly for $y \in M$.

Proof. Suppose, by contradiction, that the lemma is false. Then there exist $\gamma_0 > 0$, $(y_n) \subset M$ and $\varepsilon_n \to 0^+$ such that

$$|J_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - m_{V_0}| \geq \gamma_0. \tag{5.1}$$

For simplicity, we write only Φ_n, Ψ_n and t_n to denote $\Phi_{\varepsilon_n}(y_n)$, Ψ_{ε_n,y_n} and t_{ε_n}, respectively.

We claim that, for some subsequence, $t_n \to t_0 > 0$. Assume that this is true and set $\widehat{w}_n(z) := \eta(|\varepsilon_n z|)\omega(z)$. Since $t_n\Psi_n \in \mathcal{N}_{\varepsilon_n}$, we can use the change of variables $\varepsilon_n z := \varepsilon_n x - y_n$ to write

$$\int_{\mathbb{R}^N} |\nabla(t_n\widehat{w}_n(z)|^2 dz + \int_{\mathbb{R}^N} V(\varepsilon_n z + y_n)\frac{G^{-1}(t_n\widehat{w}_n(z))}{g(G^{-1}(t_n\widehat{w}_n(z)))}(t_n\widehat{w}_n(z))dz$$
$$= \int_{\mathbb{R}^N} \frac{h(G^{-1}(t_n\widehat{w}_n(z)))}{g(G^{-1}(t_n\widehat{w}_n(z)))}(t_n\widehat{w}_n(z))dz.$$

Since $\widehat{w}_n(z) \to \omega(z)$ for a.e. $z \in \mathbb{R}^N$, $y_n \to y \in M$ and $t_0 > 0$, we can take the limit in the above expression and use the Lebesgue Theorem to conclude that $t_0\omega \in \mathcal{M}_{V_0}$.

But, by (N_1), the projection on $\mathcal{M}_{V_0}$ is unique, and therefore we obtain $t_0 = 1$. Thus, taking the limit at the equality

$$\begin{aligned} J_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) &= \frac{t_{\varepsilon_n}^2}{2}\int_{\mathbb{R}^N} |\nabla \eta(|\varepsilon_n z|)w(z)|^2 dx \\ &+ \frac{1}{2}\int_{\mathbb{R}^N} V(\varepsilon_n z + y_n) G^{-1}(t_{\varepsilon_n}\eta(|\varepsilon_n z|)w(z))^2 dz \\ &- \int_{\mathbb{R}^N} H(G^{-1}(t_{\varepsilon_n}\eta(|\varepsilon_n z|)w(z)))dz, \end{aligned}$$

we conclude that $J_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) \to I_{V_0}(\omega) = m_{V_0}$, which contradicts (5.1).

It remains to check the claim. We first prove that $(t_n) \subset \mathbb{R}$ is bounded. Suppose, by contradiction, that for some subsequence (still denote (t_n)) we have that $|t_n| \to +\infty$. Since the map $s \mapsto h(s)/(g(s)G(s))$ is non-negative, for n large there holds

$$\begin{aligned} \int \frac{h(G^{-1}(t_n\Psi_n))}{g(G^{-1}(t_n\Psi_n))}(t_n\Psi_n) &= \int \frac{h(G^{-1}(t_n\Psi_n))}{g(G^{-1}(t_n\Psi_n))G(G^{-1}(t_n\Psi_n))}(t_n\Psi_n)^2 \\ &\geq \int_{B_{\delta/2}(0)} \frac{h(G^{-1}(t_n\omega(z)))}{g(G^{-1}(t_n\omega(z)))G(G^{-1}(t_n\omega(z)))} t_n^2 \omega(z)^2 dz \end{aligned}$$

Now notice that the set $\Omega := \{z \in \mathbb{R}^N : \omega(z) \neq 0\} \cap B_{\delta/2}(0)$ has positive measure. Moreover, by (g_3), we have that $|G^{-1}(t_n\omega(z))| \to +\infty$, for a.e. $z \in \Omega$. So, we can use Fatou's Lemma to conclude that

$$\liminf_{n\to+\infty} \frac{1}{t_n^2}\int \frac{h(G^{-1}(t_n\Psi_n))}{g(G^{-1}(t_n\Psi_n))}(t_n\Psi_n) = +\infty. \tag{5.2}$$

