EXISTENCE AND MULTIPLICITY OF POSITIVE SOLUTIONS FOR A FOURTH-ORDER ELLIPTIC EQUATION

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Abstract. We prove existence and multiplicity of solutions for the problem

$$\begin{cases} \Delta^2 u + \lambda \Delta u = |u|^{2^*-2}u, \text{ in } \Omega, \\ u, -\Delta u > 0, \text{ in } \Omega, \quad u = \Delta u = 0, \text{ on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 5$, is a bounded regular domain, $\lambda > 0$ and $2^* = 2N/(N-4)$ is the critical Sobolev exponent for the embedding of $W^{2,2}(\Omega)$ into the Lebesgue spaces.

1. Introduction

For a bounded smooth domain $\Omega \subset \mathbb{R}^N$, consider the critical problem

$$(BN_\lambda) \qquad \begin{cases} -\Delta u = \lambda |u|^{s-2}u + |u|^{4/(N-2)}u, \text{ in } \Omega, \\ u > 0, \text{ in } \Omega, \qquad u = 0, \text{ on } \partial\Omega, \end{cases}$$

for $2 \leq s < 2N/(N-2)$. In their celebrated paper [5], Brezis and Nirenberg proved that, for $s = 2$, the existence of solution is related with the interaction of λ with the first eigenvalue of $(-\Delta, W_0^{1,2}(\Omega))$, namely

$$\lambda_1(\Omega) := \inf \left\{ \int_\Omega |\nabla u|^2 dx : u \in W_0^{1,2}(\Omega), \int_\Omega |u|^2 dx = 1 \right\}.$$

More specifically, they proved the following:

- there is no solution for $\lambda \geq \lambda_1(\Omega)$ and, if Ω is starshaped, for $\lambda \leq 0$;
- if $N \geq 4$, there is a solution for $0 < \lambda < \lambda_1(\Omega)$;
- if $N = 3$, there is a solution for $\lambda < \lambda_1(\Omega)$ close to $\lambda_1(\Omega)$. Moreover, if Ω is a ball, there is no solution for $\lambda > 0$ close to 0.

Since we cannot solve the problem in the entire range $(0, \lambda_1(\Omega))$ when $N = 3$, we say that it is the critical dimension for problem (BN_λ) (see [17] for the notion of critical dimension for the polyharmonic operator). In the case $2 < s < 2N/(N-4)$, they obtained solution for any $\lambda > 0$. After this, a lot of papers concerning critical nonlinearities appeared. In particular, we recall that Rey [19] and Lazzo [12] proved, for $s = 2$, that the problem has at least $\mathrm{cat}(\Omega)$ solutions if $\lambda > 0$ is close to 0 (see [6, 7, 1] for related results). Here, $\mathrm{cat}(\Omega)$ stands for the usual Ljusternik-Schnirelmann category of $\overline{\Omega}$ in itself.

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Some of the aforementioned results were extend to the fourth-order problem

$$(V_\mu) \qquad \begin{cases} \Delta^2 u = \mu|u|^{s-2}u + |u|^{2^*-2}u, \text{ in } \Omega, \\ u,\ -\Delta u > 0, \text{ in } \Omega, \qquad u = \Delta u = 0, \text{on } \partial\Omega, \end{cases}$$

where $2 \le s < 2^* := 2N/(N-4)$. In this case, if we denote by $\mu_1(\Omega)$ the first eigenvalue of

$$\Delta^2 u = \mu u, \text{ in } \Omega, \qquad u = \Delta u = 0, \text{ on } \partial\Omega,$$

it was proved by van der Vorst in [22] that, for $s = 2$,

- there is no solution for $\mu \ge \mu_1(\Omega)$ and, if Ω is starshaped, for $\mu \le 0$;
- if $N \ge 8$, there is a solution for $0 < \mu < \mu_1(\Omega)$;
- if $N \in \{5, 6, 7\}$, there is a solution for $\mu < \mu_1(\Omega)$ close to $\mu_1(\Omega)$.

In the same paper the author conjectured that these former dimensions are critical. This conjecture was considered by Gazzola, Grunau and Squassina in [9], where they proved that

- if $N \in \{5, 6, 7\}$ and Ω is a ball, then (V_μ) has no solution for $\mu > 0$ close to 0.

The case $2 < s < 2^*$ was treated in [13], where the authors obtained a solution if $N \ge 8$ and $2 \le s < 2^*$, or $N \in \{5, 6, 7\}$ and $2^* - 2 < s < 2^*$. They also proved that the problem has $\text{cat}(\Omega)$ solutions for $\mu > 0$ close to 0.

In this paper we address the problem

$$(P_\lambda) \qquad \begin{cases} \Delta^2 u + \lambda\Delta u = |u|^{2^*-2}u, \text{ in } \Omega, \\ u,\ -\Delta u > 0, \text{ in } \Omega, \quad u = \Delta u = 0, \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$, $N \ge 5$, is a bounded domain and $\lambda > 0$. Differently from (V_μ), existence and non existence are related with the first eigenvalue $\lambda_1(\Omega)$ of $(-\Delta, W_0^{1,2}(\Omega))$. Indeed, our first aim is to identify the range of (possible) existence for the parameter λ and therefore establish the notion of critical dimension. We prove the following:

Theorem 1.1. *The following holds:*

1. *if $\lambda \ge \lambda_1(\Omega)$, then (P_λ) has no solution;*
2. *if Ω is star shaped with respect to the origin and $\lambda \le 0$, then (P_λ) has no solutions in $C^4(\overline{\Omega})$;*
3. *if $N = 5$ and Ω is a ball, then there exists $\lambda_* > 0$ such that (P_λ) has no solution if $\lambda < \lambda_*$.*

The proof of the first two items rely on classical arguments and a Pohozaev identity. For the last one, we take advantage of the radiality of the domain to proceed with an ODE approach. The restriction on the dimension is closely related with the existence of the embedding $W^{3,2}(\Omega) \hookrightarrow L^\infty(\Omega)$, which holds if $N < 6$. The above nonexistence result complement those of [17, 15, 16, 9] which deal with problem (V_μ) under Dirichlet or Navier boundary conditions.

In our second result we follow the ideas of Brezis and Nirenberg [5] for obtaining existence of solution:

Theorem 1.2. *Suppose that $\Omega \in C^{4,\alpha}$, for some $0 < \alpha < 1$. Then (P_λ) has at least one $C^{4,\alpha}(\overline{\Omega})$ solution if*

1. *$N \ge 6$ and $\lambda \in (0, \lambda_1(\Omega))$;*

(2) $N = 5$ *and* $\lambda \in (\lambda^*, \lambda_1(\Omega))$*, with* $\lambda^* \in (0, \lambda_1(\Omega))$.

Roughly speaking, Theorem 1.2 says that, differently from the biharmonic version of the Brezis-Nirenberg problem (V_μ), the critical dimension for (P_λ) is only $N = 5$. Actually, as obseved in [14], the notion of critical dimension is also related with the integrability of the L^2-norm of the gradient of the functions which realize the best constant of the embedding $W^{2,2}(\Omega) \hookrightarrow L^{2^*}(\Omega)$, namely $U_\varepsilon(x) = c_N[\varepsilon(\varepsilon^2 + |x|^2)^{-1}]^{(N-4)/2}$, for an appropriated value of $c_N > 0$. It is clear that, if $N = 5$ and Ω is a ball, then $\lambda_* \leq \lambda^*$. Unfortunately, we do not know what happens if $\lambda \in [\lambda_*, \lambda^*]$ or even the sharp value of λ_*.

