TWO SOLUTIONS FOR A PLANAR EQUATION WITH COMBINED NONLINEARITIES AND CRITICAL GROWTH

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Abstract. We prove the existence of two nonnegative nontrivial solutions for the equation

$$-\Delta u - \frac{1}{2}(x \cdot \nabla u) = \lambda a(x)|u|^{q-2}u + f(u), \quad x \in \mathbb{R}^2,$$

where $1 < q < 2$, a is indefinite in sign and the function $f(s)$ behaves like $e^{\alpha s^2}$ at infinity. The results holds for small values of the parameter $\lambda > 0$.

1. Introduction

In this paper, we address the existence of nonegative solutions for the equation

$$(P_\lambda) \qquad -\Delta u + \frac{1}{2}(x \cdot \nabla u) = \lambda a(x)|u|^{q-2}u + f(u), \quad x \in \mathbb{R}^2,$$

where $1 < q < 2$, a is a radial function which can change sign and the function $f \in C(\mathbb{R}, \mathbb{R})$ has critical growth, that is,

(f_0) there exists $\alpha_0 > 0$ such that

$$\lim_{s \to +\infty} \frac{f(s)}{e^{\alpha s^2}} = \begin{cases} 0, & \text{if } \alpha > \alpha_0, \\ +\infty, & \text{if } \alpha < \alpha_0. \end{cases}$$

As it is well known, in dimension two the concept of criticality is related with the so callled Trudinger-Moser inequality which appears in the pioneer works [18, 24]. After then, there is a vast literature concerning this kind of critical nonlinearities (see [1, 6, 11, 19, 22, 14] and references therein).

Before presenting our asumptions let us recall that, as quoted by Escobedo and Kavian in [10], the operator in (P_λ) naturally appears when we consider the existence of self-similar solutions for homogeneous heat equations. Actually, when one seek for solutions of the form $\omega(t,x) = t^{-1/(p-2)}u(t^{-1/2}x)$ for the evolution equation

$$\omega_t - \Delta\omega = |\omega|^{p-2}\omega, \quad t > 0,\, x \in \mathbb{R}^N,$$

we are lead to consider the elliptic equation

$$-\Delta u - \frac{1}{2}(x \cdot \nabla u) = \lambda u + |u|^{p-2}u, \qquad x \in \mathbb{R}^N.$$

In [10] the authors noticed that, if $K(x) := \exp(|x|^2/4)$, then

$$\operatorname{div}(K(x)\nabla u) = K(x)\left[\Delta u + \frac{1}{2}(x \cdot \nabla u)\right],$$

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and it is therefore natural to seek solutions of (P_λ) in the closure of the infinitely differentiable radial functions with compact support $C^\infty_{c,rad}(\mathbb{R}^2)$ with respect to the norm

$$\|u\| := \left(\int_{\mathbb{R}^2} K(x)|\nabla u|^2 \mathrm{d}x \right)^{1/2}.$$

As we shall see in Section 2, the space X defined above has nice properties. In particular, some versions of the usual Trudinger-Moser inequalities hold in X as well as continuous emdebbeding in the weighted Lebesgue spaces $L^p_K(\mathbb{R}^N)$ defined as the set of measurable and radial functions $u : \mathbb{R}^2 \to \mathbb{R}$ such that the integral $\int_{\mathbb{R}^2} K(x)|u|^p \,\mathrm{d}x$ is finite. Thus, for any $p \geq 2$, it is well defined

$$S_p := \inf \left\{ \int_{\mathbb{R}^2} K(x)|\nabla u|^2 \,\mathrm{d}x \,:\, u \in X,\ \int_{\mathbb{R}^2} K(x)|u|^p \,\mathrm{d}x = 1 \right\}. \tag{1.1}$$

We denote by $s' := s/(s-1)$ the conjugated exponent of $s > 1$. The basic assumptions on the potential a are the following:

(a_0) $a(x) = a(|x|)$ for a.e. $x \in \mathbb{R}^2$;
(a_1) $a \in L^{\sigma_q}_K(\mathbb{R}^N)$ for some $2 \leq \sigma_q \leq (2/q)'$;
(a_2) the set $\Omega^+_a := \{x \in \mathbb{R}^N : a(x) > 0\}$ has an interior point.

Concerning the nonlinearity f, besides the critical growth condition (f_0), we also assume the folllowing:

(f_1) $\lim_{s \to 0^+} f(s)/s = 0$;
(f_2) there exists $\theta_0 > 2$ such that

$$0 \leq \theta_0 F(s) := \theta_0 \int_0^s f(t)\,\mathrm{d}t \leq sf(s), \quad \forall\, s \geq 0.$$

(f_3) for each $\theta > 2$, there exists $s_\theta > 0$ such that

$$0 \leq \theta F(s) \leq sf(s), \quad \forall\, s \geq s_\theta.$$

(f_4) there exists $p_0 > 2$ such that

$$f(s) \geq C_{p_0} s^{p_0 - 1}, \quad \forall\, s \geq 0,$$

where

$$C_{p_0} > \left[\frac{(p_0-2)}{2p_0} \frac{\alpha_0}{2\pi} \right]^{(p_0-2)/2} S^{p_0/2}_{p_0}$$

and S_{p_0} is defined in (1.1).

