POSITIVE SOLUTION FOR AN INDEFINITE FOURTH-ORDER NONLOCAL PROBLEM

MARCELO F. FURTADO AND JOÃO PABLO P. DA SILVA

Abstract. We prove the existence of positive solution for the problem

$$\gamma\Delta^2 u - m(u)\Delta u = \mu a(x)u^q + b(x)u^p, \quad \text{in } \Omega, \qquad u = \gamma\Delta u = 0, \quad \text{on } \partial\Omega,$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $\gamma \in \{0,1\}$, $0 < q < 1 < p$, m is weakly continuous in $H^2(\Omega) \cap H_0^1(\Omega)$, $a \in L^\infty(\Omega)$ is nonnegative and b is a bounded potential which can change sign. The solution is obtained via a sub-supersolution approach when the parameter $\mu > 0$ is small.

1. Introduction

In this paper we consider the equation

$$\gamma\Delta^2 u - m(u)\Delta u = f_\mu(x,u), \quad x \in \Omega,$$

where $\gamma \in \{0,1\}$, the nonlinearity f_μ depends on the parameter $\mu > 0$ and m is assumed to be weakly continuous in $H^2(\Omega) \cap H_0^1(\Omega)$. Due to the presence of this last function the equation is not a pointwise identity and therefore the problem is called nonlocal.

In what follows we make some comments on the physical importance of this kind of problem. When $\gamma = 1$, the equation is related to the so called Berger plate model (see [3, 7])

$$u_{tt} + \Delta^2 u + \left(Q + \int_\Omega |\nabla u|^2 \mathrm{d}x \right) \Delta u = f(x,u,u_t),$$

and it is a simplification of the von Karman plate equation that describes large deflection of plate. The parameter Q describes in-plane forces applied to the plate and the function f represents transverse loads which may depend on the displacement u and the velocity u_t. The equation is also related with some models which describe the bending equilibrium states of a beam subjected to a force f and other elastic force (see [34]), namely

$$u_{tt} + \frac{EI}{\rho} u_{xxxx} - \left(\frac{h}{\rho} + \frac{EA}{2\rho L} \int_0^L |u_x|^2 \mathrm{d}x \right) u_{xx} = f(x,u).$$

When $\gamma = 0$, the equation has its origin in the theory of nonlinear vibration, specially with the following model for the modified d'Alembert wave equation

$$\rho\, u_{tt} - \left(\frac{P_0}{h} + \frac{E}{2L} \int_0^L |u_x|^2\, dx \right) u_{xx} = f(x,u),$$

1991 *Mathematics Subject Classification.* Primary 35J35; Secondary 35J30.

Key words and phrases. Fourth-order problems; concave and convex problems;sub and supersolution methods.

The first author was partially supported by CNPq and FAPDF/Brazil.

proposed by Kirchoff in [21]. Its main feature is to consider the effects of changes on the length of the string during vibrations. In the two above models, the parameters E, I, ρ, h, A, L and P_0 are positive and have specific physical meanings.

We are interest here in the case that f_μ is a combined nonlinearity. More specifically, we shall consider the following nonlocal fourth-order problem

$$(P_\mu) \qquad \begin{cases} \gamma\Delta^2 u - m(u)\Delta u = \mu a(x)u^q + b(x)u^p, & \text{in } \Omega, \\ u > 0, \quad \text{in } \Omega, \\ u = \gamma\Delta u = 0, \quad \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $\gamma \in \{0, 1\}$, $\mu > 0$ is a parameter, $0 < q < 1 < p$ and the potentials a and b verify the following basic assumptions:

(a_1) $a \in L^\infty(\Omega)$ is nonnegative;

(a_2) there exist an open set $\Omega_a \subset \Omega$ and $\delta > 0$ such that

$$\inf_{x\in\Omega_a} a(x) \geq \delta;$$

(b_1) $b \in L^\infty(\Omega)$.

In the celebrated paper [2], Ambrosetti, Brezis and Cerami supposed that $\gamma = 0$, m, a and b are constant and equal to 1 and obtained two positive solutions if $\mu > 0$ is small. In [10], de Figueiredo, Gossez and Ubilla generalized this result by considering nonconstant sign-changing potentials. In this setting, the Maximum Principle can fail and therefore the solutions obtained were only nonnegative. Some other results for the Laplacian, s-Laplacian, fractional Laplacian and Kirchhoff operator can be found in [1, 28, 12, 35, 5, 23] and references therein. In all this paper, only the second order case $\gamma = 0$ was considered. Concerning the fourth-order one, we notice that, in [15], the authors supposed that m is increasing, $a \equiv 1$, $b \equiv 1$ and obtained infinitely many solutions, for $1 < q < 2 < p = 2N/(N-4)$ and $\mu > 0$ small. This result was partially extended in [29], where the authors assumed that $b \equiv 1$, the (nonautonomous) concave term were of type $\mu h(x, u)$ with some technical assumptions on h and the growth of the function m.

To present our main results, we denote by H the Hilbert space $H^2(\Omega) \cap H^1_0(\Omega)$ endowed with the norm

$$\|u\| := \left(\int_\Omega (\Delta u)^2 \mathrm{d}x\right)^{1/2}, \quad \forall\, u \in H.$$

If g is a measurable function, we set $g^+(x) := \max\{g(x), 0\}$ and $g^- := g^+ - g$. The first result of this paper can be stated as follows:

Theorem 1.1. *Suppose that $\gamma = 1$, $0 < q < 1 < p$ and the potentials a, b satisfy $(a_1) - (a_2)$ and (b_1). If m verifies*

(m_1) $m : H \to \mathbb{R}$ is weakly continuous;

(m_2) $m(0) > 0$,

then there exists $\mu^* = \mu^*(\Omega, \|a^+\|_{L^\infty(\Omega)}, \|b^+\|_{L^\infty(\Omega)}, N)$ such that, for each $\mu \in (0, \mu^*)$, the problem (P_μ) has a solution.

In our second result, we consider the second order case, namely $\gamma = 0$. In this new setting, we need to consider a global sign assumption on m. More specifically, we prove the following:

Theorem 1.2. *Suppose that $\gamma = 0$, $0 < q < 1 < p$ and the potentials a, b satisfy $(a_1) - (a_2)$ and (b_1). If m verifies (m_1) and*

$(\widehat{m_2})$ $\inf_{u \in H} m(u) > 0$,

then there exists $\mu^* = \mu^*(\Omega, \|a^+\|_{L^\infty(\Omega)}, \|b^+\|_{L^\infty(\Omega)}, N)$ such that, for each $\mu \in (0, \mu^*)$, the problem (P_μ) has a solution.