On the other hand, since we may suppose that $t_n \geq 1$, it follows from (3.11) and the Lebesgue Theorem that

$$\begin{aligned} \int \frac{h(G^{-1}(t_n\Psi_n))}{g(G^{-1}(t_n\Psi_n))}(t_n\Psi_n) &= t_n^2 \int |\nabla \Psi_n|^2 + \int V(\varepsilon_n x)\frac{G^{-1}(t_n\Psi_n)}{g(G^{-1}(t_n\Psi_n))}(t_n\Psi_n) \\ &\leq t_n^2 A_n, \end{aligned}$$

where

$$A_n := \int |\nabla \Psi_n|^2 + V(\varepsilon_n x)[G^{-1}(\Psi_n)]^2.$$

By the Lebesgue Theorem we have that $A_n \to Q_{V_0}(\omega) > 0$. Hence, we can use Fatou's Lemma to obtain

$$Q_{V_0}(\omega) \geq \liminf_{n\to+\infty} \frac{1}{t_n^2}\int \frac{h(G^{-1}(t_n\Psi_n))}{g(G^{-1}(t_n\Psi_n))}(t_n\Psi_n),$$

which contradicts (5.2).

Since we have proved that (t_n) is bounded we may suppose that $t_n \to t_0 \geq 0$. It remains to discard the possibility $t_0 = 0$. Suppose, by contradiction, that $t_0 = 0$.

Then, we may suppose that $t_n \leq 1$ and it follows from (g_7), (3.11) and (2.6) that

$$\begin{aligned}\frac{t_n^2}{2}A_n &\leq t_n^2 \int |\nabla \Psi_n|^2 + \frac{1}{2}\int V(\varepsilon_n x)[G^{-1}(t_n\Psi_n)]^2 \\ &\leq \int |\nabla (t_n\Psi_n)|^2 + \int V(\varepsilon_n x)\frac{G^{-1}(t_n\Psi_n)}{g(G^{-1}(t_n\Psi_n))}(t_n\Psi_n) \\ &= \int \frac{h(G^{-1}(t_n\Psi_n))}{g(G^{-1}(t_n\Psi_n))}(t_n\Psi_n) \\ &\leq \varepsilon t_n^2 \int |\Psi_n|^2 + C_1 t_n^p \int |\Psi_n|^p.\end{aligned}$$

Dividing the above expression by t_n^2, taking the limit as $n \to +\infty$ and using the Lebesgue Theorem we obtain $0 < Q_{V_0}(\omega) \leq 2\varepsilon \int \omega^2$, for any $\varepsilon > 0$, which does not make sense. Hence, $t_0 > 0$ and the proof is concluded. □

5.2. **The map β_ε.** Given $\delta > 0$, we consider the set

$$M_\delta := \{x \in \mathbb{R}^N : \mathrm{dist}(x, M) \leq \delta\}.$$

We may assume that δ in chosen in such way that M_δ and M are homotopically equivalent. Moreover, we can pick $\rho > 0$ such that $M_\delta \subset B_\rho(0)$. Let $\chi : \mathbb{R}^N \to \mathbb{R}^N$ be defined as $\chi(x) := x$ if $|x| < \rho$, $\chi(x) := \rho x/|x|$ if $|x| \geq \rho$. Finally, consider the baricenter map $\beta_\varepsilon : \mathcal{N}_\varepsilon \to \mathbb{R}^N$ given by

$$\beta_\varepsilon(v) := \frac{\int \chi(\varepsilon x) v^2(x)dx}{\int v^2(x)dx}.$$

Since $M \subset B_\rho(0)$, we can use the definition of χ and the Lebesgue Theorem to conclude that

$$\lim_{\varepsilon \to 0^+} \beta_\varepsilon(\Phi_\varepsilon(y)) = y, \quad \text{uniformly for } y \in M. \tag{5.3}$$

The following compactness result is the key stone to prove that the range of the map β_ε is near to the set M.

Proposition 5.5. *Let $(\varepsilon_n) \subset (0, +\infty)$ and $(v_n) \subset \mathcal{N}_{\varepsilon_n}$ be such that $\varepsilon_n \to 0$ and $J_{\varepsilon_n}(v_n) \to m_{V_0}$. Then there exists a sequence $(\widetilde{y}_n) \subset \mathbb{R}^N$ such that $\widetilde{v}_n := v_n(\cdot + \widetilde{y}_n)$ has a convergent subsequence in $H^1(\mathbb{R}^N)$. Furthermore, up to a subsequence, $y_n := \varepsilon_n \widetilde{y}_n \to y \in M$.*