In the main result of this paper we prove the following multiplicity result:

Theorem 1.1. *Suppose that $1 < q < 2$, a and f satisfy $(a_0)-(a_2)$ and $(f_0)-(f_4)$, respectively. Then there exists $\lambda_* > 0$ such that, for any $\lambda \in (0, \lambda_*)$, Problem (P_λ) has at least two nonzero nonnegative solutions.*

In the proofs, we apply variational methods. The first solution is obtained by a minimization argument and the second one as an application of the Mountain Pass Theorem. We are going to use the variational framework introduced in [13] to deal with the critical range of the function f. The hypothesis (f_3) is important to get some convergence results and it has already appeared in [20, 25]. Moreover, this condition is a consequence of

$(\widehat{f_3})$ there exist constants $R_0, M_0 > 0$ such that

$$0 < F(s) \leq M_0 f(s), \quad \forall\, s \geq R_0,$$

which has been used for instance in the papers [11, 12]. Condition (f_4) is a version of another one introduced in [6] and it is used to correctly localize the minimax level of the energy functional associated to (P_λ).

The main motivation for our result comes from the concave-convex equation

$$-\Delta u = \lambda a(x)|u|^{q-2}u + b(x)|u|^{p-2}u, \quad u \in H_0^1(\Omega),$$

with $1 < q < 2$, $\Omega \subset \mathbb{R}^N$ open and bounded, $N \geq 3$ and $2 < p \leq 2N/(N-2)$. In a celebrated work Ambrosetti, Brezis and Cerami [3] supposed that $a(x) \equiv b(x) \equiv 1$ and prove that the problem has at least two positive solutions provided $\lambda \in (0, \Lambda)$. After this work, many results with combined nonlinearities have appeared. Since it impossible to give a complet list of reference we cite [5, 7, 8, 20, 9, 21, 16] and the references therein. There are also some results for the unbounded case $\Omega = \mathbb{R}^N$. In this setting, we need to require some integrability conditions on a and b in order to deal with the problem variationally. We can cite, among other results, the papers [2, 23, 4, 17]. We also cite the recent paper [15] where the authors considered the version of (P_λ) for higher dimensions $N \geq 3$. The main result of this paper complement the aforementioned works since we deal with the operator $u \mapsto \Delta u + (1/2)(x \cdot \nabla u)$ and consider the 2-dimensional case.

The paper contains two more sections: in the next one we present the variational setting to deal with (P_λ) and obtain the first solution. In Section 3, we prove that Problem (P_λ) has a second solution.

2. Variational setting and the first solution

Throughout the paper we write $\int u$ instead of $\int_{\mathbb{R}^2} u(x)\mathrm{d}x$. Since we are looking for nonnegative solutions we may assume that $f(s) = 0$, for any $s \leq 0$. By (f_1), this assumption does not affect the contintuity of f.

In order to present the functional space to deal with our problem we consider $C^\infty_{c,rad}(\mathbb{R}^2)$ the space of infinitely differentiable radial functions with compact support and denote by X the closure of $C^\infty_{c,rad}(\mathbb{R}^2)$ with respect to the norm

$$\|u\| := \left(\int K(x)|\nabla u|^2 \right)^{1/2},$$

where

$$K(x) := e^{|x|^2/4}, \quad \forall\, x \in \mathbb{R}^2.$$

For each $p \geq 2$, we also consider the weighted Lebesgue space $L^p_K(\mathbb{R}^2)$ of all the radial measurable functions $u : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\|u\|_p := \left(\int K(x)|u|^p \right)^{1/p} < \infty.$$

As proved in [13, Lemma 2.1], the space X is compactally embedded into the Lebesgue spaces $L^p_K(\mathbb{R}^2)$ for any $p \in [2, \infty)$. Moreover, the following version of the Trudinger-Moser inequality holds:

Theorem 2.1. *For any $p \geq 2$, $u \in X$ and $\alpha > 0$ we have that the function $K(x)|u|^p(e^{\alpha u^2} - 1) \in L^1(\mathbb{R}^2)$. Moreover, if $\|u\| \leq M$ and $\alpha M^2 < 4\pi$, then there exists $C = C(M, \alpha, p) > 0$ such that*

$$\int K(x)|u|^p(e^{\alpha u^2} - 1) \leq C(M, \alpha, p)\|u\|^p.$$

Proof. See [13, Theorem 1.1 and Corollary 1.2]. □

Actually, in paper [13] the authors established the so called Trudinger-Moser inequalities for the space X. The above result is a counterpart of the following well known result (see [6, 19]): for any $u \in W^{1,2}(\mathbb{R}^2)$ and $\alpha > 0$ it holds $(e^{\alpha u^2} - 1) \in L^1(\mathbb{R}^2)$. Moreover, if $\||\nabla u|\|_{L^2(\mathbb{R}^2)} \le 1$, $\|u\|_{L^2(\mathbb{R}^2)} \le M < \infty$ and $\alpha < 4\pi$, then there exists $C = C(M, \alpha)$ such that

$$\int (e^{\alpha u^2} - 1) \le C(M, \alpha). \tag{2.1}$$

Moreover, we also have the following improvement of the Trudinger-Moser inequality:

