MULTIPLICITY OF SOLUTIONS FOR A NONLINEAR BOUNDARY VALUE PROBLEM IN THE UPPER HALF-SPACE

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Abstract. We obtain multiple solutions for the nonlinear boundary value problem

$$-\Delta u-\frac{1}{2}\,(x\cdot\nabla u)=\lambda a(x)|u|^{q-2}u,\ \text{in}\ \mathbb{R}^N_+,\qquad \frac{\partial u}{\partial\nu}=b(x')|u|^{p-2}u,\ \text{on}\ \partial\mathbb{R}^N_+,$$

where $\mathbb{R}^N_+=\{(x',x_N):x'\in\mathbb{R}^{N-1},\,x_N>0\}$ is the upper half-space, $N\geq 3$, $\lambda>0$ is a parameter, $1<q<2<p\leq 2_*=2(N-1)/(N-2)$. The potentials a and b satisfy mild conditions which allow us to use variational methods. In some results, they can be indefinite in sign.

1. Introduction and main results

Let $\mathbb{R}^N_+:=\big\{(x',x_N):x'\in\mathbb{R}^{N-1},\,x_N>0\big\}$ be the upper half-space and consider the following heat equation with nonlinear boundary condition

$$v_t-\Delta v=0,\ \text{ in}\ \mathbb{R}^N_+\times(0,+\infty),\qquad \frac{\partial v}{\partial\eta}=|v|^{p-2}v,\ \text{ on}\ \partial\mathbb{R}^N_+\times(0,+\infty),$$

where $2<p\leq 2_*:=2(N-1)/(N-2)$ and $\partial u/\partial\eta$ denotes the partial outward normal derivative. Solutions of type

$$v(x,t)=t^{-\lambda}u(t^{-1/2}x),$$

with $\lambda=1/(2(p-2))>0$, are called self-similar solutions. Besides preserve the PDE scaling, they carry simultaneously information about small and large scale behaviors, providing also qualitative properties like global existence, blow-up and asymptotic behavior (see e.g. [24, 26, 25]).

An easy computation shows that the profile u above needs to satisfy

$$-\Delta u-\frac{1}{2}\,(x\cdot\nabla u)=\lambda u,\ \text{ in}\ \mathbb{R}^N_+,\qquad \frac{\partial u}{\partial\eta}=|u|^{p-2}u,\ \text{ on}\ \partial\mathbb{R}^N_+.$$

Such problem was recently considered in [19, 20], where existence results were presented according to the range of λ. Actually, these papers were strongly motivated by the vast literature concerning the version of the problem for the whole space $\mathbb{R}^N$ with different types of nonlinearities. We could quote [4, 11, 28, 13, 27, 22, 21] and their references for results about existence, nonexistence, multiplicity, decay rate, among other properties of solutions.

2010 *Mathematics Subject Classification.* Primary 35J66; Secondary 35J20.

Key words and phrases. Nonlinear boundary conditions; concave-convex problems; critical trace problems; half-space; self-similar solutions.

The first author was partially supported by FAP-DF/Brazil and CNPq/Brazil. The second author was supported by CNPq/Brazil.

Here, we are going to study the effect of replacing the linear term λu in the above equation by a sublinear indefinite function. Our main motivation comes from the problem

$$-\Delta u = \lambda a(x)|u|^{q-2}u + b(x)|u|^{r-2}u, \text{ in } \Omega, \qquad u = 0, \text{ on } \partial\Omega,$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $1 < q < 2 < r \leq 2N/(N-2)$ and the potentials a, b satisfy natural regularity conditions. In the celebrated paper [3], Ambrosetti, Brezis and Cerami considered the constant case $a \equiv 1$, $b \equiv 1$ and obtained $\Lambda > 0$ such that the problem admits at least two positive solutions whenever $\lambda \in (0, \Lambda)$, at least one if $\lambda = \Lambda$ and no solution if $\lambda > \Lambda$. Variable and indefinite potentials were considered in [15] (see also [16]). In [23], the authors obtained for

$$-\Delta u + u = |u|^{r-2}u, \text{ in } \Omega, \qquad \frac{\partial u}{\partial \eta} = \lambda |u|^{q-2}u, \text{ on } \partial\Omega,$$

results which are analogous to that of [3]. Some of their results were extended in [32] for the indefinite potential case (see also [29]). All the aforementioned works belong to a huge class of problems which are now called of concave-convex type.