In view of the results proved in [12, 19, 13], it is natural to ask if, as in (BN_λ) and (V_μ), we have more solutions if Ω has rich topology. In our last result we give a positive answer to this questions by proving the following multiplicity result:

Theorem 1.3. *Suppose that* $\Omega \in C^{4,\alpha}$*, for some* $0 < \alpha < 1$ *and* $N \geq 6$*. Then there exists* $\lambda_{**} \in (0, \lambda_1(\Omega))$ *such that* (P_λ) *has at least* $cat_\Omega(\Omega)$ *solutions if* $\lambda \in (0, \lambda_{**})$.

In the proof we apply classical Ljusternik-Schnirelmann theory, as done in [12] for the Brezis-Nirenberg problem (see also [2] for the fourth order problem with Dirichlet boundary conditions). One of the key points is how to extend functions of $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ to $\mathcal{D}^{2,2}(\mathbb{R}^N)$. This kind of problem has already appeared in [9] in the proof of a splitting lemma for problem (V_μ). Here, we borrow an extension procedure from [4] which works if we have regularity for the solutions of the problem (P_λ). Since we do not find in the literature the appropriated regularity theorems we show in Section 2 how we can use the L^p-regularity theory even in the case $\lambda > 0$. The same arguments show that, if $u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ is a critical point of the associated energy functional, it verifies the boundary condition of (P_λ) in the trace sense.

The main results of this paper complement the aforementioned works, since we deal here with a different equation. It seems that the perturbation Δu gives a new nature for the problem, since the critical dimensions are different from (V_μ). We believe that many others situations can be considered, for instance the existence of nodal solutions, high-energy solutions or even other type of multiplicity results.

The paper is organized in the following way: Section 2 is devoted to the proof of the non existence results. In Section 3 we present the proof of Theorem 1.2 and, in Section 4, we prove our multiplicity result. The paper also contains an appendix concerning the regularity of the solutions.

2. The nonexistence results

Throughout the paper we denote by H the Hilbert space $W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)$ endowed with the norm

$$\|u\| := \left(\int_\Omega (\Delta u)^2 dx \right)^{1/2}.$$

We denote by $\|u\|_r$ the $L^r(\Omega)$-norm of a function $u \in H$ and write only $\int_\Omega u$ instead of $\int_\Omega u(x)dx$.

The following result is an easy consequence of the spectral theory of the Laplacian and Holder's inequality. We include the proof just for completeness.

Lemma 2.1. *The space* H *is compactly embedded into* $W_0^{1,2}(\Omega)$.

Proof. Let $(\varphi_k)_{k\in\mathbb{N}}$ be the eigenfunctions of $(-\Delta, W_0^{1,2}(\Omega))$ and $(\lambda_k)_{k\in\mathbb{N}}$ it associated eigenvalues. Since they are orthogonal in $W_0^{1,2}(\Omega)$ and $L^2(\Omega)$, the same occurs in H. Hence, if $u=\sum_{k=1}^{\infty}\alpha_k\varphi_k\in H$, we can compute

$$\|u\|^2=\sum_{k=1}^{\infty}\alpha_k^2\|\varphi_k\|^2=\sum_{k=1}^{\infty}\alpha_k^2\lambda_k\|\nabla\varphi_k\|_2^2\geq\lambda_1(\Omega)\sum_{k=1}^{\infty}\alpha_k^2\|\nabla\varphi_k\|_2^2=\lambda_1(\Omega)\|\nabla u\|_2^2, \tag{2.1}$$

which proves the continuous embedding $H\hookrightarrow W_0^{1,2}(\Omega)$. The compactness follows from the inequality

$$\|\nabla u\|_2^2=\int(\nabla u\cdot\nabla u)=-\int(u\Delta u)\leq\|u\|_2\|u\|$$

and the compactness of $W_0^{1,2}(\Omega)\hookrightarrow L^2(\Omega)$. □

We notice that, if $u\in C^4(\overline{\Omega})$ verifies $u=\Delta u=0$ on $\partial\Omega$, then

$$\int_\Omega\Delta^2u(x\cdot\nabla u)\,dx=-\frac{N-4}{2}\int_\Omega(\Delta u)^2dx+\int_{\partial\Omega}\frac{\partial u}{\partial\nu}\frac{\partial\Delta u}{\partial\nu}(x\cdot\nu)\,d\sigma, \tag{2.2}$$

$$\int_\Omega|u|^{2^*-2}u(x\cdot\nabla u)\,dx=-\frac{N}{2^*}\int_\Omega|u|^{2^*}dx, \tag{2.3}$$

$$\int_\Omega\Delta u(x\cdot\nabla u)\,dx=\frac{1}{2}\int_{\partial\Omega}|\nabla u|^2(x\cdot\nu)\,d\sigma+\frac{N-2}{2}\int_\Omega|\nabla u|^2dx, \tag{2.4}$$

where ν stands for the outward normal unitary vector. Actually, the last two equalities are standard (see [23, Appendix B]) and the first one is proved in [10, Section 7.4].

As a consequence of the above expression, we can prove our non existence result.

Proof of Theorem 1.1. Let $\varphi_1>0$ be an eigenfunction associated to $\lambda_1(\Omega)$. By using it as a test function in the weak formulation of (P_λ) we get

$$\lambda_1(\Omega)^2\int_\Omega u\varphi_1=\int_\Omega\Delta u\Delta\varphi_1=\lambda\int_\Omega\nabla u\cdot\nabla\varphi_1+\int_\Omega u^{2^*-1}\varphi_1>\lambda\lambda_1(\Omega)\int_\Omega u\varphi_1,$$

and therefore $\lambda<\lambda_1(\Omega)$ and the first item is proved.

If $u\in C^4(\overline{\Omega})$ is a classical solution of (P_λ) we can set $w:=\Delta u$ to get

$$\Delta(\Delta u+\lambda u)=\Delta w+\lambda w>0,\text{ in }\Omega,\qquad w=0,\text{ on }\partial\Omega.$$

Since $\lambda\leq 0$, it follows from Hopf's lemma that $\frac{\partial(\Delta u)}{\partial\nu}>0$ on $\partial\Omega$. By using the Maximum Principle we obtain

$$\Delta u+\lambda u<0,\text{ in }\Omega,\qquad u=0,\text{ on }\partial\Omega,$$

and therefore $\frac{\partial u}{\partial\nu}<0$ on $\partial\Omega$. If Ω is starshaped with respect to the origin we have that $x\cdot\nu>0$ on $\partial\Omega$, and therefore

$$A_1:=\int_{\partial\Omega}\frac{\partial u}{\partial\nu}\frac{\partial\Delta u}{\partial\nu}(x\cdot\nu)\,d\sigma<0,\qquad A_2:=\frac{1}{2}\int_{\partial\Omega}|\nabla u|^2(x\cdot\nu)\,d\sigma\geq 0.$$

By multiplying first equation in (P_λ) by $(x\cdot\nabla u)$, integrating by parts and using $(2.2)-(2.4)$, we obtain

$$-\frac{N-4}{2}\|u\|^2+A_1=-\lambda A_2-\frac{\lambda(N-2)}{2}\|\nabla u\|_2^2-\frac{N-4}{2}\|u\|_{2^*}^{2^*}.$$

Since $\|u\|^2 = \lambda\|\nabla u\|_2^2 + \|u\|_{2^*}^{2^*}$, $A_1 < 0$ and $A_2 \geq 0$, the above expression implies that

$$0 = \lambda\|\nabla u\|_2^2 + A_1 + \lambda A_2 < 0,$$

which is a contradiction.