Theorem 2.2. *Let $(v_n) \subset X$ be such that $\|v_n\| = 1$ and $v_n \rightharpoonup v$ weakly in X, with $\|v\| < 1$. Then, for each $0 < p < 4\pi/(1 - \|v\|^2)$, up to a subsequence it holds*

$$\sup_{n \in \mathbb{N}} \int K(x) v_n^2 (e^{p v_n^2} - 1) < \infty.$$

Proof. See [13, Theorem 1.3]. □

Finnaly, we quote an auxiliar result which will be useful (see [13, equation (2.4)]): for any $p \ge 1$, there exists $C_p > 0$ such that

$$\left(\int K(x)^p |u|^{2p} \right) \le C_p \|u\|^2, \quad \forall\, u \in X. \tag{2.2}$$

Moreover, the space X is continuosly embedded into $W^{1,2}(\mathbb{R}^2)$.

In the sequel we show how we can use the Trundinger-Moser inequality to define the energy functional associated to the problem (P_λ). Let $\alpha > \alpha_0$ be given by (f_1) and $p \ge 1$. By using the critical growth condition (f_0) we obtain

$$\lim_{|s| \to +\infty} \frac{f(s)}{|s|^{p-1}(e^{\alpha s^2} - 1)} = 0.$$

This and (f_1) imply that, for any given $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\max\{|f(s)s|, |F(s)|\} \le \varepsilon s^2 + C_\varepsilon |s|^p (e^{\alpha s^2} - 1), \quad \forall\, s \in \mathbb{R}. \tag{2.3}$$

This inequality with $p = 2$ and Theorem 2.1 imply that the functional $u \mapsto \int K(x) F(u)$ belongs to $C^1(X, \mathbb{R})$.

Given $u \in X$, we set $u^+(x) := \max\{u(x), 0\}$. By Hölder's inequality and (a_1), we get

$$\left| \int K(x) a(x) (u^+)^q \right| \le \|a\|_{\sigma_q} \left(\int K(x) |u|^{q\sigma_q'} \right)^{1/\sigma_q'}.$$

Since $q\sigma_q' \ge 2$, the right-hand side above is finite. Thus, by using some standard calculations we can show that the functional $I_\lambda : X \to \mathbb{R}$ given by

$$I_\lambda(u) := \frac{1}{2} \int K(x) |\nabla u|^2 - \frac{\lambda}{q} \int K(x) a(x) (u^+)^q - \int K(x) F(u)$$

is well defined, it belongs to $C^1(X, \mathbb{R})$ and its critical points are exactly the weak solutions of the equation (P_λ). If $I_\lambda'(u) = 0$ and $u^-(x) := \max\{-u(x), 0\}$, then $0 = I_\lambda(u) u^- = -\|u^-\|^2$, and therefore we conclude that $u \ge 0$ a.e. in $\mathbb{R}^2$.

Since $1 < q < 2$, we can find the first solution for our problem by using a minization argument in a small ball centered at the origin. More specifically, we have the following:

Lemma 2.3. *Suppose that f satisfies $(f_0)-(f_1)$. Then there exists $\lambda^*, \rho > 0$ such that, for any $\lambda \in (0, \lambda^*)$, there hold*

$$I_\lambda(u) \geq \rho^2/8, \text{ if } \|u\| = \rho, \qquad I_\lambda(u) \geq -\rho^2/8, \text{ if } \|u\| \leq \rho. \tag{2.4}$$

Proof. Since the map $J(u) := \int K(x)a(x)(u^+)^q$ is continuos at $u = 0$, for any given $\varepsilon > 0$, there exists $\rho_1 > 0$ such that $|J(u)| \leq q\varepsilon$, whenever $\|u\| \leq \rho_1$. Thus, we can pick $\alpha > \alpha_0$ and use (2.3) to obtain

$$I_\lambda(u) \geq \frac{1}{2}\|u\|^2 - \lambda\varepsilon - \varepsilon \int K(x)u^2 - C_\varepsilon \int K(x)|u|^p(e^{\alpha u^2} - 1), \quad \forall \|u\| \leq \rho_1.$$

By taking ρ_1 small if necessary, we may assume that $\alpha\rho_1^2 < 4\pi$, and therefore it follows from Theorem 2.1 with $p > 2$ and the Sobolev embedding $X \hookrightarrow L^2_K(\mathbb{R}^2)$ that

$$I_\lambda(u) \geq \frac{1}{2}\left(1 - \varepsilon C_1 - C(\rho_1, \alpha, p)\|u\|^{p-2}\right)\|u\|^2 - \lambda\varepsilon, \quad \forall \|u\| \leq \rho_1.$$

Since $p > 2$, we can take $\varepsilon > 0$ small and obtain $0 < \rho < \rho_1$ such that

$$I_\lambda(u) \geq \frac{1}{4}\|u\|^2 - \lambda\varepsilon, \quad \forall \|u\| \leq \rho.$$

A straightforward computation shows that the lemma holds for $\lambda^* := \rho^2/(8\varepsilon)$. □

We are able to obtain our first solution.