In this paper, we deal with the concave-convex boundary value problem

$$(P_\lambda) \qquad \begin{cases} -\Delta u - \dfrac{1}{2}(x \cdot \nabla u) & = & \lambda a(x)|u|^{q-2}u, \quad x \in \mathbb{R}^N_+, \\ \dfrac{\partial u}{\partial \eta} & = & b(x')|u|^{p-2}u, \quad x' \in \mathbb{R}^{N-1}, \end{cases}$$

where $N \geq 3$, $\lambda > 0$ is a parameter, $1 < q < 2 < p \leq 2_*$ and we have identified $\partial\mathbb{R}^N_+ \simeq \mathbb{R}^{N-1}$. For describing the assumptions on the potentials we need first to present the functional space to deal with (P_λ). This is done in what follows.

As we shall see, the function $K(x) = \exp(|x|^2/4)$ is closely related with the appropriated space to look for solutions of our problem. In order to present the assumptions on the coefficients, we denote for any $2 \leq r \leq 2^*$ the weighted Lebesgue space

$$(1.1) \qquad L^r_K(\mathbb{R}^N_+) = \left\{ u \in L^r(\mathbb{R}^N_+) : \|u\|_r = \left(\int_{\mathbb{R}^N_+} K(x)|u|^r dx \right)^{1/r} < \infty \right\}.$$

If we denote by $r' = r/(r-1)$ the conjugated exponent of $r > 1$, we can present the basic hypothesis on a, b in the following way:

(a_0) $a \in L^{\sigma_q}_K(\mathbb{R}^N_+) \cap L^{N/2}_{loc}(\mathbb{R}^N_+)$ for some

$$\left(\frac{p}{q}\right)' < \sigma_q \leq \left(\frac{2}{q}\right)';$$

(b_0) $b \in L^\infty(\mathbb{R}^{N-1})$.

Since they can change it sign, we may define the sets

$$\Omega_a^+ := \{x \in \mathbb{R}^N_+ : \ a(x) > 0\}, \quad \Omega_b^+ := \{x' \in \mathbb{R}^{N-1} : \ b(x') > 0\}.$$

In our first results we obtain existence of two nonnegative solutions when roughly speaking the closure of the set Ω_a^+ intersects Ω_b^+ and the parameter $\lambda > 0$ approaches zero. More specifically, denoting by $B_\delta(0)$ the open ball centered at origin with radii $\delta > 0$, we prove the following:

Theorem 1.1. *Suppose that a, b satisfy (a_0) and (b_0). If $1 < q < 2 < p < 2_*$, then there exists $\lambda_* > 0$ such that, for any $\lambda \in (0, \lambda_*)$, problem (P_λ) has at least two nonnegative nonzero solutions provided*

(ab) *there exists $\delta > 0$ such that*

$$(B_\delta(0) \cap \mathbb{R}^N_+) \subset \Omega_a^+, \quad (B_\delta(0) \cap \partial\mathbb{R}^N_+) \subset \Omega_b^+.$$

In our second result, we consider the critical case by adding a flatness condition on the potential b:

Theorem 1.2. *Suppose that $N \geq 7$, $p = 2_*$ and the other conditions of Theorem 1.1 are verified. Then there exists $\lambda_* > 0$ such that, for any $\lambda \in (0, \lambda_*)$, problem (P_λ) has at least two nonnegative nonzero solutions provided*

(b_1) *there exist $M > 0$ and $\sigma > N - 1$ such that*

$$\|b\|_\infty - b(x') \leq M|x'|^\sigma, \quad \text{for a.e. } x' \in B_\delta(0) \cap \partial\mathbb{R}^N_+.$$

The first solution will be obtained with a standard minimization argument while the second one requires finer arguments. This is specially true when $p = 2_*$, since the trace embedding we are going to use fails to be compact. Two points are important to overcome this difficulty: a trick regularization study of the first solution on the boundary and the application of an idea of Brezis and Nirenberg [10], together with fine estimates of a modification of the *instanton functions* founded by Escobar [17] and Beckner [8].

