KIRCHHOFF ELLIPTIC PROBLEMS WITH ASYMPTOTICALLY LINEAR OR SUPERLINEAR NONLINEARITIES

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Abstract. We establish the existence and multiplicity of solutions for Kirchhoff elliptic problems of type

$$-m\left(\int_{\mathbb{R}^3}|\nabla u|^2dx\right)\Delta u = f(x,u), \quad x\in\mathbb{R}^3,$$

where $m:\mathbb{R}_+\to\mathbb{R}$ is continuous, positive and satisfies appropriate growth and/or monotonicity conditions. We consider the cases that f is asymptotically $3-$linear or $3-$superlinear at infinity, in an appropriated sense. By using variational methods, we obtain our results under crossing assumptions of the functions m and f with respect to limit eigenvalues problems. In the model case $m(t)=a+bt$, we also prove a concentration result for some solutions when $b\to 0^+$.

1. Introduction

Consider the problem

$$-m\left(\int_{\Omega}|\nabla u|^2dx\right)\Delta u = g(x,u) \quad \text{in} \quad \Omega, \qquad u=0 \quad \text{on} \quad \partial\Omega,$$

where $\Omega\subset\mathbb{R}^N$ is a bounded smooth domain, $m:\mathbb{R}_+\to\mathbb{R}$ is a positive function and the nonlinear function g has polynomial growth. It is called nonlocal due to the presence of the term $m\left(\int_\Omega|\nabla u|^2dx\right)$ in the equation and it has its origin in the theory of nonlinear vibrations. For instance, in the case $m(t)=a+bt$, with a, $b>0$, it comes from the following model for the modified d'Alembert wave equation

$$\rho\frac{\partial^2u}{\partial t^2}-\left(\frac{P_0}{h}+\frac{E}{2L}\int_0^L\left|\frac{\partial u}{\partial x}\right|^2dx\right)\frac{\partial^2u}{\partial x^2}=g(x,u),$$

for free vibrations of elastic strings. Here, L is the length of the string, h is the area of the cross-section, E is the Young modulus of the material, ρ is the mass density and P_0 is the initial tension. This kind of nonlocal equation was first proposed by Kirchhoff [20] and it was considered theoretically or experimentally by several physicists after that (see [11, 24, 25, 27]). Nonlocal problems also appear in other fields as, for example, biological systems where u describes a process which depends on the average of itself (for instance, population density). We refer the reader to

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[16, 23], and references therein, for more examples on the physical motivation of this problem.

In the present work, we are interested in the case that the problem is settled in the entire space $\mathbb{R}^3$. More specifically, we consider

$$(P) \qquad \begin{cases} -m\left(\|u\|^2\right)\Delta u = f(x,u), & x \in \mathbb{R}^3, \\ u \in \mathcal{D}^{1,2}(\mathbb{R}^3), \end{cases}$$

where $\|u\| = (\int_{\mathbb{R}^3} |\nabla u|^2 dx)^{1/2}$ and $\mathcal{D}^{1,2}(\mathbb{R}^3)$ is the closure of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm $\|\cdot\|$. We have some structural assumptions on $m \in C(\mathbb{R})$ and we shall consider two different classes of functions $f \in C(\mathbb{R}^3 \times \mathbb{R})$ depending on the growth at infinity: the asymptotically 3−linear and the 3-superlinear case.

In our first results, we consider the following set of conditions on m:

(m_1) there exists $m_0 > 0$ such that $m(t) \geq m_0$ for all $t \geq 0$;
(m_2) there holds

$$\lim_{t\to+\infty} \frac{m(t)}{t} = m_\infty > 0.$$

Since the term $\int_{\mathbb{R}^3} u^2 dx$ does not appear in the left-hand side of our equation, we are not able to model the problem in $H^1(\mathbb{R}^3)$. Actually, the natural space is $\mathcal{D}^{1,2}(\mathbb{R}^3)$. Unfortunately, it is not embedded into the Lebesgue spaces $L^p(\mathbb{R}^3)$ for $p \in [2,6)$, and therefore we need to impose suitable growth conditions on f. So, given $\alpha > 1$, we consider the class of functions

$$\Gamma_\alpha := \left\{ g \in L^\alpha(\mathbb{R}^3) \,:\, g^+ := \max\{g,0\} \not\equiv 0 \text{ and } g \in L^s_{loc}(\mathbb{R}^3) \text{ for some } s > \alpha \right\}.$$

In the first part of the paper we shall assume that f satisfies

(f_1) there exist $A \in \Gamma_{3/2}$ and $B \in \Gamma_3$ such that

$$|f(x,t)| \leq A(x)|t| + B(x)|t|^3, \quad \text{for all } (x,t) \in \mathbb{R}^3 \times \mathbb{R};$$

(f_2) if $F(x,t) := \int_0^t f(x,s)ds$, then there exists $g_0 \in \Gamma_{3/2}$ such that

$$\lim_{t\to 0} \frac{2F(x,t)}{t^2} = g_0(x), \quad \text{uniformly in } \mathbb{R}^3;$$

(f_3) there exists $g_\infty \in \Gamma_3$ such that

$$\lim_{|t|\to\infty} \frac{f(x,t)}{t^3} = g_\infty(x), \quad \text{uniformly in } \mathbb{R}^3.$$

Under the above conditions, we can easily show that the weak solutions of (P) are precisely the critical points of the functional $I : \mathcal{D}^{1,2}(\mathbb{R}^3) \to \mathbb{R}$ given by

$$I(u) = \frac{1}{2} M(\|u\|^2) - \int_{\mathbb{R}^3} F(x,u)dx.$$

As it is well known, the existence of such critical points is affected by the interaction of m and f with the spectrum of some eigenvalue problems. Due to the presence of the nonlocal term m, the principal part of I has different degrees near the origin and the infinity. So, we need to consider two different eigenvalue problems. More specifically, let $g_\infty \in \Gamma_3$ be given by condition (f_3) and consider the nonlinear eigenvalue problem

$$-\|u\|^2 \Delta u = \mu g_\infty(x) u^3, \quad x \in \mathbb{R}^3, \qquad u \in \mathcal{D}^{1,2}(\mathbb{R}^3). \tag{1.1}$$

As we will see in Section 2, it has a first eigenvalue given by

$$\mu_1(g_\infty) = \inf\left\{\|u\|^4 \,:\, u \in \mathcal{D}^{1,2}(\mathbb{R}^3),\ \int_{\mathbb{R}^3} g_\infty(x)u^4 dx = 1\right\} > 0.$$

By the same reason, if we consider $g_0 \in \Gamma_{3/2}$ given by condition (f_2), we can deal with the (linear) eigenvalue problem

$$-\Delta u = \lambda g_0(x)u, \quad x \in \mathbb{R}^3, \qquad u \in \mathcal{D}^{1,2}(\mathbb{R}^3),$$

and obtain a first eigenvalue

$$\lambda_1(g_0) = \inf\left\{\|u\|^2 \,:\, u \in \mathcal{D}^{1,2}(\mathbb{R}^3),\ \int_{\mathbb{R}^3} g_0(x)u^2 dx = 1\right\} > 0.$$

Since this last problem is linear, we can proceed inductively to define, for each $k \in \mathbb{N}$, the positive eigenvalues

$$\lambda_{k+1}(g_0) = \inf\left\{\|u\|^2 \,:\, u \in \operatorname{span}\{\varphi_1, \ldots, \varphi_k\}^\perp,\ \int_{\mathbb{R}^3} g_0(x)u^2 dx = 1\right\},$$

where φ_j is an eigenfunction associated to $\lambda_j(g_0)$, $j = 1, \ldots, k$.

In our first result we prove the following:

Theorem 1.1 (Asymptotic linear case 1)**.** *Suppose that m and f satisfy (m_1)–(m_2) and $(f_1) - (f_3)$, respectively. If, for some $k \geq 1$,*

$$\lambda_k(g_0) < \frac{1}{m(0)} < \lambda_{k+1}(g_0), \qquad \mu_1(g_\infty) > \frac{1}{m_\infty},$$

then problem (P) has at least two nonzero solutions.