Proposition 2.4. *Suppose that f satisfies $(f_0)-(f_1)$ and let $\lambda^* > 0$ be given by the previous lemma. Then, for any $\lambda \in (0, \lambda^*)$, the infimun*

$$b_\lambda := \inf_{u \in \overline{B_\rho(0)}} I_\lambda(u) < 0,$$

is achievied by a nonzero solution u_λ of (P_λ).

Proof. It follows from (2.4) that b_λ is well defined. In order to verify that $b_\lambda < 0$ we consider the set Ω_a^+ given by (a_2) and $\phi \in C_c^\infty(\Omega_a^+)$ such that $\int K(x)a(x)\phi^q > 0$. Given $\varepsilon > 0$, by (f_1), there exists $\delta > 0$ such that $|F(s)| \leq \varepsilon s^2$, for any $|s| \leq \delta$. Thus,

$$I_\lambda(t\phi) \leq \frac{t^2}{2}\|\phi\|^2 - \lambda\frac{t^q}{q}\int K(x)a(x)\phi^q - \varepsilon^2 t^2 \int K(x)\phi^2,$$

whenever $0 < t\|\phi\|_{L^\infty(\mathbb{R}^2)} \leq \delta$. Then $I_\lambda(t\phi) < 0$ if $t > 0$ is small and we conclude that $b_\lambda < 0$. It follows from (2.4) and the Ekeland Variational Principle that, for each $\lambda \in (0, \lambda^*)$ fixed, there exists a sequence $(u_n) \subset B_\rho(0)$ such that

$$I_\lambda(u_n) \to b_\lambda < 0, \qquad I'_\lambda(u_n) \to 0.$$

We claim that, along a subsequence, $u_n \to u_\lambda$ strongly in X. If this is true, it follows that $I_\lambda(u_\lambda) = b_\lambda < 0$ and therefore $u_\lambda \neq 0$ is a nonegative critical point of I_λ.

It remains to prove the claim. Since $(u_n) \subset X$ is bounded we may suppose that $u_n \rightharpoonup u_\lambda$ weakly in X. We set $w_n := u_n - u_\lambda$ and notice that, since $w_n \rightharpoonup 0$ weakly in X, we have that

$$I'_\lambda(u_n)w_n = \|u_n\|^2 - \|u_\lambda\|^2 - \lambda \int K(x)a(x)(u_n^+)^{q-1}w_n - \int K(x)f(u_n)w_n. \tag{2.5}$$

We claim that

$$\lim_{n\to+\infty}\int K(x)a(x)(u_n^+)^{q-1}w_n=0,\quad \lim_{n\to\infty}\int K(x)f(u_n)w_n=0. \tag{2.6}$$

If this is true, it follows from (2.5) that $\|u_n\|\to\|u_\lambda\|$ and therefore the weak convergence of (u_n) implies that $u_n\to u_\lambda$ strongly in X.

In order to verify (2.6) we recall that $\sigma_q\le 2/(2-q)$ to obtain $p\ge 2$ such that

$$\frac{1}{\sigma_q}+\frac{1}{2/(q-1)}+\frac{1}{p}=1.$$

This and Hölder's inequality provide

$$\left|\int K(x)a(x)(u_n^+)^{q-1}w_n\right|\le\|a\|_{\sigma_q}\|u_n\|_2^{q-1}\|w_n\|_p^p.$$

The first statament in (2.6) follows from this enquality and the compactness of the embedding $X\hookrightarrow L_K^p(\mathbb{R}^2)$. The proof of the second one is more envolved. We first apply (2.3) with $p=3$ and Hölder's inequality to get

$$\begin{aligned}\left|\int K(x)f(u_n)w_n\right|&\le\varepsilon\int K(x)|u_n||w_n|+C_\varepsilon\int K(x)|u_n|^2|w_n|(e^{\alpha u_n^2}-1)\\&\le\varepsilon\|u_n\|_2\|w_n\|_2+C_\varepsilon D_n,\end{aligned}$$

where

$$D_n:=\int K(x)|u_n|^2|w_n|(e^{\alpha u_n^2}-1).$$

Since $w_n\to 0$ strongly in $L_K^2(\mathbb{R}^2)$ it is enough to verify that $D_n\to 0$. By picking $r_i>1$, $i=1,2,3$, such that $1/r_1+1/r_2+1/r_3=1$ and $r_2>2$, we can use Hölder inequality again to get

$$\begin{aligned}D_n&\le\left(\int K(x)^{r_1}|u_n|^{2r_1}\right)^{1/r_1}\|w_n\|_{L^{r_2}(\mathbb{R}^2)}\left(\int(e^{\alpha r_3\|u_n\|^2(u_n/\|u_n\|)^2}-1)\right)^{1/r_3}\\&\le C_{r_1}\|u_n\|^2\|w_n\|_{r_2}\left(\int(e^{\alpha r_3\|u_n\|^2(u_n/\|u_n\|)^2}-1)\right)^{1/r_3},\end{aligned}$$

where we have used (2.2), $K(x)\ge 1$ and the inequality

$$(e^s-1)^r\le(e^{sr}-1),\quad\forall\, s\ge 0,\ r>1. \tag{2.7}$$

Since $\alpha\|u_n\|^2\le\alpha\rho^2<4\pi$, we can choose r_3 close to 1 in such way that $\alpha r_3\|u_n\|^2\le\gamma<4\pi$, and therefore it follows from (2.1) that

$$\sup_{n\in\mathbb{N}}\int(e^{\alpha r_3\|u_n\|^2(u_n/\|u_n\|)^2}-1)\le C_1.$$

Thus, since $w_n\to 0$ in $L_K^{r_2}(\mathbb{R}^2)$, we conclude that $D_n\to 0$. □