Proof. We first prove that $(\|v_n\|_{\varepsilon_n}) \subset \mathbb{R}$ is bounded. Indeed, suppose by contradiction that this is not the case. Let $w_n := v_n/\|v_n\|_{\varepsilon_n}$ and notice that, arguing as in the proof of item (8) of Proposition 2.2, we have that

$$\int \left(|\nabla w_n|^2 + V(\varepsilon_n x)[G^{-1}(w_n)]^2\right) \leq 16 \max\{\|w_n\|_{\varepsilon_n}, \|w_n\|_{\varepsilon_n}^2\} = 16$$

and therefore

$$\begin{aligned}\int |w_n|^2 &\leq \frac{1}{(V_0 G^{-1}(1))^2} \int\limits_{\{|w_n| \leq 1\}} V(\varepsilon_n x)[G^{-1}(w_n)]^2 dx + \int\limits_{\{|w_n| \geq 1\}} |w_n|^{2^*} dx \\ &\leq \frac{16}{(V_0 G^{-1}(1))^2} + C_1 \left(\int |\nabla w_n|^2\right)^{2^*/2} \leq C_2,\end{aligned}$$

for some C_1, $C_2 > 0$. Hence, the sequence (w_n) is bounded in $H^1(\mathbb{R}^N)$ and we may assume that $w_n \rightharpoonup w$ weakly in $H^1(\mathbb{R}^N)$ and $w_n(x) \to w(x)$ for a.e. $x \in \mathbb{R}^N$. Suppose that there exists $(z_n) \subset \mathbb{R}^N$ and r, $\eta > 0$ such that

$$\liminf_{n\to+\infty} \int_{B_r(z_n)} |w_n|^2 \geq \eta > 0 \tag{5.4}$$

Then we may assume that $\widehat{w}_n(x) := w_n(x + z_n)$ is such that $\widehat{w}_n \rightharpoonup \widehat{w}$ weakly in $H^1(\mathbb{R}^N)$, with $\widehat{w} \neq 0$. But

$$m_{V_0} + o_n(1) = J_{\varepsilon_n}(v_n) - \frac{1}{2} J'_{\varepsilon_n}(v_n)v_n \geq \frac{1}{2} \int \mathcal{L}(G^{-1}(\widehat{w}_n(x)\|v_n\|_{\varepsilon_n})).$$

Since the set $\{x \in \mathbb{R}^N : \widehat{w}(x) \neq 0\}$ has positive Lebesgue measure, we can use (g_3), Lemma 2.3 and Fatou's Lemma to get

$$m_{V_0} \geq \liminf_{n\to+\infty} \frac{1}{2} \int \mathcal{L}(G^{-1}(\widehat{w}_n(x)\|v_n\|_{\varepsilon_n})) = +\infty,$$

which is absurd. Hence, (5.4) does not occur and therefore we have that $w_n \to 0$ in $L^p(\mathbb{R}^N), 2 < p < 2^*$. Thus, for any $K > 1$, we have that $\int H(G^{-1}(Kw_n)) \to 0$ and we can use (N_5), (g_6) and the same argument of the proof of item (8) of Proposition 2.2 to get

$$\begin{aligned}
m_{V_0} + o_n(1) &= J_{\varepsilon_n}(v_n) \geq J_{\varepsilon_n}\left(\frac{K}{\|v_n\|_{\varepsilon_n}} v_n\right) = J_{\varepsilon_n}(Kw_n) \\
&= \frac{K^2}{2} \int |\nabla w_n|^2 + \frac{1}{2} \int V(\varepsilon_n x)[G^{-1}(Kw_n)]^2 + o_n(1) \\
&\geq \frac{K}{2} \left(\int |\nabla w_n|^2 + V(\varepsilon_n x)[G^{-1}(w_n)]^2 \right) + o_n(1) \\
&\geq \frac{K}{2} \left(\frac{1}{16} \min\{\|w_n\|_{\varepsilon_n}, \|w_n\|^2_{\varepsilon_n}\} \right) + o_n(1) = \frac{K}{32} + o_n(1).
\end{aligned}$$

Since $K > 1$ is arbitrary, we obtain a contradiction taking $n \to +\infty$.