In our next results we consider the complementary case $m(0)^{-1} < \lambda_1(g_0)$ and $m_\infty^{-1} > \mu_1(g_\infty)$. In this new setting, the functional has a different geometry and is no longer coercive. In order to get compactness, we impose a nonquadratic condition at infinity (see [14]), namely,

(m_3) $[2M(t) - m(t)t] \geq 0$, for all $t \geq 0$;

(f_4) there exits $D \in L^1(\mathbb{R}^3)$ such that

$$\begin{cases} [f(x,t)t - 4F(x,t)] \geq D(x), & \text{for all } (x,t) \in \mathbb{R}^3 \times \mathbb{R}; \\ \lim\limits_{|t| \to +\infty} [f(x,t)t - 4F(x,t)] = +\infty, & \text{for all } x \in \mathbb{R}^3. \end{cases}$$

The second result of this paper reads as:

Theorem 1.2 (Asymptotic linear case 2)**.** *Suppose that m and f satisfy (m_1)–(m_3) and $(f_1) - (f_4)$, respectively. If*

$$\frac{1}{m(0)} < \lambda_1(g_0), \qquad \mu_1(g_\infty) < \frac{1}{m_\infty},$$

then problem (P) has at least one nonzero solution.

In our last result for the asymptotically 3−linear case, we consider a version of the classical nonresonance hypothesis at infinity. Hence, we drop the condition (m_3) and replace (f_4) by

(f_5) $w = 0$ is the only solution of

$$-\|w\|^2 \Delta w = \frac{1}{m_\infty} g_\infty(x) w^3, \quad x \in \mathbb{R}^3,$$

and prove the following:

Theorem 1.3 (Asymptotic linear case 2'). *Suppose that m and f satisfy $(m_1)-(m_2)$ and $(f_1)-(f_3)$, (f_5), respectively. If*

$$\frac{1}{m(0)} < \lambda_1(g_0) \quad \text{and} \quad \mu_1(g_\infty) < \frac{1}{m_\infty},$$

then problem (P) has at least one nonzero solution.

In the second part of the paper, we consider the 3-superlinear case at infinity. Naturally, we need to impose some different assumptions on m and f. More specifically, we assume that:

$(m_2)'$ there holds

$$\lim_{t\to+\infty} \frac{2M(t)}{t^2} = m_\infty > 0;$$

(m_4) the function $t \mapsto [2M(t) - m(t)t]$ is nondecreasing in $[0,+\infty)$;

$(f_1)'$ there exist $A \in \Gamma_{3/2}$ and $B \in \Gamma_{6/(6-p)}$, with $4 < p < 6$, such that

$$|f(x,t)| \le A(x)|t| + B(x)|t|^{p-1}, \quad \text{for all } (x,t) \in \mathbb{R}^3 \times \mathbb{R};$$

$(f_3)'$ there holds

$$\lim_{t\to+\infty} \frac{F(x,t)}{t^4} = +\infty, \quad \text{unifomly in } x \in \mathbb{R}^3;$$

(f_6) for each $x \in \mathbb{R}^3$, the function $t \mapsto [f(x,t)t - 4F(x,t)]$ is decreasing in $(-\infty,0)$ and increasing in $(0,+\infty)$.

Under the above set of hypotheses, we prove the following:

Theorem 1.4 (Superlinear case). *Suppose that m and f satisfy (m_1), $(m_2)'$, (m_4) and $(f_1)'$, (f_2), $(f_3)'$, (f_6), respectively. If*

$$\frac{1}{m(0)} \neq \lambda_k(g_0), \quad \textit{for all } k \in \mathbb{N},$$

then problem (P) has at least one nonzero solution.

In the final result of this paper, we consider the model case $m(t) = a + bt$, with a, $b > 0$. If we impose conditions on f in such way that the negative infimum d_b of the energy functional is achieved for any $b \in (0,1]$, it is not difficult to use the monotonicity of the map $b \mapsto d_b$ to prove that the solution u_b strongly converges, as $b \to 0^+$, to a solution of the local limit problem. As a matter of fact, after proving that the main point for concentration is the boundedness of the family of solutions (see Lemma 5.1), we finish the paper by showing that concentration occurs for a special class of solutions given by the above theorems. More specifically, we prove the following:

Theorem 1.5 (Asymptotic behavior of solutions). *Let $m(t) = a+bt$, with $a, b > 0$. Suppose that f satisfies $(f_1)-(f_4)$ and (f_6) with*

$$\frac{1}{a} < \lambda_1(g_0), \qquad \mu_1(g_\infty) < 1.$$

Then, for each $b \in (0,1)$, problem (P) has a solution $u_b \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ such that $u_b \to u_0$ strongly in $\mathcal{D}^{1,2}(\mathbb{R}^3)$, where $u_0 \neq 0$ is a weak solution of the local problem

$$\begin{cases} -a\Delta u = f(x,u), \quad x \in \mathbb{R}^3, \\ u \in \mathcal{D}^{1,2}(\mathbb{R}^3). \end{cases}$$

The same result holds if f satisfies the hypotheses of Theorem 1.4.

In the proofs of all of our theorems we use variational methods. Actually, we look for critical points of the energy functional I. In order to do that we use Local Linking Theorems besides the Mountain Pass Theorem, depending on the position of the numbers $m(0)^{-1}$ and m_∞^{-1} in the spectrum of the eigenvalue problems. The lack of compactness inherit to elliptic problems defined in the whole space is overcame by assuming that the appropriate eigenvalue problem at infinity has a weak iteration with the nonlinearity f.

We present now examples of functions f which satisfy our assumptions. For the 3-asymptotically linear case we can consider the nonlinearity

$$f(x,t) = A(x)\frac{t}{1+t^2} + B(x)\frac{t^5}{1+t^2}, \qquad \text{for all } (x,t) \in \mathbb{R}^3 \times \mathbb{R},$$

with $A \in L^1(\mathbb{R}^N) \cap \Gamma_{3/2}$ and $B \in L^1(\mathbb{R}^N) \cap \Gamma_3$ being nonnegative functions. We can prove that it satisfies conditions $(f_1) - (f_4)$. Concerning the $3-$superlinear case, we may pick

$$f(x,t) = a_0(x)t + a_1(x)|t|^{p-2}t + a_2(x)|t|^{q-2}t, \qquad \text{for all } (x,t) \in \mathbb{R}^3 \times \mathbb{R},$$

with $4 < p \leq q < 6$. This function satisfies $(f_1)', (f_2), (f_3)'$ and (f_6) for any nonnegative functions $a_0 \in \Gamma_{3/2}$, $a_1 \in \Gamma_{6/(6-p)}$ and $a_2 \in \Gamma_{6/(6-p)}$. In the same setting, we can also consider $f(x,t) = a(x)t^3 \ln(1+|t|)$, with $a \in \Gamma_{6/(6-p)}$.

We refer to [6, 7, 12] for results concerning the evolution equation associated to the Kirchhoff equations. For its stationary version, as far as we know, the first paper dealing with variational methods was [1]. Since then, there is a vast literature concerning the existence, nonexistence, multiplicity and concentration behavior of solutions for such kind of problems (see [2, 3, 4, 13, 17, 18, 19] and the references therein). The main tools used are variational methods, genus theory and topological methods and the author consider the (most studied) case that f is $3-$superlinear at infinity. In most of them, some sort of Ambrowsetti-Rabinowitz superlinear condition (see [5]) was imposed. The literature for asymptotically $3-$linear nonlinearities is not so huge. In [26], the authors consider the bounded domain case for $m(t) = a + bt$. They obtained the existence of one nontrivial solution when $\lambda_k(1) < (1/a) < \lambda_{k+1}(1)$, $\mu_m(1) < (1/b) < \mu_{m+1}$ and $k \neq m$. It is worth mention that, in this last paper, the authors obtained an increasing and unbounded sequence of eigenvalues for the bounded domain version of (1.1) via the Yang Index. However, they are able to consider resonant cases. This last paper was complemented in [28], where it was considered the sublinear case at the origin and also a version of the $3-$superlinear case at infinity with a sort of Ambrosetti-Rabinowitz condition. In [22], the authors used topological and variational methods to extend some of the results of [26] for the resonant case. We finally mention [9], where a Schrödinger-Kirchhoff problem was considered for a large class of functions f.

The article is organized as follows: in the forthcoming section, we present some preliminary results. Section 3 is devoted to the asymptotically $3-$linear case, Section 4 to the $3-$superlinear case and the final one to the proof of asymptotic behavior of the solutions.

