KIRCHHOFF-SCHRÖDINGER EQUATIONS IN $\mathbb{R}^2$ WITH CRITICAL EXPONENTIAL GROWTH AND INDEFINITE POTENTIAL

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Abstract. We prove the existence of ground state solution for the nonlocal problem

$$m\left(\int_{\mathbb{R}^2}(|\nabla u|^2+b(x)u^2)\mathrm{d}x\right)(-\Delta u+b(x)u)=A(x)f(u)\quad\text{in}\quad\mathbb{R}^2,$$

where m is a Kirchhoff type function, b may be negative and noncoercive, A is locally bounded and the function f has critical exponential growth. We also obtain new results for the classical Schrödinger equation, namely the local case $m\equiv 1$. In the proofs, we apply Variational Methods besides a new Trudinger-Moser type inequality.

1. Introduction

We study the problem

$$(P)\qquad m\left(\int_{\mathbb{R}^2}(|\nabla u|^2+b(x)u^2)\mathrm{d}x\right)(-\Delta u+b(x)u)=A(x)f(u)\quad\text{in}\quad\mathbb{R}^2,$$

where $m:[0,\infty)\to(0,\infty)$ and $f:\mathbb{R}\to[0,\infty)$ are continuous functions and $b,A\in L^\infty_{\mathrm{loc}}(\mathbb{R}^2)$. The potential b may vanish on sets of positive measure or even be negative and the nonlinearity f has critical growth. We look for solutions in the subspace of $W^{1,2}(\mathbb{R}^2)$ given by

$$H:=\left\{u\in W^{1,2}(\mathbb{R}^2):\int_{\mathbb{R}^2}b(x)u^2\mathrm{d}x<\infty\right\}.$$

Due to the presence of the term $m(\int_{\mathbb{R}^2}(|\nabla u|^2+b(x)u^2)\mathrm{d}x)$ the equation in (P) is no longer a pointwise identity and therefore the problem is called nonlocal. In [20], G. Kirchhoff presented his study on transverse vibrations of elastic strings and proposed a hyperbolic equation of the type

$$\frac{\partial^2u}{\partial t^2}-\left(k_1+k_2\int_0^L\left|\frac{\partial u}{\partial x}\right|^2\mathrm{d}x\right)\frac{\partial^2u}{\partial x^2}=0,\tag{1.1}$$

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where k_1, k_2 and L are positive constants. This extends the classical D'Alembert wave equation by considering the effects of changes in the length of the strings during vibrations. So, more general versions of (1.1) and the corresponding stationary equations have been called Kirchhoff equations and became subject of intense research mainly after the works of S.I. Pohozaev [28] and J.-L. Lions [24]. Variational Methods have been used by many authors to obtain results of existence and multiplicity of solutions for stationary Kirchhoff equations since the pioneering work of C.O. Alves et al. [3].

In order to present the conditions on the nonlocal term m we first define $M(t) := \int_0^t m(\tau)\mathrm{d}\tau$, $t \geq 0$. The hypotheses on $m : [0,\infty) \to (0,\infty)$ are:

(m_1) $m_0 := \inf_{t\geq 0} m(t) > 0$;

(m_2) for any $t_1, t_2 \geq 0$, it holds

$$M(t_1 + t_2) \geq M(t_1) + M(t_2);$$

(m_3) $\dfrac{m(t)}{t}$ is decreasing in $(0,\infty)$.

Condition (m_2) is valid, for instance, if m is non-decreasing. The typical example of function satisfying $(m_1) - (m_3)$ is $m(t) = \alpha + \beta t$, with $\alpha > 0$ and $\beta \geq 0$. Other examples are $m(t) = \alpha + \beta t^{\delta}$, with $\delta \in (0,1)$, $m(t) = \alpha(1 + \log(1+t))$ or $m(t) = \alpha + \beta e^{-t}$.

Concerning the potential $b \in L^{\infty}_{\mathrm{loc}}(\mathbb{R}^2)$, we set

$$\lambda_1^b := \inf\left\{ \int_{\mathbb{R}^2} (|\nabla u|^2 + b(x)u^2)\mathrm{d}x : u \in H \text{ and } \|u\|_{L^2(\mathbb{R}^2)} = 1 \right\}$$

and, for each $\Omega \subset \mathbb{R}^2$ open and nonempty,

$$\nu_b(\Omega) := \inf\left\{ \int_{\mathbb{R}^2} (|\nabla u|^2 + b(x)u^2)\mathrm{d}x : u \in W_0^{1,2}(\Omega) \text{ and } \|u\|_{L^2(\Omega)} = 1 \right\}$$

and $\nu_b(\varnothing) = \infty$. The hypotheses on b are:

(b_1) $\lambda_1^b > 0$;

(b_2) $\lim_{r\to\infty} \nu_b\left(\mathbb{R}^2 \backslash \overline{\{x \in \mathbb{R}^2 : |x| < r\}}\right) = \infty$;

(b_3) there exists $B_0 > 0$ such that

$$b(x) \geq -B_0, \quad \forall\, x \in \mathbb{R}^2.$$

For the function $A \in L^{\infty}_{\mathrm{loc}}(\mathbb{R}^2)$, we suppose that

(A_1) $A(x) \geq 1$ for any $x \in \mathbb{R}^2$;

(A_2) there exists $\beta_0 > 1$, $C_0 > 0$ and $R_0 > 0$ such that

$$A(x) \leq C_0 \left[1 + (b^+(x))^{1/\beta_0} \right], \quad \forall\, x \in \mathbb{R}^2 \setminus B_{R_0}(0),$$

where $b^+(x) := \max\{0, b(x)\}$.

Conditions $(b_1) - (b_3)$ and $(A_1) - (A_2)$ were first considered by B. Sirakov [30] in the study of a class of subcritical Schrödinger equations in dimension $N \geq 3$. These hypotheses ensure that H is a Hilbert space with inner product given by

$$\langle u, v\rangle_H = \int_{\mathbb{R}^2} (\nabla u \cdot \nabla v + b(x)uv)\mathrm{d}x, \quad \forall\, u,\, v \in H,$$

and norm $\|u\|_H = \sqrt{\langle u,u\rangle_H}$. Moreover, H is continuously embedded into $W^{1,2}(\mathbb{R}^2)$ and, for every $p \geq 2$, compactly embedded into the weighted Lebesgue space

$$L^p_A(\mathbb{R}^2) := \left\{ u : \mathbb{R}^2 \to \mathbb{R} \text{ measurable } : \int_{\mathbb{R}^2} A(x)|u|^p \mathrm{d}x < \infty \right\},$$

which is a Banach space when endowed with the norm

$$\|u\|_{L^p_A(\mathbb{R}^2)} = \left(\int_{\mathbb{R}^2} A(x)|u|^p \mathrm{d}x \right)^{1/p}.$$

For the proof of these embeddings, see [30, Sections 2 and 3]. By (A_1), $L^p_A(\mathbb{R}^2) \hookrightarrow L^p(\mathbb{R}^2)$ and, consequently, the embedding $H \hookrightarrow L^p(\mathbb{R}^2)$ is also compact. In order to guarantee the compactness of this last embedding, one normally use the conditions $b(x) \geq b_0 > 0$ and

$$\lim_{|x|\to\infty} b(x) = \infty, \text{ or } 1/b \in L^1(\mathbb{R}^2), \text{ or } \mathrm{meas}(\Omega_{b,K}) < \infty \;\forall\, K > 0, \tag{1.2}$$

where $\Omega_{b,K} := \{x \in \mathbb{R}^2 : b(x) < K\}$. One weaker geometric condition which implies on (b_2) is (see [30, Theorem 1.4]): for any $K > 0$, any $r > 0$ and any sequence $(x_n) \subset \mathbb{R}^2$ with $\lim_{n\to\infty} |x_n| = \infty$, we have

$$\lim_{n\to\infty} \mathrm{meas}(\Omega_{b,K} \cap B_r(x_n)) = 0.$$

A potential satisfying the above condition is $b(x) = b(x_1, x_2) = |x_1x_2|$. Since (b_2) and (b_3) are sufficient conditions for λ_1^b to be achieved (see [30, Proposition 2.2]), it is easy to see that this potential also satisfies (b_1). Moreover, since for any constant $C \in \mathbb{R}$ we have $\Omega_{b-C,K} = \Omega_{b,K+C}$ and $\lambda_1^{b-C} = \lambda_1^b - C$, other potential satisfying $(b_1)-(b_3)$ is $b(x) = |x_1x_2| - C$, for certain values of C. Notice that these two examples do not satisfy (1.2).

Embedding $H \hookrightarrow W^{1,2}(\mathbb{R}^2)$ implies that, for some constant $\zeta > 0$,

$$\|u\|_H \geq \zeta \|\nabla u\|_{L^2(\mathbb{R}^2)}, \quad \forall\, u \in H. \tag{1.3}$$

If $b \leq 0$ on some set with positive measure, then we cannot have $\zeta > 1$. However, we can consider $\zeta = 1$ if

$(\widehat{b_3})$ $b(x) \geq 0$ for any $x \in \mathbb{R}^2$.