For proving our results we use the sub-supersolution method. It is important to emphasize that, for nonlocal problems, this is not a simple issue. Actually, as quoted in [25, 13], the comparison principle may fail for the operator $u \mapsto m(u)\Delta u$ unless we impose some (nonnatural) restrictions on the function m. In [25], the author assumed that $m(u) = m(\int_\Omega |\nabla u|^2 \mathrm{d}x)$ and the function $m(t)t^{1/2}$ was increasing. For the same type of functions m, the authors in [13] assumed that $m(t)t$ was invertible. Notice that we have no invertibility nor monotonicity assumptions, and therefore our hypothesis are weaker than those considered in these two papers. For example, besides the Kirchoff case $m(u) = a + b\int_\Omega |\nabla u|^2 \mathrm{d}x$, we can also consider, among others, $m(u) = a + b\int_\Omega |u(x)|^q \mathrm{d}x$, for any subcritical power $1 \leq q < 2N/(N-4)$, and $a > 0$, $b \geq 0$. This kind of nonlocal term appears in the study of population of bacteria subject to spreading when $q = 1$ (see [6]) and, for $q = 2$, the problem reduces to Carrier's equation which is related to nonlinear deflection of beams (see [17]). Actually, our assumptions on m are weaker than those of [16, 14, 25, 24, 13, 20] and many others. Moreover, even in the local second order case, our result complement those of [10, 11] since our solution is positive and we have no upper bound on p.

Roughly speaking, the difficulties presented in the above paragraph relies on the nonvalidity of the Maximum Principle for operators of fourth order. This also reflects on the strategy of proving the positivity of solutions of (P_μ). Some well-known arguments do not work in our setting. For example, we cannot use the positive part u^+ of a function $u \in H$. Also, we cannot argue as in [31, 32, 4, 26] since we deal here with an indefinite nonlinearity. The idea of replace $-\Delta u$ by $-\Delta u + Bu$, with $B > 0$ large (see the condition $(H_0)'$ in [11]) does not work for the biharmonic operator (see [27, Theorem 7.1] and [30, Theorem 5.5]). Finally, some extension arguments used in the second order case cannot be used here because, if Ω_0 is a proper subset of Ω and $u \in H_0^1(\Omega)$, then the usual zero extension of u to the entire set Ω can be outside H.

To overcome the difficulties pointed above, we use the Fixed Point Theorem together with a sub-super solution approach without monotone iteration (see [8]). The main problem relies on obtaining the subsolution and, to do that, we prove a Krein-Rutman type result for an eigenvalue problem with sign-changing weight and fourth-order operator (see Proposition 2.3). We think this result has an interest in itself and it could be used to improve some other results which involve indefinite nonlinearities. For the second-order problem, besides the former approach, we also use a simple and instructive idea of working in $H^2(\Omega) \cap H_0^1(\Omega)$. This enables us to consider a condition on m which is weaker than those assumed in the previous works.

The rest of this paper is organized as follows: in the next section, we develop the sub-supersolution method. Theorems 1.1 and 1.2 are proved in Section 3 and 4, respectively.

2. The sub-supersolution framework

In this section, we present the sub-super solution method to deal with problem (P_μ). In what follows we denote by $\|g\|_\infty$ the $L^\infty(\Omega)$-norm of a bounded function g. For any $\mu > 0$, we define

$$f_\mu(x,s) := \mu a(x)s^q + b(x)s^p, \qquad x \in \Omega,\, s \geq 0.$$

We also consider the numbers

$$r := \frac{1}{2}\mathrm{diam}(\Omega), \qquad l := \frac{r^4}{4N^2}, \tag{2.1}$$

and

$$\mu_* := \begin{cases} \dfrac{1}{2\|a^+\|_\infty l^q}\left(\dfrac{1}{2\|b^+\|_\infty l^p}\right)^{(1-q)/(p-1)}, & \text{if } \|b^+\|_\infty > 0, \\ +\infty, & \text{otherwise.} \end{cases} \tag{2.2}$$

For each $R > 0$, we denote $H_R := H \cap \overline{B_R(0)}$. In view of $(m_1) - (m_2)$, there exists $R_0 > 0$ such that

$$m_{R_0} := \inf_{u \in H_{R_0}} m(u) > 0. \tag{2.3}$$

In our first result we obtain a supersolution for the problem (P_μ), in the following sense:

Lemma 2.1. *For each $\mu \in (0,\mu_*)$ there exists $\overline{u} \in H_0^1(\Omega) \cap C^\infty(\Omega)$ such that*

$$\begin{cases} \Delta^2 \overline{u} - m(\psi)\Delta \overline{u} \geq \mu a(x)\overline{u}^q + b(x)\overline{u}^p, & \forall\, \psi \in H_{R_0}, \\ \overline{u},\, -\Delta\overline{u} > 0, & \text{in } \Omega, \\ \overline{u} = \Delta \overline{u} = 0, & \text{on } \partial\Omega. \end{cases}$$

Moreover, the function $\overline{u} = \overline{u}(\mu, \|a^+\|_\infty, \|b^+\|_\infty, \Omega, N, R_0)$ is such that $\|\overline{u}\|_\infty \to 0$ uniformly as $\mu \to 0^+$.

Proof. Let $e_i \in H_0^1(\Omega) \cap C^\infty(\Omega)$, $i = 1,2$, be such that

$$-\Delta e_1 = 1, \qquad -\Delta e_2 = e_1, \qquad \text{in } \Omega.$$

From the Maximum Principle, we get

$$\begin{cases} \Delta^2 e_2 = 1, & \text{in } \Omega, \\ e_2,\, -\Delta e_2 > 0, & \text{in } \Omega, \\ e_2 = \Delta e_2 = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.4}$$

We now consider $x_0 \in \Omega$ such that $\Omega \subset B_r(x_0)$, where $r > 0$ was defined in (2.1). Then, if

$$\widehat{e_1}(x) := -\frac{1}{2N}|x-x_0|^2 + \frac{r^2}{2N}, \qquad x \in B_r(x_0),$$

we have that $-\Delta\widehat{e_1} = 1 = -\Delta e_1$ in Ω, and $\widehat{e_1} \geq e_1$ on $\partial\Omega$, and therefore $r^2/(2N) = \|\widehat{e_1}\|_\infty \geq \|e_1\|_\infty$. Moreover, if

$$\widehat{e_2}(x) := -\frac{r^2}{4N^2}|x-x_0|^2 + \frac{r^4}{4N^2}, \quad x \in B_r(x_0),$$

we obtain $-\Delta\widehat{e_2} = \|\widehat{e_1}\|_\infty \geq e_1 = -\Delta e_2$ in Ω, and $\widehat{e_2} \geq e_2$ on $\partial\Omega$. Thus,

$$\|e_2\|_\infty \leq \|\widehat{e_2}\|_\infty = \frac{r^4}{4N^2} = l.$$

Let $\mu \in (0, \mu_*)$, $K > 0$ to be choosed later and fix $\psi \in H_{R_0}$. By using (2.4) and $m(\psi) \geq 0$, we obtain

$$\Delta^2(Ke_2) - m(\psi)\Delta(Ke_2) \geq K\Delta^2 e_2 = K. \tag{2.5}$$

On the other hand, since $\|e_2\|_\infty \leq l$, we get

$$f_\mu(x, Ke_2) \leq \mu\|a^+\|_\infty K^q l^q + \|b^+\|_\infty K^p l^p. \tag{2.6}$$

So, if we pick

$$K := (2\mu\|a^+\|_\infty l^q)^{1/(1-q)}, \tag{2.7}$$

a straightfoward computation and $\mu < \mu_*$ provide

$$\mu\|a^+\|_\infty K^q l^q = \frac{K}{2}, \qquad \|b^+\|_\infty K^p l^p \leq \frac{K}{2}.$$

Hence, we can use (2.5)-(2.6) to o btain

$$\Delta^2(Ke_2) - m(\psi)\Delta(Ke_2) \geq \mu a(x)(Ke_2)^q + b(x)(Ke_2)^p,$$

and the lemma holds for the function $\overline{u} := Ke_2$. □

We devote the rest of this section to the construction of a subsolution. This process is more involved and we start by proving a variant of the Maximum Principle presented in [33, Lemma 3.1].