We now prove the last statement of Theorem 1.1. Without loss of generality we suppose that $\Omega = B := \{x \in \mathbb{R}^N : \|x\| < 1\}$. By elliptic regularity Δ^{-1} is continuous from $W_0^{1,2}(B)$ to $W^{3,2}(B) \cap W_0^{1,2}(B)$. Since $N = 5$, this last space is embedded into $L^\infty(B)$ and therefore $(\Delta^{-1})^* : (L^\infty(B))^* \to (W_0^{1,2}(B))^* \simeq W_0^{1,2}(B)$ is also continuous. Hence, there exists $C_1 > 0$ such that

$$|\langle(\Delta^{-1})^* f, \phi\rangle| \leq C_1\|f\|_{(L^\infty(B))^*}\|\phi\|_{W_0^{1,2}(B)}, \tag{2.5}$$

for any $f \in (L^\infty(B))^*$ and $\phi \in W_0^{1,2}(B)$.

If $u \in C^4(\overline{B})$ is a solution of (P_λ) we can define $f_u \in (L^\infty(B))^*$ by $f_u(v) := \int_B v(\Delta^2 u)dx$ and compute

$$\langle(\Delta^{-1})^* f_u, u\rangle = f_u(\Delta^{-1}u) = \int_B (\Delta^{-1}u)(\Delta^2 u)dx = \int_B u(\Delta u)dx = -\int_B |\nabla u|^2 dx.$$

Since $\|f_u\|_{(L^\infty(B))^*} = \int_B (\Delta^2 u)dx$, it follows from the above equality and (2.5) that, for some $C_2 > 0$, there holds

$$C_2\int_B |\nabla u|^2 dx \leq \left(\int_B (\Delta^2 u)dx\right)^2. \tag{2.6}$$

On the other hand, we can use and the fact that u is a solution of (P_λ) to write

$$\lambda\int_B |\nabla u|^2 dx = -\int_{\partial B} \frac{\partial u}{\partial \nu}\frac{\partial \Delta u}{\partial \nu}(x\cdot\nu)d\sigma - \frac{\lambda}{2}\int_{\partial B} |\nabla u|^2(x\cdot\nu)d\sigma$$

Notice that $(u, -\Delta u) \in C^2(\overline{B}) \times C^2(\overline{B})$ satisfies the system $-\Delta u_i = f_i(u_1, u_2)$ in B, $u_i = 0$ on ∂B, $i = 1, 2$, with $f_1(u_1, u_2) := u_2$ and $f_2(u_1, u_2) := \lambda u_2 + u_1^{2^*-1}$. By [20, Theorem 1], we conclude that u and Δu are radially symmetric. Hence, writing $u = u(r)$, we obtain

$$\begin{aligned}\lambda\|\nabla u\|_{L^2(B)}^2 &= -\int_{\partial B} u'(1)(\Delta u)'(1)d\sigma - \frac{\lambda}{2}\int_{\partial B}(u'(1))^2 d\sigma\\ &= -\frac{1}{5\omega_5}\left(\int_{\partial B} u'(1)d\sigma\right)\left(\int_{\partial B}(\Delta u)'(1)d\sigma\right) - \frac{\lambda}{10\omega_5}\left(\int_{\partial B} u'(1)d\sigma\right)^2,\end{aligned}$$

where ω_5 is volume of B_1. If follows from Divergence's Theorem that

$$\lambda\|\nabla u\|_{L^2(B)}^2 = \frac{1}{5\omega_5}\left(\int_B(-\Delta u)dx\right)\left(\int_B(\Delta^2 u)dx\right) - \frac{\lambda}{10\omega_5}\left(\int_B(\Delta u)dx\right)^2. \tag{2.7}$$

Since the function $w(x) := (1 - |x|^2)/(2N)$ satisfies $-\Delta w = 1$ in B, $w = 0$ on ∂B, we have that

$$\int_B -\Delta u\, dx = \int_B w(\Delta^2 u)\, dx. \tag{2.8}$$

As proved in [10, p. 278], there exists $C_3 > 0$ such that

$$\int_B (\Delta^2 u)dx \leq C_3\int_B w(\Delta^2 u)dx \leq C_3\int_B (\Delta^2 u)dx.$$

This, (2.7), (2.8) and (2.6) imply that

$$\lambda\|\nabla u\|^2_{L^2(B)} \geq \frac{1}{\omega_5}\left(\frac{1}{5C_3}-\frac{\lambda}{10}\right)\left(\int_B (\Delta^2 u)dx\right)^2 \geq \frac{C_2}{\omega_5}\left(\frac{1}{5C_3}-\frac{\lambda}{10}\right)\|\nabla u\|^2_{L^2(B)},$$

for any $\lambda < 2/C_3$. Hence, if we set

$$\lambda_* := \min\left\{\frac{2}{C_3}, \frac{2C_2}{C_3(C_2+10\omega_5)}\right\},$$

we can easily conclude that $u = 0$ whenever $\lambda < \lambda_*$. □

3. The existence result

For each $\lambda \geq 0$, we define the functional $I_\lambda : H \to \mathbb{R}$ as

$$I_\lambda(u) = \frac{1}{2}\int_\Omega (\Delta u)^2 - \frac{\lambda}{2}\int_\Omega |\nabla u|^2 - \frac{1}{2^*}\int_\Omega (u^+)^{2^*},$$

where $u^+(x) := \max\{u(x), 0\}$. Standard arguments show that the critical points of $I_\lambda \in C^1(H, \mathbb{R})$ weakly satisfy the equation in (P_λ).

We notice that the boundary condition $\Delta u = 0$ on $\partial\Omega$ is not satisfied for a general function of the space H. However, we are able to adapt the L^p-regularity theory to prove that the it holds for the critical points of I_λ. Actually, we shall prove in the final section of this paper the following regularity result.

Proposition 3.1. *Let $q := 2N/(N+4)$ and suppose that $u \in H$ is such that $I'_\lambda(u) = 0$. Then,*

(i) *if $\Omega \in C^{3,\alpha}$, for some $0 < \alpha < 1$, then $u \in W^{4,q}(\Omega) \cap W^{1,q}_0(\Omega)$, $\Delta u \in W^{2,q}(\Omega) \cap W^{1,q}_0(\Omega)$ and $\Delta^2 u = -\lambda\Delta u + u^{2^*-1}$ a.e. in Ω;*

(ii) *if $\Omega \in C^{4,\alpha}$, for some $0 < \alpha < 1$, then $u \in C^{4,\alpha}(\overline{\Omega})$.*

Proof. We present the proof in the appendix since the case $\lambda > 0$ requires an adaptation of the classical elliptic regularity arguments. □

In what follows we are intending to apply the Mountain Pass Theorem to obtain a nonzero critical point of I_λ.

Lemma 3.2. *If $\lambda < \lambda_1(\Omega)$ and $(u_n) \subset H$ is such that*

$$I_\lambda(u_n) \to c < \frac{2}{N}S^{N/4}, \quad I'_\lambda(u_n) \to 0,$$

then (u_n) has a convergent subsequence

Proof. Since $I'_\lambda(u_n) \to 0$, we can use (2.1) to obtain

$$c + o_n(1) + o_n(1)\|u_n\| \geq I_\lambda(u_n) - \frac{1}{2^*}I'_\lambda(u_n)u_n \geq \frac{2}{N}\left(\frac{\lambda_1(\Omega)-\lambda}{\lambda_1(\Omega)}\right)\|u_n\|^2,$$

where $o_n(1)$ stands for a quantity converging zero as $n \to +\infty$. Hence, $(u_n) \subset H$ is bounded and we may suppose that $u_n \rightharpoonup u$ weakly in H for some $u \in H$. This, $I'_\lambda(u_n) \to 0$ and Lemma 2.1 imply that $I'_\lambda(u) = 0$ and therefore we can compute $I_\lambda(u) = I_\lambda(u) - (1/2)I'_\lambda(u)u = (2/N)\|u^+\|^{2^*}_{2^*} \geq 0$.