3. The second solution

We devote this section to the proof that (P_λ) has a second solution of Mountain Pass type. We recall that a sequence $(u_n)\subset X$ is called a $(PS)_c$ sequence for I_λ if $I_\lambda(u_n)\to c\in\mathbb{R}$ and $I'_\lambda(u_n)\to 0$. We say that I_λ satisfies the Palais-Smale condition at level c ($(PS)_c$ for short) if any $(PS)_c$ sequence has a convergente subsequence.

Lemma 3.1. *Suppose that f satisfies $(f_0)-(f_3)$ and let $(u_n)\subset X$ be a (PS)$_c$ sequence for I. Then, up to a subsequece, $u_n \rightharpoonup u$ weakly in X, with $I'(u)=0$. Moreover,*

$$\int K(x)F(u_n)\to\int K(x)F(u),\qquad \limsup_{n\to+\infty}\int_{B_R(0)^c}K(x)f(u_n)u_n\,\mathrm{d}x=0,$$

for any $R>0$.

Proof. By using (f_2), Hölder's inequality and the embedding $X\hookrightarrow L_K^{q\sigma_q'}(\mathbb{R}^{\mathbb{N}})$, we obtain

$$\begin{aligned} c+o_n(1)\|u_n\|+o_n(1) &= I_\lambda(u_n)-\frac{1}{\theta}I_\lambda'(u_n)u_n\\ &\geq \left(\frac{1}{2}-\frac{1}{\theta}\right)\|u_n\|^2-C_1\left(\frac{1}{q}-\frac{1}{\theta}\right)\|a\|_{\sigma_q}\|u_n\|^q,\end{aligned}$$

where $o_n(1)$ stands for a quantity approchng zero as $n\to+\infty$. Since $1<q<2$ the above inequality implies that (u_n) is bounded and therefore, up to a subsequence, $u_n\rightharpoonup u$ weakly in X.

In order to verify that $I_\lambda'(u)=0$ we consider $\phi\in C_{c,rad}^\infty(\mathbb{R}^2)$. Arguing as in the proof of Proposition 2.4 we can prove that

$$\lim_{n\to+\infty}\int K(x)a(x)(u_n^+)^{q-1}\phi=\int K(x)a(x)(u^+)^{q-1}\phi.$$

Moreover, $I_\lambda'(u_n)u_n=o_n(1)$ and Hölder's inequality show that $(\int K(x)f(u_n)u_n)$ is bounded. Thus, since $K\geq 1$, we obtain

$$\int|f(u_n)u_n|=\int f(u_n)u_n\leq\int K(x)f(u_n)u_n\leq C_2$$

and it follows from [11, Lemma 2.1] that $f(u_n)\to f(u)$ in $L^1_{loc}(\mathbb{R}^2)$. Hence, since K is bounded in the support of ϕ, we have that

$$\lim_{n\to+\infty}\int K(x)f(u_n)\phi=\int K(x)f(u)\phi.$$

Altogether, these convergences show that

$$0=\lim_{n\to+\infty}I_\lambda'(u_n)\phi=I_\lambda'(u)\phi,\qquad\forall\phi\in C_{c,rad}^\infty(\mathbb{R}^2).$$

By density we conclude that $I_\lambda'(u)=0$.

The other two convergences stated in the lemma can be proved arguing along the same lines of [13, Lemma 4.5]. We omite the details. □

As a consequence of the above lemma, we have the following local compactness result:

Proposition 3.2. *Suppose that f satisfies $(f_0)-(f_3)$. For any $\lambda\in(0,\lambda^*)$, let $u_\lambda\in X$ be the solution given by Proposition 2.4. If $u=0$ and $u=u_\lambda$ are the only critical points of I_λ then this functional satisfies the $(PS)_c$ condition for any*

$$c<I_\lambda(u_\lambda)+\frac{2\pi}{\alpha_0}.$$

Proof. Let $(u_n) \subset X$ be such that $I'_\lambda(u_n) \to 0$ and $I_\lambda(u_n) \to c < I_\lambda(u_\lambda) + 2\pi/\alpha_0$. Acoording to Lemma 3.1, we may suppose that $u_n \rightharpoonup u$ weakly in X, with $I'_\lambda(u) = 0$. It follows from Young's inequality that, for a.e. $x \in \mathbb{R}^2$,

$$K(x)a(x)(u_n^+)^q \leq \frac{1}{\sigma_q} K(x)a(x)^{\sigma_q} + \frac{1}{\sigma'_q} K(x)|u_n|^{q\sigma'_q}$$

Recalling that the embedding $X \hookrightarrow L^{q\sigma'_q}(\mathbb{R}^2)$ is compact, we obtain an integrable function which dominates the left-hand side above. Since we also have pointwise convergence we can use Lebesgue's Theorem to get

$$\lim_{n\to+\infty} K(x)a(x)(u_n^+)^q = \int K(x)a(x)(u^+)^q. \tag{3.1}$$

So, we infer from Lemma 3.1 and $I_\lambda(u_n) \to c$ that

$$\lim_{n\to+\infty} \|u_n\|^2 = 2c + 2\left[\frac{\lambda}{q}\int K(x)a(x)(u^+)^q + \int K(x)F(u)\right].$$