Concerning the nonlinearity $f : \mathbb{R} \to [0, \infty)$, we first suppose that $f(s) = 0$, for any $s \leq 0$, and define $F(s) := \int_0^s f(\tau)\mathrm{d}\tau$, $s \in \mathbb{R}$. The main hypotheses on f are:

(f_1) there exists $\alpha_0 > 0$ such that

$$\lim_{s\to\infty} \frac{f(s)}{e^{\alpha s^2}} = \begin{cases} 0, & \text{if } \alpha > \alpha_0, \\ \infty, & \text{if } \alpha < \alpha_0; \end{cases}$$

(f_2) there exists $s_0, K_0 > 0$ such that

$$F(s) \leq K_0 f(s), \quad \forall\, s \geq s_0;$$

(f_3) there exists $\theta_0 > 4$ such that

$$\theta_0 F(s) \leq s f(s), \quad \forall\, s > 0;$$

(f_4) $\dfrac{f(s)}{s^3}$ is positive and non-decreasing in $(0, \infty)$.

If $\theta > 4$, an example of function f satisfying $(f_1) - (f_4)$ is

$$f(s) = \frac{d}{ds}\left(\frac{s^\theta}{\theta}(e^{s^2} - 1)\right) = s^{\theta-1}(e^{s^2} - 1) + \frac{2s^{\theta+1}}{\theta}e^{s^2}.$$

According to (f_1) we are dealing with a function having critical growth. This notion of criticality was originally motivated by the Trudinger-Moser inequality (see [26, 31]), which states that $W_0^{1,2}(\Omega)$ is continuously embedded into the Orlicz space $L_{\phi_\alpha}(\Omega)$ associated with the function $\phi_\alpha(t) := e^{\alpha t^2} - 1$, $t \in \mathbb{R}$, for $0 < \alpha \leq 4\pi$ and any bounded domain $\Omega \subset \mathbb{R}^2$. This result has been generalized in many ways (see [6, 13, 29, 2, 22, 10, 11] and references therein). Here, we prove a version of that result for functions belonging to the space H (see Lemma 2.3).

The main difficulty in dealing with critical growth is the lack of compactness from the embeddings of the Sobolev spaces into Orlicz spaces L_{ϕ_α}. In [25, subsection I.7], P.-L. Lions proved a concentration-compactness result that allows us to overcome this trouble in $W_0^{1,2}(\Omega)$, $\Omega \subset \mathbb{R}^2$ bounded domain. This result has had many generalizations and applications in recent years (see [21, 32, 33, 7, 15] and references therein). Corollary 2.4 in next section is a version of the result of P.-L. Lions for space H.

Before stating our results, we introduce some notations:

$$S_p := \inf_{u \in H\setminus\{0\}} \frac{\|u\|_H}{\|u\|_{L^p(\mathbb{R}^2)}}, \quad p \geq 2, \tag{1.4}$$

$$C_p := \inf\left\{C > 0 : pM(t^2 S_p^2) - 2Ct^p \leq pM\left(\frac{4\pi\zeta^2}{\alpha_0}\right),\ \forall\ t > 0\right\}, \quad p > 4.$$

The values S_p and C_p are finite, for $p \geq 2$ and $p > 4$ respectively, due to the embedding $H \hookrightarrow L^p(\mathbb{R}^2)$ and the hypothesis (m_3), which implies that $m(t) < m(1)t$ for any $t > 1$.

Our main results for the problem (P) can be stated as follows:

Theorem 1.1. *Suppose that $(m_1) - (m_3)$, $(b_1) - (b_3)$, $(A_1) - (A_2)$ and $(f_1) - (f_4)$ are satisfied. Suppose also that*

(f_5) *there exists $p_0 > 4$ such that*

$$f(s) > C_{p_0} s^{p_0 - 1}, \quad \forall\ s > 0.$$

Then problem (P) has a nonnegative ground state solution.

Theorem 1.2. *Suppose that $(m_1) - (m_3)$, $(b_1) - (b_2)$, $(\widehat{b_3})$, $(A_1) - (A_2)$ and $(f_1) - (f_4)$ are satisfied. Suppose also that*

(f_6) *there exists $\gamma_0 > 0$ such that*

$$\liminf_{s\to\infty} \frac{sf(s)}{e^{\alpha_0 s^2}} \geq \gamma_0 > 4\alpha_0^{-1} m\left(\frac{4\pi}{\alpha_0}\right) \inf_{R>0}\left\{R^{-2} e^{R^2 M_R/2}\right\},$$

where $M_R := \|b\|_{L^\infty(B_R(0))}$.

Then problem (P) has a nonnegative ground state solution.

Hypotheses (m_3) and (f_4) ensure that the solutions given by Theorems 1.1 and 1.2 are ground state solutions. However, as we will see in the proofs, we still obtain nonnegative nontrivial solution for the problem (P), not necessarily ground state, if we replace (m_3) and (f_4) by weaker conditions, namely:

(m_3^*) there exist constants $a_1 > 0$ and $T > 0$ such that

$$m(t) \leq a_1 t, \quad \forall\, t \geq T;$$

(f_4^*) $\displaystyle\lim_{s\to 0^+} \frac{f(s)}{s} = 0$

and the conditions of monotonicity given in the conclusion of Lemma 2.5 in the next section. Specifically, in the case of Theorem 1.2, this replacement allow us to consider functions f that vanish on some neighborhood of origin.

The ideas used here permit us to obtain new results even in the local case. Actually, if $m \equiv 1$, equation in (P) is reduced to the Schrödinger equation

$$(\widehat{P}) \qquad -\Delta u + b(x)u = A(x)f(u) \quad \text{in} \quad \mathbb{R}^2.$$

In this case, instead of (f_3) and (f_4), we consider the hypotheses

$(\widehat{f}_3)$ there exists $\widehat{\theta}_0 > 2$ such that

$$\widehat{\theta}_0 F(s) \leq s f(s), \quad \forall\ s > 0;$$

$(\widehat{f}_4)$ $\dfrac{f(s)}{s}$ is positive and non-decreasing in $(0, \infty)$.

In contrast to (f_4), hypothesis $(\widehat{f}_4)$ does not imply on (f_4^*). Setting, for $q > 2$,

$$\widehat{C}_q := \inf\left\{C > 0 : qS_q^2 t^2 - 2Ct^q \leq \frac{4\pi q\zeta^2}{\alpha_0},\ \forall\, t > 0\right\} = S_q^q \left(\frac{\alpha_0(q-2)}{4\pi q\zeta^2}\right)^{(q-2)/2},$$

the main results for problem $(\widehat{P})$ can be stated as follows:

Theorem 1.3. *Suppose that* $(b_1) - (b_3)$, $(A_1) - (A_2)$, $(f_1) - (f_2)$, $(\widehat{f}_3)$ *and* (f_4^*) *are satisfied. Suppose also that*

$(\widehat{f}_5)$ *there exists* $q_0 > 2$ *such that*

$$f(s) > \widehat{C}_{q_0} s^{q_0 - 1}, \quad \forall\ s > 0.$$

Then problem $(\widehat{P})$ *has a nonnegative nontrivial weak solution. If, in addition,* f *satisfies* $(\widehat{f}_4)$*, the solution is ground state.*

Theorem 1.4. *Suppose that* $(b_1) - (b_2)$, $(\widehat{b}_3)$, $(A_1) - (A_2)$, $(f_1) - (f_2)$, $(\widehat{f}_3)$ *and* (f_4^*) *are satisfied. Suppose also that*

$(\widehat{f}_6)$ *there exists* $\widehat{\gamma}_0 > 0$ *such that*

$$\liminf_{s\to\infty} \frac{sf(s)}{e^{\alpha_0 s^2}} \geq \widehat{\gamma}_0 > 4\alpha_0^{-1} \inf_{R>0}\left\{R^{-2} e^{R^2 M_R/2}\right\}.$$

Then problem $(\widehat{P})$ *has a nonnegative nontrivial weak solution. If, in addition,* f *satisfies* $(\widehat{f}_4)$*, the solution is ground state.*

As far we know, there is no paper on Kirchhoff equations in unbounded domains under $(b_1) - (b_3)$, even with nonlinearity having polynomial growth. But on Schrödinger equations involving exponential growth, we can cite [9, 12]. In [9], the author studied the nonhomogeneous singular problem

$$(1.5) \qquad -\Delta u + b(x)u = \frac{g(x)f(u)}{|x|^a} + h(x), \quad x \in \mathbb{R}^2,$$

with b satisfying $(b_1)-(b_3)$, f having subcritical exponential growth and $a \in [0,2)$. In [12], the authors studied the nonhomogeneous quasilinear problem

$$-\Delta_N u + b(x)|u|^{N-2}u = c(x)|u|^{N-2}u + g(x)f(u) + \varepsilon h(x), \quad x \in \mathbb{R}^N, \tag{1.6}$$

where $\Delta_N u = \mathrm{div}(|\nabla u|^{N-2}\nabla u)$, $N \geq 2$, with b and f satisfying hypotheses similar to $(b_1),(b_2),(\widehat{b_3})$ and $(f_1),(f_2),(\widehat{f_3}),(f_4^*),(\widehat{f_5})$, respectively. The potential c was taken nonnegative and belonging to an appropriated Lebesgue space, with norm, in this space, bounded by a suitable constant. Notice that, for certain sign-changing potentials b, this hypothesis does not include the case in which b is replaced by b^+ and $c(x) = b^-(x) := \max\{0, -b(x)\}$ in equation (1.6). Actually, although b^+ satisfies $(b_1),(b_2),(\widehat{b_3})$ whenever b satisfies $(b_1)-(b_3)$, powers of b^- may not be integrable, as for example $b(x) = |x_1x_2| - C$ given previously. For $h \not\equiv 0$ with small norm in an apropriated dual space, two solutions were obtained in [9] and [12] for problems (1.5) and (1.6), respectively.