In the second part of the paper, we take advantage of the symmetry to get more and more solutions (with no prescribed sign). Unfortunately, in this case we do not assume that both the potentials are indefinite.

We prove the following:

Theorem 1.3. *Suppose that $1 < q < 2$, $a \geq 0$ and $b \not\equiv 0$ satisfiy (a_0) and (b_0), respectively. Then problem (P_λ) has infinitely many solutions in each of the following cases:*

1. *$2 < p < 2_*$ and $\lambda > 0$;*
2. *$p = 2_*$, $b \leq 0$ and $\lambda > 0$;*
3. *$p = 2_*$ and $\lambda > 0$ is small.*

Theorem 1.4. *Suppose that $1 < q < 2 < p < 2_*$, $a \not\equiv 0$ and $b \geq 0$ satisfy (a_0) and (b_0), respectively. Then, for any $\lambda > 0$, problem (P_λ) has infinitely many solutions.*

The above theorems will be proved as application of suitable versions of the Symmetric Mountain Pass Theorem [2]. They were proved by Tonkes in the paper [30] which strongly motivated the second part of our work (see also [6, 7] for some earlier results). In the critical case, when $b \leq 0$, the boundary term is related with a semi-norm and therefore we can argue as in the subcritical case. When $p = 2_*$ and b is indefinite in sign, we borrow an argument from [5]. It can be proved that, when $b \leq 0$, the energy of the solutions given by Theorem 1.3 are negative and goes to zero. On the other hand, in Theorem 1.4, this energy goes to infinity, the same occurring with the norm of the solutions.

The paper is organized as follows: in Section 2 we present the variational framework to deal with our problem and obtained the first solution; in Section 3 we finish the proof of the first two theorems; Section 4 is devoted to the proof of Theorems 1.3 and 1.4.

2. Variational setting

Throughout the paper we assume that $1 < q < 2 < p \leq 2_*$ and conditions (a_0), (b_0) hold. Following Escobedo and Kavian [18], we first set

$$K(x) := \exp(|x|^2/4), \quad x \in \mathbb{R}^N_+,$$

and notice that the first equation in (P_λ) is equivalent to

$$-\mathrm{div}(K(x)\nabla u) = \lambda K(x)a(x)|u|^{q-2}u, \quad x \in \mathbb{R}^N_+.$$

Hence, it is natural looking for solutions in the space X defined as the closure of $C_0^\infty(\overline{\mathbb{R}^N_+})$ with respect to the norm

$$\|u\| := \left(\int_{\mathbb{R}^N_+} K(x)|\nabla u|^2 dx \right)^{1/2}.$$

Recall the definition of $L^r_K(\mathbb{R}^N_+)$ in (1.1) and define, for each $2 \leq s \leq 2_*$, the space

$$L^s_K(\mathbb{R}^{N-1}) := \left\{ u \in L^s(\mathbb{R}^{N-1}) : \| u \|_s := \left(\int_{\mathbb{R}^{N-1}} K(x',0)|u|^s dx' \right)^{1/s} < \infty \right\}.$$

We collect in the next proposition the abstract results proved in [19, 20].

Proposition 2.1. *For any $r \in [2, 2^*)$ and $s \in [2, 2_*)$, the embeddings $X \hookrightarrow L^s_K(\mathbb{R}^N_+)$ and $X \hookrightarrow L^s_K(\mathbb{R}^{N-1})$ are compact. In the critical cases $r = 2^*$ and $p = 2_*$, we have only continuous embeddings.*

Given $2 \leq r \leq 2^*$ and $2 \leq s \leq 2_*$, we can use the above result to define the following embedding constants:

$$S_r := \inf_{u \in X/\{0\}} \frac{\int_{\mathbb{R}^N_+} K(x)|\nabla u|^2 dx}{\left(\int_{\mathbb{R}^N_+} K(x)|u|^r dx \right)^{2/r}},$$

$$S_{s,\partial} := \inf_{u \in X/\{0\}} \frac{\int_{\mathbb{R}^N_+} K(x)|\nabla u|^2 dx}{\left(\int_{\mathbb{R}^{N-1}} K(x',0)|u|^s dx' \right)^{2/s}}.$$