Lemma 2.2. *Let $\lambda_1 := \lambda_1(\Omega)$ be the first eigenvalue of $(-\Delta, H_0^1(\Omega))$ and supoose that β, $\theta \in \mathbb{R}$ satisfy*

$$\beta^2 \geq 4\theta, \quad \beta > -2\lambda_1, \quad \lambda_1^2 + \beta\lambda_1 + \theta > 0.$$

If u, $\Delta u \in H$, $u \neq 0$ and

$$\Delta^2 u - \beta\Delta u + \theta u \geq 0, \text{ in } \Omega,$$

then $u > 0$ in Ω.

Proof. Let

$$g(t) := t^2 - \beta t + \theta, \qquad t \in \mathbb{R}.$$

From $\beta^2 \geq 4\theta$, $\beta > -2\lambda_1$ and $g(-\lambda_1) > 0$, we infer that the roots $t^\pm$ of the function g, namely

$$t^- := \frac{\beta - \sqrt{\beta^2 - 4\theta}}{2}, \quad t^+ := \frac{\beta + \sqrt{\beta^2 - 4\theta}}{2},$$

verify $t^+ \geq t^- > -\lambda_1$. Moreover, if we set $v := -\Delta u + t^- u \in H_0^1(\Omega)$, a direct calcultion povides

$$-\Delta v + t^+ v = \Delta^2 u - \beta u + \theta u \geq 0, \text{ in } \Omega.$$

By picking $\varphi = v^- \in H_0^1(\Omega)$ as a test function in the above inequality and using $t^+ > -\lambda_1$, we obtain

$$\int_\Omega |\nabla v^-|^2 \mathrm{d}x \leq -t^+ \int_\Omega (v^-)^2 \mathrm{d}x < \lambda_1 \int_\Omega (v^-)^2 \mathrm{d}x \leq \int_\Omega |\nabla v^-|^2 \mathrm{d}x,$$

and therefore $v \geq 0$ in Ω. The same argument and $t^- > -\lambda_1$ imply that $u \geq 0$ in Ω. Since $u \not\equiv 0$ the result follows from Harnack's inequality (see [18, Theorem 8.20]). □

The next result combines an idea introduced by Hess [19] with the standard theory of Krein-Rutman.

Proposition 2.3 (Principal eigenvalue). *Suppose that* $\lambda_1^2+\beta\lambda_1+\theta>0$ *and* $\beta>-2\lambda_1$. *Let* $c\in L^\infty(\Omega)$ *be a weight verifying*

$$c(x)\geq -1 \text{ for a.e. } x\in\Omega, \qquad \alpha_B:=\inf_{x\in B}c(x)>0,$$

where $B\subset\overline{B}\subset\Omega$. *Set*

$$d_B:=\sup_{\varphi\in C_0^\infty(B)\setminus\{0\}}\frac{\alpha_B\displaystyle\int_B\varphi^2\mathrm{d}x}{\displaystyle\int_B\left[(\Delta\varphi)^2+\beta|\nabla\varphi|^2+\theta\varphi^2\right]\mathrm{d}x}>0.$$

If $\beta^2-4\theta>4d_B^{-1}$, *then the eigenvalue problem*

$$\begin{cases}\Delta^2u-\beta\Delta u+\theta u=\lambda c(x)u, & \text{in } \Omega,\\ u=\Delta u=0, & \text{on } \partial\Omega,\end{cases}$$

has a principal eigenvalue $\lambda_1^c>0$ *with associated positive eigenfunction* φ_1 *belonging to* $W_0^{1,2}(\Omega)\cap W^{4,2}(\Omega)\cap C^3(\Omega)$.

Proof. For u, $v\in H$, we define

$$(u,v)_*:=\int_\Omega\left[\Delta u\Delta v+\beta(\nabla u\cdot\nabla v)+\theta uv\right]\mathrm{d}x$$

and notice that

$$\|u\|_*^2:=(u,u)_*\geq(\lambda_1^2+\beta\lambda_1+\theta)\int_\Omega u^2\mathrm{d}x.$$

Hence $(\cdot,\cdot)_*$ is an inner product in H and, since $\|u\|\leq\|u\|_*$ for any $u\in H$, we have that $H^2(\Omega)\cap H_0^1(\Omega)$ endowed with this inner product is a Hilbert space. Thus, we can apply the Riesz Theorem to obtain $\phi_1\in H$ such that

$$(\phi_1,v)_*=\lambda_1^c\int_\Omega c(x)\phi_1 v\,\mathrm{d}x, \qquad \forall\, v\in H, \tag{2.8}$$

where the number λ_1^c is given by

$$\frac{1}{\lambda_1^c}:=\sup_{\varphi\in H\setminus\{0\}}\frac{\displaystyle\int_\Omega c(x)\varphi^2\mathrm{d}x}{\displaystyle\int_\Omega\left[(\Delta\varphi)^2+\beta|\nabla\varphi|^2+\theta\varphi^2\right]\mathrm{d}x}>0. \tag{2.9}$$

This shows that $d_B>0$.

For any $\alpha\in(0,1)$ we denote

$$C_0^\alpha(\overline{\Omega}):=\{u\in C^\alpha(\overline{\Omega}):u\equiv 0 \text{ on } \partial\Omega\},$$

and define the operator $T:C_0^\alpha(\overline{\Omega})\to C^{3,\alpha}(\overline{\Omega})\cap C_0^\alpha(\overline{\Omega})$ in the following way:

$$Tu=v \quad\Longleftrightarrow\quad \begin{cases}\Delta^2v-\beta\Delta v+(\theta+\lambda_1^c)v=(1+c(x))u, & \text{in } \Omega,\\ v=\Delta v=0, & \text{on } \partial\Omega.\end{cases}$$

Since (2.9) implies that $\beta^2>4(\theta+\lambda_1^c)$, the same argument used in the proof of Lemma 2.2 provide

$$\Delta^2v-\beta\Delta v+(\theta+\lambda_1^c)v=(-\Delta+t^-\mathrm{Id})(-\Delta-t^+\mathrm{Id})v,$$

with $t^+ \geq t^- > -\lambda_1$. So, the equality $Tu = v$ can be rewritten as

$$\begin{cases} -\Delta v + t^+ v = w, & \text{in } \Omega, \\ v = 0, & \text{on } \partial\Omega. \end{cases},$$

and

$$\begin{cases} -\Delta w + t^- w = (c(x)+1)u, & \text{in } \Omega, \\ w = 0, & \text{on } \partial\Omega, \end{cases}$$

Recalling that $c \in L^\infty(\Omega)$, we can apply the standard L^p-theory to ensure that the operator T is well defined and compact.