With the potential b satisfying hypotheses similar to (1.2), we also refer to [23], for a Kirchhoff equation, and [14, 32], for Schrödinger equations. Other related results can be founded in [4, 5, 15, 16, 17]. On Kirchhoff equations in bounded domains, we refer to [18, 19, 27]. All of these papers deal with critical or subcritical exponential growth of Trudinger-Moser type.

In addition to the aspects already mentioned, our results complement the aforementioned works in other ways: with the exception of [18], in the other papers it was not proved the existence of *ground state* solutions; differently from [4, 5, 14, 15, 16, 17, 23, 32], we consider a potential that may change sign or vanish; in these same papers and in [9], the regularity of the potential is stronger than that considered here; in [9, 12], it was assumed that the weight function g in equations (1.5) and (1.6) satisfies hypotheses similar to (A_1) and (A_2), but the regularity on A is stronger than here; finally, although in [12] it has been considered a potential b of the same type of ours, the Trudinger-Moser inequality proved here is more general and allow us to consider the more natural hypotheses (f_6) and $(\widehat{f_6})$, instead of (f_5) and $(\widehat{f_5})$.

The rest of this paper is organized as follows: in Section 2 we prove preliminary results related with the Trudinger-Moser inequality; in Section 3 we detail the variational framework of problem (P); in Section 4 we prove estimates for the Mountain Pass level of the energy functional; finally, in Section 5, we prove our main results.

2. Preliminary results

Hereafter, we write $\int_\Omega u$ instead of $\int_\Omega u(x)\mathrm{d}x$, for any $\Omega \subset \mathbb{R}^2$ and $u \in L^1(\Omega)$. Norms in H, in $W^{1,2}(\mathbb{R}^2)$ and in $L^p(\mathbb{R}^2)$, $1 \leq p \leq \infty$, are denoted by $\|\cdot\|$, $\|\cdot\|_{1,2}$ and $\|\cdot\|_p$, respectively. Notations $C_1, C_2, \ldots$ represent positive constants whose exact values are irrelevant. Hypotheses $(b_1)-(b_3)$, $(A_1)-(A_2)$ are always be assumed from now on.

The next result was proved in [13] (see also [6]).

Lemma 2.1. *If $\alpha > 0$ and $v \in W^{1,2}(\mathbb{R}^2)$, then $\int_{\mathbb{R}^2}(e^{\alpha v^2}-1) < \infty$. Moreover, if $\alpha < 4\pi$, $\|\nabla v\|_2 \leq 1$ and $\|v\|_2 \leq M$, then there exists $C = C(\alpha, M) > 0$ such that*

$$\int_{\mathbb{R}^2}(e^{\alpha v^2}-1) \leq C.$$

We need a version of this last result adapted to our variational framework. We start with a technical result.

Lemma 2.2. *Let β_0 be given by hypothesis (A_2) and $\alpha > 0$. For any $v \in H$ and $r \in [1, \beta_0)$, the function $A(\cdot)^r(e^{\alpha v^2} - 1)^r$ belongs to $L^1(\mathbb{R}^2)$.*

Proof. Since $(e^{\alpha s^2} - 1)^r \leq e^{r\alpha s^2} - 1$, for any $s \in \mathbb{R}$, and $A \in L^\infty_{\mathrm{loc}}(\mathbb{R}^2)$ we get

$$\int_{\mathbb{R}^2} A(x)^r(e^{\alpha v^2} - 1)^r \leq \int_{\mathbb{R}^2\setminus B_{R_0}(0)} A(x)^r(e^{r\alpha v^2} - 1) + C_1 \int_{B_{R_0}(0)} (e^{r\alpha v^2} - 1), \tag{2.1}$$

where $R_0 > 0$ is given by hypothesis (A_2). From Lemma 2.1, we conclude that the last integral above is finite. In order to estimate the first one, notice that

$$\int_{\mathbb{R}^2\setminus B_{R_0}(0)} A(x)^r(e^{r\alpha v^2} - 1) = \sum_{m=1}^{\infty} \frac{(r\alpha)^m}{m!} \int_{\mathbb{R}^2\setminus B_{R_0}(0)} A(x)^r v^{2m}. \tag{2.2}$$

Now, by (A_2) and Hölder's inequality, we have that

$$\begin{aligned}
&\int_{\mathbb{R}^2\setminus B_{R_0}(0)} A(x)^r v^{2m} \leq \\
&\leq C_2 \|v\|_{2m}^{2m} + C_3 \int_{\mathbb{R}^2\setminus B_{R_0}(0)} (b^+(x))^{r/\beta_0} v^{2m} \qquad (2.3)\\
&\leq C_2 \|v\|_{2m}^{2m} + C_3 \left(\int_{\mathbb{R}^2} b^+(x)v^2\right)^{r/\beta_0} \left(\int_{\mathbb{R}^2} v^{2(m\beta_0 - r)/(\beta_0 - r)}\right)^{(\beta_0 - r)/\beta_0}.
\end{aligned}$$

But, by (b_3) and (b_1),

$$\begin{aligned}
\int_{\mathbb{R}^2} b^+(x)v^2 &= \int_{\mathbb{R}^2} b(x)v^2 - \int_{\{b(x)\leq 0\}} b(x)v^2 \leq \|v\|^2 + B_0 \|v\|_2^2 \\
&\leq \|v\|^2 + B_0 \frac{\|v\|^2}{\lambda_1^b} = C_4 \|v\|^2.
\end{aligned}$$

This and (2.3) imply that

$$\begin{aligned}
\int_{\mathbb{R}^2\setminus B_{R_0}(0)} A(x)^r v^{2m} &\leq C_5 \|v\|^{2m} + C_6 \|v\|^{2r/\beta_0} \|v\|^{2(m\beta_0 - r)/\beta_0} \qquad (2.4)\\
&= C_7 \|v\|^{2m},
\end{aligned}$$

where we have used that $\min\{2m, 2(m\beta_0 - r)/(\beta_0 - r)\} \geq 2$ and H is continuously embbeded into $L^p(\mathbb{R}^2)$, for any $p \geq 2$. Therefore, from (2.1), (2.2) and (2.4) we obtain

$$\begin{aligned}
\int_{\mathbb{R}^2} A(x)^r(e^{\alpha v^2} - 1)^r &\leq C_7 \sum_{m=1}^{\infty} \frac{1}{m!}(r\alpha \|v\|^2)^m + C_1 \int_{B_{R_0}(0)} (e^{r\alpha v^2} - 1) \\
&= C_7(e^{r\alpha\|v\|^2} - 1) + C_1 \int_{B_{R_0}(0)} (e^{r\alpha v^2} - 1) \qquad (2.5)\\
&< \infty,
\end{aligned}$$

which completes the proof. □

The following lemma is a version of Lemma 2.1 for our framework.

Lemma 2.3. *Let $\alpha > 0$, $q > 0$ and $\omega, v \in H$. Then*

$$\int_{\mathbb{R}^2} A(x)|\omega|^q(e^{\alpha v^2} - 1) < \infty.$$

Moreover, if $\alpha < 4\pi\zeta^2$ and $\|v\| \le 1$, then there exists $C = C(\alpha, q) > 0$ such that

$$\int_{\mathbb{R}^2} A(x)|\omega|^q(e^{\alpha v^2} - 1) \le C \|\omega\|^q.$$

Proof. Let $r \in (1, \beta_0)$ be such that $qr' \ge 2$, where $r' := r/(r-1)$. By Hölder's inequality, embbeding $H \hookrightarrow L^{qr'}(\mathbb{R}^2)$ and Lemma 2.2 we obtain

$$\begin{aligned} \int_{\mathbb{R}^2} A(x)|\omega|^q(e^{\alpha v^2} - 1) &\le \|\omega\|^q_{qr'} \left(\int_{\mathbb{R}^2} A(x)^r(e^{\alpha v^2} - 1)^r\right)^{1/r} \\ &\le C_1 \|\omega\|^q \left(\int_{\mathbb{R}^2} A(x)^r(e^{\alpha v^2} - 1)^r\right)^{1/r}, \end{aligned} \tag{2.6}$$

and the first statement is proved.

If $\alpha < 4\pi\zeta^2$ and $\|v\| \le 1$, take $r \in (1, \beta_0)$ such that $r\alpha < 4\pi\zeta^2$. By using (2.5)-(2.6) and writing $v^2 = \zeta^{-2}(\zeta v)^2$, we have that

$$\int_{\mathbb{R}^2} A(x)|\omega|^q(e^{\alpha v^2} - 1) \le C_2 \|\omega\|^q \left(e^{r\alpha\|v\|^2} - 1 + \int_{B_{R_0}(0)} (e^{r\alpha\zeta^{-2}(\zeta v)^2} - 1)\right)^{1/r}.$$

Since $\|v\| \le 1$, by (1.3) we have $\|\nabla(\zeta v)\|_2 \le 1$. Furthermore, $\|\zeta v\|_2 \le C_3\zeta \|v\| \le M$, for some $M > 0$ independent of v. The result follows from Lemma 2.1, the above inequality and $r\alpha\zeta^{-2} < 4\pi$. □

We present now a version of a famous result of Lions [25, subsection I.7] to our space H.