If we set $v_n := u_n - u$, it follows from a version of the Brezis-Lieb's lemma (see [23, Lemma 1.32]) that $\|v_n^+\|^{2^*}_{2^*} = \|u_n^+\|^{2^*}_{2^*} - \|u^+\|^{2^*}_{2^*} + o_n(1)$. Thus,

$$I'_\lambda(v_n)v_n = I'_\lambda(u_n)u_n - I'_\lambda(u)u + o_n(1) = o_n(1).$$

Since $v_n \to 0$ in $W_0^{1,2}(\Omega)$, we conclude that, for some $b \geq 0$,

$$\lim_{n\to+\infty} \|v_n\|^2 = b = \lim_{n\to+\infty} \|v_n^+\|_{2^*}^{2^*}. \tag{3.1}$$

By definition, $\|v_n\|^2 \geq S\|v_n\|_{2^*}^2 \geq S(\|v_n^+\|_{2^*}^{2^*})^{2^*/2}$, and therefore the above equations imply that $b \geq Sb^{2^*/2}$. If $b > 0$, we conclude that $b \geq S^{N/4}$. However, since

$$I_\lambda(v_n) = I_\lambda(u_n) - I_\lambda(u) + o_n(1)$$

and $I_\lambda(u) \geq 0$, we can take limit as $n \to +\infty$, use (3.1) and $v_n \to 0$ in $W_0^{1,2}(\Omega)$ again to get

$$\frac{2}{N}S^{N/4} \leq \left(\frac{1}{2} - \frac{1}{2^*}\right) b \leq c < \frac{2}{N}S^{N/4},$$

which does not make sense. Hence $b = 0$, that is, $u_n \to u$ in H. $\square$

Let $\mathcal{D}^{2,2}(\mathbb{R}^N)$ be the closure of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm $\|\cdot\|$ and consider

$$S := \inf_{u\in\mathcal{D}^{2,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N}(\Delta u)^2}{\left(\int_{\mathbb{R}^N}|u|^{2^*}\right)^{2/2^*}}.$$

We know from [21] that $S > 0$ is attained by the family of functions

$$U_\varepsilon(x) := c_N\left(\frac{\varepsilon}{\varepsilon^2 + |x|^2}\right)^{(N-4)/2},$$

with $\varepsilon > 0$ and $c_N := [N(N-4)(N^2-4)]^{(N-4)/8}$.

Lemma 3.3. *If $N \geq 6$, there exists $u \in H \setminus \{0\}$ such that*

$$\max_{t\geq 0} I_\lambda(tu) < \frac{2}{N}S^{N/4}.$$

The same holds if $N = 5$ and $\lambda < \lambda_1(\Omega)$ is sufficiently close to $\lambda_1(\Omega)$.

Proof. We first consider $N \geq 6$ and assume, without loss of generality , that $0 \in \Omega$. Let $\phi \in C_0^\infty(\Omega, [0,1])$ be such that $\phi \equiv 1$ in $B_r(0)$ and $\phi \equiv 0$ outside $B_{2r}(0)$, where $r > 0$ is such that $B_{2r}(0) \subset \Omega$. For $\varepsilon > 0$, we set $u_\varepsilon(x) := \phi(x)U_\varepsilon(x)$ and use the calculations of [3, p. 236] to write

$$\|u_\varepsilon\|^2 = S^{N/4} + O(\varepsilon^{N-4}), \qquad \|u_\varepsilon\|_{2^*}^{2^*} = S^{N/4} + O(\varepsilon^N), \tag{3.2}$$

as $\varepsilon \to 0^+$. Moreover, we can compute

$$\begin{aligned}
\|\nabla u_\varepsilon\|_2^2 &= \int_{B_{2r}(0)} |\nabla U_\varepsilon|^2 \\
&\quad + \int_{B_{2r}(0)} \left[(\phi^2 - 1)|\nabla U_\varepsilon|^2 + 2\rho U_\varepsilon(\nabla\phi\cdot\nabla U_\varepsilon) + U_\varepsilon^2|\nabla\phi|^2\right] \\
&= \int_{B_{2r}(0)} |\nabla U_\varepsilon|^2 + O(\varepsilon^{N-4}).
\end{aligned}$$

Since $\int_{B_{2r}(0)} |\nabla U_\varepsilon|^2 = c_N^2(4-N)^2\varepsilon^2 A_\varepsilon$, with

$$A_\varepsilon := \int_{B_{(2r)/\varepsilon}(0)} \frac{|y|^2}{(1+|y|^2)^{N-2}}dy,$$

we have that

$$\|\nabla u_\varepsilon\|_2^2 = c_N^2(4-N)^2\varepsilon^2 A_\varepsilon + O(\varepsilon^{N-4}).$$

We infer from the above expression and (3.2)

$$\begin{aligned}\max_{t\geq 0} I_\lambda(tu_\varepsilon) &= \frac{2}{N}\left[\frac{\|u_\varepsilon\|^2-\lambda\|\nabla u_\varepsilon\|_2^2}{\|u_\varepsilon\|_{2^*}^2}\right]^{N/4}\\ &= \frac{2}{N}\left[S+\varepsilon^2\left(O(\varepsilon^{N-6})-\lambda c_N^2(4-N)^2A_\varepsilon\right)\right]^{N/4}.\end{aligned}$$

Since $A_\varepsilon \to d > 0$ if $N > 6$, and $A_\varepsilon \to +\infty$ if $N = 6$, the lemma holds with $u := u_\varepsilon$, $\varepsilon > 0$ small.

The above argument does not hold if $N = 5$. In this case we consider $\varphi_1 > 0$ the first eigenfunction of $(-\Delta, W_0^{1,2}(\Omega))$, and notice that

$$\lim_{\lambda\to\lambda_1(\Omega)^-}\frac{\|\varphi_1\|^2-\lambda\|\nabla\varphi_1\|_2^2}{\|\varphi_1\|_{10}^2}=\lim_{\lambda\to\lambda_1(\Omega)^-}\lambda_1(\Omega)(\lambda_1(\Omega)-\lambda)\frac{\|\varphi_1\|_2^2}{\|\varphi_1\|_{10}^2}=0.$$

Hence, there exists $\lambda^* \in (0, \lambda_1(\Omega))$ such that,

$$\max_{t\geq 0} I_\lambda(t\varphi_1)=\frac{2}{5}\left[\frac{\|\varphi_1\|^2-\lambda\|\nabla\varphi_1\|_2^2}{\|\varphi_1\|_{10}^2}\right]^{5/4}<\frac{2}{5}S^{5/4},$$

for any $\lambda \in (\lambda^*, \lambda_1(\Omega))$. The lemma is proved. □

We are ready to present the proof of our existence result.

Proof of Theorem 1.2. Suppose that λ belongs to an interval such that Lemma 3.3 holds. By using the definition of S we obtain

$$I_\lambda(u)\geq\|u\|^2\left(\frac{\lambda_1(\Omega)-\lambda}{2\lambda_1(\Omega)}-\frac{S^{-2/2^*}}{2^*}\|u\|^{2^*-2}\right),$$

and therefore there exists ρ, $\alpha > 0$ such that $\inf_{\partial B_\rho(0)} I_\lambda \geq \alpha$. If we consider $\varepsilon > 0$ small and $u_\varepsilon \in H$ as in the proof of Lemma 3.3, we have that $I_\lambda(tu_\varepsilon) \to -\infty$ as $t \to +\infty$. Hence, the set $\Gamma := \{\gamma \in C([0, 1], H) : \gamma(0) = 0,\ I_\lambda(\gamma(1)) < 0\}$ is non empty and we obtain from the Mountain Pass Theorem a sequence $(u_n) \subset H$ such that $I'_\lambda(u_n) \to 0$ and

$$I_\lambda(u_n)\to c_\lambda:=\inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I_\lambda(\gamma(t)). \tag{3.3}$$

The definition of Γ and Lemma 3.3 imply that $c_\lambda < (2/N)S^{N/4}$. Hence, by Lemma 3.2, we may suppose that $u_n \to u$ strongly in H. By the regularity of I_λ we get $I'_\lambda(u) = 0$ and $I_\lambda(u) \geq \alpha > 0$, that is, u is a nonzero weak solution of (P_λ).