2. Preliminaries

Throughout the paper, the norms in $L^p(\mathbb{R}^3)$ and $L^\infty(\mathbb{R}^3)$ are denoted by $\|\cdot\|_p$ and $\|\cdot\|_\infty$, respectively. For each $R>0$, we denote $B_R := \{x\in\mathbb{R}^3 : \|x\|_{\mathbb{R}^3} < R\}$ and by B_R^c its complement. For an integrable function g, we write only $\int g$ to denote $\int_{\mathbb{R}^3} g(x)dx$. Finally, we denote by C_1, $C_2,\ldots$ positive constants (possibly different).

Let X be a real Banach space and $I\in C^1(X,\mathbb{R})$. A sequence $(u_n)\subset X$ is said to be a Cerami sequence at level $c\in\mathbb{R}$ for I if

$$\lim_{n\to+\infty} I(u_n)=c,\qquad \lim_{n\to+\infty}\|I'(u_n)\|_{X^*}(1+\|u_n\|_X)=0.$$

The functional I satisfies the Cerami condition at the level $c\in\mathbb{R}$ if any such sequence has a convergent subsequence. When this condition holds for any $c\in\mathbb{R}$, we only say that I satisfies the Cerami condition.

In a similar way, we can define the Palais-Smale compactness condition just replacing Cerami sequences by Palais-Smale sequences, that is, sequences $(u_n)\subset X$ such that

$$\lim_{n\to+\infty} I(u_n)=c,\qquad \lim_{n\to+\infty}\|I'(u_n)\|_{X^*}=0.$$

We state in the sequel the abstract results we are going to use in our proofs. We consider only the Cerami condition, since it is more general. For the proofs we refer to [21, Theorem 1], [21, Theorem 2] and [5, Theorem 2.1], respectively (see also [8] for the verification that the deformation lemma holds with Cerami instead of Palais-Smale condition).

Theorem 2.1 (Local Linking for asymptotically quadratic functionals)**.** *Let $X = X_1\oplus X_2$ be a real Banach space with $\dim X_1<\infty$. Suppose that $I\in C^1(X,\mathbb{R})$ satisfies the following:*

- (I_1) *I has a local linking at the origin, that is, there exists $\rho>0$ such that*

$$\begin{cases} I(u)\le 0, & \forall\, u\in X_1\cap B_\rho(0),\\ I(u)\ge 0, & \forall\, u\in X_2\cap B_\rho(0);\end{cases}$$

- (I_2) *I maps bounded sets into bounded sets;*
- (I_3) *I satisfies the Cerami condition;*
- (I_4) *there holds*

$$-\infty<\inf_{u\in X} I(u)<0.$$

Then, the functional I has at least two nonzero critical points.

Theorem 2.2 (Local Linking for superquadratic functionals)**.** *Let $X = X_1\oplus X_2$ be a real Banach space with $\dim X_1<\infty$. Suppose that $I\in C^1(X,\mathbb{R})$ satisfies $(I_1)-(I_3)$ and*

- (I_5) *for any finite dimensional subspace $\widetilde{X}\subset X$ there holds*

$$\lim_{u\in\widetilde{X},\,\|u\|\to+\infty} I(u)=-\infty.$$

Then, the functional I has at least one nonzero critical point.

Theorem 2.3 (Mountain Pass)**.** *Let X be a real Banach space. Suppose that $I\in C^1(X,\mathbb{R})$ satisfies $I(0)=0$ and*

(I_6) *there exists* α, $\rho > 0$ *such that*

$$I(u) \geq \alpha, \quad \forall\, u \in X \cap \partial B_\rho(0);$$

(I_7) *there exists* $e \in X$ *such that* $\|e\| > \rho$ *and* $I(e) < 0$.

Let

$$c := \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} I(\gamma(t)),$$

where $\Gamma := \{\gamma \in C([0,1], X), \gamma(0) = 0, \gamma(1) = e\}$. *If* I *satisfies the Cerami condition at level* c*, then* I *has a nonzero critical point.*

We now present the abstract framework to deal with problem (P). Hereafter, we shall consider the Hilbert space

$$X := \mathcal{D}^{1,2}(\mathbb{R}^3) = \left\{ u \in L^6(\mathbb{R}^3) \,:\, \|u\| := \left(\int |\nabla u|^2 \right)^{1/2} < +\infty \right\}.$$

Setting $M(t) := \int_0^t m(s)ds$ and $F(x,t) := \int_0^t f(x,s)ds$, we can use (f_1) and standard calculations to prove that the energy functional $I : X \to \mathbb{R}$ given by

$$I(u) := \frac{1}{2} M(\|u\|^2) - \int F(x,u),$$

is well defined. Actually, $I \in C^1(X,\mathbb{R})$ and the critical points of I are precisely the weak solutions of (P).

We start with a technical result.

Lemma 2.4. *If* $g \in \Gamma_3$ *and* $(u_n) \subset X$ *is such that* $u_n \rightharpoonup u$ *weakly in* X*, then* $\int g(x)u_n^4 \to \int g(x)u^4$*, as* $n \to +\infty$.

Proof. We claim that $\int g(x)|u_n - u|^4 \to 0$, as $n \to +\infty$. Indeed, given $\varepsilon > 0$, there exists $R = R(\varepsilon) > 0$ such that $\int_{B_R^c} |g(x)|^3 dx \leq \varepsilon^3$. It follows from Hölder's inequality, the choice of R and the boundedness of (u_n) in $L^6(\mathbb{R}^3)$ that

$$\int_{B_R^c} |g(x)||u_n - u|^4 dx \leq \|g\|_{L^3(B_R^c)} \left(\int_{B_R^c} |u_n - u|^6 dx \right)^{2/3} \leq C_2 \varepsilon.$$

On the other hand, since $g \in L^s(B_R)$ for some $s > 3$, we can use Hölder's inequality again to get

$$\int_{B_R} |g(x)||u_n - u|^4 dx \leq \|g\|_{L^s(B_R)} \left(\int_{B_R} |u_n - u|^{4s'} dx \right)^{1/s'}.$$

Since $4s' < 6$, the right-hand side above goes to zero as $n \to +\infty$. The claim follows from the above expressions.

Now, we write $g = g^+ + g^-$, with $g^+ := \max\{g(x), 0\}$ and $g^- := g - g^+$. The sequence (ψ_n) defined as $\psi_n := (g^+)^{1/4} u_n$ is bounded in $L^4(\mathbb{R}^3)$ and $\psi_n(x) \to \psi(x) := g^+(x)|u(x)|^p$, for a.e. $x \in \mathbb{R}^3$. It follows from Brezis and Lieb's lemma [10, Theorem 1] that

$$\lim_{n\to+\infty} \int \left(g^+(x)|u_n|^4 - g^+(x)|u_n - u|^4 \right) = \int g^+(x)|u|^4,$$

and therefore we infer from the first part of the proof that $\int g^+(x)u_n^4 \to \int g^+(x)u^4$, as $n \to +\infty$. Since the same argument holds if we replace g^+ by g^-, the result follows. □

We now take $g_\infty \in \Gamma_3$ given by condition (f_3) and consider the nonlinear eigenvalue problem

$$(LP)_\infty \qquad \begin{cases} -\|u\|^2\Delta u = \mu g_\infty(x)u^3, & x \in \mathbb{R}^3, \\ u \in \mathcal{D}^{1,2}(\mathbb{R}^3). \end{cases}$$

We first prove that its first eigenvalue

$$\mu_1(g_\infty) := \inf\left\{\|u\|^4 \, : \, u \in X, \, \int g_\infty(x)u^4 = 1\right\},$$

is positive and it is achieved. Indeed, since $g_\infty^+ \not\equiv 0$ the set $\Sigma := \{u \in X : \int g_\infty(x)u^4 = 1\}$ is nonempty. If $u \in \Sigma$, it follows from Hölder's inequality that

$$1 = \int g_\infty(x)u^4 \leq \|g_\infty\|_3\|u\|_6^4 \leq C\|g_\infty\|_3\|u\|^4,$$

and therefore $\mu_1(g) > 0$. Let $(u_n) \subset \Sigma$ be such that $\|u_n\|^4 \to \mu_1(g)$. Up to a subsequence, we may suppose that $u_n \rightharpoonup \phi_1$ weakly in X for some ϕ_1 in X. We infer from Lemma 2.4 that $\phi_1 \in \Sigma$. Hence, since the norm is sequentially weakly continuous, it follows that $\|\phi_1\|^4 = \mu_1(g_\infty)$. A simple application of the Lagrange Theorem shows that ϕ_1 verifies the eigenvalue problem $(LP)_\infty$.