Corollary 2.4. *Let $q > 0$ and let $(\omega_n), (v_n) \subset H$ be such that (ω_n) is bounded in H, $v_n \rightharpoonup v$ weakly in H and $\|v_n\| = 1$, for any $n \in \mathbb{N}$. Then, if $\|v\| < 1$, for any $0 < p < 4\pi\zeta^2/(1 - \|v\|^2)$ it holds*

$$\sup_{n\in\mathbb{N}} \int_{\mathbb{R}^2} A(x)|\omega_n|^q(e^{pv_n^2} - 1) < \infty.$$

The same holds if $\|v\| = 1$ and $0 < p < \infty$.

Proof. First of all notice that, given $a, b \in \mathbb{R}$ and $\varepsilon > 0$, by Young's inequality we have

$$\begin{aligned} a^2 &= (a-b)^2 + b^2 + 2\varepsilon(a-b)b\varepsilon^{-1} \\ &\le (a-b)^2 + b^2 + 2\left(\frac{\varepsilon^2(a-b)^2}{2} + \frac{b^2\varepsilon^{-2}}{2}\right) \\ &= (1+\varepsilon^2)(a-b)^2 + (1+\varepsilon^{-2})b^2. \end{aligned}$$

Thus, if $r_1, r_2 > 1$ are such that $1/r_1 + 1/r_2 = 1$, by using Young's inequality again we obtain

$$\begin{aligned} A(x)|\omega_n|^q e^{pv_n^2} &\le (A(x)|\omega_n|^q)^{1/r_1} e^{p(1+\varepsilon^2)(v_n-v)^2} (A(x)|\omega_n|^q)^{1/r_2} e^{p(1+\varepsilon^{-2})v^2} \\ &\le \frac{1}{r_1} A(x)|\omega_n|^q e^{r_1 p(1+\varepsilon^2)(v_n-v)^2} + \frac{1}{r_2} A(x)|\omega_n|^q e^{r_2 p(1+\varepsilon^{-2})v^2}. \end{aligned}$$

So,

$$\begin{aligned}\int_{\mathbb{R}^2} A(x)|\omega_n|^q(e^{pv_n^2}-1) \quad &\leq \quad \frac{1}{r_1}\int_{\mathbb{R}^2} A(x)|\omega_n|^q\left(e^{r_1p(1+\varepsilon^2)(v_n-v)^2}-1\right)\\ &\quad + \frac{1}{r_2}\int_{\mathbb{R}^2} A(x)|\omega_n|^q\left(e^{r_2p(1+\varepsilon^{-2})v^2}-1\right).\end{aligned}$$

Since (ω_n) is bounded in H, inequality (2.6) with $\alpha = r_2p(1+\varepsilon^{-2})$ and Lemma 2.2 guarantee that the second integral on the right-hand side above is bounded independently of n. In order to estimate the other integral notice that, since $\|v_n\| = 1$ and $v_n \rightharpoonup v$ weakly in H, we get

$$\lim_{n\to\infty} p\,\|v_n - v\|^2 = p(1-\|v\|^2) < 4\pi\zeta^2.$$

Then, by taking $r_1 > 1$ sufficiently close to 1 and $\varepsilon > 0$ small, there exists $n_0 \in \mathbb{N}$ such that

$$r_1p(1+\varepsilon^2)\,\|v_n - v\|^2 < 4\pi\zeta^2, \quad \forall\ n > n_0.$$

Observing that $(v_n-v)^2 = \|v_n-v\|^2((v_n-v)/\|v_n-v\|)^2$, from the above inequality and Lemma 2.3 it follows that

$$\int_{\mathbb{R}^2} A(x)|\omega_n|^q\left(e^{r_1p(1+\varepsilon^2)(v_n-v)^2}-1\right) \leq C_1\,\|\omega_n\|^q \leq C_2, \quad \forall\ n > n_0,$$

which concludes the proof. □

The next result is an easy consequence of the monotonicity conditions (m_3) and (f_4).

Lemma 2.5. *Suppose that* (m_3) *and* (f_4) *hold. Then*

- (i) *the function* $L(t) := (1/2)M(t) - (1/4)m(t)t$ *is increasing in* $[0,\infty)$*; in particular,* $L(t) > L(0) = 0$*, for any* $t > 0$*;*
- (ii) *the function* $G(s) := sf(s) - 4F(s)$ *is non-decreasing in* $[0,\infty)$*; in particular,* $G(s) \geq G(0) = 0$*, for any* $s > 0$*.*

Proof. We only prove the first item since the other one is analogous. Let $t_1, t_2 \in \mathbb{R}$ be such that $0 < t_1 < t_2$. By (m_3), we have

$$\begin{aligned}2M(t_1) - m(t_1)t_1 \quad &= \quad 2M(t_2) - 2\int_{t_1}^{t_2}\frac{m(\tau)}{\tau}\tau\ \mathrm{d}\tau - \frac{m(t_1)}{t_1}t_1^2\\ &< \quad 2M(t_2) - \frac{m(t_2)}{t_2}(t_2^2-t_1^2) - \frac{m(t_2)}{t_2}t_1^2\\ &= \quad 2M(t_2) - m(t_2)t_2.\end{aligned}$$

and therefore the function $\widehat{L}(t) = 4L(t) = 2M(t) - m(t)t$ is increasing in $(0,\infty)$. Continuity in $t = 0$ implies that this property holds in $[0,\infty)$. □

We finish this section by presenting a convergence result proved in [8].

Lemma 2.6. *Let* $\Omega \subset \mathbb{R}^2$ *be a bounded domain. If* $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ *is a continuous function and* $(u_n) \subset L^1(\Omega)$ *is a sequence such that*

$$u_n \to u \ \text{in}\ L^1(\Omega), \quad f(\cdot,u_n), f(\cdot,u) \in L^1(\Omega), \quad \int_\Omega |f(x,u_n)u_n| \leq C,$$

where $C > 0$ *is a constant, then* $f(\cdot,u_n) \to f(\cdot,u)$ *in* $L^1(\Omega)$*.*

3. Variational framework

Given $\varepsilon > 0$, $\alpha > \alpha_0$ and $q \geq 1$, by (f_1) and (f_4^*) there exists a constant $C = C(\varepsilon, \alpha, q) > 0$ such that

$$\max\{|F(s)|, |sf(s)|\} \leq \varepsilon s^2 + C|s|^q(e^{\alpha s^2} - 1), \quad \forall\, s \in \mathbb{R}. \tag{3.1}$$

This, the embbeding $H \hookrightarrow L^2_A(\mathbb{R}^2)$ and Lemma 2.3 show that the functional $I : H \to \mathbb{R}$ given by

$$I(u) := \frac{1}{2}M(\|u\|^2) - \int_{\mathbb{R}^2} A(x)F(u), \quad u \in H,$$

is well defined. Moreover, Lemmas 2.2, 2.3 and standard arguments show that $I \in C^1(H, \mathbb{R})$ and, for any $u, v \in H$, there holds

$$I'(u)v = m(\|u\|^2) \int_{\mathbb{R}^2} (\nabla u \cdot \nabla v + b(x)uv) - \int_{\mathbb{R}^2} A(x)f(u)v,$$

and therefore critical points of I are precisely the weak solutions of problem (P).

Lemma 3.1. *Suppose that (m_1), (f_1) and (f_4^*) hold. Then there exists $\rho > 0$ and $\sigma > 0$ such that*

$$I(u) \geq \sigma\ , \quad \forall\ u \in H,\ \|u\| = \rho.$$

Proof. Let $\varepsilon > 0$, $\alpha > \alpha_0$ and $q > 2$. By (3.1), the embbeding $H \hookrightarrow L^2_A(\mathbb{R}^2)$ and Lemma 2.3, if $0 < \rho_1 < (4\pi\zeta^2/\alpha)^{1/2}$, then for $u \in H$ with $\|u\| \leq \rho_1$ we have that

$$\begin{aligned} \int_{\mathbb{R}^2} A(x)F(u) &\leq \varepsilon \int_{\mathbb{R}^2} A(x)u^2 + C\int_{\mathbb{R}^2} A(x)|u|^q(e^{\alpha u^2} - 1) \\ &\leq \varepsilon C_1 \|u\|^2 + C\int_{\mathbb{R}^2} A(x)|u|^q \left(e^{\alpha\rho_1^2(u/\|u\|)^2} - 1\right) \\ &\leq \varepsilon C_1 \|u\|^2 + C_2 \|u\|^q . \end{aligned}$$

Let $m_0 > 0$ be given by the hypothesis (m_1). Since $M(t) \geq m_0 t$, for any $t \geq 0$, we obtain

$$I(u) \geq \|u\|^2 \left(\frac{m_0}{2} - \varepsilon C_1 - C_2 \|u\|^{q-2}\right),$$

whenever $\|u\| \leq \rho_1$. Now choose $\varepsilon > 0$ and $0 < \rho \leq \rho_1$ such that $(m_0/2) - \varepsilon C_1 - C_2\rho^{q-2} > 0$. This choice is possible because $q > 2$. Thereby, for any $u \in H$ with $\|u\| = \rho$, we have that $I(u) \geq \sigma$, where

$$\sigma := \rho^2 \left(\frac{m_0}{2} - \varepsilon C_1 - C_2\rho^{q-2}\right) > 0.$$

This concludes the proof. □

Lemma 3.2. *Suppose that $(m_1), (m_3^*), (f_1), (f_3)$ and (f_4^*) hold. If $\rho > 0$ is given by Lemma 3.1, then there exists $v_0 \in H$ such that $I(v_0) < 0$ and $\|v_0\| > \rho$.*