Since $(\|v_n\|_{\varepsilon_n}) \subset \mathbb{R}$ is bounded, we obtain $C_3 > 0$ verifying

$$\int \left(|\nabla v_n|^2 + V(\varepsilon_n x)[G^{-1}(v_n)]^2 \right) \leq 16 \max\{\|v_n\|_{\varepsilon_n}, \|v_n\|^2_{\varepsilon_n}\} \leq C_3.$$

As in the beginning of the proof, this implies that (v_n) is bounded in $H^1(\mathbb{R}^N)$. Again, we cannot have $v_n \to 0$ in $L^p(\mathbb{R}^N)$, and therefore there exist $(\widetilde{y}_n) \subset \mathbb{R}^N$ and constants r, $\eta > 0$ verifying

$$\liminf_{n\to\infty} \int_{B_r(\widetilde{y}_n)} |v_n|^2 \geq \eta.$$

Setting $\widetilde{v}_n := v_n(\cdot + \widetilde{y}_n)$ we may assume that $\widetilde{v}_n \rightharpoonup \widetilde{v} \neq 0$ weakly in $H^1(\mathbb{R}^N)$. Let $t_n > 0$ be such that $w_n = t_n \widetilde{v}_n \in \mathcal{M}_{V_0}$. After a change of variable, it is easy to see that

$$m_{V_0} \leq I_{V_0}(w_n) = I_{V_0}(t_n v_n) \leq J_{\varepsilon_n}(t_n v_n) \leq J_{\varepsilon_n}(v_n),$$

and therefore

$$\lim_{n\to\infty} I_{V_0}(w_n) = m_{V_0}.$$

Since the sequence $(\|\widetilde{v}_n\|_{H^1(\mathbb{R}^N)})$ is far away from zero, the above limit and Proposition 4.3 imply that $t_n \to t_0 \geq 0$. Recalling that the manifold $\mathcal{M}_{V_0}$ verifies (N_2), we conclude that $t_0 > 0$. Hence, $w_n \rightharpoonup w := t_0 \widetilde{v} \neq 0$ weakly in $H^1(\mathbb{R}^N)$.

According to Proposition 4.4 the sequence (w_n) strongly converges in $H^1(\mathbb{R}^N)$ and therefore $\widetilde{v}_n \to \widetilde{v}$ in $H^1(\mathbb{R}^N)$.

It remains to prove that $(y_n) := (\varepsilon_n \widetilde{y}_n) \to y$ is such that $V(y) = V_0$. We first prove that $(y_n) \subset \mathbb{R}^N$ is bounded. Suppose by contradiction that, along a subsequence, $|y_n| \to +\infty$ and first assume that $V(x) \to +\infty$, as $|x| \to +\infty$. Since $(v_n) \subset \mathcal{N}_{\varepsilon_n}$ is bounded, we can use (3.11) and (2.6) to obtain $C_4 > 0$ such that

$$\begin{aligned} \int V(\varepsilon_n x + y_n)[G^{-1}(\widetilde{v}_n)]^2 &\leq 2\int |\nabla v_n|^2 + V(\varepsilon_n z)\frac{G^{-1}(v_n)v_n}{g(G^{-1}(v_n))} \\ &= 2\int \frac{h(G^{-1}(v_n))}{g(G^{-1}(v_n))}v_n \leq C_4, \end{aligned}$$

where $z = x + \widetilde{y}_n$. On the other hand, since $V(\varepsilon_n x + y_n) \to +\infty$ for all $x \in \mathbb{R}^N$ and $\widetilde{v}_n \to \widetilde{v} \neq 0$, we can use Fatou's lemma to obtain

$$\liminf_{n\to+\infty} \int V(\varepsilon_n x + y_n)[G^{-1}(\widetilde{v}_n)]^2 = +\infty,$$

which is a contradiction. In the case $V_\infty < +\infty$, we can use (V_3) and the strong convergence of (w_n), to get

$$\begin{aligned} m_{V_0} &< \frac{1}{2}\int |\nabla w|^2 + \frac{1}{2}\int V_\infty [G^{-1}(w)]^2 - \int H(G^{-1}(w)) \\ &\leq \liminf_{n\to+\infty}\left(\frac{1}{2}\int \left(|\nabla w_n|^2 + V(\varepsilon_n x + y_n)[G^{-1}(w_n)]^2\right) - \int H(G^{-1}(w_n))\right) \\ &= \liminf_{n\to+\infty}\left(\frac{1}{2}\int \left(|\nabla (t_n v_n)|^2 + V(\varepsilon_n z)[G^{-1}(t_n v_n)]^2\right) - \int H(G^{-1}(t_n v_n))\right). \end{aligned}$$

It follows that

$$m_{V_0} < \liminf_{n\to+\infty} J_{\varepsilon_n}(t_n v_n) = \liminf_{n\to+\infty} J_{\varepsilon_n}(v_n) = m_{V_0},$$

which does not make sense.