It clear from the definition of $\mu_1(g_\infty)$ that the following inequality holds

$$\mu_1(g_\infty)\int g_\infty(x)u^4 \leq \|u\|^4, \quad \forall\, u \in X. \tag{2.1}$$

We now take the function $g_0 \in \Gamma_{3/2}$ given by condition (f_2) and study the linear eigenvalue problem

$$(LP)_0 \qquad \begin{cases} -\Delta u = \lambda g_0(x)u, & x \in \mathbb{R}^3, \\ u \in \mathcal{D}^{1,2}(\mathbb{R}^3). \end{cases}$$

Arguing as above, we can prove that its first eigenvalue

$$\lambda_1(g_0) := \inf\left\{\|u\|^2 \, : \, u \in X, \, \int g_0(x)u^2 = 1\right\} > 0$$

is also achieved by an eigenfunction $\varphi_1 \in X$. Moreover, following the same ideas developed in [15], we can proceed inductively and define, for each $k \in \mathbb{N}$, the positive eigenvalues

$$\lambda_{k+1}(g_0) = \inf\left\{\|u\|^2 \, : \, u \in \operatorname{span}\{\varphi_1, \ldots, \varphi_k\}^\perp, \, \int g_0(x)u^2 = 1\right\}.$$

which are such that

$$\lambda_{k+1}(g_0)\int g_0(x)u^2 \leq \|u\|^2, \quad \forall\, u \in \operatorname{span}\{\varphi_1, \ldots, \varphi_k\}^\perp \tag{2.2}$$

and

$$\lambda_k(g_0)\int g_0(x)u^2 \geq \|u\|^2, \quad \forall\, u \in \operatorname{span}\{\varphi_1, \ldots, \varphi_k\}.$$

3. Asymptotically linear case

In the first part of this section we shall assume that, for some $k \in \mathbb{N}$, there hold

$$\lambda_k(g_0) < \frac{1}{m(0)} < \lambda_{k+1}(g_0), \qquad \mu_1(g_\infty) > \frac{1}{m_\infty}, \tag{3.1}$$

where m_∞, g_0 and g_∞ are given in (m_2), (f_2) and (f_3), respectively. In order to obtain the local linking at the origin, we define

$$X_1 := \operatorname{span}\{\varphi_1, \dots, \varphi_k\}, \qquad X_2 := X_1^\perp, \tag{3.2}$$

where φ_j are the eigenfunctions of the weighted linear problem $(LP)_0$.

Lemma 3.1. *Suppose that (f_1), (f_2) and (3.1) hold. Then, I satisfies the condition (I_1) of Theorem 2.1.*

Proof. We first prove the second statement of (I_1). Suppose, by contradiction, that there exists $(u_n) \subset X_2$ such that $\|u_n\| \to 0$ and $I(u_n) \le (1/n)\|u_n\|^2$. If we define $w_n := u_n/\|u_n\|$, we have that

$$\frac{M(\|u_n\|^2)}{\|u_n\|^2} \le \int \frac{2F(x,u_n)}{u_n^2} w_n^2 + o_n(1),$$

where $o_n(1)$ stands for a quantity approaching zero as $n \to +\infty$. We may suppose that $w_n \rightharpoonup w$ with $\|w\| \le 1$. We claim that the integral on the right-hand side above converges to $\int g_0(x)w^2$. If this is true, we can take the limit and use (2.2) to get

$$m(0) \le \int g_0(x)w^2 \le \frac{1}{\lambda_{k+1}(g_0)}\|w\|^2 \le \frac{1}{\lambda_{k+1}(g_0)},$$

which contradicts (3.1). In the left-hand side above equation we have used that

$$\lim_{t\to 0} \frac{M(t)}{t} = \lim_{t\to 0} m(t) = m(0). \tag{3.3}$$

It remains to prove the claim. First notice that, since $u_n(x) \to 0$ for a.e. $x \in \mathbb{R}^3$, we can use (f_2) to obtain

$$\lim_{n\to+\infty} \frac{2F(x,u_n(x))}{u_n^2(x)} w_n^2(x) = g_0(x)w^2(x), \quad \text{for a.e. } x \in \mathbb{R}^3.$$

Moreover, since $\|u_n\| \le 1$ for n large, we can use (f_1) to get

$$\left|\frac{2F(x,u_n(x))}{u_n^2(x)} w_n^2(x)\right| \le A(x)w_n^2(x) + \frac{1}{2}B(x)w_n^4(x).$$

Recalling that $A \in \Gamma_{3/2}$, we can argue as in the proof of Lemma 2.4 to conclude that $\int |A(x)||w_n|^2 \to \int |A(x)||w|^2$ and therefore there exists $\psi_1 \in L^1(\mathbb{R}^3)$ such that $|A(x)w_n^2(x)| \le \psi_1(x)$ for a.e. $x \in \mathbb{R}^3$. The same holds for the function $B(x)w_n^4$. Therefore, the second statement of the lemma result follows from the Lebesgue Dominated Convergence Theorem.

The proof of the first statement in (I_1) can be done in a similar way, just observing that, in this case, $1 = \|w_n\|^2 \to \|w\|^2$, since we have strong convergence in the finite dimensional subspace X_1. We omit the details. $\square$

Lemma 3.2. *Suppose that $(m_2)'$, (f_1), (f_3) and (3.1) hold. Then, the functional I is coercive.*

Proof. Suppose, by contradiction, that there exists $(u_n) \subset X$ such that $\|u_n\| \to +\infty$, but $I(u_n) \le C_1$. If we define $w_n := u_n/\|u_n\|$, it follows from the above equation and $(m_2)'$ that

$$m_\infty + o_n(1) = \frac{2M(\|u_n\|^2)}{\|u_n\|^4} \le o_n(1) + \int \frac{4F(x,u_n)}{u_n^4} w_n^4. \tag{3.4}$$

We may assume that $w_n \rightharpoonup w$ weakly in X. Since $\|u_n\| \ge 1$ for n large, if $w = 0$ it follows from (f_1) that, for a.e. $x \in \mathbb{R}^3$, there holds

$$\left| \frac{4F(x,u_n(x))}{u_n^4(x)} w_n^4(x) \right| \le 2A(x)w_n^2(x) + B(x)w_n^4(x) \to 0, \quad \text{as } n \to \infty.$$

This and the same arguments used in the proof of Lemma 3.1 show that we can take the limit in (3.4) and use the Lebesgue Theorem to conclude that $m_\infty \le 0$, which contradicts (m_2).

In the case $w \ne 0$, we can use (f_3) and L'Hospital's rule again to get

$$\lim_{n\to+\infty} \frac{4F(x,u_n(x))}{u_n^4(x)} w_n^4(x) = g_\infty(x)w^4(x), \quad \text{for a.e. } x \in \{w \ne 0\}.$$

Thus, taking the limit as before, we obtain from (2.1)

$$m_\infty \le \int g_\infty(x)w^4 \le \frac{1}{\mu_1(g_\infty)}\|w\|^4 \le \frac{1}{\mu_1(g_\infty)},$$

which contradicts (3.1). The lemma is proved. □

We prove now that, under the setting of Theorem 1.1, we have compactness.