It is well known that $K := \{u \in C_0^\alpha(\overline{\Omega}) : u \geq 0 \text{ in } \Omega\}$ is a total cone in $C_0^\alpha(\overline{\Omega})$. Moreover, $\beta^2 > 4(\theta + \lambda_1^c)$ and Lemma 2.2 imply that $T(K) \subset K$. Hence, we can apply the Krein-Rutman Theorem (see [22] or [9, Theorem 19.2]) to obtain a principal eigenvalue $\lambda_0 > 0$ and a positive eigenfunction $\varphi_1 \in C^{3,\alpha}(\overline{\Omega}) \cap C_0^\alpha(\overline{\Omega})$ such that

$$\begin{cases} \Delta^2\varphi_1 - \beta\Delta\varphi_1 + (\theta + \lambda_1^c)\varphi_1 = \lambda_0(1 + c(x))\varphi_1, & \text{in } \Omega, \\ \varphi_1 = \Delta\varphi_1 = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.10}$$

The eigenvalue λ_0 can be characterized as

$$\frac{1}{\lambda_0} := \sup_{\varphi \in H\setminus\{0\}} \frac{\displaystyle\int_\Omega (1 + c(x))\varphi^2 \mathrm{d}x}{\displaystyle\int_\Omega \left[(\Delta\varphi)^2 + \beta|\nabla\varphi|^2 + (\theta + \lambda_1^c)\varphi^2\right] \mathrm{d}x}.$$

In what follows we shall verify that $\lambda_0 = \lambda_1^c$. If this is true, it follows from (2.10) that $\varphi_1 > 0$ is an eigenfunction of the linear problem presented in the statement of the lemma.

In order to check that $\lambda_0 = \lambda_1^c$, we first use equality (2.8) with $v = \phi_1$ to get $\|\phi_1\|_*^2 = \lambda_1^c \int_\Omega c(x)\phi_1^2 \mathrm{d}x$. Hence, we infer from the characterization of λ_0 that

$$\frac{1}{\lambda_0} \geq \frac{\int_\Omega c(x)\phi_1^2 \mathrm{d}x + \|\phi_1\|_{L^2(\Omega)}^2}{\|\phi_1\|_*^2 + \lambda_1^c\|\phi_1\|_{L^2(\Omega)}^2} = \frac{1}{\lambda_1^c},$$

and therefore $\lambda_1^c \geq \lambda_0$. On the other hand, since

$$\|\varphi_1\|_*^2 + \lambda_1^c\|\varphi_1\|_{L^2(\Omega)}^2 = \lambda_0\|\varphi_1\|_{L^2(\Omega)}^2 + \lambda_0 \int_\Omega c(x)\varphi_1^2 \, \mathrm{d}x,$$

we obtain from (2.9) that

$$\frac{1}{\lambda_1^c} \geq \frac{\int_\Omega c(x)\varphi_1^2 \, \mathrm{d}x}{\|\varphi_1\|_*^2} = \frac{\lambda_0^{-1}\left[\|\varphi_1\|_*^2 + (\lambda_1^c - \lambda_0)\|\varphi_1\|_{L^2(\Omega)}^2\right]}{\|\varphi_1\|_*^2} \geq \frac{1}{\lambda_0}.$$

Thus, the reverse inequality $\lambda_0 \geq \lambda_1^c$ holds and we conclude that $\lambda_0 = \lambda_1^c$. □

We are ready to construct our supersolution.

Lemma 2.4. *Let μ_* be defined in* (2.2), *$\mu \in (0, \mu_*)$ and $\overline{u} \in H_0^1(\Omega) \cap C^\infty(\Omega)$ as in Lemma 2.1. Then, for some $R > 0$, there exists $\underline{u} \in W_0^{1,2}(\Omega) \cap W^{4,2}(\Omega) \cap C^3(\Omega)$ such that*

$$\begin{cases} \Delta^2\underline{u} - m(\psi)\Delta\underline{u} \leq \mu a(x)\underline{u}^q + b(x)\underline{u}^p, & \forall\, \psi \in H_R, \\ 0 < \underline{u} < \overline{u}, & \textit{in } \Omega, \\ \underline{u} = \Delta\underline{u} = 0, & \textit{on } \partial\Omega. \end{cases} \tag{2.11}$$

Proof. For any $R > 0$, we define

$$M_R := \sup_{u \in H_R} m(u) > 0.$$

By continuity, $\lim_{R\to 0^+} M_R = m(0) > 0$, and therefore we can choose δ, $R > 0$ small in such way that

$$\frac{M_R^2 - \delta}{4(M_R - m(\psi))} > \frac{M_R}{2}, \qquad \forall\, \psi \in H_R. \tag{2.12}$$

Let $\Omega_a \subset \Omega$ be given by (a_2) and set $C_1 = C_1(\Omega_a, M_R, \delta)$ as

$$C_1 := \sup_{\varphi \in C_0^\infty(B)\setminus\{0\}} \frac{\int_B \varphi^2 \mathrm{d}x}{\int_B \left[(\Delta\varphi)^2 + M_R|\nabla\varphi|^2 + \left(\frac{M_R^2-\delta}{4}\right)\varphi^2\right]\mathrm{d}x} > 0.$$

Pick $\alpha > 0$ in such way that

$$d_{\Omega_a} := \alpha C_1 > \frac{4}{M_R^2}$$

and define $c \in L^\infty(\Omega)$ in the following way:

$$c(x) := \begin{cases} \alpha, & \text{if } x \in \Omega_a, \\ -1, & \text{if } x \in \Omega \setminus \Omega_a. \end{cases}$$

By invoking Proposition 2.3 with the above weight and $B = \Omega_a$, we obtain $\lambda_1^c > 0$ and $\varphi_1 \in W_0^{1,2}(\Omega) \cap W^{4,2}(\Omega) \cap C^3(\Omega)$ such that

$$\begin{cases} \Delta^2\varphi_1 - M_R\Delta\varphi_1 + \left(\frac{M_R^2-\delta}{4}\right)\varphi_1 = \lambda_1^c c(x)\varphi_1, & \text{in } \Omega, \\ \varphi_1 > 0, & \text{in } \Omega, \\ \varphi_1 = \Delta\varphi_1 = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.13}$$

Since $c \equiv \alpha$ in Ω_a, the variational characterization of the first eigenvalue easily gets $1/\lambda_a^c \geq \alpha C_1$, and therefore we can assume that $\delta - 4\lambda_1^c > 0$. Thus, we can use (2.12) to obtain

$$\frac{M_R^2 - \delta}{4(M_R - m(\psi))} > \frac{M_R - \sqrt{\delta - 4\lambda_1^c}}{2}, \qquad \forall\, \psi \in H_R. \tag{2.14}$$