From Theorem 3.1 we conclude that $u \in C^{4,\alpha}(\overline{\Omega})$ and $\Delta u = 0$ on $\partial\Omega$. We shall prove that $u \geq 0$ in Ω. Indeed, let $\psi \in H$ be such that

$$\int_\Omega \nabla\psi\cdot\nabla\phi=\int_\Omega u^-\phi,\qquad \forall\,\phi\in W_0^{1,2}(\Omega),$$

where $u^-(x) := \max\{-u(x), 0\}$. By picking $\phi = \psi^-$, we obtain $-\int_\Omega |\nabla\psi^-|^2 = \int_\Omega u^-\psi^- \geq 0$. Thus, $\psi^- \equiv 0$ and we get

$$0=I'_\lambda(u)\psi\leq\int_\Omega \Delta u\Delta\psi-\lambda\int_\Omega\nabla u\cdot\nabla\psi=-\int_\Omega|\nabla u^-|^2+\lambda\int_\Omega(u^-)^2,$$

from which it follows that $\left(1-\frac{\lambda}{\lambda_1(\Omega)}\right)\|\nabla u^-\|_2^2$, that is, $u^- \equiv 0$. We can now decompose the equation of (P_λ) in two second-order equations and use $0 < \lambda <$

$\lambda_1(\Omega)$ and the strong Maximum Principle to conclude that $u > 0$ in Ω, and $-\Delta u > 0$ in Ω. The theorem is proved. $\square$

4. The multiplicity result

In this section we prove Theorem 1.3. We start by introducing the following minimax level

$$\widehat{c}_\lambda := \inf_{u\in\mathcal{N}_\lambda} I_\lambda(u),$$

where $\mathcal{N}_\lambda := \{u \in H \setminus \{0\} : I'_\lambda(u)u = 0\}$ is the Nehari manifold of I_λ. If $\lambda \in (0, \lambda_1(\Omega))$ and $N \geq 6$, we may invoke Theorem 1.2 to obtain $u_\lambda \in H \cap \mathcal{N}_\lambda$ such that $I'_\lambda(u_\lambda) = 0$ and $I_\lambda(u_\lambda) = c_\lambda$. Thus, we get (see [18, Proposition 3.11])

$$c_\lambda \leq \inf_{u\in H\setminus\{0\}} \sup_{t\geq 0} I_\lambda(tu) \leq \inf_{u\in\mathcal{N}_\lambda} \sup_{t\geq 0} I_\lambda(tu) = \inf_{u\in\mathcal{N}_\lambda} I_\lambda(u) = \widehat{c}_\lambda \leq I_\lambda(u_\lambda) = c_\lambda.$$

and therefore $c_\lambda = \widehat{c}_\lambda$. Thus, arguing along the same lines of [13, Lemma 2.4], we can prove that

$$\lim_{\lambda\to 0^+} c_\lambda = c_0 = \frac{2}{N} S^{N/4}. \tag{4.1}$$

Given a function $f \in C^{0,\alpha}(\overline{\Omega})$ such that $f \equiv 0$ on $\partial\Omega$, it is well defined its $C^{0,\alpha}$-extension given by

$$\overline{f}(x) = \begin{cases} f(x), & \text{if } x \in \Omega, \\ 0, & \text{otherwise.} \end{cases}$$

The proof of the next lemma can be found in [14, Lemma 3.2].

Lemma 4.1. *Let $(v_n) \subset H$ be such that*

$$\int_\Omega |v_n|^{2^*} \geq 1, \qquad \int_\Omega (\Delta v_n)^2 = S + o_n(1),$$

and w_n be the Newtonian potential of $\overline{-\Delta v_n}$, that is,

$$w_n(x) := \frac{1}{N(N-2)\omega_N} \int_{\mathbb{R}^N} \frac{\overline{-\Delta v_n}(z)}{|x-z|^{N-2}} dz,$$

with ω_N being the volume of the unit ball $B_1(0) \subset \mathbb{R}^N$. Then there exists $(y_n, \mu_n) \subset \mathbb{R}^N \times (0, +\infty)$ such that $y_n \to y \in \overline{\Omega}$, $\mu_n \to 0$ and

$$\phi_n(x) := \mu_n^{(N-4)/2} w_n(\mu_n x + y_n)$$

strongly converges to ϕ in $\mathcal{D}^{2,2}(\mathbb{R}^N)$, with $\|\Delta\phi\|_2^2 = S$.

For $r > 0$, we define

$$\Omega_r^+ := \left\{x \in \mathbb{R}^N : \operatorname{dist}(x, \Omega) \leq r\right\}, \qquad \Omega_r^- := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \geq r\}.$$

We suppose that r is small in such way that $B_{2r}(0) \subset \Omega$ and $\operatorname{cat}_\Omega(\Omega) = \operatorname{cat}_{\Omega_r^+}(\Omega_r^-)$. We also set $H_r := W^{2,2}(B_r(0)) \cap W_0^{1,2}(B_r(0))$,

$$J_\lambda(u) := \frac{1}{2}\int_{B_r(0)} (\Delta u)^2 - \frac{\lambda}{2}\int_{B_r(0)} |\nabla u|^2 - \frac{1}{2^*}\int_{B_r(0)} (u^+)^{2^*}, \quad \forall\, u \in H_r,$$

and

$$m_\lambda := \inf_{u\in\mathcal{M}_\lambda} J_\lambda(u) = \inf_{u\in H_r\setminus\{0\}} \max_{t\geq 0} J_\lambda(tu),$$

where $\mathcal{M}_\lambda$ is the Nehari manifold of the functional J_λ.

Lemma 4.2. *If* $0 < \lambda < \lambda_1(\Omega)$*, then* $c_\lambda < m_\lambda$*.*

Proof. The first eigenvalue $\lambda_1(B_r(0))$ of $(-\Delta, W_0^{1,2}(B_r(0))$ verifies $\lambda_1(\Omega) < \lambda_1(B_r(0))$. Thus, we obtain from Theorem 1.2 a function $v_\lambda \in C^{4,\alpha}(\overline{B_r(0)})$ such that $J_\lambda(v_\lambda) = m_\lambda$.

Since $\Delta v_\lambda \in C^{2,\alpha}(\overline{B_r(0)})$, we can consider $u \in C^{2,\alpha}(\overline{\Omega})$ as being the solution of

$$\begin{cases} -\Delta u = \overline{-\Delta v_\lambda}, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

Recalling that $-\Delta v_\lambda > 0$ in $B_r(0)$ and using the Maximum Principle we conclude that $u > 0$ in Ω. Moreover, since $v_\lambda = 0$ on $\partial B_r(0)$, the Maximum Principle also implies that $u \geq v_\lambda$ in $B_r(0)$, with strict inequality in a subset of positive measure. Hence,

$$\int_\Omega |\nabla u|^2 = \int_\Omega u(\overline{-\Delta v_\lambda}) = \int_{B_r(0)} u(-\Delta v_\lambda) > \int_{B_r(0)} v_\lambda(-\Delta v_\lambda) = \int_{B_r(0)} |\nabla v_\lambda|^2$$

and

$$\int_\Omega (u^+)^{2^*} > \int_{B_r(0)} (v_\lambda^+)^{2^*},$$

and therefore

$$\begin{aligned} m_\lambda = J_\lambda(v_\lambda) &= \frac{2}{N}\left[\frac{\int_{B_r(0)}(\Delta v_\lambda)^2 - \lambda \int_{B_r(0)} |\nabla v_\lambda|^2}{\left(\int_{B_r(0)}(v_\lambda^+)^{2^*}\right]^{2/2^*}}\right]^{N/4} \\ &> \frac{2}{N}\left[\frac{\|u\|^2 - \lambda\|\nabla u\|_2^2}{\|u^+\|_{2^*}^2}\right]^{N/4} = \max_{t>0} I_\lambda(tu) \geq c_\lambda. \end{aligned}$$