Lemma 3.3. *Suppose that (m_1) and (f_1) hold. Then any bounded sequence $(u_n) \subset X$ such that $\|I'(u_n)\|_{X^*} \to 0$ has a convergent subsequence.*

Proof. We may assume that $u_n \rightharpoonup u$ weakly in X and $u_n \to u$ strongly in $L^s_{loc}(\mathbb{R}^3)$, for any $2 \le s < 6$. This local convergence, (f_1) and Lebesgue Dominated Convergence Theorem imply that

$$\lim_{n\to+\infty} \int_{B_R} f(u_n)u_n\,dx = \int_{B_R} f(u)u\,dx = \lim_{n\to+\infty} \int_{B_R} f(u_n)u\,dx,$$

for any $R > 0$. Given $\varepsilon > 0$, we can use (f_1), Hölder's inequality, the boundedness of (u_n) in $L^6(\mathbb{R}^3)$ and the same argument of the proof of Lemma 2.4 to conclude that

$$\left| \int_{B_R^c} f(u_n)u_n\,dx \right| \le C_1\varepsilon,$$

for some $R > 0$ sufficiently large. Since analogous inequalities hold for the integrals $\int_{B_R^c} f(u)u\,dx$ and $\int_{B_R^c} f(u_n)u\,dx$, we conclude that

$$\lim_{n\to+\infty} \int f(x,u_n)u_n = \int f(x,u)u = \lim_{n\to+\infty} \int f(x,u_n)u.$$

If $\rho_0 \ge 0$ is such that $\|u_n\|^2 \to \rho_0^2$, we have

$$o_n(1) = I'(u_n)u_n = m(\rho_0^2)\rho_0^2 - \int f(x,u)u + o_n(1). \tag{3.5}$$

Thus,

$$\begin{aligned} o_n(1) = I'(u_n)u &= m(\|u_n\|^2)\int (\nabla u_n \cdot \nabla u) - \int f(x,u_n)u \\ &= m(\rho_0^2)\|u\|^2 - \int f(x,u)u + o_n(1). \end{aligned}$$

Using the last identity and (3.5) we get $m(\rho_0^2)\|u\|^2 = m(\rho_0^2)\rho_0^2$. It follows from (m_1) that $\rho_0^2 = \|u\|^2$ and the weak convergence implies that $u_n \to u$ strongly in X. $\square$

We are ready to present the proof of our first theorem.

Proof of Theorem 1.1. By Lemma 3.1, I satisfies the conditions (I_1) of Theorem 2.1 with respect to the decomposition (3.2). In the same way, we can prove also that condition (I_4) is also satisfied. The proof of (I_2) easily follows from (f_1) and Hölder's inequality. If $(u_n) \subset X$ is a Palais-Smale sequence, by Lemma 3.2, it is bounded and therefore it follows from Lemma 3.3 that I satisfies the Palais-Smale condition. So, we may now invoke Theorem 2.1 to get two nonzero solutions for (P). $\square$

From now on we shall assume that

$$\frac{1}{m(0)} < \lambda_1(g_0), \qquad \mu_1(g_\infty) < \frac{1}{m_\infty}, \tag{3.6}$$

where m_∞, g_0 and g_∞ are given in (m_2), (f_2) and (f_3), respectively.

Lemma 3.4. *Suppose that $(m_1)-(m_3)$, (f_1) and (f_4) hold. Assume also that (3.6) holds. Then I satisfies the Palais-Smale condition at any level $c \in \mathbb{R}$.*

Proof. Let $(u_n) \subset X$ be such that $I(u_n) \to c$ and $\|I'(u_n)\|_{X^*} \to 0$. According to Lemma 3.3, it is sufficient to prove that (u_n) has a bounded subsequence. Suppose, by contradiction, that $\|u_n\| \to +\infty$. Then it follows from (m_2) that

$$\begin{aligned} o_n(1) = \frac{I'(u_n)u_n}{\|u_n\|^4} &= \frac{m(\|u_n\|^2)}{\|u_n\|^2} - \int \frac{f(x,u_n)u_n}{\|u_n\|^4} \\ &= m_\infty - \int \frac{f(x,u_n)u_n}{\|u_n\|^4} + o_n(1). \end{aligned} \tag{3.7}$$

If we define $w_n := u_n/\|u_n\|$, along a subsequence we have that $w_n \rightharpoonup w$ weakly in X. If $w = 0$, then we can use (f_1) and $\|u_n\| \geq 1$ to get

$$\left| \int \frac{f(x,u_n)u_n}{\|u_n\|^4} \right| \leq \int A(x)w_n^2 + \int B(x)w_n^4 = o_n(1).$$

Taking the limit in (3.7), we conclude that $m_\infty = 0$, which contradicts (m_2). Thus, $w \neq 0$.

Now, using (m_3) we get

$$c + o_n(1) = I(u_n) - \frac{1}{4}I'(u_n)u_n \geq \frac{1}{4}\int \left[f(x,u_n)u_n - 4F(x,u_n)\right].$$

Since $|u_n(x)| \to +\infty$ for a.e. $x \in \{w \neq 0\}$ and this set has positive measure, it follows from (f_4) and Fatou's Lemma that

$$4c \geq -\|D\|_{L^1(\{w=0\})} + \int_{\{w\neq 0\}} \liminf_{n\to+\infty} \left[f(x,u_n)u_n - 4F(x,u_n)\right] dx = +\infty,$$

which does not make sense. Thus, (u_n) is bounded in X and the result is proved. $\square$

Lemma 3.5. *Suppose that (m_1), (m_2), (f_1) and (f_5) hold. Then, I satisfies the Palais-Smale condition at any level $c \in \mathbb{R}$.*

Proof. Let $(u_n) \subset X$ be such that $I(u_n) \to c$ and $\|I'(u_n)\|_{X^*} \to 0$. In order to prove that (u_n) is bounded in X we suppose, by contradiction, that $\|u_n\| \to +\infty$ along a subsequence. If we define $w_n := u_n/\|u_n\|$, we can argue as in the above lemma we conclude that $w_n \rightharpoonup w \neq 0$ weakly in X.

For any $v \in C_0^\infty(\mathbb{R}^3)$, we have that

$$o_n(1) = \frac{I'(u_n)v}{\|u_n\|^3} = \frac{m(\|u_n\|^2)}{\|u_n\|^2}\int (\nabla w_n \cdot \nabla v) - \int \frac{f(x,u_n)}{\|u_n\|^3}v.$$

We claim that the last integral above converges to $\int g_\infty(x)w^3 v$ and therefore, taking the limit as $n \to +\infty$ and using (m_2) we conclude that $w \neq 0$ is a solution to

$$-m_\infty \Delta w = g_\infty(x)w^3, \quad x \in \mathbb{R}^3. \tag{3.8}$$

In order to prove the claim we notice that, in the set $\{w \neq 0\}$, there holds $|u_n(x)| \to +\infty$. Hence, it follows from (f_3) that

$$\lim_{n\to+\infty} \frac{f(x,u_n(x))}{\|u_n\|^3}v(x) = \lim_{n\to+\infty} \frac{f(x,u_n(x))}{u_n(x)^3}w_n(x)^3 v(x) = g_\infty(x)w(x)^3 v(x),$$

for a.e. $x \in \{w \neq 0\}$. Moreover, using (f_1) and $\|u_n\| \geq 1$, we get

$$\left|\frac{f(x,u_n(x))}{\|u_n\|^3}v(x)\right| \leq \|v\|_\infty \left(A(x)|w_n(x)| + B(x)|w_n|^3\right), \tag{3.9}$$

and therefore

$$\lim_{n\to+\infty} \frac{f(x,u_n(x))}{\|u_n\|^3}v(x) = 0 = g_\infty(x)w(x)^3 v(x),$$

for a.e. $x \in \{w = 0\}$.

Let Ω be the compact support of the function v and notice that $w_n \to w$ strongly in $L^s(\Omega)$ for any $1 \leq s < 6$. Hence, $|w_n(x)| \leq \psi_s(x)$ for a.e. $x \in \Omega$, with $\psi_s \in L^s(\Omega)$. Recalling that $A \in L^s(\Omega)$ with $s > (3/2)$, we can use Young's inequality to get

$$|A(x)w_n(x)| \leq \frac{1}{s}|A(x)|^s + \frac{1}{s'}|\psi_{s'}(x)|^{s'},$$

with $s' < 3$. Thus, it follows from the Lebesgue Dominated Convergence Theorem that $\int_\Omega A(x)|w_n| \to \int_\Omega A(x)|w|$. Analogously, $\int_\Omega B(x)|w_n|^3 \to \int_\Omega B(x)|w|^3$. Thus, we infer from (3.9) that there exists $\psi \in L^1(\Omega)$ such that $|\|u_n\|^{-3}f(x,u_n)v| \leq \psi(x)$ for a.e. $x \in \Omega$. The claim now follows from the Lebesgue Theorem.