Since $I'_\lambda(u) = 0$, we have that $u = 0$ or $u = u_\lambda$. If $u = 0$, it follows from the above equation and $I_\lambda(u_\lambda) < 0$ that

$$\lim_{n\to+\infty} \|u_n\|^2 = 2c < 2I_\lambda(u_\lambda) + \frac{4\pi}{\alpha_0} < \frac{4\pi}{\alpha_0}.$$

and therefore we can argue as in the proof of Proposition 2.4 to conclude that $u_n \to 0$ strongly in X. Actually, in the final part of the argument we need to choose $\alpha > \alpha_0$ and $r_3 > 1$ sufficienttly close to α_0 and 1, respectively, in order to guarantee that $\alpha r_3 \|u_n\|^2 \leq \gamma < 4\pi$.

It remains to consider the case $u = u_\lambda$. First notice that

$$o_n(1) = I'_\lambda(u_n)u_n = \|u_n\|^2 - \lambda \int K(x)a(x)(u_n^+)^q - \int K(x)f(u_n)u_n. \tag{3.2}$$

We claim that

$$\lim_{n\to+\infty} \int K(x)f(u_n)u_n = \int K(x)f(u)u.$$

If this is true, we can use (3.1)-(3.2) to obtain

$$o_n(1) = I'_\lambda(u_n)u_n = \|u_n\|^2 - \|u\|^2 + I'_\lambda(u)u + o_n(1).$$

Recalling that $I'_\lambda(u)u = 0$, we conclude that $\|u_n\| \to \|u\|$ and therefore $u_n \to u$ strongly in X.

In order to prove the claim we first notice that, by Lemma 3.1, it is sufficient to show that, for any $R > 0$, there holds

$$\lim_{n\to+\infty} \int_{B_R(0)} K(x)f(u_n)u_n \,\mathrm{d}x = \int_{B_R(0)} K(x)f(u)u \,\mathrm{d}x.$$

As in the first case, we have that

$$\lim_{n\to+\infty} \|u_n\|^2 = 2(c + c_0) > 0, \tag{3.3}$$

with

$$c_0 := \frac{\lambda}{q}\int K(x)a(x)(u^+)^q + \int K(x)F(u).$$

Hence, if we set $v_n := u_n/\|u_n\|$, we conclude that $v_n \rightharpoonup v := u_\lambda[2(c+c_0)]^{-1/2}$ weakly in X. If we pick $\alpha > \alpha_0$ in such way that $c < I_\lambda(u_\lambda) + (2\pi)/\alpha$ a straightforward computation provides

$$2\alpha(c + c_0) < \frac{4\pi}{1 - \|v\|^2}.$$

From (3.3), we obtain $\gamma > 0$ such that $\alpha\|u_n\|^2 < \gamma < (4\pi)/(1 - \|v\|^2)$. We now pick $1 < \beta < 2$ close to 1 in such way that

$$\alpha\beta\|u_n\|^2 < \gamma\beta < \frac{4\pi}{1 - \|v\|^2}.$$

By using Theorem 2.2 with $p = \gamma\beta$, we conclude that

$$\sup_{n\in\mathbb{N}} \int K(x)v_n^2(e^{\alpha\beta\|u_n\|^2 v_n^2} - 1) < \sup_{n\in\mathbb{N}} \int K(x)v_n^2(e^{\gamma\beta v_n^2} - 1) < \infty. \tag{3.4}$$

Up to a subsequence, we have that $u_n \to u$ strongly in $L^2(B_R(0))$, and therefore there exists $\psi \in L^2(B_R(0))$ such that $|u_n(x)|^2 \le \psi(x)^2$ a.e. in $B_R(0)$. By (2.3), we get

$$\int_A K(x)f(u_n)u_n \,\mathrm{d}x \le C_1 \int_A \psi(x)^2 \,\mathrm{d}x + C_2 \int_A K(x)|u_n|^{2/\beta}(e^{\alpha u_n^2} - 1)\mathrm{d}x, \tag{3.5}$$

for any measurable subset $A \subset B_R(0)$. Hölder's inequality, (2.7) and the definition of v_n provide

$$\begin{aligned}\int_A K(x)|u_n|^{2/\beta}&(e^{\alpha u_n^2} - 1)\,\mathrm{d}x \\ &\le \left(\int_A K(x)\,\mathrm{d}x\right)^{1/\beta'} \left(\int_A K(x)u_n^2(e^{\alpha\beta u_n^2} - 1)\,\mathrm{d}x\right)^{1/\beta} \\ &\le \|u_n\|^{2/\beta}\|K\|_{L^1(A)}^{1/\beta'} \left(\int K(x)v_n^2(e^{\alpha\beta\|u_n\|^2 v_n^2} - 1)\right)^{1/\beta}.\end{aligned}$$