By condition (a_0), we have that $2 \leq q\sigma_q' < 2^*$, and therefore we can use Hölder's inequality to get

$$\left| \int_{\mathbb{R}^N_+} K(x)a(x)(u^+)^q dx \right| \leq \|a\|_{\sigma_q} \left(\int_{\mathbb{R}^N_+} K(x)|u|^{q\sigma_q'} dx \right)^{1/\sigma_q'} < +\infty, \tag{2.1}$$

for any $u \in X$. Hence, condition (b_0) and standard arguments show that the functional

$$I_\lambda(u) := \frac{1}{2}\|u\|^2 - \frac{\lambda}{q} \int_{\mathbb{R}^N_+} K(x)a(x)(u^+)^q \, dx - \frac{1}{p} \int_{\mathbb{R}^{N-1}} K(x',0)b(x')(u^+)^p \, dx'$$

belongs to $C^1(X, \mathbb{R})$. Here and in what follows we will denote $u^+ := \max\{u, 0\}$ and $u^- := u^+ - u$. If $I_\lambda'(u) = 0$, then we can compute $0 = I_\lambda'(u)u^-$ to conclude that $\|u^-\| = 0$, and therefore the critical points of I_λ are nonnegative solutions of problem (P_λ)

The first step in the proof of Theorem 1.1 is the study of I_λ near origin.

Lemma 2.2. *There exist* $\rho = \rho(q, p, |b|_\infty) > 0$, $\alpha = \alpha(\rho) > 0$ *and* $\lambda_* = \lambda_*(q, \rho) > 0$ *such that* $I_\lambda(u) \geq \alpha > 0$, *for any* $u \in X$ *verifying* $\|u\| = \rho$, *and* $\lambda \in (0, \lambda_*)$.

Proof. By using (2.1) and Proposition 2.1, we get

$$\begin{aligned} I_\lambda(u) &\geq \frac{1}{2}\|u\|^2 - \frac{\lambda}{q}\|a\|_{\sigma_q}\|u\|^q_{q\sigma'_q} - \frac{1}{p}|b|_\infty |u|_p^p \\ &= \frac{\|u\|^q}{2}\left[\|u\|^{2-q} - \frac{2}{p}S_{p,\partial}^{-p/2}|b|_\infty\|u\|^{p-q} - \lambda\frac{2}{q}S_{q\sigma'_q}^{-q/2}\|a\|_{\sigma_q}\right]. \end{aligned}$$

The function $g : (0,\infty) \to \mathbb{R}$ given by $g(t) := t^{2-q} - C_1 t^{p-q}$, with $C_1 := 2S_{p,\partial}^{-p/2}|b|_\infty/p$, achieves its maximum value at

$$\rho := \left[\frac{(2-q)}{C_1(p-q)}\right]^{1/(p-2)}.$$

Thus, for any $u \in X$ satisfying $\|u\| = \rho$, there holds

$$I_\lambda(u) \geq \frac{\rho^q}{2}\left(g(\rho) - \lambda\frac{2}{q}S_{q\sigma'_q}^{-q/2}\|a\|_{\sigma_q}\right) \geq \frac{\rho^q}{2}\frac{g(\rho)}{2} = \alpha > 0,$$

whenever

$$\lambda < \lambda_* := \frac{qS_{q\sigma'_q}^{q/2}}{4\|a\|_{\sigma_q}}g(\rho),$$

and the result follows. □

We obtain in the next proposition our first solution.

Proposition 2.3. *Let* λ_*, $\rho > 0$ *be as in the above lemma. For any* $\lambda \in (0, \lambda_*)$, *we have that*

$$-\infty < c_0 := \inf_{u\in\overline{B_\rho(0)}} I_\lambda(u) < 0$$

and the infimum is attained at $u_0 \in B_\rho(0)$ *such that* $u_0 \in L^\nu_{loc}(\mathbb{R}^N_+) \cap L^\nu_{loc}(\mathbb{R}^{N-1})$ *for any* $\nu \geq 1$.