Since (y_n) is bounded, we may assume that $y_n \to y$. If $y \notin M$, then $V(y) > V_0$ and we can use $\varepsilon_n \to 0$ and the same argument above to get a contradiction. The proposition is proved. □

Following [5], we introduce the set

$$\Sigma_\varepsilon := \{v \in \mathcal{N}_\varepsilon : J_\varepsilon(u) \leq m_{V_0} + f(\varepsilon)\},$$

where $f : \mathbb{R}^+ \to \mathbb{R}^+$ is such that $f(\varepsilon) \to 0$ as $\varepsilon \to 0^+$. Given $y \in M$, we can use Lemma 5.4 to conclude that $f(\varepsilon) := |J_\varepsilon(\Phi_\varepsilon(y)) - m_{V_0}|$ is such that $f(\varepsilon) \to 0$ as $\varepsilon \to 0^+$. Thus, $\Phi_\varepsilon(y) \in \Sigma_\varepsilon$ and therefore $\Sigma_\varepsilon \neq \varnothing$ for any $\varepsilon > 0$ small.

Lemma 5.6. *For any $\delta > 0$ we have that*

$$\lim_{\varepsilon\to 0^+} \sup_{v\in\Sigma_\varepsilon} \operatorname{dist}(\beta_\varepsilon(v), M_\delta) = 0.$$

Proof. Let $(\varepsilon_n) \subset \mathbb{R}^+$ be such that $\varepsilon_n \to 0$. By definition there exists $(v_n) \subset \Sigma_{\varepsilon_n}$ such that

$$\operatorname{dist}(\beta_{\varepsilon_n}(v_n), M_\delta) = \sup_{u\in\Sigma_{\varepsilon_n}} \operatorname{dist}(\beta_{\varepsilon_n}(v), M_\delta) + o_n(1).$$

Thus, it sufficient to find a sequence $(y_n) \subset M_\delta$ such that

$$|\beta_{\varepsilon_n}(v_n) - y_n| = o_n(1). \tag{5.5}$$

In order to obtain such sequence, we notice that $(u_n) \subset \Sigma_{\varepsilon_n} \subset \mathcal{N}_{\varepsilon_n}$, and we can use (N_5) to get

$$m_{V_0} \leq \max_{t\geq 0} I_{V_0}(tv_n) \leq \max_{t\geq 0} J_{\varepsilon_n}(tv_n) = J_{\varepsilon_n}(v_n) \leq m_{V_0} + f(\varepsilon_n),$$

from which follows that $J_{\varepsilon_n}(v_n) \to m_{V_0}$. Thus, we may invoke Proposition 5.5 to obtain a sequence $(\widetilde{y}_n) \subset \mathbb{R}^N$ such that $(y_n) := (\varepsilon_n \widetilde{y}_n) \subset M_\delta$ for n sufficiently large. Hence,

$$\beta_{\varepsilon_n}(v_n) = y_n + \frac{\int_{\mathbb{R}^N} [\chi(\varepsilon_n z + y_n) - y_n] \widetilde{v}_n(z)^2 dz}{\int_{\mathbb{R}^N} \widetilde{v}_n(z)^2 dz}$$

and it follows from the strong convergence of $\widetilde{v}_n$ that (5.5) holds. □

We are now ready to present the proof of Theorem 5.1

Proof of Theorem 5.1. Let $\delta > 0$ be such that M_δ and M are homotopically equivalent. We can use (5.3), Lemmas 5.4 and 5.6, and argue as in [5, Section 6] to obtain $\varepsilon^* > 0$ such that, for any $\varepsilon \in (0, \varepsilon_\delta)$, the diagram

$$M \xrightarrow{\Phi_\varepsilon} \Sigma_\varepsilon \xrightarrow{\beta_\varepsilon} M_\delta$$

is well defined and $\beta_\varepsilon \circ \Phi_\varepsilon$ is homotopical equivalent to the embedding $\iota : M \to M_\delta$. In view of item (3) of Proposition 2.2 and the proof of (N_4), we can use a standard argument to check that J_ε restricted to $\mathcal{N}_\varepsilon$ verifies the Palais-Smale condition. Thus, we can use Theorem 5.2 and Lemma 5.3 to obtain at least $\mathrm{cat}_{M_\delta}(M)$ critical points of J_ε restricted to $\mathcal{N}_\varepsilon$. By (N_4), each of these critical points is a non-zero solution of $(\widetilde{P}_\varepsilon)$. The theorem is proved. □

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