We now notice that, since $c(x) \geq -1$ for a.e. $x \in \Omega$, (2.13) implies that

$$\Delta^2\varphi_1 - M_R\Delta\varphi_1 + \left(\frac{M_R^2-\delta}{4} + \lambda_1^c\right)\varphi_1 \geq 0, \quad \text{in } \Omega.$$

Arguing as in the proof of Lemma 2.2, we can write

$$(-\Delta + t^-\mathrm{Id})(-\Delta - t^+\mathrm{Id})\varphi_1 \geq 0, \quad \text{in } \Omega,$$

with

$$t^- := \frac{M_R - \sqrt{\delta - 4\lambda_1^c}}{2}, \quad t^+ := \frac{M_R + \sqrt{\delta - 4\lambda_1^c}}{2},$$

and therefore it follows from the Maximum Principle that

$$-\Delta\varphi_1 + \frac{M_R - \sqrt{\delta - 4\lambda_1^c}}{2}\varphi_1 \geq 0, \quad \text{in } \Omega.$$

Since $\varphi_1 > 0$, the above expression and (2.14) imply that, for any $\psi \in H_R$,

$$-\Delta\varphi_1 + \frac{M_R^2 - \delta}{4(M_R - m(\psi))}\varphi_1 \geq 0, \quad \text{in } \Omega.$$

which is equivalent to

$$\Delta^2\varphi_1 - M_R\Delta\varphi_1 + \left(\frac{M_R^2 - \delta}{4}\right)\varphi_1 \geq \Delta^2\varphi_1 - m(\psi)\Delta\varphi_1. \tag{2.15}$$

We now define

$$\underline{u} := \varepsilon\varphi_1,$$

with $\varepsilon > 0$ satisfying

$$\varepsilon < \frac{K}{\alpha\lambda_1^c\|\varphi_1\|_\infty}.$$

For any $\psi \in H_R$, it follows from (2.15) and (2.13) that

$$\Delta^2\underline{u} - m(\psi)\Delta\underline{u} \leq \varepsilon\lambda_1^c c(x)\varphi_1 < K, \tag{2.16}$$

where $K > 0$ was defined in (2.7). The above expression and the definition of the function $\overline{u}$ in that former lemma provide

$$\Delta^2\underline{u} - m(\psi)\Delta\underline{u} < K \leq \Delta^2\overline{u} - m(\psi)\Delta\overline{u}, \quad \text{in } \Omega.$$

Recalling that $\underline{u} = \overline{u} = \Delta\underline{u} = \Delta\overline{u} = 0$ on Ω, we infer from the above inequality and Lemma 2.2 that $\underline{u} < \overline{u}$ in Ω. Hence, the last two statements in (2.11) hold.

We now claim that, for some $\varepsilon > 0$ small, we have that

$$\varepsilon\lambda_1^c c(x)\varphi_1 \leq f_\mu(x, \underline{u}), \quad \text{in } \Omega. \tag{2.17}$$

If this is true, we can use (2.16) to conclude that $\underline{u}$ verifies the first statement of (2.11).

It remains to verify (2.17). We fist suppose that $b^- \not\equiv 0$ and consider $x \in \Omega \setminus \Omega_a$. If

$$\varepsilon^{p-1} < \frac{\lambda_1^c}{\|b^-\|_\infty\|\varphi_1\|_\infty^{p-1}}, \tag{2.18}$$

we can recall that $a \geq 0$ and $c \equiv -1$ in $\Omega \setminus \Omega_a$ to get

$$f_\mu(x, \underline{u}) \geq -\varepsilon^p\|b^-\|_\infty\varphi_1(x)^p \geq -\varepsilon\lambda_1^c\varphi_1(x) = \varepsilon\lambda_1^c c(x)\varphi_1(x), \quad \text{for a.e. } x \in \Omega \setminus \Omega_a,$$

which implies (2.17). If $b^- \equiv 0$ and $x \in \Omega \setminus \Omega_a$, the above inequality holds independently of the value of ε. The proof for $x \in \Omega_a$ is more involved. We first set

$$g(\varepsilon) := \mu\delta - \varepsilon^{p-q}\|b^-\|_\infty\|\varphi_1\|_\infty^{p-q} - \varepsilon^{1-q}\alpha\lambda_1^c\|\varphi_1\|_\infty^{1-q}, \qquad \varepsilon > 0.$$

Since g is continuos and $g(0) = \mu\delta$, there exists $\varepsilon > 0$ small such that

$$g(\varepsilon) \geq \frac{\mu\delta}{2}. \tag{2.19}$$

Hence, recalling that $a \geq \delta$ and $c \equiv \alpha$ in Ω_a, we get

$$\begin{aligned} 0 &\leq \varepsilon^q\varphi_1(x)^q\frac{\mu\delta}{2} \\ &\leq \varepsilon^q\varphi_1(x)^q g(\varepsilon) \\ &\leq \varepsilon^q\varphi_1(x)^q\left[\mu\delta + b(x)\varepsilon^{p-q}\varphi_1(x)^{p-q} - \varepsilon^{1-q}\alpha\lambda_1^c\varphi_1(x)^{1-q}\right] \\ &\leq f_\mu(x, \underline{u}) - \varepsilon\lambda_1^c c(x)\varphi_1(x), \end{aligned}$$

which is exactly (2.17). We now conclude the proof by picking $\varepsilon > 0$ small in such way that (2.18) and (2.19) hold. □

3. The fourth-order case

We devote this section to the proof of our first theorem. Let $\mu_* > 0$ be defined in (2.2), $\mu \in (0, \mu_*)$ and $\overline{u} \in H_0^1(\Omega) \cap C^\infty(\Omega)$, $\underline{u} \in W_0^{1,2}(\Omega) \cap W^{4,2}(\Omega) \cap C^3(\Omega)$ be given by Lemmas 2.1 and 2.4. For each $u \in H$, we define the truncated function

$$\widehat{u}(x) := \begin{cases} \overline{u}(x), & \text{if } u(x) \geq \overline{u}(x), \\ u(x), & \text{if } \underline{u}(x) \leq u(x) \leq \overline{u}(x), \\ \underline{u}(x), & \text{if } u(x) \leq \underline{u}(x), \end{cases}$$

For $\theta > 0$, we can use $a \geq 0$ to obtain

$$\frac{d}{dt}\left(f_\mu(x,t) + \theta t\right) = \mu q a(x) t^{q-1} + p b(x) t^{p-1} + \theta \geq -p\|b^-\|_\infty t^{p-1} + \theta,$$

for any $x \in \Omega$, $t \geq 0$. Hence, if we set

$$\theta := p\|b^-\|_\infty \|\overline{u}\|_\infty^{p-1} \tag{3.1}$$

we conclude that

$$\text{the map } t \mapsto (f_\mu(x,t) + \theta t) \text{ is nondecreasing in } [0, \|\overline{u}\|_\infty], \text{ for any } x \in \Omega. \tag{3.2}$$