Proof. The inequality $c_0 > -\infty$ is obvious, since I_λ maps bounded sets in bounded sets. Let $\delta > 0$ given by (ab) and consider $\varphi \in C_0^\infty(B_\delta(0))$ such that $\int_{\mathbb{R}^N_+} K(x)a(x)\varphi^q dx > 0$. Then,

$$\frac{I_\lambda(t\varphi)}{t^q} \leq \frac{t^{2-q}}{2}\|\varphi\|^2 - \frac{\lambda}{q}\int_{\mathbb{R}^N_+} K(x)a(x)\varphi^q dx,$$

and therefore

$$\limsup_{t\to 0^+}\frac{I_\lambda(t\varphi)}{t^q} \leq -\frac{\lambda}{q}\int_{\mathbb{R}^N_+} K(x)a(x)\varphi^q dx < 0,$$

which proves that $I_\lambda(t\varphi) < 0$, for any $t > 0$ small. This implies that $c_0 < 0$.

Let $(u_n) \subset \overline{B_\rho(0)}$ be a minimizing sequence for c_0. We may assume that, for some $u_0 \in X$,

$$\begin{cases} u_n \rightharpoonup u_0 \text{ weakly in } X, \\ u_n \to u_0 \text{ strongly in } L^r_K(\mathbb{R}^N_+), \\ u_n^+(x) \to u_0^+(x),\ |u_n(x)| \leq h_r(x) \text{ for a.e. } x \in \mathbb{R}^N_+, \end{cases} \tag{2.2}$$

for any $2 \le r < 2^*$ and $h_r \in L^r_K(\mathbb{R}^N_+)$. Moreover, since $I_\lambda \ge \alpha > 0$ on $\partial B_\rho(0)$, we can use $2 \le q\sigma_q' < 2_*$ and the Ekeland Variational Principle to also assume that

$$\lim_{n\to+\infty} I_\lambda(u_n) = c_0, \qquad \lim_{n\to+\infty} I_\lambda'(u_n) = 0.$$

We claim that $I_\lambda'(u_0) = 0$. Indeed, pick $\phi \in C_0^\infty(\overline{\mathbb{R}^N_+})$ and call Ω its support. Since $\sigma_q > (p/q)' = p/(p-q)$, its possible to choose $p_0 \in (2,p)$ close to p and such that

$$\sigma_q > \frac{p_0}{p_0 - q} > \frac{p_0}{p_0 + 1 - q}.$$

Thus, there exists $t > 1$ satisfying

$$\frac{1}{\sigma_q} + \frac{1}{p_0/(q-1)} + \frac{1}{t} = 1.$$

Using Young's inequality we get

$$|K(x)a(x)(u_n^+)^{q-1}\phi(x)| \le C_1 \left(|a(x)|^{\sigma_q} + |h_{p_0}|^{p_0} + |\phi(x)|^t\right)$$

for a.e. $x \in \Omega$. It follows from the pointwise convergence in (2.2) and the Lebesgue's Theorem that

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N_+} K(x)a(x)(u_n^+)^{q-1}\phi\, dx = \int_{\mathbb{R}^N_+} K(x)a(x)(u_0^+)^{q-1}\phi\, dx.$$

A simpler argument shows that

$$\lim_{n\to+\infty} \int_{\mathbb{R}^{N-1}} K(x')b(x')(u_n^+)^{p-1}\phi\, dx' = \int_{\mathbb{R}^{N-1}} K(x')b(x')(u_0^+)^{p-1}\phi\, dx'.$$

So, the claim follows from the weak convergence of (u_n) and the density of $C_0^\infty(\overline{\mathbb{R}^N_+})$ in X.