We now notice that, from Lemma 2.4, (f_1) and again the Lebesgue Dominated Convergence Theorem, we obtain

$$\begin{aligned}
\int g_\infty(x)w^4 &= o_n(1) + \frac{I'(u_n)u_n}{\|u_n\|^4} + \int g_\infty(x)w_n^4 \\
&= o_n(1) + \frac{m(\|u_n\|^2)}{\|u_n\|^2} - \int \left[\frac{f(x,u_n)}{u_n^3} - g_\infty(x)\right] w_n^4 \\
&= o_n(1) + m_\infty.
\end{aligned}$$

Thus, $m_\infty = \int g_\infty(x)w^4$ and it follows from (3.8) that $\|w\| = 1$. But this contradicts (f_5). $\square$

We can now obtain the existence of solution in the Mountain Pass setting.

Proof of Theorems 1.2 and 1.3. Since $\lambda_1(g_0) > 1/m(0)$, we can argue along the same lines of the proof of Lemma 3.1 to obtain α, $\rho > 0$ such that

$$I(u) \geq \alpha, \quad \forall\, u \in B_\rho(0) \cap X.$$

Moreover, according to the discussion in the beginning of Section 2, we can choose $\phi_1 \in X$ such that $\|\phi_1\| = 1$ and

$$-\Delta\phi_1 = \mu_1(g_\infty)g_\infty(x)\phi_1^3, \quad x \in \mathbb{R}^3.$$

From (m_1) we conclude that $M(t) \to +\infty$, as $t \to +\infty$. Hence, it is clear that (m_2) implies $(m_2)'$. Thus, we can use (f_3) to get

$$\begin{aligned}\lim_{t\to+\infty} \frac{4I(t\phi_1)}{t^4} &= \lim_{t\to+\infty}\left[\frac{2M(t^2)}{t^4} - \int \frac{4F(x,t\phi_1)}{(t\phi_1)^4}\phi_1^4\right] \\ &= m_\infty - \int g_\infty(x)\phi_4 = \left[m_\infty - \frac{1}{\mu_1(g_\infty)}\right] < 0,\end{aligned}$$

from which we conclude that $I(t\phi_1) \to -\infty$ as $t \to +\infty$.

By the above considerations, the functional I satisfies the geometric conditions (I_6) and (I_7) of Theorem 2.3. Hence, we can use this former theorem together with Lemma 3.4 and Lemma 3.5 to obtain a nonzero critical point of I. □

4. The superlinear case

We devote this section to the proof of Theorem 1.4. We first notice that, under the condition $(f_1)'$, the functional I belongs to $C^1(X,\mathbb{R})$. Moreover, we have the following compactness property:

Lemma 4.1. *Suppose that (m_1), $(m_2)'$, (m_4), $(f_1)'$ and (f_6) hold. Then I satisfies the Cerami condition at any level $c > 0$.*

Proof. Let $(u_n) \subset X$ be such that $I(u_n) \to c > 0$ and $\|I'(u_n)\|_{X^*}(1 + \|u_n\|) \to 0$. Arguing as in the proof of Lemma 3.3, it is sufficient to prove that (u_n) is bounded. Suppose, by contradiction, that $\|u_n\| \to +\infty$ along a subsequence. If we define $w_n := u_n/\|u_n\|$, we can use $(m_2)'$ to get

$$\begin{aligned}o_n(1) &= \frac{4I(u_n)}{\|u_n\|^4} = \frac{2M(\|u_n\|^2)}{\|u_n\|^4} - \int \frac{4F(x,u_n)}{u_n^4}w_n^4 \\ &= m_\infty + o_n(1) - \int \frac{4F(x,u_n)}{u_n^4}w_n^4.\end{aligned} \tag{4.1}$$

We may suppose that $w_n \rightharpoonup w$ weakly in X and $w_n(x) \to w(x)$ for a.e. $x \in \mathbb{R}^3$.

From (f_6), we obtain $\partial_t F(x,t)/t^4 > 0$ for all $t > 0$ and $x \in \mathbb{R}^3$, and therefore

$$\frac{F(x,t)}{t^4} > \frac{F(x,s)}{s^4}, \quad \forall\, t > s > 0.$$

Since an analogous argument holds if $t < 0$, we conclude that

$$\frac{F(x,u_n)}{u_n^4}w_n^4 \geq \frac{F(x,su_n)}{(su_n)^4}w_n^4, \quad \forall\, x \in \mathbb{R}^3,\ s \in (0,1).$$

Picking $s = 1/\|u_n\|$ and using (f_1'), we get

$$\frac{F(x,u_n)}{u_n^4}w_n^4 \geq F(x,w_n) \geq -A(x)|w_n|^2 - B(x)|w_n|^p,$$

and therefore Fatou's lemma gives

$$\begin{aligned}&\liminf_{n\to\infty}\int \frac{4F(x,u_n)}{u_n^4}w_n^4 + A(x)|w_n|^2 + B(x)|w_n|^p\\ \geq\ &\int \liminf_{n\to\infty}\left[\frac{4F(x,u_n)}{u_n^4}w_n^4 + A(x)|w_n|^2 + B(x)|w_n|^p\right].\end{aligned}$$

However, using the same ideas discussed in the proof of Lemma 2.4, we can prove that

$$\lim_{n\to\infty}\int \left(A(x)|w_n|^2 + B(x)|w_n|^p\right) = \int \left(A(x)|w|^2 + B(x)|w|^p\right).$$

So,

$$\liminf_{n\to\infty}\int \frac{4F(x,u_n)}{u_n^4}w_n^4 \geq \int \liminf_{n\to\infty}\frac{4F(x,u_n)}{u_n^4}w_n^4.$$

If the set $\{w\neq 0\}$ has positive measure, then we have that $|u_n(x)|\to+\infty$, as $n\to+\infty$, for a.e. $x\in\{w\neq 0\}$. Using (4.1), $(f_3)'$ and the above expression, one has

$$m_\infty = \liminf_{n\to\infty}\int \frac{4F(x,u_n)}{u_n^4}w_n^4 \geq \int_{\{w\neq 0\}} \liminf_{n\to\infty}\frac{4F(x,u_n)}{u_n^4}w_n^4\,dx = +\infty,$$

which is a contradiction. So, $w=0$ and we can use $(f_1)'$ and the Lebesgue Theorem to conclude that

$$\lim_{n\to\infty}\int F(x,Lw_n) = 0, \quad \forall\, L\geq 0. \tag{4.2}$$

Let $t_n\in[0,1]$ be such that

$$I(t_nu_n) = \max_{t\in[0,1]} I(tu_n).$$

If $t_n=0$ for all $n\geq n_0$, we have that $c+o_n(1)=I(u_n)\leq I(0)=0$ and therefore $c\leq 0$, which does not hold. Suppose that $t_n=1$ for all $n\geq n_0$. Since $M(t)\to+\infty$, as $t\to+\infty$, there exists $L>0$ such that $M(L)>2c$. We may assume that $0<L<\|u_n\|$, for all $n\geq n_0$, and therefore

$$c+o_n(1) = I(u_n) \geq I(Lw_n) = \frac{1}{2}M(L) - \int F(x,Lw_n).$$

Taking the limit and using (4.2) we obtain $2c\geq M(L)$, which is a contradiction with $M(L)>2c$.

From the above remarks, we may assume, without loss of generality, that $t_n\in(0,1)$, in such way that $\frac{d}{dt}I(tu_n)=0$ for $t=t_n$. Thus, using (m_4) and (f_6), we reach

$$\begin{aligned}4I(t_nu_n) &= 4I(t_nu_n) - I'(t_nu_n)(t_nu_n)\\ &= 2M(\|t_nu_n\|^2) - m(\|t_nu_n\|^2)\|t_nu_n\|^2 +\\ &\quad + \int (f(x,t_nu_n)t_nu_n - 4F(x,t_nu_n))\\ &\leq 2M(\|u_n\|^2) - m(\|u_n\|^2)\|u_n\|^2 + \int (f(x,u_n)u_n - 4F(x,u_n))\\ &= 4I(u_n) - I'(u_n)(u_n) = 4I(u_n) = 4c+o_n(1).\end{aligned}$$

Hence, picking L as before, we obtain $4c+o_n(1)\geq I(t_nu_n)\geq I(Lw_n)$. Taking the limit we obtain again a contradiction. This finishes the proof. □

We are ready to obtain the solution in the superlinear case.