This, (3.5), (3.4) and the boundedness of (u_n) imply that

$$\int_A K(x)f(u_n)u_n \,\mathrm{d}x \le C_1\|\psi\|_{L^2(A)} + C_3\|K\|_{L^1(A)}^{1/\beta'}$$

and therefore the first integral above is uniformly small provided the measure of A is small. Hence, the set $\{K(x)f(u_n)u_n\}$ is uniformly integrable and therefore a standard application of Egoroff's Theorem implies that $K(x)f(u_n)u_n \to K(x)f(u)u$ in $L^1(B_R(0))$. The proposition is proved. □

Before presenting the proof of our main theorem, we shall verify that, for any $p \ge 2$, the constant S_p defined in (1.1) is attained by a nonnegative function $\omega_p \in X$ such that $\|\omega_p\|_p = 1$. Indeed, let $(u_n) \subset X$ be such that $\|u_n\|_p = 1$ and $\|u_n\|^2 \to S_p$. Up to a subsequence, $u_n \rightharpoonup \omega_p$ weakly in X and therefore $\|\omega_p\|^2 \le \liminf_{n\to+\infty} \|u_n\|^2 = S_p$. Due to the compactness of the embedding $X \hookrightarrow L_K^p(\mathbb{R}^2)$, we have that $u_n \to \omega_p$ strongly in $L_K^p(\mathbb{R}^2)$, and therefore $\|\omega_p\|_p = 1$. Hence, $S_p \le \|\omega_p\|^2$ and we conlude that S_p is attained by ω_p. Since we may replace u_n by $|u_n|$ in the former argument, the strong convergence in $L_K^p(\mathbb{R}^2)$ show that we may assume $\omega_p \ge 0$.

Proof of Theorem 1.1 . Let $\lambda_* > 0$ be given by Proposition 2.4. For any $\lambda \in (0, \lambda^*)$ there exists a solution u_λ such that $I_\lambda(u_\lambda) < 0$. Recall that such solution was obtained by a minimization argument on the ball $B_\rho(0)$. Hence, by considering a small ball if necessary, we may assume that the solutions $(u_\lambda)_{\lambda\in(0,\lambda^*)}$ are close to zero.

Consider $p_0 > 2$ given by (f_4) and ω_{p_0} the function obtained before the beginning of this proof. By integrating the inequality in (f_4) we obtain $F(s) \geq (C_{p_0}/p_0)s^{p_0}$, for any $s \geq 0$. Thus

$$I(t\omega_{p_0}) \leq \left[\frac{t^2}{2}\|\omega_{p_0}\|^2 - C_{p_0}\frac{t^p}{p}\right] - \lambda\frac{t^q}{q}\int K(x)a(x)\omega_{p_0}^q \tag{3.6}$$

from which it follows that $I_\lambda(t\omega_{p_0}) \to -\infty$ as $t \to +\infty$. Hence, there exists $t_0 > 0$ large such that $e := t_0\omega_{p_0}$ verifies $\|e\| > \rho$ and $I_\lambda(e) < 0$, for any $\lambda \in (0, \lambda^*)$. This and (2.4) show that we can define the Mountain Pass level

$$c_\lambda := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I_\lambda(\gamma(t)),$$

where $\Gamma := \{\gamma \in C([0,1], X) : \gamma(0) = 0,\ \gamma(1) = e\}$. It is clear from this definition that

$$c_\lambda \leq \max_{t\geq 0} I_\lambda(t\omega_{p_0}). \tag{3.7}$$

In order to estimate c_λ we call $g(t)$ the function into the brackets in (3.6) and use (f_4) again to obtain, for any $t \geq 0$,

$$g(t) \leq \max_{\tau\geq 0}\left[\frac{\tau^2}{2}\|\omega_{p_0}|^2 - C_{p_0}\frac{\tau^p}{p}\right] = \gamma := \frac{(p_0-2)}{2p_0}\frac{S_{p_0}^{p_0/(p_0-2)}}{C_{p_0}^{2/(p_0-2)}} < \frac{2\pi}{\alpha_0}.$$

Notice that γ is independent of λ. So, since $I_\lambda(u_\lambda) \to 0^-$ as $\lambda \to 0^+$, we can find $\lambda_* \in (0, \lambda^*)$ such that

$$\max_{t\geq 0} I_\lambda(t\omega_{p_0}) \leq \max_{t\geq 0}\left\{g(t) - \lambda\frac{t^q}{q}\int K(x)a(x)\omega_{p_0}^p\right\} < I_\lambda(u_\lambda) + \frac{2\pi}{\alpha_0}, \quad \forall\,\lambda \in (0, \lambda_*).$$

Thus, we infer from (3.7) that

$$c_\lambda < I_\lambda(u_\lambda) + \frac{2\pi}{\alpha_0}, \quad \forall\,\lambda \in (0, \lambda_*).$$

We can now obtain a second nonzero solution, for $\lambda \in (0, \lambda_*)$, in the following way: suppose, by contradiction, that the only critical points of I_λ are $u = 0$ and $u = u_\lambda$. Then, it follows from the above inequality and Proposition 3.2 that I_λ satisfies the Palais-Smale condition at the level c_λ. The Mountain Pass Theorem provides a critical point $u_M \in X$ such that $I_\lambda(u_M) > 0$. Since $I_\lambda(0) = 0$ and $I_\lambda(u_\lambda) < 0$, we have that $u_M \notin \{0, u_\lambda\}$, which is a contradiction. Hence, there is another critical point different from 0 and u_λ. The theorem is proved. □.