From Young's inequality, we obtain

$$|K(x)a(x)(u_n^+)^q| \le |K(x)| \left(\frac{|a|^{\sigma_q}}{\sigma_q} + \frac{|u_n^+|^{q\sigma_q'}}{\sigma_q'}\right) \le K(x)\left(|a|^{\sigma_q} + h_{q\sigma_q'}^{q\sigma_q'}(x)\right),$$

for a.e. $x \in \mathbb{R}^N_+$. Since $2 \le q\sigma_q' < p \le 2^*$, we can use Hölder's inequality to conclude that this last function is integrable and we infer from Lebesgue's Theorem again that

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N_+} K(x)a(x)(u_n^+)^q\, dx = \int_{\mathbb{R}^N_+} K(x)a(x)(u_0^+)^q\, dx.$$

Thus,

$$\begin{aligned} c_0 &= \liminf_{n\to+\infty} \left[I_\lambda(u_n) - \frac{1}{p} I_\lambda'(u_n)u_n\right] \\ &= \liminf_{n\to+\infty} \left[\left(\frac{1}{2} - \frac{1}{p}\right)\|u_n\|^2 - \lambda\left(\frac{1}{q} - \frac{1}{p}\right)\int_{\mathbb{R}^N_+} K(x)a(x)(u_n^+)^q\, dx\right] \\ &\ge \left[\left(\frac{1}{2} - \frac{1}{p}\right)\|u_0\|^2 - \lambda\left(\frac{1}{q} - \frac{1}{p}\right)\int_{\mathbb{R}^N_+} K(x)a(x)(u_0^+)^q\, dx\right] \\ &= I_\lambda(u_0) - \frac{1}{p} I_\lambda'(u_0)u_0 = I_\lambda(u_0). \end{aligned}$$

Hence $I(u_0) = c_0 < 0$ and it follows from Lemma 2.2 that $u_0 \in B_\rho(0)$.

In order to obtain regularity for the solution, we set $w := \exp(|x|^2/8)u_0 \in W^{1,2}_{loc}(\mathbb{R}^N_+)$ and notice that w weakly solves

$$\begin{cases} -\Delta w = f(x,w), & \text{in } \mathbb{R}^N_+, \\ \dfrac{\partial w}{\partial \eta} = g(x',w), & \text{on } \partial\mathbb{R}^N_+, \end{cases}$$

where

$$f(x,t) := a(x)\exp((2-q)|x|^2/8)|t|^{q-2}t - \left[(|x|^2+4N)/16\right]t$$

and

$$g(x',t) := b(x')\exp((2-p)|x'|^2/8)|t|^{p-2}t,$$

for $x \in \mathbb{R}^N_+$, $x' \in \mathbb{R}^{N-1}$ and $t \in \mathbb{R}$. It is easy to check that

$$|f(x,t)| \le \Gamma_1(x)(1+|t|), \qquad |g(x',t)| \le \Gamma_2(x')(1+|t|)$$

for the functions

$$\Gamma_1(x) := |a(x)|\exp((2-q)|x|^2/8) + \left[(|x|^2+4N)/16\right], \qquad \Gamma_2(x') := b(x').$$

Using (a_0) and (b_0) we conclude that $\Gamma_1 \in L^{N/2}_{loc}(\mathbb{R}^N_+)$ and $\Gamma_2 \in L^{N-1}_{loc}(\mathbb{R}^{N-1})$. Hence, we can use a version of Brezis-Kato's Theorem [12] (see also [1, Appendix 4]) to conclude that $u_0 \in L^{\nu}_{loc}(\mathbb{R}^N_+) \cap L^{\nu}_{loc}(\mathbb{R}^{N-1})$ for any $\nu \ge 1$. The proposition is proved. □

3. Proof of Theorems 1.1 and 1.2

Recall that, if E is a Banach space, $\Phi \in C^1(E,\mathbb{R})$ and $c \in \mathbb{R}$, the functional Φ satifies the $(PS)_c$ condition if any sequence $(u_n) \subset E$ such that

$$\lim_{n\to+\infty} \Phi(u_n) = c, \qquad \lim_{n\to+\infty} \Phi'(u_n) = 0,$$

has a convergent subsequence. From now on, any such sequence will be called $(PS)_c$-sequence.