Proof of Theorem 1.4. Let g_0 given by (f_2) and suppose that, for some $k \geq 1$, there holds
$$\lambda_k(g_0) < \frac{1}{m(0)} < \lambda_{k+1}(g_0).$$
Consider the decomposition fo X given in (3.2). By Lemmas 3.1 and 4.1, I satisfies the conditions (I_1) and (I_3). The proof of (I_2) easily follows from $(f_1)'$ and Hölder's inequality. If we can prove (I_5), we may invoke Theorem 2.2 to obtain the desired nonzero solution of (P).

In order to prove (I_5) we fix a finite dimensional subspace $\widetilde{X} \subset X$ and suppose, by contradiction, that there exists $(u_n) \subset \widetilde{X}$ such that $\|u_n\| \to +\infty$ but $I(u_n) \geq -C_1$, for some $C_1 > 0$. If we define $w_n := u_n/\|u_n\|$ we may suppose that $w_n \to w$ strongly in X with $\|w\| = 1$, since $\dim \widetilde{X} < \infty$. We have that
$$o_n(1) \leq \frac{4I(u_n)}{\|u_n\|^4} = m_\infty + o_n(1) - \int \frac{4F(x,u_n)}{u_n^4} w_n^4$$
and therefore we can use (f_3') and the same argument used in the proof of Lemma 4.1 to conclude that $w \neq 0$ cannot hold. This is a contradiction and therefore the theorem is proved in the case $m(0)^{-1} \in (\lambda_k(g_0), \lambda_{k+1}(g_0))$.

We now consider $\lambda_1(g_0) > m(0)^{-1}$. Arguing along the same lines of the proof of Lemma 3.1 we obtain α, $\rho > 0$ such that
$$I(u) \geq \alpha, \quad \forall\, u \in B_\rho(0) \cap X,$$
and therefore I satisfies (I_6). Let $\phi \in C_0^\infty(\mathbb{R}^3)$ be a nonnegative function with support Ω and $\|\phi\| = 1$. Using (3.3) and condition (m_2), we obtain $C_2 > 0$ such that
$$M(t) \leq C_2 t^2 + C_2 t, \quad \forall\, t \geq 0.$$
Moreover, given $R > 0$ such that $R \int_\Omega |\phi|^4 dx > C_2/2$, we can use $(f_3)'$ to get
$$F(x,t) \geq Rt^4 - C_3, \quad \forall\, x \in \Omega,\, t \geq 0,$$
for some $C_3 > 0$. From the above expressions we obtain
$$I(t\phi) \leq \frac{C_2}{2} t^4 + \frac{C_2}{2} t^2 - Rt^4 \int_\Omega |\phi|^4\, dx + C_3 |\Omega|,$$
where $|\Omega|$ denotes the Lebesque measure of Ω. Hence, $I(t\phi) \to -\infty$, as $t \to +\infty$, and therefore I verifies (I_7). It follows from Lemma 4.1 and Theorem 2.3 that I has a nonzero critical point. □

5. Concentration of solutions

In this section, we study the concentration of solutions of the problem
$$(P)_b \qquad \begin{cases} -\left(a + b\|u\|^2\right) \Delta u = f(x,u), & x \in \mathbb{R}^3, \\ u \in \mathcal{D}^{1,2}(\mathbb{R}^3), \end{cases}$$
where $a > 0$ and $b > 0$. As we know, the solutions are the critical points of the functional
$$I_b(u) := \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \int F(x,u), \quad u \in X.$$
The next lemma shows that the concentration of solutions is a direct consequence of boundedness:

Lemma 5.1. *Suppose (f_1) or $(\widetilde{f_1})$ holds. Let $(b_n) \subset (0,1)$ and $(u_{b_n}) \subset X$ be such that u_{b_n} is a solution to $(P)_{b_n}$, for each $n \in \mathbb{N}$. If (u_{b_n}) is bounded in X and $b_n \to 0^+$, then (u_n) strongly converges in X to a weak solution of the local problem $(P)_0$.*

Proof. For saving notation, we write only I_n and u_n to denote I_{b_n} and u_{b_n}, respectively. Since (u_n) is bounded, up to a subsequence, $u_n \rightharpoonup u_0$ weakly in X. For any $v \in X$, we have

$$0 = I_n'(u_n)v = (a + b_n\|u_n\|^2)\int (\nabla u_n \cdot \nabla v) - \int f(x,u_n)v.$$

We know that $b_n\|u_n\|^2 \to 0$, as $n \to +\infty$. Hence, taking the limit in the above expression, using (f_1) and the Lebesgue Dominated Convergence Theorem, we can argue as in the proof of Lemma 3.3 to conclude that

$$0 = a\int (\nabla u_0 \cdot \nabla v) - \int f(x,u)v, \qquad \forall\, v \in X,$$

and therefore u_0 weakly solves $(P)_0$.

In order to prove the strong convergent of (u_n) we first notice that, as before,

$$\lim_{n\to\infty}\int f(x,u_n)u_n = \int f(x,u_0)u_0 = \lim_{n\to\infty}\int f(x,u_n)u_0.$$

Hence, we can use Cauchy-Schwarz's and Young's inequality to get

$$\begin{aligned} a\|u_n - u_0\|^2 &= (I_n'(u_n) - I_0'(u_0))\,(u_n - u_0) + \int (f(x,u_n) - f(x,u_0))(u_n - u_0) \\ &- \quad b_n\|u_n\|^2\int \nabla(u_n - u_0)\cdot\nabla u_n \\ &= o_n(1) + \frac{b_n}{2}\|u_n\|^4 + \frac{b_n}{2}\|u_n\|^2\|u_n - u_0\|^2, \end{aligned}$$

or, equivalently,

$$(a - (b_n/2)\|u_n\|^2)\|u_n - u_0\|^2 \le o_n(1).$$

From the boundedness of (u_n) and $b_n \to 0^+$, it follows that $u_n \to u_0$ strongly in X. This ends the proof. □

We are ready to finish the paper presenting the proof of the concentration result.

Proof of Theorem 1.5. For any $t > 1$, we can use (f_6) to get

$$\begin{aligned} \frac{F(x,t)}{t^4} - F(x,1) &= \int_1^t \frac{d}{ds}\left\{\frac{F(x,s)}{s^4}\right\}ds = \int_1^t \frac{f(x,s)s - 4F(x,s)}{s^5}ds \\ &\le [f(x,t)t - 4F(x,t)]\left(1 - \frac{1}{t^4}\right), \end{aligned}$$

and therefore

$$f(x,t)t - 4F(x,t) \ge \frac{F(x,t)}{t^4} - F(x,1), \quad \forall\, x \in \mathbb{R}^3,\ t > 1.$$

Since we can prove an analogous inequality holds for $t < -1$, we conclude that f satisfies (f_4) with $D = 0$, whenever f satisfies $(f_3)'$ and (f_6). So, without loss of generality, we assume from now on that (f_4) holds.

Suppose that all the conditions of Theorem 1.5 are satisfied. Since $\lambda_1(g_0) > a^{-1}$, we can argue along the same lines of the proof of Lemma 3.1 to obtain α, $\rho > 0$ such that

$$\frac{a}{2}\|u\|^2 - \int F(x,u) \geq \alpha\|u\|^2, \quad \forall\, u \in X \cap B_\rho(0).$$

Thus, the functional I_b satisfies condition (I_6) of Theorem 2.3 with the numbers α, ρ being independent of $b \in [0,1]$. As we can infer from the proof of Theorems 1.2 and 1.4 (in this last case for $\lambda_1(g_0) > m(0)^{-1}$), there exists $e \in X$ such that $\|e\| > \rho$ and $I_1(e) < 0$. Hence, since $I_b(e) \leq I_1(e)$ for any $b \in (0,1)$, we conclude that the Mountain Pass level of the functional I_b is given by

$$c_b := \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} I_b(\gamma(t)) \geq \alpha > 0,$$

where $\Gamma := \{\gamma \in C([0,1],X), \gamma(0) = 0,\, \gamma(1) = e\}$. The main point here is that the set Γ is independent of b, in such way that the monotonicity of $b \mapsto I_b(u)$ implies that the function $b \mapsto c_b$ is nondecreasing.