Proof. By the continuity of m and (m_3^*), there exists $a_0 > 0$ such that

$$M(t) \leq a_0 t + a_1 \frac{t^2}{2}, \quad \forall\ t \geq 0. \tag{3.2}$$

On the other hand, by (f_3), there exist constants $C_1, C_2 > 0$ such that

$$F(s) \geq C_1 s^{\theta_0} - C_2, \quad \forall\ s \geq 0.$$

Now choose $v \in C_0(\mathbb{R}^2)\backslash\{0\}$ with $v \geq 0$ in $\mathbb{R}^2$. If $\Omega \subset \mathbb{R}^2$ contains the support of v, the above inequalities and (A_1) provide, for any $t \geq 0$,

$$I(tv) \leq a_0 t^2 \frac{\|v\|^2}{2} + a_1 t^4 \frac{\|v\|^4}{4} - C_1 t^{\theta_0} \int_\Omega v^{\theta_0} + C_2|\Omega|.$$

Since $\int_\Omega v^{\theta_0} > 0$ and $\theta_0 > 4$, we conclude that $I(tv) \to -\infty$, as $t \to \infty$. Hence the result holds for $v_0 = t_0 v$, with $t_0 > 0$ large enough. □

Remark 3.3. *For future reference we notice that the above lemma can be proved with a different argument if $f(s) > 0$ for any $s > 0$. In this case, for any $w \in H$ with $w^+ \not\equiv 0$, we have $\int_{\mathbb{R}^2} A(x)F(w) > 0$. On the other hand, defining, for any $s \in \mathbb{R}$,*

$$\phi_s(t) := t^{-\theta_0} F(ts) - F(s), \quad t > 0,$$

by (f_3) we have that $\phi_s'(t) \geq 0$, for any $t > 0$. This implies that $\phi_s(t) \geq \phi_s(1) = 0$ for any $t \geq 1$. That is,

$$F(ts) \geq t^{\theta_0} F(s), \quad \forall\ t \geq 1.$$

So, for $t \geq 1$, by (3.2) and the above inequality we have

$$I(tw) \leq a_0 t^2 \frac{\|w\|^2}{2} + a_1 t^4 \frac{\|w\|^4}{4} - t^{\theta_0} \int_{\mathbb{R}^2} A(x)F(w)$$

and the conclusion follows as before.

Lemmas 3.1 and 3.2 show that the energy functional I has the geometry of Mountain Pass Theorem. Thus, there exists a sequence $(u_n) \subset H$ such that

$$I(u_n) \longrightarrow c^* := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)) \ \text{ and } \ I'(u_n) \longrightarrow 0$$

as $n \to \infty$, where $\Gamma := \{\gamma \in C([0,1], H) : \gamma(0) = 0 \text{ and } I(\gamma(1)) < 0\}$. It is worth noticing that, by the definition of c^* and the proof of Lemma 3.1, we easily see that $c^* \geq \sigma > 0$.

4. Minimax estimates

In the first part of this section we will obtain an estimate for c^* in terms of the parameters ζ and α_0, given in the inequality (1.3) and the hypothesis (f_1), respectively.

We first consider the case $\zeta < 1$ and observe that S_p defined in (1.4) is the best constant of the compact embedding $H \hookrightarrow L^p(\mathbb{R}^2)$. Hence, there exists $v_p \in H$ such that $\|v_p\|_p = 1$ and $S_p = \|v_p\| > 0$. Without loss of generality, we may assume that $v_p \geq 0$ a.e. in $\mathbb{R}^2$.

Proposition 4.1. *Suppose that (m_3^*), (f_1), (f_3), (f_4^*) and (f_5) hold. If $\zeta < 1$ then*

$$c^* < \frac{1}{2} M\left(\frac{4\pi\zeta^2}{\alpha_0}\right).$$

Proof. Let $p_0 > 4$ be given in hypothesis (f_5) and $v_{p_0} \in H$ be such that $\|v_{p_0}\| = S_{p_0}$ and $\|v_{p_0}\|_{p_0} = 1$. Recalling that (f_5) implies that $f(s) > 0$ for any $s > 0$, by Remark 3.3 we have that $I(tv_{p_0}) \to -\infty$ as $t \to \infty$. Thus, it follows from the definition of c^* that

$$c^* \leq \max_{t > 0} I(tv_{p_0}).$$

By (A_1) and (f_5),

$$I(tv_{p_0}) < \frac{1}{2}M(t^2\,\|v_{p_0}\|^2) - t^{p_0}\frac{C_{p_0}}{p_0}\int_{\mathbb{R}^2}|v_{p_0}|^{p_0}, \quad \forall\, t>0.$$

Hence, from the definition of C_{p_0} we obtain

$$\max_{t>0} I(tv_{p_0}) < \max_{t>0}\left\{\frac{1}{2}M(t^2 S_{p_0}^2) - t^{p_0}\frac{C_{p_0}}{p_0}\right\} \le \frac{1}{2}M\left(\frac{4\pi\zeta^2}{\alpha_0}\right),$$

which concludes the proof. □

In order to deal with the case $\zeta = 1$ we define, for $n \ge 2$ and $R > 0$, the following sequence of scaled and truncated Green's functions (see Moser [26]):

$$\widetilde{G}_n(x) = \frac{1}{\sqrt{2\pi}}\begin{cases} (\log n)^{1/2}, & \text{if } |x| \le R/n,\\ \dfrac{\log(R/|x|)}{(\log n)^{1/2}}, & \text{if } R/n \le |x| \le R,\\ 0, & \text{if } |x| \ge R. \end{cases}$$

Notice that $\widetilde{G}_n \in W^{1,2}(\mathbb{R}^2)$ and $\operatorname{supp}(\widetilde{G}_n) = \overline{B_R(0)}$. Consequently, $\widetilde{G}_n \in H$. Furthermore,

$$\int_{\mathbb{R}^2}|\nabla \widetilde{G}_n|^2 = \frac{1}{2\pi\log n}\int_{\{R/n<|x|<R\}}|x|^{-2} = \frac{1}{\log n}\int_{R/n}^{R} s^{-1}\mathrm{d}s = 1$$

and, recalling the notation $M_R = \|b\|_{L^\infty(B_R(0))}$,

$$\begin{aligned}\int_{\mathbb{R}^2} b(x)|\widetilde{G}_n|^2 &= \frac{\log n}{2\pi}\int_{B_{R/n}(0)} b(x) + \frac{1}{2\pi\log n}\int_{\{R/n\le|x|\le R\}} b(x)\log^2\left(\frac{R}{|x|}\right)\\ &\le \frac{R^2M_R\log n}{2n^2} + \frac{M_R}{\log n}\int_{R/n}^{R} s\log^2\left(\frac{R}{s}\right)\mathrm{d}s\\ &= \frac{R^2M_R\log n}{2n^2} + \frac{R^2M_R}{\log n}\left(\frac{n^2-1}{4n^2} - \frac{\log^2(n)+\log n}{2n^2}\right)\\ &\le \frac{R^2M_R}{4\log n}.\end{aligned}$$

Then, if we set $\xi_n := \|\widetilde{G}_n\|$, we have $\xi_n^2 \le 1 + R^2M_R/(4\log n)$ and $\xi_n \to 1$ as $n \to \infty$.

We now consider the sequence of functions

$$G_n := \frac{\widetilde{G}_n}{\xi_n}$$

and prove the following technical result:

Lemma 4.2. *We have that*

$$\liminf_{n\to\infty}\int_{B_R(0)} e^{4\pi G_n^2} \ge \pi R^2 e^{-R^2M_R/2} + \pi R^2.$$

Proof. Since $\xi_n^2 \le 1 + R^2M_R/(4\log n)$, then

$$2(\xi_n^{-2}-1)\log n = 2\xi_n^{-2}(1-\xi_n^2)\log n \ge -\xi_n^{-2}\frac{R^2M_R}{2}$$

and therefore

$$\begin{aligned}\int_{B_{R/n}(0)} e^{4\pi G_n^2} &= \int_{B_{R/n}(0)} e^{2\xi_n^{-2}\log n} \\ &= \pi R^2 e^{2(\xi_n^{-2}-1)\log n} \geq \pi R^2 e^{-\xi_n^{-2}R^2 M_R/2}.\end{aligned} \tag{4.1}$$

On the other hand, by using the change of variable $t = \xi_n^{-1}\log(R/s)/\log n$, we get

$$\begin{aligned}\int_{\{R/n\leq|x|\leq R\}} e^{4\pi G_n^2} &= \int_{\{R/n\leq|x|\leq R\}} e^{2\xi_n^{-2}\log^2(R/|x|)/\log n} \\ &= 2\pi\int_{R/n}^{R} s e^{2(\xi_n^{-1}\log(R/s)/\log n)^2\log n}\mathrm{d}s \\ &= 2\pi R^2\xi_n\log n\int_0^{\xi_n^{-1}} e^{2(t^2-\xi_n t)\log n}\mathrm{d}t \\ &\geq 2\pi R^2\xi_n\log n\int_0^{\xi_n^{-1}} e^{-2\xi_n t\log n}\mathrm{d}t \\ &= -\pi R^2 e^{-2\log n} + \pi R^2.\end{aligned}$$