□

For $\lambda > 0$, we define the level set $I_\lambda^{m_\lambda} := \{u \in H : I_\lambda(u) \leq m_\lambda\}$. By Theorem 1.2 and the last lemma, this set is nonempty whenever $0 < \lambda < \lambda_1(\Omega)$. We also consider the barycenter map $\beta : \mathcal{N}_\lambda \to \mathbb{R}^N$ given by

$$\beta(u) := \frac{\displaystyle\int_\Omega (\Delta u)^2 x dx}{\displaystyle\int_\Omega (\Delta u)^2 dx}, \qquad u \in \mathcal{N}_\lambda.$$

Lemma 4.3. *There exists* $\lambda_{**} \in (0, \lambda_1(\Omega))$ *such that* $\beta(u) \in \Omega_r^+$*, whenever* $\lambda \in (0, \lambda_{**})$ *and* $u \in \mathcal{N}_\lambda \cap I_\lambda^{m_\lambda}$*.*

Proof. Suppose, by contradiction, that there exists $\lambda_n \to 0$ and $u_n \in \mathcal{N}_{\lambda_n} \cap I^{m_{\lambda_n}}$ such that $\beta(u_n) \notin \Omega_r^+$. Then,

$$c_{\lambda_n} \leq \frac{1}{2}\|u_n\|^2 - \frac{\lambda_n}{2}\|\nabla u_n\|_2^2 - \frac{1}{2^*}\|u_n^+\|_{2^*}^{2^*} \leq m_{\lambda_n}.$$

and

$$0 = I'_{\lambda_n}(u_n)u_n = \|u_n\|^2 - \lambda_n\|\nabla u_n\|_2^2 - \|u_n^+\|_{2^*}^{2^*}.$$

As in Lemma 3.2, (u_n) is bounded, and therefore $\lambda_n|\nabla u_n|_2^2 = o_n(1)$. Since (4.1) remains true c_λ replaced by m_λ, we can take the limit in the two above expressions to get

$$\frac{1}{2}\|u_n\|^2 - \frac{1}{2^*}\|u_n^+\|_{2^*}^{2^*} \to \frac{2}{N}S^{N/4}, \qquad \|u_n\|^2 - \|u_n^+\|_{2^*}^{2^*} \to 0.$$

Thus, $\|u_n\|^2 \to S^{N/4}$ and $\|u_n^+\|_{2^*}^{2^*} \to S^{N/4}$. If we set $v_n := u_n/\|u^+\|_{2^*}^{2^*}$ we obtain

$$\int_\Omega |v_n|^{2^*} \geq 1, \qquad \int_\Omega (\Delta v_n)^2 = S + o_n(1).$$

Let (y_n, μ_n), (w_n), (ϕ_n) and ϕ given by Lemma 4.1 . By picking $\psi \in C_0^\infty(\mathbb{R}^N, \mathbb{R}^N)$ such that $\psi(z) = z$ in $\overline{\Omega}$, we obtain

$$\begin{aligned}
\beta(u_n) &= \frac{\|u_n^+\|_{2^*}^2}{\|u_n\|^2}\int_\Omega [\overline{\Delta v_n}(z)]^2\psi(z)dz = \frac{\|u_n^+\|_{2^*}^2}{\|u_n\|^2}\int_{\mathbb{R}^N} [\Delta w_n(z)]^2\psi(z)dz \\
&= \left(\frac{(S^{N/4})^{2/2^*}}{S^{N/4}} + o_n(1)\right)\int_{\mathbb{R}^N} \mu_n^N[\Delta w_n(\mu_n x + y_n)]^2\psi(\mu_n x + y_n)dx \\
&= (S^{-1} + o_n(1))\int_{\mathbb{R}^N} [\Delta\phi_n(x)]^2\psi(\mu_n x + y_n)dx.
\end{aligned}$$

Since $\mu_n \to 0$, $y_n \to y \in \overline{\Omega}$, $\psi(y) = y$ and $\|\Delta\phi\|_2^2 = S$, we can use the Lebesgue Theorem to obtain

$$\lim_{n\to+\infty}\beta(u_n) = S^{-1}y\|\Delta\phi\|_2^2 = y \in \overline{\Omega},$$

which contradicts $\beta(u_n) \notin \Omega_r^+$. The lemma is proved. $\square$

We assume from now on that $\lambda \in (0, \lambda_{**})$. Let $v_\lambda \in H_r$ be such that $J_\lambda(v_\lambda) = m_\lambda$ and recall that v_λ and Δv_λ are radially symmetric. For each $y \in \Omega_r^-$, let $\gamma_\lambda(y) : \Omega \to \mathbb{R}$ be such that

$$\begin{cases} -\Delta\gamma_\lambda(y) = \overline{-\Delta v_\lambda}, & \text{in } \Omega, \\ \gamma_\lambda(y) = 0, & \text{on } \partial\Omega. \end{cases}$$

Arguing as in the proof of Lemma 4.2 we obtain

$$\int_\Omega |\nabla\gamma_\lambda(y)|^2 > \int_{B_r(0)} |\nabla v_\lambda|^2, \qquad \int_\Omega (\gamma_\lambda(y)^+)^{2^*} > \int_{B_r(0)} (v_\lambda^+)^{2^*}.$$

Thus, since $\int_\Omega (\Delta\gamma_\lambda(y))^2 = \int_{B_r(0)} (\Delta v_\lambda)^2 dx$, we have that

$$I_\lambda(\gamma_\lambda(y)) < I_\lambda(\overline{v_\lambda}) = m_\lambda, \qquad I_\lambda'(\gamma_\lambda(y))\gamma_\lambda(y) < I_\lambda'(\overline{v_\lambda})\overline{v_\lambda} = 0,$$

and therefore $\gamma_\lambda(y) \notin \mathcal{N}_\lambda$. Hence, if we set

$$t_y := \left[\frac{\|\gamma_\lambda(y)\|^2 - \lambda\|\nabla\gamma_\lambda(y)\|_2^2}{\|\gamma_\lambda(y)^+\|_{2^*}^{2^*}}\right]^{1/(2^*-2)},$$

we conclude that $t_y\gamma_\lambda(y) \in \mathcal{N}_\lambda$. All these remarks show that the map $\widehat{\gamma}_\lambda : \Omega_r^- \to \mathcal{N}_\lambda \cap I_\lambda^{m_\lambda}$ given by

$$\widehat{\gamma}_\lambda(y) := t_y\gamma_\lambda(y), \qquad y \in \Omega_r^-$$

is well defined and continuous.

We can now present the proof of our multiplicity result.

Proof of Theorem 1.3. Suppose that $\lambda \in (0, \lambda_{**})$, where λ_{**} comes from Lemma 4.3. The same argument used in the proof of Lemma 3.2 shows that I_λ constrained to $\mathcal{N}_\lambda$ satisfies the Palais-Smale condition at any level $c < (2/N)S^{N/4}$. Moreover, arguing as in the proof of Theorem 1.2, we can check that $m_\lambda < (2/N)S^{N/4}$. Hence, if we set $\Sigma_\lambda := \mathcal{N}_\lambda \cap I_\lambda^{m_\lambda}$, it follows from standard Lusternik–Schnirelmann Theory (see [23, Theorem 5.19]) that I_λ constrained to $\mathcal{N}_\lambda$ has at least $\mathrm{cat}_{\Sigma_\lambda}(\Sigma_\lambda)$ critical points. If $u \in H$ is one of these critical points, we can easily prove that $I'_\lambda(u) = 0$. Since $u \in \mathcal{N}_\lambda$ it is nonzero and, as in our first theorem, it is a classical solution of (P_λ).