References

[1] Adimurthi, *Existence of positive solutions of the semilinear Dirichlet problem with critical growth for the n-Laplacian*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **17** (1990), 393-413.

[2] C.O. Alves, *Multiple positive solutions for equations involving critical Sobolev exponent in* $\mathbb{R}^N$, Electron. J. Differential Equations (1997), no. 13, 10 pp. (electronic).

[3] A. Ambrosetti, H. Brezis and G. Cerami, *Combined effects of concave and convex nonlinearities in some elliptic problems*, J. Funct. Anal. **122** (1994), 519-543.

[4] A. Ambrosetti, J. Garcia Azorero and I. Peral, *Elliptic variational problems in* $\mathbb{R}^N$ *with critical growth*, J. Differential Equations **168** (2000), 10-32.

[5] J.G. Azorero and I.P. Alonso, *Some results about the existence of a second positive solution in a quasilinear critical problem*, Indiana Univ. Math. J. **43** (1994), 941- 957
[6] D. M. Cao, *Nontrivial solution of semilinear elliptic equation with critical exponent in* $\mathbb{R}^2$, Comm. Partial Differential Equations **17** (1992), 407-435.
[7] P. Drábek and Y.X. Huang, *Multiplicity of positive solutions for some quasilinear elliptic equation in* $\mathbb{R}^N$ *with critical Sobolev exponent*, J. Differential Equations **140** (1997), 106-132.
[8] D.G. deFigueiredo, J.P. Gossez, P. Ubilla, *Local superlinearity and sublinearity for indefinite semilinear elliptic problems*, J. Functional Analysis **199** (2003) 452-467.
[9] D.G. deFigueiredo, J.P. Gossez and P. Ubilla, *Multiplicity results for a family of semilinear elliptic problems under local superlinearity and sublinearity*, J. Eur. Math. Soc. **8** (2006), 269-286.
[10] M. Escobedo and O. Kavian, Variational problems related to self-similar solutions of the heat equation, Nonlinear Anal. **11** (1987), 1103-1133.
[11] D.G. deFigueiredo, O.H. Miyagaki and B. Ruf, *Elliptic equations in* $\mathbb{R}^2$ *with nonlinearities in the critical growth range*, Calc. Var. Partial Differential Equations **4** (1995), 139-153.
[12] D.G. deFigueiredo, J.M.B. do Ó and B. Ruf, *On an inequality by N. Trudinger and J. Moser and related elliptic equations*, Comm. Pure Appl. Math. **55** (2002), 135-152.
[13] M.F. Furtado, E.S. Medeiros and U.B. Severo, *A Trudinger-Moser inequality in a weighted Sobolev space and applications*, Math. Nach. **287** (2104), 1255-1473.
[14] M.F. Furtado, E.S. Medeiros and U.B. Severo, *AOn a Class of Semilinear Elliptic Eigenvalue Problems in* $\mathbb{R}^2$, Proc. Edinburgh Math. Society **60** (2107), 107-126.
[15] M.F. Furtado, R. Ruviaro and J.P.P. Silva, *Two solutions for an elliptic equation with fast increasing weight and concave-convex nonlinearities*, Journal of Mathematical Analysis and Applications **416** (2014), 698-709.
[16] M.F. Furtado, J.P.P. Silva and B.N. Souza, *Elliptic equations with weight and combined nonlinearities*, Advanced Nonlinear Studies **16** (2016), 509-517.
[17] Z. Liu and Z.Q. Wang, *Schrödinger equations with concave and convex nonlinearities*, Z. Angew. Math. Phys. **56** (2005), 609-629.
[18] J. Moser, *A sharp form of an inequality by N. Trudinger*, Indiana Univ. Math. J. **20** (1985), 185-201.
[19] J.M.B. do Ó, *N-Laplacian equations in* $\mathbb{R}^N$ *with critical growth*, Abstr. Appl. Anal. **2** (1997), 301-315.
[20] J.C. Pádua, E.A.B. Silva and S.H.M. Soares, *Positive solutions of critical semilinear problems involving a sublinear term on the origin*, Indiana Univ. Math. J. **55** (2006), 1091-1111.
[21] F.O.V. de Paiva, *Nonnegative solutions of elliptic problems with sublinear indefinite nonlinearity*, J. Functional Analysis **261** (2011), 2569-2586.
[22] B. Ruf, *A sharp Trudinger-Moser type inequality for unbounded domains in* $\mathbb{R}^2$, J. Funct. Anal. **219** (2005), 340-367.
[23] E. Tonkes, *A semilinear elliptic equation with convex and concave nonlinearities*, Topol. Methods Nonlinear Anal. **13** (1999), 251-271.
[24] N.S. Trudinger, *On the imbedding into Orlicz spaces and some applications*, J. Math. Mech. **17** (1967), 473-484.
[25] Z. Zhang, M. Calanchi, B. Ruf, *Elliptic equations in* $\mathbb{R}^2$ *with one-sided exponential growth*, Commun. Contemp. Math. **6** (2004), 947-971.

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