We now define the operator $T : H \to H \cap C^{3,\alpha}(\overline{\Omega})$ by

$$Tu = v \quad \Longleftrightarrow \quad \begin{cases} \Delta^2 v - m(u)\Delta v + \theta v = f_\mu(x, \widehat{u}) + \theta \widehat{u}, & \text{in } \Omega, \\ v = \Delta v = 0, & \text{on } \partial\Omega. \end{cases}$$

and prove the following:

Lemma 3.1. *Let μ_*, R_0, $R > 0$ be given by (2.2), (2.3) and Lemma 2.1, respectively. Then there exist $R^* \in (0, R)$ and $\mu^* \in (0, \mu_*)$ such that, for any $\mu \in (0, \mu^*)$, the operator T above is well defined, compact and satisfies $T(\overline{B_{R^*}(0)}) \subset \overline{B_{R^*}(0)}$.*

Proof. Since $m(0) > 0$ and m is continuous, there exists $R^* \leq \min\{R_0, R\}$ such that

$$m_{R^*} := \inf_{u \in H_{R^*}} m(u) > 0.$$

Moreover, since $\lim_{\mu \to 0^+} \|\overline{u}\|_\infty = 0$, the equation (3.1) implies that $\theta \to 0$, as $\mu \to 0^+$. Hence, we may also assume that μ is small in such way that $m(u)^2 \geq m_{R^*}^2 \geq 4\theta$, for any $u \in H_{R^*}$.

As in the proof the of Lemma 2.2, the equality $Tu = v$ can be written as

$$\begin{cases} -\Delta v + t_u^+ v = w, & \text{in } \Omega, \\ v = 0, & \text{on } \partial\Omega, \end{cases}$$

and

$$\begin{cases} -\Delta w + t_u^- w = f_\mu(x, \widehat{u}) + \theta \widehat{u}, & \text{in } \Omega, \\ w = 0, & \text{on } \partial\Omega. \end{cases},$$

with

$$t_u^\pm := \frac{m(u) \pm \sqrt{m(u)^2 - 4\theta}}{2}.$$

By using the usual L^p-theory we can check that T is well defined. Moreover, by (m_1), for any boudend sequence $(u_n) \subset H$, we may assume that $m(u_n) \to m(u)$, as $n \to +\infty$, where $u \in H$ is the weak limit of (u_n). Hence, we can use standard arguments to prove that T is compact. Finally, the definitions of T, $\widehat{u}$ and (3.2) show that $Tu \geq 0$, for any $u \in H_{R^*}$.

Let $C_1 > 0$ be such that $C_1 \int_\Omega |u| \mathrm{d}x \leq \|u\|$, for any $u \in H$. By denoting $v = Tu$ and recalling that $0 \leq \widehat{u} \leq \overline{u}$ in Ω, we infer from the definition of θ that

$$\Delta^2 v - m(u)\Delta v + \theta v \leq \mu \|a\|_\infty \|\overline{u}\|_\infty^q + \|b^+\|_\infty \|\overline{u}\|_\infty^p + p\|b^-\|_\infty \|\overline{u}\|_\infty^p.$$

Since $\|\overline{u}\|_\infty \to 0$ as $\mu \to 0^+$, for small values of $\mu > 0$ we have that

$$\Delta^2 v - m(u)\Delta v + \theta v \leq R^* C_1.$$

By multiplying the above inequality by $v = Tu \geq 0$, integrating over Ω and recalling that $v = \Delta v = 0$ on $\partial\Omega$, we get

$$\|v\|^2 + m(u)\||\nabla v|\|_{L^2(\Omega)}^2 + \theta \|v\|_{L^2(\Omega)}^2 \leq R^* C_1 \|v\|_{L^1(\Omega)} \leq R^* \|v\|.$$

Hence, we conclude that $T(\overline{B_{R^*}(0)}) \subset \overline{B_{R^*}(0)}$. □

We are ready to prove our first theorem.

Proof of Theorem 1.1. Let μ^*, $R^* > 0$ as in the previous lemma and fix $\mu \in (0, \mu^*)$. Since the compact operador T is such that $T(\overline{B_{R^*}(0)}) \subset \overline{B_{R^*}(0)}$, by Schauder's Fixed Point Theorem there exists $u \in \overline{B_{R^*}(0)}$ such that $Tu = u$. Hence, $\widehat{u} \leq \overline{u}$, (3.2) and Lemma 2.1 imply that

$$\Delta^2 u - m(u)\Delta u + \theta u = f_\mu(x, \widehat{u}) + \theta \widehat{u} \leq f_\mu(x, \overline{u}) + \theta \overline{u} \leq \Delta^2 \overline{u} - m(u)\Delta \overline{u} + \theta \overline{u},$$

which is equivalent to

$$\Delta^2(\underline{u} - u) - m(u)\Delta(\underline{u} - u) + \theta(\underline{u} - u) \leq 0.$$

Since $m(u)^2 \geq 4\theta$, we can use Lemma 2.2 to conclude that $u \leq \overline{u}$ in Ω. Analagously, we can use $\underline{u} \leq \widehat{u}$, (3.2) and Lemma 2.4, to get

$$\Delta^2 \underline{u} - m(u)\Delta \underline{u} + \theta \underline{u} \leq f_\mu(x, \underline{u}) + \theta \underline{u} \leq f_\mu(x, \widehat{u}) + \theta \widehat{u} = \Delta^2 u - m(u)\Delta u + \theta u,$$

from which we obtain $u \geq \underline{u}$ in Ω. Thus, $\underline{u} \leq u \leq \overline{u}$ in Ω and it follows from the definition of $\widehat{u}$ that $\widehat{u} = u$. Since $Tu = u$, this implies that $u \in H$ is a solution of the problem. □

4. The second-order case

From now on we deal with the problem (P_μ) with $\gamma = 0$. In this new setting, instead of (m_2), we shall assume that

$$m_0 := \inf_{u \in H} m(u) > 0.$$

The proof follows the same lines of that presented in the last section.

Lemma 4.1. *Let*

$$\mu_* := \begin{cases} \dfrac{m_0}{2\|a^+\|_\infty \|e\|_\infty^p} \left(\dfrac{m_0}{2\|b^+\|_\infty \|e\|_\infty^q} \right)^{(1-q)/(p-1)}, & \textit{if } \|b^+\|_\infty > 0, \\ +\infty, & \textit{otherwise}, \end{cases}$$

For each $\mu \in (0,\mu_*)$ there exists $\overline{u} \in H_0^1(\Omega)\cap C^\infty(\Omega)$ such that

$$\begin{cases} -m(\psi)\Delta\overline{u} \geq \mu a(x)\overline{u}^q + b(x)\overline{u}^p, & \forall\,\psi \in H, \\ \overline{u} > 0, & \text{in } \Omega, \\ \overline{u} = 0, & \text{on } \partial\Omega. \end{cases}$$

Moreover, the function $\overline{u} = \overline{u}(\mu, \|a^+\|_\infty, \|b^+\|_\infty, \Omega, N)$ is such that $\|\overline{u}\|_\infty \to 0$ uniformly as $\mu \to 0^+$.