Lemma 3.1. *If $2 < p < 2_*$, then the functional I_λ satisfies the $(PS)_c$ condition for any $c \in \mathbb{R}$.*

Proof. Let $(u_n) \subset X$ be a $(PS)_c$-sequence. Computing $I'_\lambda(u_n) - (1/p)I'_\lambda(u_n)u_n$, using (a_0) and Hölders's inequality, we can check that (u_n) is bounded. Then, up to a subsequence, we have that $u_n \rightharpoonup u$ weakly in X and $u_n \to u$ strongly in $L^r_K(\mathbb{R}^N_+)$ and $L^s_K(\mathbb{R}^{N-1})$, for any $r \in [2,2^*)$ and $s \in [2,2_*)$, respectively. Setting $q_0 := q\sigma'_q \in [2,p)$ and applying Hölder's inequality with exponents σ_q, $q_0/(q-1)$ and q_0, we get

$$\left| \int_{\mathbb{R}^N_+} K(x)a(x)(u_n^+)^{q-1}(u_n-u)\,dx \right| \le \|a\|_{\sigma_q}\|u_n\|_{q_0}^{q-1}\|u_n-u\|_{q_0} \to 0,$$

as $n \to +\infty$. Analogously,

$$\left| \int_{\mathbb{R}^{N-1}} K(x',0)b(x')(u_n^+)^{p-1}(u_n-u)\,dx' \right| \le |b|_\infty |u_n|_p^{p-1}|u_n-u|_p \to 0.$$

From the two above expressions and the weak convergence we obtain

$$o(1) = I'_\lambda(u_n)(u_n-u) = \|u_n\|^2 - \|u\|^2 + o(1),$$

as $n \to +\infty$. The result is now a consequence of the weak convergence. □

When dealing with the critical case, we need the following local compactness result:

Lemma 3.2. *If $p = 2^*$ and the function u_0 given by Proposition 2.3 is the only nonzero critical point of I_λ, then I_λ satisfies the Palais-Smale condition at any level*

$$c < \overline{c} := I_\lambda(u_0) + \frac{1}{2(N-1)} \frac{1}{\|b\|_\infty^{N-2}} S_{2_*,\partial}^{N-1}.$$

Proof. Let $(u_n) \subset X$ be a $(PS)_c$-sequence. As in Lemma 3.1, we may assume that $u_n \rightharpoonup u$ weakly in X and $u_n \to u$ strongly in $L_K^{q\sigma_q'}(\mathbb{R}_+^N)$. Hence, we infer from the Lebesgue Theorem that, as $n \to +\infty$,

$$\int_{\mathbb{R}_+^N} K(x)a(x)(u_n^+)^q \, dx = \int_{\mathbb{R}_+^N} K(x)a(x)(u^+)^q \, dx + o(1).$$

If $z_n := (u_n - u)$, we can use $I_\lambda'(u_n)u_n = o(1)$ and Brezis-Lieb's lemma [9] to obtain

$$\begin{aligned} o(1) &= \|u_n\|^2 - \lambda \int_{\mathbb{R}_+^N} K(x)a(x)(u_n^+)^q \, dx - \int_{\mathbb{R}^{N-1}} K(x',0)b(x')(u_n^+)^{2_*} \, dx' \\ &= I_\lambda'(u)u + \|z_n\|^2 - \int_{\mathbb{R}^{N-1}} K(x',0)b(x')(z_n^+)^{2_*} \, dx' + o(1). \end{aligned}$$

As in the proof of Proposition 2.3, we have that $I_\lambda'(u) = 0$. So, passing the above expression to the limit, we obtain $\gamma \geq 0$ such that

$$\lim_{n\to+\infty} \|z_n\|^2 = \gamma = \lim_{n\to+\infty} \int_{\mathbb{R}^{N-1}} K(x',0)b(x')(z_n^+)^{2_*} \, dx'.$$