Therefore, since $\lim\limits_{n\to\infty}\xi_n = 1$, it follows from (4.1) and the above inequality that

$$\liminf_{n\to\infty}\int_{B_R(0)} e^{4\pi G_n^2} \geq \pi R^2 e^{-R^2 M_R/2} + \pi R^2,$$

as stated. □

Now, for $\zeta = 1$, we can use the previous lemma to obtain the same estimate of Proposition 4.1 with condition (f_6) instead of (f_5):

Proposition 4.3. *Suppose that* (m_3^*), (f_1), (f_3), (f_4^*) *and* (f_6) *hold. Then*

$$c^* < \frac{1}{2}M\left(\frac{4\pi}{\alpha_0}\right).$$

Proof. As in the proof of Lemma 3.2, we have that $I(tG_n)\to-\infty$ as $t\to\infty$. By definition of c^*, it follows that

$$c^* \leq \max_{t>0} I(tG_n), \quad \forall\, n\geq 2.$$

Since the functional I has the Mountain Pass geometry, for each n there exists $t_n > 0$ such that

$$I(t_n G_n) = \max_{t>0} I(tG_n).$$

Thus, it is enough to prove that, for some $n\in\mathbb{N}$, we have

$$I(t_n G_n) < \frac{1}{2}M\left(\frac{4\pi}{\alpha_0}\right).$$

Suppose, by contradiction, that the above inequality is false. Since $\|G_n\| = 1$, we have that

$$I(t_n G_n) = \frac{1}{2}M(t_n^2) - \int_{\mathbb{R}^2} A(x)F(t_n G_n) \geq \frac{1}{2}M\left(\frac{4\pi}{\alpha_0}\right), \quad \forall\, n\geq 2.$$

Since A and F are nonnegative, this implies that $M(t_n^2) \geq M(4\pi/\alpha_0)$. But M is an increasing function, because its derivative m is positive. We conclude that

$$t_n^2 \geq \frac{4\pi}{\alpha_0}. \tag{4.2}$$

On the other hand, since $I'(t_nG_n)t_nG_n = 0$, we can use (A_1), $f \geq 0$ and $\mathrm{supp}(G_n) = \overline{B_R(0)}$ to obtain

$$\begin{aligned} m(t_n^2)t_n^2 &= \int_{B_R(0)} A(x)f(t_nG_n)t_nG_n \\ &\geq \int_{B_{R/n}(0)} f(t_nG_n)t_nG_n \\ &= \int_{B_{R/n}(0)} f\left(\frac{t_n\xi_n^{-1}}{\sqrt{2\pi}}(\log n)^{1/2}\right)\frac{t_n\xi_n^{-1}}{\sqrt{2\pi}}(\log n)^{1/2}. \end{aligned} \tag{4.3}$$

But notice that, given $0 < \delta < \gamma_0$, by (f_6) there exists $s_\delta > 0$ such that

$$f(s)s \geq (\gamma_0 - \delta)e^{\alpha_0 s^2}, \quad \forall\ s \geq s_\delta. \tag{4.4}$$

Since $t_n\xi_n^{-1}(\log n)^{1/2} \to \infty$ as $n \to \infty$, because $\xi_n \to 1$ and $t_n \not\to 0$, it follows that, for n large,

$$\begin{aligned} m(t_n^2)t_n^2 &\geq \int_{B_{R/n}(0)} (\gamma_0 - \delta)e^{\alpha_0 t_n^2(\xi_n\sqrt{2\pi})^{-2}\log n} \\ &= \pi R^2(\gamma_0 - \delta)e^{\left(\alpha_0 t_n^2(\xi_n\sqrt{2\pi})^{-2} - 2\right)\log n}. \end{aligned}$$

This inequality and (m_3^*) imply that the sequence $(t_n) \subset (0, \infty)$ is bounded and, consequently, there exists $t_0 > 0$ such that, up to a subsequence, $t_n \to t_0$ as $n \to \infty$. In this case, the above inequality also implies that

$$\lim_{n\to\infty}\left(\alpha_0 t_n^2(\xi_n\sqrt{2\pi})^{-2} - 2\right) = 2\left(\frac{\alpha_0}{4\pi}t_0^2 - 1\right) \leq 0.$$

From this and (4.2), we infer that

$$\lim_{n\to\infty} t_n^2 = \frac{4\pi}{\alpha_0}. \tag{4.5}$$

Now, for each $n \geq 2$, define the sets

$$D_{n,\delta} := \{x \in B_R(0) : t_nG_n(x) \geq s_\delta\}, \quad E_{n,\delta} := B_R(0)\backslash D_{n,\delta}.$$

By hypothesis (A_1), (4.3) and (4.4), we have that

$$\begin{aligned} m(t_n^2)t_n^2 &\geq \int_{D_{n,\delta}} f(t_nG_n)t_nG_n + \int_{E_{n,\delta}} f(t_nG_n)t_nG_n \\ &\geq (\gamma_0 - \delta)\left(\int_{B_R(0)} e^{\alpha_0 t_n^2G_n^2} - \int_{E_{n,\delta}} e^{\alpha_0 t_n^2 G_n^2}\right) \\ &\quad + \int_{E_{n,\delta}} f(t_nG_n)t_nG_n. \end{aligned} \tag{4.6}$$

But $G_n(x) \to 0$ for a.e. $x \in B_R(0)$ and, therefore, $\chi_{E_{n,\delta}}(x) \to 1$ for a.e. $x \in B_R(0)$, as $n \to \infty$, where $\chi_{E_{n,\delta}}$ is the characteristic function of $E_{n,\delta}$. Moreover, $t_nG_n < s_\delta$

in $E_{n,\delta}$. Then, it follows from the Lebesgue's Theorem that

$$\int_{E_{n,\delta}} e^{\alpha_0 t_n^2 G_n^2} \longrightarrow \pi R^2, \quad \int_{E_{n,\delta}} f(t_n G_n) t_n G_n \longrightarrow 0.$$

Hence, by (4.2), (4.5), (4.6) and Lemma 4.2, we get

$$\begin{aligned} m\left(\frac{4\pi}{\alpha_0}\right)\frac{4\pi}{\alpha_0} &\geq (\gamma_0-\delta)\liminf_{n\to\infty}\left(\int_{B_R(0)} e^{\alpha_0 t_n^2 G_n^2}\right) - (\gamma_0-\delta)\pi R^2 \\ &\geq (\gamma_0-\delta)\liminf_{n\to\infty}\left(\int_{B_R(0)} e^{4\pi G_n^2}\right) - (\gamma_0-\delta)\pi R^2 \\ &\geq (\gamma_0-\delta)\pi R^2 e^{-R^2 M_R/2}. \end{aligned}$$

Since $0<\delta<\gamma_0$ is arbitrary, we can let $\delta \to 0^+$ in the above inequality to obtain

$$\gamma_0 \leq \frac{4}{\alpha_0} m\left(\frac{4\pi}{\alpha_0}\right) R^{-2} e^{R^2 M_R/2}.$$

Since $R>0$ is also arbitrary, we can take the infimum for $R>0$ in this inequality and obtain a contradiction with (f_6). This concludes the proof. $\square$

Let $\mathcal{N}$ be the Nehari manifold associated with the functional I, namely

$$\mathcal{N} := \{u \in H \setminus \{0\} : I'(u)u = 0\}$$

and define

$$d^* := \inf_{u\in\mathcal{N}} I(u).$$

The next result shows that obtaining a ground state solution is equivalent to show that there exists a critical point u_0 such that $I(u_0) = c^*$.

Lemma 4.4. *Suppose that (m_3), (f_1), (f_3) and (f_4) hold. Then $c^* \leq d^*$.*

Proof. Let $u \in \mathcal{N}$. Then, recalling that $f(s)=0$ for $s\leq 0$, the fact that $u\neq 0$ and $I'(u)u=0$ imply that $u^+ \not\equiv 0$. If $h(t) := I(tu)$, $t\geq 0$, we have

$$\begin{aligned} h'(t) = I'(tu)u &= I'(tu)u - t^3 I'(u)u \\ &= m(t^2\|u\|^2)t\|u\|^2 - \int_{\mathbb{R}^2} A(x)f(tu)u \\ &\quad - t^3 m(\|u\|^2)\|u\|^2 + t^3\int_{\mathbb{R}^2} A(x)f(u)u \\ &= t^3\|u\|^4\left(\frac{m(t^2\|u\|^2)}{t^2\|u\|^2} - \frac{m(\|u\|^2)}{\|u\|^2}\right) \\ &\quad + t^3\int_{\{u>0\}} A(x)u^4\left(\frac{f(u)}{u^3} - \frac{f(tu)}{(tu)^3}\right), \end{aligned}$$

for any $t>0$. Thus, by (m_3) and (f_4), we have that $h'(t)\geq 0$ for $0<t<1$ and $h'(t)\leq 0$ for $t>1$. Since $h'(1) = I'(u)u = 0$, then

$$I(u) = h(1) = \max_{t\geq 0} h(t) = \max_{t\geq 0} I(tu).$$

On the other hand, since $u^+ \not\equiv 0$ and (f_4) implies that $f(s) > 0$ for any $s > 0$, by Remark 3.3 there exists $t_0 > 0$ such that $I(t_0 u) < 0$. Defining $\gamma : [0,1] \to H$ by $\gamma(t) := tt_0 u$, from definition of c^* it follows that

$$c^* \leq \max_{t\in[0,1]} I(\gamma(t)) \leq \max_{t\geq 0} I(tu) = I(u).$$

Since $u \in \mathcal{N}$ is arbitrary, we conclude that $c^* \leq d^*$. □

5. Proof of the main theorems

We present in this final section the proofs for our main theorems. We first prove that Palais-Smale sequences are bounded.