Proof. Let $e \in H_0^1(\Omega)\cap C^\infty(\Omega)$ be such that $-\Delta e = 1$ in Ω, since $0<\mu<\mu_*$, we choice

$$K := \left(\frac{2\mu\|a^+\|_\infty\|e\|_\infty^q}{m_0}\right)^{1/(1-q)}, \qquad \overline{u} := Ke. \tag{4.1}$$

By using the definition of K together with $\mu<\mu_*$ we get

$$f_\mu(x,\overline{u}) \leq \mu\|a\|_\infty K^p\|e\|_\infty^p + \|b^+\|_\infty K^q\|e\|_\infty^q \leq Km_0 \leq Km(\psi) = -m(\psi)\Delta\overline{u},$$

for any ψ. The lemma is proved. □

We now turn our attention to the subsolution. As in the fourth-order case, it is important to consider an eigenvalue problem with a suitable weigth.

Lemma 4.2. *Let μ_* and $\overline{u}$ as in Lemma 4.1. Then, there exists $\underline{u} \in W_0^{1,2}(\Omega)\cap W^{2,2}(\Omega)\cap C^3(\Omega)$ such that, for any $R>0$, it hold*

$$\begin{cases} -m(\psi)\Delta\underline{u} \leq \mu a(x)\underline{u}^p + b(x)\underline{u}^q, & \forall\,\psi \in H_R, \\ 0 < \underline{u} < \overline{u}, & \text{in } \Omega, \\ \underline{u} = 0, & \text{on } \partial\Omega. \end{cases} \tag{4.2}$$

Proof. Let $\Omega_a \subset \Omega$ given by (a_2) and define the weight $c \in L^\infty(\Omega)$ as

$$c(x) := \begin{cases} 1, & \text{if } x \in \Omega_a, \\ -1, & \text{if } x \in \Omega\setminus\Omega_a. \end{cases}$$

Consider $\varphi_1 > 0$ and $\lambda_1^c > 0$ (see [19]) such that

$$-\Delta\varphi_1 = \lambda_1^c c(x)\varphi_1, \quad \text{in } \Omega,$$

and set

$$\underline{u} := \varepsilon\varphi_1,$$

with $\varepsilon>0$ such that

$$\varepsilon < \varepsilon^* := \frac{K}{\lambda_1^c\|\varphi_1\|_\infty}.$$

A simple computation shows that

$$-\Delta(\varepsilon\varphi_1) = \varepsilon\lambda_1^c c(x)\varphi_1 \leq \varepsilon\lambda_1^c\|\varphi_1\|_\infty < K = -\Delta(Ke),$$

with $K>0$ defined in (4.1), and therefore it follows from the Maximum Principle that $\underline{u} < \overline{u}$ in Ω.

In order to check the first statement in (4.2) we suppose that $b^- \not\equiv 0$ and consider $x \in \Omega\setminus\Omega_a$. If

$$\varepsilon^{p-1} < \frac{m_0\lambda_1^c}{\|b^-\|_\infty\|\varphi_1\|_\infty^{p-1}},$$

we can recall that $a \geq 0$ and $c \equiv -1$ in $\Omega\setminus\Omega_a$, to get

$$f_\mu(x,\underline{u}) \geq -\varepsilon^{p-1}\|b^-\|_\infty\|\varphi_1\|_\infty^{p-1}\underline{u} \geq -m_0\lambda_1^c\underline{u} = -m_0\Delta(\underline{u}) \geq -m(\psi)\Delta(\underline{u}),$$

for a.e. $x \in \Omega \setminus \Omega_a$ and any $\psi \in H_R$. If $b^- \equiv 0$ and $x \in \Omega_a$, then $f_\mu(x, \underline{u}) \geq 0 \geq -m_0 \Delta(\underline{u})$ and therefore the above inequality trivially holds for any $\varepsilon > 0$. It remains to consider $x \in \Omega_a$. In this case, since $c(x) = 1$, we have that

$$-m(\psi)\Delta(\underline{u}) = \varepsilon m(\psi)\lambda_1^c c(x)\varphi_1 \leq \varepsilon M_R \lambda_1^c \varphi_1, \tag{4.3}$$

with

$$M_R := \sup_{u \in H_R} m(u) > 0.$$

But

$$\begin{aligned} f_\mu(x, \underline{u}) - \varepsilon M_R \lambda_1^c \varphi_1 &= \mu a(x)\varepsilon^q \varphi_1^q + b(x)\varepsilon^p \varphi_1^p - \varepsilon M_R \lambda_1^c \varphi_1 \\ &\geq \varepsilon^q \varphi_1^q \left[\mu\delta - \varepsilon^{p-q}\|b^-\|_\infty \|\varphi_1\|_\infty^{p-q} - \varepsilon^{1-q} M_R \lambda_1^c \|\varphi_1\|_\infty^{1-q}\right]. \end{aligned}$$

If we call $g(\varepsilon)$ the continuos function into the brackets above we obtain $\varepsilon > 0$ small such that $g(\varepsilon) \geq (\mu\delta)/2$, and therefore

$$f_\mu(x, \underline{u}) - \varepsilon M_R \lambda_1^c \varphi_1 \geq \frac{\mu\delta}{2}\varepsilon^q \varphi_1^q > 0.$$

This and (4.3) imply that the fuction $\underline{u}$ verifies the first statement in (4.2) also in the set Ω_a. □

We are ready to prove our second theorem.

Proof of Theorem 1.2. As in the proof of Theorem 1.1 we define $T : H \to H \cap C^{1,\alpha}(\overline{\Omega})$ by

$$T(u) = v \quad \iff \quad \begin{cases} -m(u)\Delta v + \theta v = f_\mu(x, \widehat{u}) + \theta \widehat{u}, & \text{in } \Omega, \\ v = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\theta > 0$ is such that the function $t \mapsto f_\mu(x, t)$ is nondecreasing in $[0, \|\overline{u}\|_\infty]$, for any $x \in \Omega$. As before, T is well defined and compact.

We now set

$$c_0 := \inf_{x \in \Omega,\, t \in [0, \|\overline{u}\|_\infty]} |f_\mu(x, t) + \theta t|^2.$$

For any $u \in H$, if we denote $v = Tu$ and integrate by parts we obtain

$$\begin{aligned} c_0|\Omega| &\geq \int_\Omega (-m(u)\Delta v + \theta v)^2 \mathrm{d}x \\ &= m(u)^2\|v\|^2 + 2m(u)\theta \||\nabla v|\|_{L^2(\Omega)}^2 + \theta^2 \|v\|_{L^2(\Omega)}^2 \\ &\geq m_0^2 \|v\|^2, \end{aligned}$$

where $|\Omega|$ stands for the Lebesgue measure of Ω. Hence, if we define $R^* := \sqrt{c_0|\Omega|}/m_0$, we have that $T(\overline{B_{R^*}(0)}) \subset \overline{B_{R^*}(0)}$ and the Schauder's Fixed Point Theorem provides $u \in \overline{B_{R^*}(0)}$ such that $Tu = u$. The result now follows from the Maximum Principle as in the proof of Theorem 1.1. □

References

[1] S. Alama and G. Tarantello, *On semilinear elliptic equations with indefinite nonlinearities*, Calculus of Variations in PDE **1** (1993), 439-475.