We now take $(b_n) \subset (0,1)$ such that $b_n \to 0^+$ and call $u_{b_n} \in X$ the solution to $(P)_{b_n}$ given by the Mountain Pass Theorem. We are going to prove that this sequence of solutions is bounded in X. Using the same notation of the previous lemma, we assume, by contradiction, that $\|u_n\| \to +\infty$. If we define $w_n := u_n/\|u_n\|$, then we may suppose that $w_n \rightharpoonup w$ weakly in X and $w_n(x) \to w(x)$ for a.e. $x \in \mathbb{R}^3$. We have that

$$c_1 \geq c_n = I_n(u_n) = I_n(u_n) - \frac{1}{4}I_n'(u_n)u_n \geq \frac{1}{4}\int [f(x,u_n)u_n - 4F(x,u_n)].$$

If $w \neq 0$, we can use (f_4) and the same argument employed in the proof of Lemma 3.4 to obtain a contradiction by using Fatou's Lemma. Hence, we conclude that $w = 0$ and therefore we can use (f_1) (or $(f_1)'$) and the Lebesgue Dominated Convergence Theorem to conclude that

$$\lim_{n\to\infty} \int F(x,Lw_n) = 0, \quad \forall\, L \geq 0.$$

Let $t_n \in [0,1]$ be such that

$$I_n(t_nu_n) = \max_{t\in[0,1]} I_n(tu_n).$$

Since $c_n = I_n(u_n) \leq I_n(t_nu_n)$, we have that $t_n > 0$. Arguing as in the proof of Lemma 4.1 we can show that $t_n = 1$ cannot occur. Thus, $\frac{d}{dt}I_n(tu_n) = 0$ holds true for $t = t_n$. It follows from (f_6) that

$$\begin{aligned}
I_n(t_nu_n) &= I_n(t_nu_n) - \frac{1}{4}I_n'(t_nu_n)(t_nu_n)\\
&= \frac{a}{4}\|t_nu_n\|^2 + \frac{1}{4}\int [f(x,t_nu_n)t_nu_n - 4F(x,t_nu_n)]\\
&\leq \frac{a}{4}\|u_n\|^2 + \frac{1}{4}\int [f(x,u_n)u_n - 4F(x,u_n)]\\
&= I_n(u_n) - \frac{1}{4}I_n'(u_n)(u_n) = I(u_n) = c_n \leq c_1.
\end{aligned}$$

Arguing as in Lemma 4.1 again, we obtain a contradiction from $c_1 \geq I_n(t_nu_n) \geq I_n(Lw_n)$, where $L > 0$ is large enough.

Now we have proved that (u_n) is bounded, we may invoke Lemma 5.1 to conclude that $u_n \to u_0$ strongly in X, with u_0 being a weak solution to the local problem $(P)_0$. Passing the inequality $I_n(u_n) \geq \alpha > 0$ to the limit, we conclude that $I_0(u_0) > 0$. Hence $u_0 \neq 0$ and the theorem is proved. $\square$

References

[1] C.O. Alves, F.J.S.A. Corrêa, T.F. Ma, *Positive solutions for a quasilinear elliptic equation of Kirchhoff type*, Comput. Math. Appl. **49** (2005), 85—93. 5

[2] D. Arcoya, F. O. de Paiva, J. M. Mendoza, *Existence of solutions for a nonhomogeneous elliptic Kircchoff type equation*, J. Math. Anal. Appl. **480** (2019), 123401. 5

[3] A. Azzollini, *The elliptic Kirchhoff equation in $\mathbb{R}^N$ perturbed by a local nonlinearity*, Differential Integral Equations **25** (2012), 543-554. 5

[4] A. Azzollini, *A note on the elliptic Kirchhoff equation in $\mathbb{R}^N$ perturbed by a local nonlinearity*, Commun. Contemp. Math. **17** (2015), Art. ID 1450039, 5 pp 5

[5] A. Ambrosetti, P. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Funct. Anal. **14** (1973), 349–381. 5, 6

[6] P. D'Ancona, S. Spagnolo, *Global solvability for the degenerate Kirchhoff equation with real analytic data*, Invent. Math. **108** (1992), 247–262. 5

[7] A. Arosio, S. Panizzi, *On the well-posedness of the Kirchhoff string*, Trans. Amer. Math. Soc. **348** (1996), 305–330. 5

[8] P. Bartolo, V. Benci, D. Fortunato, *Abstract critical point theorems and applications to some nonlinear problems with strong resonance at infinity*, Nonlinear Anal. **7** (1983), 981–1012. 6

[9] A. M. Batista, M.F. Furtado, *Existence of solution for an asymptotically linear Schrödinger-Kirchhoff equation*, Potential Analysis **50** (2019), 609–619. 5

[10] H. Brezis, H. Lieb, *A relation between pointwise convergence of functions and convergence of functionals*, Proc. Amer. Math. Soc. **88** (1983), 486–490. 7

[11] G. F. Carrier, *On the nonlinear vibration problem of the elastic string*, Quart. Appl. Math. **3** (1945), 157–165. 1

[12] M. M. Cavalcanti, V.N.D. Cavalcanti, J. A. Soriano, *Global existence and uniform decay rates for the Kirchhoff–Carrier equation with nonlinear dissipation*, Adv. Differential Equations **6** (2001), 701–730. 5

[13] C.-Y. Chen, Y.-C. Kuo, T.-F. Wu, *The Nehari manifold for a Kirchhoff type problem involving sign-changing weight functions*, J. Differential Equations **250** (2011), 1876–1908. 5

[14] D.G. Costa, C.A. Magalhães, *Variational elliptic problems which are nonquadratic at infinity*, Nonlinear Analysis **23** (1994), 1401–1412. 3

[15] D.G. de Figueiredo, *Positive solutions of semilinear elliptic problems*, in: Differential equations, São Paulo, in: Lecture Notes in Math., 957, Springer, Berlin-New York, 1982, pp. 34–87. 8

[16] G. Eisley, *Nonlinear vibrations of beams and rectangular plates*, Z. Anger. Math. Phys. **15** (1964), 167–175. 2

[17] G. M. Figueiredo, J. R. Santos Junior, *Multiplicity and concentration behavior of positive solutions for a Schrödinger-Kirchhoff type problem via penalization method*, ESAIM Control Optim. Calc. Var. **20** (2014), 389–415. 5

[18] G. M. Figueiredo, J. R. Santos Junior, *Multiplicity of solutions for a Kirchhoff equation with subcritical or critical growth*, Differential Integral Equations **25** (2012), 853—868. 5

[19] G. M. Figueiredo, A. Suárez, *Some remarks on the comparison principle in Kirchhoff equations*, Rev. Mat. Iberoam. **34** (2018), 609–620. 5

[20] G. Kirchhoff, Vorlesungen über Mathematische Physik: Mechanik, Teubner, Leipzig (1876). 1

[21] S. Li, M. Willem, *Applications to local linking to critical point theory*, J. Math. Anal. Appl. **189** (1995), 6–32. 6

[22] Z. P. Liang, F. Y. Li, J. P. Shi, *Positive solutions to Kirchhoff type equations with nonlinearity having prescribed asymptotic behavior,* Ann. Inst. H. Poincaré Anal. Non Linéaire **31** (2014), 155–167. 5

[23] J. L. Lions, *On some questions in boundary value problems of mathematical physics.* International Symposium on Continuum, Mechanics and Partial Differential Equations, Rio de Janeiro(1977), Mathematics Studies, North- Holland, Amsterdam **30** (1978), 284-346. 2

[24] R. Narashima, *Nonlinear vibration of an elastic string*, J. Sound Vib. **8** (1968), 134–146. 1

[25] D. W. Oplinger, *Frequency response of a nonlinear stretched string*, J. Acoust. Soc. Am. **32** (1960), 1529–1538. 1

[26] K. Perera, Z. Zhang, *Nontrivial solutions of Kirchhoff-type problems via the Yang index*, J. Differential Equations **221** (2006), 246–255. 5

[27] K. Schlesinger, *Saitenschwingungen mit endlicher amplitude.*, Z. Techn. Phys. **12** (1931), 33–39. 1

[28] J. Sun, S. Liu, *Nontrivial solutions of Kirchhoff type problems*, Appl. Math. Lett. **25** (2012), 500–504. 5

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