Proposition 5.1. *Suppose that* (m_1), (m_3), (f_1)–(f_3) *and* (f_4^*) *hold. Let* $(u_n) \subset H$ *be a Palais-Smale sequence for the functional* I *in the level* $c \in \mathbb{R}$, *that is,*

$$I(u_n) \longrightarrow c \quad \text{and} \quad I'(u_n) \longrightarrow 0$$

as $n \to \infty$. *Then* (u_n) *is bounded in* H. *Moreover, up to a subsequence,*

(i) $\displaystyle\int_\Omega A(x) f(u_n) \longrightarrow \int_\Omega A(x) f(u)$, *for any bounded domain* $\Omega \subset \mathbb{R}^2$;

(ii) $\displaystyle\int_{\mathbb{R}^2} A(x) F(u_n) \longrightarrow \int_{\mathbb{R}^2} A(x) F(u)$.

Proof. By using Lemma 2.5(i) and (f_3), we get

$$c + o(1) + \|u_n\| \geq I(u_n) - \frac{1}{\theta_0} I'(u_n)u_n \geq \left(\frac{\theta_0 - 4}{4\theta_0}\right) m_0 \|u_n\|^2,$$

as $n \to \infty$, where m_0 is given in hypothesis (m_1). Since $\theta_0 > 4$ and $m_0 > 0$, the above inequality implies that the sequence (u_n) is bounded in H.

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. Since $u_n \rightharpoonup u$ weakly in H, it follows that $u_n \to u$ in $L^1(\Omega)$, up to a subsequence. Moreover, since $I'(u_n)u_n \to 0$ as $n \to \infty$, we get

$$\int_\Omega |f(u_n)u_n| \leq \int_{\mathbb{R}^2} A(x) f(u_n) u_n = m(\|u_n\|^2) \|u_n\|^2 - I'(u_n)u_n \leq C_1. \tag{5.1}$$

By (3.1), $f(u_n)$, $f(u) \in L^1(\Omega)$ and therefore we conclude from Lemma 2.6 that $f(u_n) \to f(u)$ in $L^1(\Omega)$. But

$$\int_\Omega A(x)|f(u_n) - f(u)| \leq \|A\|_{L^\infty(\Omega)} \int_\Omega |f(u_n) - f(u)| \longrightarrow 0,$$

which proves (i). For the second item we take $r > 0$ and use (i) to obtain $h \in L^1(B_r(0))$ such that $A(x)f(u_n(x)) \leq h(x)$ for a.e. $x \in B_r(0)$. So, by using (f_2) we get

$$\begin{aligned} A(x)F(u_n(x)) &\leq \|A\|_{L^\infty(B_r(0))} \max_{s\in[0,s_0]} F(s) + K_0 A(x) f(u_n(x)) \\ &\leq \|A\|_{L^\infty(B_r(0))} F(s_0) + K_0 h(x) \end{aligned}$$

for a.e. $x \in B_r(0)$. Since we may assume that $u_n(x) \to u(x)$ for a.e. $x \in \mathbb{R}^2$ and F is continuous, by Lebesgue's Theorem we obtain

$$\int_{B_r(0)} A(x)F(u_n) \longrightarrow \int_{B_r(0)} A(x)F(u).$$

Thus, in order to conclude the proof of item (ii), it is enough to show that, given $\delta>0$, there exists $r>0$ such that:

$$\int_{\mathbb{R}^2\setminus B_r(0)} A(x)F(u_n)<\delta,\quad \forall\, n\in\mathbb{N};\quad \int_{\mathbb{R}^2\setminus B_r(0)} A(x)F(u)<\delta. \tag{5.2}$$

Since $A(\cdot)F(u)$ is integrable, the second inequality holds for $r>0$ large. For the first one, we can use (f_2) and (f_4^*) to write

$$F(s)\le C_2|s|^2+C_3 f(s),\quad \forall\, s\in\mathbb{R}.$$

Then, given $K>0$, by the above inequality, the embbeding $H\hookrightarrow L_A^3(\mathbb{R}^2)$, the boundedness of (u_n) in H and (5.1), we have that

$$\begin{aligned}\int_{\{|u_n|>K\}\cap(\mathbb{R}^2\setminus B_r(0))} A(x)F(u_n) \ &\le\ C_2\int_{\{|u_n|>K\}\cap(\mathbb{R}^2\setminus B_r(0))} A(x)|u_n|^2\\ &\quad +C_3\int_{\{|u_n|>K\}\cap(\mathbb{R}^2\setminus B_r(0))} A(x)f(u_n)\\ &\le\ \frac{C_2}{K}\int_{\mathbb{R}^2} A(x)|u_n|^3+\frac{C_3}{K}\int_{\mathbb{R}^2} A(x)f(u_n)u_n\\ &\le\ \frac{C_4}{K}.\end{aligned}$$

Thus, we can choose K large enough such that

$$\int_{\{|u_n|>K\}\cap(\mathbb{R}^2\setminus B_r(0))} A(x)F(u_n)<\frac{\delta}{2},\quad \forall\, n\in\mathbb{N}.$$

On the other hand, by inequality (3.1) with $q=2$, for $|s|\le K$ we have that

$$F(s)\le C_5|s|^2+C_6|s|^2(e^{\alpha s^2}-1)\le \Big(C_5+C_6(e^{\alpha K^2}-1)\Big)\,|s|^2\le C_7|s|^2,$$

where $\alpha>\alpha_0$ and $C_7=C_7(\alpha,K)>0$ are constants. Then

$$\int_{\{|u_n|\le K\}\cap(\mathbb{R}^2\setminus B_r(0))} A(x)F(u_n)\le C_7\int_{\{|u_n|\le K\}\cap(\mathbb{R}^2\setminus B_r(0))} A(x)|u_n|^2.$$

Since $u_n\to u$ in $L_A^2(\mathbb{R}^2)$, there exists $g\in L^1(\mathbb{R}^2)$ such that $A(x)|u_n(x)|^2\le g(x)$ for a.e. $x\in\mathbb{R}^2$. So, by choosing $r>0$ large enough such that $C_7\int_{\mathbb{R}^2\setminus B_r(0)} g(x)<\delta/2$, we have

$$\int_{\{|u_n|\le K\}\cap(\mathbb{R}^2\setminus B_r(0))} A(x)F(u_n)<\frac{\delta}{2},\quad \forall\, n\in\mathbb{N}.$$

Combining the above estimates, we obtain (5.2), which concludes the proof of the second item. $\square$

We are ready to prove Theorems 1.1 and 1.2.

Proof of Theorem 1.1. As previously observed there exists $(u_n)\subset H$ such that

$$I(u_n)\longrightarrow c^*\ \text{ and }\ I'(u_n)\longrightarrow 0, \tag{5.3}$$

as $n\to\infty$. By Proposition 5.1, this sequence is bounded in H and therefore we may assume that, for some $u_0\in H$,

$$u_n\rightharpoonup u_0 \text{ weakly in } H,\quad u_n\to u_0 \text{ in } L_A^2(\mathbb{R}^2). \tag{5.4}$$

We claim that

$$I(u_0)\ge 0. \tag{5.5}$$

Indeed, suppose by contradiction that $I(u_0) < 0$. Then $u_0 \neq 0$ and, defining $h(t) := I(tu_0)$, $t \geq 0$, we have that $h(0) = 0$ and $h(1) < 0$. Arguing as in the proof of Lemma 3.1 we see that $h(t) > 0$, for any $t > 0$ small. Thus, there exists $t_0 \in (0,1)$ such that

$$h(t_0) = \max_{t\in[0,1]} h(t) = \max_{t\in[0,1]} I(tu_0), \quad h'(t_0) = I'(t_0u_0)u_0 = 0.$$

So, by definition of c^* and Lemma 2.5,

$$\begin{aligned} c^* \leq h(t_0) &= h(t_0) - \frac{1}{4}h'(t_0)t_0 \\ &= \frac{1}{2}M(\|t_0u_0\|^2) - \frac{1}{4}m(\|t_0u_0\|^2)\,\|t_0u_0\|^2 \\ &\quad + \frac{1}{4}\int_{\mathbb{R}^2} A(x)\,(f(t_0u_0)t_0u_0 - 4F(t_0u_0)) \\ &< \frac{1}{2}M(\|u_0\|^2) - \frac{1}{4}m(\|u_0\|^2)\,\|u_0\|^2 \\ &\quad + \frac{1}{4}\int_{\mathbb{R}^2} A(x)\,(f(u_0)u_0 - 4F(u_0))\,. \end{aligned}$$

From this inequality, the lower semicontinuity of the norm, Fatou's lemma and (5.3), it follows that

$$\begin{aligned} c^* &< \liminf_{n\to\infty}\left(\frac{1}{2}M(\|u_n\|^2) - \frac{1}{4}m(\|u_n\|^2)\,\|u_n\|^2\right) \\ &\quad + \frac{1}{4}\liminf_{n\to\infty}\int_{\mathbb{R}^2} A(x)\,(f(u_n)u_n - 4F(u_n)) \\ &\leq \liminf_{n\to\infty}\left(I(u_n) - \frac{1}{4}I'(u_n)\right) = c^*, \end{aligned}$$

which is absurd. This proves (5.5).