It remains to check that $\mathrm{cat}_{\Sigma_\lambda}(\Sigma_\lambda) \geq \mathrm{cat}_\Omega(\Omega)$. In order to do this we consider the following diagram of continuous functions

$$\Omega_r^- \xrightarrow{\widehat{\gamma}_\lambda} \Sigma_\lambda \xrightarrow{\beta} \Omega_r^+$$

and notice that, for any $y \in \Omega_r^-$, there holds

$$\beta(\widehat{\gamma}_\lambda(y)) = \frac{\int_\Omega |\Delta(t_y \gamma_\lambda(y))|^2 x\, dx}{\int_\Omega |\Delta(t_y \gamma_\lambda(y))|^2 dx} = \frac{\int_{B_r(0)} (\Delta v_\lambda(z))^2 (z+y) dz}{\int_{B_r(0)} (\Delta v_\lambda(z))^2 dz},$$

where we have used the change of variables $z := x - y$. Since Δv_λ is radial $\int_{B_r(0)} (\Delta v_\lambda(z))^2 z\, dz = 0$, and therefore $\beta(\widehat{\gamma}_\lambda(y)) = y$. Thus $\beta \circ \widehat{\gamma}_\lambda$ is the identity map and we can argue as in [23, Lemma 5.25] to prove that each cover of Σ_λ by closed and contractible sets lifts to a cover of Ω_r^- by closed sets contractible in Ω_r^+. Thus,

$$\mathrm{cat}_{\Sigma_\lambda}(\Sigma_\lambda) \geq \mathrm{cat}_{\Omega_r^+}(\Omega_r^-) = \mathrm{cat}_\Omega(\Omega),$$

and the theorem is proved. □.

5. Appendix

We devote this appendix to the proof of the regularity result for the weak solutiond of the problem (P_λ). We start with the following technical lemma.

Lemma 5.1. *Let $\lambda < \lambda_1(\Omega)$, $p > \frac{N}{N-4}$, $t := \frac{Np}{N+4p} > 1$ and $a \in L^{N/4}(\Omega)$. If $\Omega \in C^{3,1}$, then the linear operator $T_a : L^p(\Omega) \to W^{4,t}(\Omega) \cap W_0^{1,t}(\Omega) \subset L^p(\Omega)$ given by*

$$T_a(v) := w \quad \Longleftrightarrow \quad \begin{cases} \Delta^2 w + \lambda \Delta w = a(x)v, \text{ in } \Omega, \\ w \in W^{4,t}(\Omega) \cap W_0^{1,t}(\Omega), \\ \Delta w \in W^{2,t}(\Omega) \cap W_0^{1,t}(\Omega) \end{cases} \tag{5.1}$$

is well defined and there exists $C > 0$ such that

$$\|T_a(v)\|_p \leq C \|a\|_{N/4} \|v\|_p, \quad \forall\, v \in L^p(\Omega).$$

Proof. Given $v \in L^p(\Omega)$, we can use Hölder's inequality to prove that $a(x)v \in L^t(\Omega)$. Thus, the equation $\Delta u = a(x)v$ is uniquely soluble in $W^{2,t}(\Omega) \cap W_0^{1,t}(\Omega)$. Now we consider the equation $\Delta w + \lambda w = u \in W^{2,t}(\Omega)$. If $\lambda \leq 0$, it follows from [11, Theorem 5.15] that is has unique solution in $W^{2,t}(\Omega) \cap W_0^{1,t}(\Omega 0$. If $\lambda > 0$ we notice that the operator $\lambda(\Delta)^{-1} : L^t(\Omega) \to W^{2,t}(\Omega) \cap W_0^{1,t}(\Omega)$ is compact from $L^t(\Omega)$ to $L^t(\Omega)$. Since $\lambda < \lambda_1(\Omega)$, we have that $\ker(Id - \lambda(\Delta)^{-1}) = \{0\}$ and therefore the uniqueness of solution also holds. Thus, since $u \in W^{2,t}(\Omega)$ and $\partial\Omega \in C^{3,1}$, we can argue as in the proof of [11, Theorem 9.15] to conclude that T_a is well defined. Moreover, using $\lambda < \lambda_1(\Omega)$ and arguing as in the proof of [11, Lemma 9.17], we may check that

$$\|z\|_{W^{2,t}(\Omega)} \leq C_1 \|(\Delta + \lambda Id)\|_t, \quad \forall\, z \in W^{2,t}(\Omega) \cap W_0^{1,t}(\Omega),$$

and therefore Hölder's inequality provides

$$\|T_a(v)\|_p = \|w\|_p \leq C_2\|w\|_{W^{4,t}(\Omega)} \leq C_3\|av\|_t \leq C\|a\|_{N/4}\|v\|_p,$$

for any $v \in L^p(\Omega)$. □

Proof of Proposition 3.1. Let $g \in L^q(\Omega)$ and consider $g_n \subset C_0^\infty(\Omega)$ such that $g_n \to g$ in $L^q(\Omega)$. Arguing as in the proof of the last lemma we obtain a solution of $\Delta^2 v_n = g_n$, with $v_n \in W^{4,q}(\Omega) \cap W_0^{1,q}(\Omega)$, $\Delta v_n \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$ and $\|v_n\|_{W^{4,q}(\Omega)} \leq C\|g_n\|_q$, for some constant $C > 0$ (independent of n). This shows that (v_n) is a Cauchy's sequence in $W^{4,q}(\Omega)$ and therefore, up to a subsequence, $v_n \to v$ strongly in $W^{4,q}(\Omega)$ for some $v \in W^{4,q}(\Omega) \cap W_0^{1,q}(\Omega)$ such that $\Delta v \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$. So, by using the Sobolev embedding $W^{4,q}(\Omega) \hookrightarrow W^{2,2}(\Omega)$, we can write

$$\int_\Omega \Delta v \Delta \phi = \int_\Omega g(x)\phi, \quad \forall\, \phi \in H.$$

We now set $g := -\lambda\Delta u + u^{2^*-1} \in L^q(\Omega)$. Recalling that $I'_\lambda(u) = 0$, we obtain

$$\int_\Omega \Delta u \Delta \phi = \int_\Omega g(x)\phi, \quad \forall\, \phi \in H.$$

Since this problem has unique solution in H, we conclude that $u = v$, and therefore $u \in W^{4,q}(\Omega) \cap W_0^{1,q}(\Omega)$ and $\Delta u \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$. Moreover,

$$\int_\Omega (\Delta^2 u)\phi = \int_\Omega \Delta u \Delta \phi = \int_\Omega g(x)\phi, \quad \forall\, \phi \in H,$$

which implies that $\Delta^2 u = -\lambda\Delta u + u^{2^*-1}$ a.e. in Ω.