[2] A. Ambrosetti, H. Brezis H and G. Cerami, *Combined effects of concave and convex nonlinearities in some elliptic problems,* J. Funct. Anal. **122** (1994), 519-543.

[3] M. Berger, *A new approach to the large deflection of plate*, J. Appl. Mech. **22** (1955), 465-472.

[4] F. Bernis, J. García Azorero and I. Peral, *Existence and multiplicity of nontrivial solutions in semilinear critical problems of fourth order.* Adv. Differential Equations **1** (1996), 219–240.

[5] M. Bhakta and D. Mukherjee, *Multiplicity results and sign changing solutions of non-local equations with concave-convex nonlinearities*, Differential Integral Equations **30** (2017), 387–422.

[6] M. Chipot and B. Lovat, *Some remarks on nonlocal elliptic and parabolic problems*,Proceedings of the Second World Congress of Nonlinear Analysts, Part 7 (Athens, 1996), Nonlinear Anal. 30 (1997), 4619–4627.

[7] I. Chueshov and I. Lasiecka, *Long-time behavior of second order evolution equations with nonlinear damping*, Mem. Amer. Math. Soc. **185**, no. 912, Amer. Math. Soc., Providence, RI, 2008.

[8] P. Clément, and G. Sweers,*Getting a solution between sub- and supersolutions without monotone iteration,* Rend. Istit. Mat. Univ. Trieste **19** (1987), 189–194.

[9] K. Deimling, *Nonlinear functional analysis.* Springer-Verlag, Berlin, 1985

[10] D.G. de Figueiredo, J. P. Gossez and P. Ubilla, *Local superlinearity and sublinearity for indefinite semilinear elliptic problems*, J. Funct. Anal. **199** (2003), 452-467.

[11] D.G. de Figueiredo, J. P. Gossez and P. Ubilla, *Multiplicity results for a family of semilinear elliptic problems under local superlinearity and sublinearity*, J. Eur. Math. Soc. **8** (2006), 269–286.

[12] D.G. de Figueiredo, J. P. Gossez and P. Ubilla, *Local "superlinearity" and "sublinearity" for the p-Laplacian*, J. Funct. Anal. **257** (2009), 721–752.

[13] G.M. Figueiredo and A. Suárez, *Some remarks on the comparison principle in Kirchhoff equations*, Rev. Mat. Iberoam. **34** (2018), 609–620.

[14] G.M. Figueiredo, N. Ikoma and J.S. Santos Júnior, *Existence and concentration result for the Kirchhoff type equations with general nonlinearities*, Arch. Ration. Mech. Anal. **213** (2014), 931–979.

[15] G.M. Figueiredo and R.G. Nascimento, *Multiplicity of solutions for equations involving a nonlocal term and the biharmonic operator*, Electron. J. Differential Equations 2016, Paper No. 217, 15 pp.

[16] G.M. Figueiredo; J.R. Santos Junior, *Multiplicity of solutions for a Kirchhoff equation with subcritical or critical growth*, Differential Integral Equations **25** (2012), 853–868.

[17] C.L. Frota and J.A. Goldstein, *Some nonlinear wave equations with acoustic boundary conditions*, J. Differential Equations **164** (2000), 92-109.

[18] D. Gilbarg and N. Trudinger, *Elliptic Partial Differential Equations of Second Order,* Classics Math., Springer-Verlag,Berlin, 2001, reprint of the 1998 edition.

[19] P. Hess, *On the Principal Eigenvalue of a Second Order Linear Elliptic Problem with an Indefinite Weight Function*, Math. Z. **179** (1982), 237–239.

[20] L. Iturriaga and E. Massa, *On necessary conditions for the comparison principle and the sub-and supersolution method for the stationary Kirchhoff equation*, J. Math. Phys. **59** (2018), no. 1, 011506, 6 pp.

[21] G. Kirchhoff, Vorlesungen über Mathematische Physik: Mechanik, Teubner, Leipzig (1876).

[22] M. G. Krein and M. A. Rutman, *Linear operators leaving invariant a cone in a Banach space*, Uspekhi Mat. Nauk, 3:1(23) (1948), 3–95

[23] J-F. Liao, Y. Pu, X-F. Ke and C-L. Tang, *Multiple positive solutions for Kirchhoff type problems involving concave-convex nonlinearities*, Comm. Pure Applied Anal. **16** (2017), 2157-2175.

[24] D. Lü and S. Peng, *Existence and asymptotic behavior of vector solutions for coupled nonlinear Kirchhoff-type systems*, J. Differential Equations **263** (2017), 8947-8978

[25] J. G. Melián and L. Iturriaga, *Some counterexamples related to the stationary Kirchhoff equation*, Proc. Amer. Math. Soc. **144** (2016), 3405–3411.

[26] J.L.F. Melo and E.M. dos Santos, *Positive solutions to a fourth-order elliptic problem by the Lusternik–Schnirelmann category*, J. Math. Anal. Applications **420** (2014), 532–550.

[27] E. Mitidieri and G. Sweers, *Weakly coupled elliptic systems and positivity*, Math. Nachr. **173** (1995), 259–286

[28] J.C.N. Pádua, E.A.B. Silva and S.H.M. Soares, *Positive solutions of critical semilinear problems involving a sublinear term on the origin*, Indiana Univ. Math. J. **55** (2006), 1091-1111.

[29] Y. Song and S. Shi, *Multiplicity of Solutions for Fourth-Order Elliptic Equations of Kirchhoff Type with Critical Exponent*, Journal of Dynamical and Control Systems **23** (2017), 375-386.

[30] G. Sweers and K. Vassi, *Positivity for a hinged convex plate with stress*, SIAM J. Math. Anal. **50** (2018), 1163–1174.

[31] R.C.A.M. van der Vorst, *Best constant for the embedding of the space $H^2 \cap H_0^1(\Omega)$ into $L^{2N/(N-4)}(\Omega)$*, Differential Integral Equations **6** (1993), 259–276.

[32] R.C.A.M. van der Vorst, *Fourth-order elliptic equations with critical growth*, C.R. Math. Acas. Sci. Paris **320** (1995), 295-299.

[33] Y-M. Wang, *On fourth-order elliptic boundary value problems with nonmonotone nonlinear function*, J. Math. Anal. Appl. **307** (2005), 1–11.

[34] S. Woinowsky-Krieger, *The effect of axial force on the vibration of hinged bars*, J. Appl. Mech. **17** (1950), 35-36.

[35] T.F. Wu, *Three positive solutions for Dirichlet problems involving critical Sobolev exponent and sign-changing weight*, J. Differential Equations **249** (2010), 1549-1578.

Universidade de Brasília, Departamento de Matemática, 70910-900, Brasília-DF, Brazil
Email address: mfurtado@unb.br

Universidade Federal do Pará, Departamento de Matemática, 66075-110, Belém-PA, Brazil
Email address: jpabloufpa@gmail.com