Now we will show that $I'(u_0) = 0$ and $I(u_0) = c^*$. Let $\rho_0 \geq 0$ such that $\|u_n\| \to \rho_0$. Clearly $\|u_0\| \leq \rho_0$ and we shall prove that the equality holds. Suppose, by contradiction, that $\|u_0\| < \rho_0$. Defining $v_n := u_n/\|u_n\|$ and $v_0 := u_0/\rho_0$, we have that $v_n \rightharpoonup v_0$ weakly in H and $\|v_0\| < 1$. So, by Corollary 2.4, it follows that

$$\sup_n \int_{\mathbb{R}^2} A(x)|u_n - u_0|^q (e^{pv_n^2} - 1) < \infty, \quad \forall\, q > 0, \quad \forall\, p < \frac{4\pi\zeta^2}{1 - \|v_0\|^2}. \tag{5.6}$$

On the other hand, by using (5.3), Proposition 5.1(ii), Proposition 4.1, (5.5) and hypothesis (m_2), we have that

$$\begin{aligned} M(\rho_0^2) &= \lim_{n\to\infty} M(\|u_n\|^2) = \lim_{n\to\infty} 2\left(I(u_n) + \int_{\mathbb{R}^2} A(x)F(u_n)\right) \\ &= 2c^* + 2\int_{\mathbb{R}^2} A(x)F(u_0) = 2c^* + M(\|u_0\|^2) - 2I(u_0) \\ &< M\left(\frac{4\pi\zeta^2}{\alpha_0}\right) + M(\|u_0\|^2) \leq M\left(\frac{4\pi\zeta^2}{\alpha_0} + \|u_0\|^2\right). \end{aligned}$$

Since M is increasing, it follows that $\rho_0^2 < (4\pi\zeta^2/\alpha_0) + \|u_0\|^2$. Hence, by observing that $\rho_0^2 = (\rho_0^2 - \|u_0\|^2)/(1 - \|v_0\|^2)$, we get

$$\alpha_0 \rho_0^2 < \frac{4\pi\zeta^2}{1 - \|v_0\|^2}.$$

Then, there exists $\eta > 0$ such that $\alpha_0 \|u_n\|^2 < \eta < 4\pi\zeta^2/(1 - \|v_0\|^2)$ for any n large enough. Thus, we can choose $r \in (1,2)$ close to 1 and $\alpha > \alpha_0$ close to α_0 such that we still have $r\alpha \|u_n\|^2 < \eta < 4\pi\zeta^2/(1 - \|v_0\|^2)$ and, by (5.6),

$$\begin{aligned} \int_{\mathbb{R}^2} A(x)|u_n - u_0|^{2-r}(e^{r\alpha u_n^2} - 1) &= \int_{\mathbb{R}^2} A(x)|u_n - u_0|^{2-r}(e^{r\alpha \|u_n\|^2 v_n^2} - 1) \\ &\le \int_{\mathbb{R}^2} A(x)|u_n - u_0|^{2-r}(e^{\eta v_n^2} - 1) \le C_1, \end{aligned}$$

for any n large. Therefore, by using inequality (3.1) with $q = 1$, Hölder's inequality, $H \hookrightarrow L_A^2(\mathbb{R}^2)$, Lemma 2.2($i$) and (5.4), we obtain

$$\begin{aligned} &\left| \int_{\mathbb{R}^2} A(x) f(u_n)(u_n - u_0) \right| \le \\ &\le C_2 \int_{\mathbb{R}^2} A(x)|u_n||u_n - u_0| + \ C_3 \int_{\mathbb{R}^2} A(x)|u_n - u_0|(e^{\alpha u_n^2} - 1) \\ &= C_2 \int_{\mathbb{R}^2} \sqrt{A(x)}|u_n|\sqrt{A(x)}|u_n - u_0| \\ &\quad + C_3 \int_{\mathbb{R}^2} (A(x)|u_n - u_0|^2)^{(r-1)/r}(A(x)|u_n - u_0|^{2-r})^{1/r}(e^{\alpha u_n^2} - 1) \\ &\le C_4 \|u_n\| \, \|u_n - u_0\|_{L_A^2(\mathbb{R}^2)} \\ &\quad + C_3 \|u_n - u_0\|_{L_A^2(\mathbb{R}^2)}^{2(r-1)/r} \left(\int_{\mathbb{R}^2} A(x)|u_n - u_0|^{2-r}(e^{r\alpha u_n^2} - 1) \right)^{1/r} \\ &\le C_5 \|u_n - u_0\|_{L_A^2(\mathbb{R}^2)} + C_6 \|u_n - u_0\|_{L_A^2(\mathbb{R}^2)}^{2(r-1)/r} \longrightarrow 0, \end{aligned}$$

as $n \to \infty$. Since $I'(u_n)(u_n - u_0) \to 0$ as $n \to \infty$, we conclude that

$$\begin{aligned} 0 &= \lim_{n\to\infty} \left(I'(u_n)(u_n - u_0) + \int_{\mathbb{R}^2} A(x) f(u_n)(u_n - u_0) \right) \\ &= \lim_{n\to\infty} m(\|u_n\|^2) \left\langle u_n, u_n - u_0 \right\rangle_H \\ &= m(\rho_0^2)(\rho_0^2 - \|u_0\|^2) \\ &> 0, \end{aligned}$$

which does not make sense. Thus, we have that $\|u_0\| = \rho_0 = \lim_{n\to\infty} \|u_n\|$ and therefore $u_n \to u_0$ strongly in H. Since $I \in C^1(H, \mathbb{R})$, from (5.3) we conclude that $I(u_0) = c^* \neq 0$ and $I'(u_0) = 0$. Recalling that $f(s) = 0$, for $s \le 0$, we can use Lemma 4.4 to conclude that $u_0 \ge 0$ is a ground state solution. □

Proof of Theorem 1.2. It is sufficient to argue as in the proof of Theorem 1.1, considering now $\zeta = 1$ and using Proposition 4.3 instead of Proposition 4.1. □

From now on we suppose that $m \equiv 1$. Hence, the equation in (P) becomes the Schrödinger equation

$$(\widehat{P}) \qquad -\Delta u + b(x)u = A(x)f(u) \quad \text{in} \quad \mathbb{R}^2.$$

The energy functional associated to this problem is

$$J(u) := \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^2} A(x)F(u), \quad u \in H.$$

Under hypotheses (f_1), $(\widehat{f}_3)$ and (f_4^*), we can prove that $J \in C^1(H, \mathbb{R})$,

$$J'(u)v = \int_{\mathbb{R}^2} (\nabla u \cdot \nabla v + b(x)uv) - \int_{\mathbb{R}^2} A(x)f(u)v, \quad \forall\ u, v \in H,$$

and J has the geometry of Mountain Pass Theorem. This ensure the existence of a sequence $(u_n) \subset H$ such that

$$(5.7) \qquad J(u_n) \longrightarrow c^{**} \quad \text{and} \quad J'(u_n) \longrightarrow 0,$$

as $n \to \infty$, where

$$c^{**} := \inf_{\lambda \in \Lambda} \max_{t \in [0,1]} J(\lambda(t)) > 0$$

and $\Lambda := \{\lambda \in C([0,1], H) : \lambda(0) = 0 \text{ and } J(\lambda(1)) < 0\}$.

Evidently, estimates for the minimax level c^{**} analogous to that of Section 4 are valid, with hypotheses $(\widehat{f}_3) - (\widehat{f}_6)$ instead of $(f_3) - (f_6)$, where necessary. Under hypotheses (f_1), (f_2), $(\widehat{f}_3)$ and (f_4^*), we also obtain the same conclusions of Proposition 5.1 for the functional J.

Proof of Theorem 1.3. Let $(u_n) \subset H$ be the sequence given in (5.7). As in the proof of Theorem 1.1, the boundedness of (u_n) in H implies on the existence of $u_0 \in H$ such that, up to a subsequence,

$$(5.8) \qquad u_n \rightharpoonup u_0 \text{ weakly in } H, \quad u_n \to u_0 \text{ in } L^2_A(\mathbb{R}^2).$$

Moreover, as we learned from the proof of Proposition 5.1, we have that $Af(u_n) \to Af(u_0)$ in $L^1(\Omega)$, for any bounded domain $\Omega \subset \mathbb{R}^2$. By this, by the weak convergence in (5.8) and the convergence $J'(u_n) \to 0$, we get

$$J'(u_0)\phi = \langle u_0, \phi \rangle_H - \int_{\mathbb{R}^2} A(x)f(u_0)\phi = 0, \quad \forall\ \phi \in C_0^\infty(\mathbb{R}^2).$$

By the same arguments of [1, Theorem 3.22], we can verify that $C_0^\infty(\mathbb{R}^2)$ is dense in H. Hence $J'(u_0)u_0 = 0$. Since, by $(\widehat{f}_3)$, we have $J(u_0) \geq (1/\widehat{\theta}_0)J'(u_0)u_0$, it follows that $J(u_0) \geq 0$. Hence, we can use the estimate $c^{**} < (2\pi\zeta^2)/\alpha_0$ and proceed as in the proof of Theorem 1.1. □

Proof of Theorem 1.4. It is sufficient to argue as in the proof of Theorem 1.3, considering now $\zeta = 1$ and using the estimate $c^{**} < (2\pi)/\alpha_0$. □

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