In order to prove that u is regular, we consider $p > N/(N-4)$ and set

$$a(x) := \begin{cases} 0, & \text{if } u(x) < 1, \\ u^{2^*-2}(x), & \text{if } u(x) \geq 1, \end{cases} \qquad b(x) := \begin{cases} u(x)^{2^*-1}, & \text{if } u(x) < 1, \\ 0, & \text{if } u(x) \geq 1. \end{cases}$$

Then $a \in L^{N/4}(\Omega)$, $b \in L^\infty(\Omega)$ and $a(x)u + b(x) = u^{2^*-1}$. Acording to [21], for any given $\varepsilon > 0$ there exists $a_\varepsilon \in L^{N/4}(\Omega)$ and $b_\varepsilon \in L^\infty(\Omega)$ such that

$$\|a_\varepsilon\|_{N/4} < \varepsilon, \qquad a_\varepsilon(x)u + b_\varepsilon(x) = a(x)u + b(x), \text{ a.e. in } \Omega.$$

Thus, we have that

$$\int_\Omega (\Delta u \Delta \phi + \lambda \nabla u \cdot \nabla \phi) = \int_\Omega (a_\varepsilon u \phi + b_\varepsilon \phi), \quad \forall\, \phi \in H. \tag{5.2}$$

Let $h_\varepsilon \in W^{4,q}(\Omega) \cap W_0^{4,q}(\Omega)$ be such that $\Delta^2 h_\varepsilon + \lambda\Delta h_\varepsilon = b_\varepsilon$. By taking $\varepsilon > 0$ smaller if necessary and using Lemma 5.1 we conclude that $\|T_{a_\varepsilon}(v)\|_p < (1/2)\|v\|_p$, for any $v \in L^p(\Omega)$. Hence, $(Id - T_{a_\varepsilon})^{-1}$ is well defined and we can set

$$v_\varepsilon := (Id - T_{a_\varepsilon})^{-1} h_\varepsilon.$$

Then $T_{a_\varepsilon} v_\varepsilon = (v_\varepsilon - h_\varepsilon)$ and it follows from (5.1) that $\Delta^2 v_\varepsilon + \lambda\Delta v_\varepsilon = a_\varepsilon v_\varepsilon + b_\varepsilon$ in Ω. Since $b_\varepsilon \in L^\infty(\Omega)$ we can use standard elliptic regularity to conclude that $v_\varepsilon \in H$, and therefore

$$\int_\Omega (\Delta v_\varepsilon \Delta \phi + \lambda \nabla v_\varepsilon \cdot \nabla \phi) = \int_\Omega (a_\varepsilon v_\varepsilon \phi + b_\varepsilon \phi), \quad \forall\, \phi \in H.$$

This and (5.2) provide

$$\int_\Omega (\Delta z_\varepsilon \Delta\phi + \lambda \nabla z_\varepsilon \cdot \nabla\phi) = \int_\Omega a_\varepsilon z_\varepsilon, \quad \forall\, \phi \in H,$$

with $z_\varepsilon := v_\varepsilon - u$. By picking $\phi = z_\varepsilon \in H$, using the Sobolev embedding and Hölder's inequality, we get

$$S\left(1 - \frac{\lambda}{\lambda_1(\Omega)}\right)\|z_\varepsilon\|_{2^*}^2 \leq \|z_\varepsilon\|^2 + \lambda\|\nabla z_\varepsilon\|_2^2 = \int a_\varepsilon z_\varepsilon^2 \leq \|a_\varepsilon\|_{N/4}\|z_\varepsilon\|_{2^*}^2.$$

This implies that $z_\varepsilon = 0$ and therefore $u = v_\varepsilon \in L^p(\Omega)$.

Since we have concluded that $u \in L^p(\Omega)$ for any $p > N/(N-4)$, we can argue as in the proof of Lemma 5.1 to conclude that $u \in W^{4,p}(\Omega)$ for any $p > N$ and therefore, by the Sobolev embedding, $u \in C^{3,\alpha}(\overline{\Omega})$. We now invoke the high-order regularity result [11, Theorem 9.19] to conclude that $u \in C^{4,\alpha}(\overline{\Omega})$. □

References

[1] C.O. Alves and Y.H. Ding, *Multiplicity of positive solutions to a p-Laplacian equation involving critical nonlinearity*, J. Math. Anal. Appl. **279** (2003), 508-521.

[2] C.O. Alves and G.M. Figueiredo, *Multiplicity of nontrivial solutions to o biharmonic equation via Lusternik-Schnirelman theory*, Math. Methods Appl. Sci. **36** (2013), 683–694.

[3] F. Bernis, J. Garcia Azorero and I. Peral, *Existence and multiplicity of nontrivial solutions in semilinear critical problems of fourth order,* Adv. Differential Equations **1** (1996), 219–240.

[4] D. Bonheure, E.M. dos Santos and M. Ramos, *Ground state and non-ground state solutions of some strongly coupled elliptic systems* Trans. Amer. Math. Soc. **364** (2011), 447-491.

[5] H. Brezis and L. Nirenberg, *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents,* Comm. Pure Appl. Math. **36** (1983), 437-477.

[6] J. Garcia Azorero and I. Peral Alonso, *Existence and non-uniqueness for the p-Laplacian: Nonlinear eigenvalues*, Comm. Partial Diff. Eq. **12** (1987), 1389-1430.

[7] J. Garcia Azorero and I. Peral Alonso, *Multiplicity of solutions for elliptic problem with critical exponent or with a nonsymmetric term*, Trans. Amer. Math. Soc. **323** (1991), 877-895.

[8] F. Gazzola, *Critical growth problems for polyharmonic operators*, Proc. Royal Soc. Edinburgh Sect. A **128** (1998), 251-263.

[9] F. Gazzola, H.C. Grunau and M. Squassina, *Existence and nonexistence results for critical growth biharmonic elliptic equations*, Calc. Var. Partial Differential Equations **18** (2003), 117-143.

[10] F. Gazzola, H.C. Grunau and G. Sweers, *Polyharmonic boundary value problems. Positivity preserving and nonlinear higher order elliptic equations in bounded domains,* Lecture Notes in Mathematics, 1991. Springer-Verlag, Berlin, 2010.

[11] D. Gilbarg and N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order,* Classics Math., Springer-Verlag,Berlin, 2001, reprint of the 1998 edition.

[12] M. Lazzo, *Solutions positives multiples pour une équation elliptique non linéaire avec léxposant critique de Sobolev*, C. R. Acad. Sci. Paris **314** (1992), 61-64.

[13] J.L.F. Melo and E.M. dos Santos, *Positive solutions to a fourth-order elliptic problem by the Lusternik–Schnirelmann category.* J. Math. Anal. Applications **420** (2014), 532–550.

[14] J.L.F. Melo and E.M. dos Santos, *A fourth-order equation with critical growth: the effect of the domain topology*, Topol. Methods Nonlinear Anal. **45** (2015), 551–574.

[15] E. Mitidieri, *A Rellich type identity and applications*, Comm. Partial Differential Equations **18** (1993), 125-151.

[16] P. Oswald, *On a priori estimates for positive solutions of a semilinear biharmonic equations in a ball*, Comment. Math. Univ. Carolinae **26** (1985), 565-577.

[17] P. Pucci and J. Serrin, *Critical exponents and critical dimensions for polyharmonic operators*, J. Math. Purres Appl. **69** (1990), 55-83.

[18] P.H. Rabinowitz, *On a class of nonlinear Schrödinger equations.*, Z. Angew. Math. Phys. **43** (1992), 270-291.

[19] O. Rey, *A multiplicity result for a variational problem with lack of compactness*, Nonlinear Anal. **13** (1989), 1241-1249.

[20] W.C. Troy,*Symmetry properties in systems of semilinear elliptic equations*, J. Differential Equations **42** (1981), 400–413.

[21] R.C.A.M. van der Vorst, *Best constant for the embedding of the space* $H^2 \cap H_0^1(\Omega)$ *into* $L^{2N/(N-4)}(\Omega)$*,* Differential Integral Equations **6** (1993), 259–276.

[22] R.C.A.M. van der Vorst, *Fourth-order elliptic equations with critical growth*, C.R. Math. Acas. Sci. Paris **320** (1995), 295-299.

[23] M. Willem, *Michel Minimax theorems.* Progress in Nonlinear Differential Equations and their Applications, 24. Birkhäuser Boston, Inc., Boston, MA, 1996.

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