We need to prove that $\gamma = 0$. In order to do this, we first take the limit in the inequality

$$\int_{\mathbb{R}^{N-1}} K(x',0)b(x')(z_n^+)^{2_*} \, dx' \leq \|b\|_\infty S_{2_*,\partial}^{-2_*/2} \left(\int_{\mathbb{R}_+^N} K(x)|\nabla z_n|^2 dx \right)^{2_*/2},$$

to obtain $\gamma \leq \|b\|_\infty S_{2_*,\partial}^{-2_*/2} \gamma^{2_*/2}$. Suppose, by contradiction, that $\gamma > 0$. Then

$$\gamma \geq \frac{1}{\|b\|_\infty^{N-2}} S_{2_*,\partial}^{N-1}. \tag{3.1}$$

On the other hand, using Brezis-Lieb again, we obtain

$$c + o(1) = I_\lambda(u_n) = I_\lambda(u) + \frac{1}{2}\|z_n\|^2 - \frac{1}{2_*} \int_{\mathbb{R}^{N-1}} K(x',0)b(x')(z_n^+)^{2_*} \, dx' + o(1).$$

Passing to the limit and using (3.1), we conclude that

$$c = I_\lambda(u) + \frac{1}{2(N-1)}\gamma \geq I_\lambda(u) + \frac{1}{2(N-1)} \frac{1}{\|b\|_\infty^{N-2}} S_{2_*,\partial}^{N-1}.$$

Recalling that u is a critical point of I_λ, we conclude from the hypotheses that $u = 0$ or $u = u_0$. Since $\max\{I_\lambda(0), I_\lambda(u_0)\} \leq 0$, the above expression contradicts $c < \overline{c}$. So, $\gamma = 0$ and we have done. □

Let $\delta > 0$ be as in assumption (ab) and take $\phi \in C^\infty(\overline{\mathbb{R}_+^N}, [0,1])$ such that $\phi \equiv 1$ in $\overline{\mathbb{R}_+^N} \cap B_{\delta/2}(0)$ and $\phi \equiv 0$ in $\overline{\mathbb{R}_+^N} \backslash B_\delta(0)$. Set, for each $\varepsilon > 0$,

$$u_\varepsilon(x) := K(x)^{-1/2}\phi(x)U_\varepsilon(x), \quad x \in \mathbb{R}_+^N,$$

where

$$U_\varepsilon(x', x_N) := \frac{\varepsilon^{(N-2)/2}}{[|x'|^2 + (x_N + \varepsilon)^2]^{(N-2)/2}}.$$

If $N \geq 7$, it is proved in [20] that,

$$\|u_\varepsilon\|^2 = A_N + O(\varepsilon^2), \qquad |u_\varepsilon|_{2_*}^{2_*} = B_N^{2_*/2} + O(\varepsilon^2), \tag{3.2}$$

as $\varepsilon \to 0^+$. Moreover, the constants A_N, B_N are such that$A_N/B_N = S_{2_*,\partial}$ and the following holds:

Lemma 3.3. *If* $\psi_\varepsilon := u_\varepsilon/|u_\varepsilon|_{2_*}$ *and* $(N-1)/(N-2) < \tau < 2_*$*, then*

$$\|\psi_\varepsilon\|^{2(N-1)} = S_{2_*,\partial}^{N-1} + O(\varepsilon^2), \qquad |\psi_\varepsilon|_\tau^\tau = O(\varepsilon^{(N-1)-\tau(N-2)/2}), \tag{3.3}$$

as $\varepsilon \to 0^+$.

Proof. Using the Mean Value theorem for $g(r) = r^s$ and a simple computation, we can check that

$$\left[A + O(\varepsilon^t)\right]^s = A^s + O(\varepsilon^t),$$

for any A, s, $t > 0$. Hence, we infer from (3.2) and the defintion of 2_* that

$$\|\psi_\varepsilon\|^{2(N-1)} = \frac{\left[A_N + O(\varepsilon^2)\right]^{N-1}}{\left[B_N^{2_*/2} + O(\varepsilon^2)\right]^{N-2}} = \frac{A_N^{N-1} + O(\varepsilon^2)}{B_N^{2_*(N-2)/2} + O(\varepsilon^2)} = \left(\frac{A_N}{B_N}\right)^{N-1} + O(\varepsilon^2).$$

Since $A_N/B_N = S$, we conclude that the first statement in (3.3) holds.

For the second one, we first notice that

$$\begin{aligned} |u_\varepsilon|_\tau^\tau &= \varepsilon^{-\tau(N-2)/2} \int_{\mathbb{R}^{N-1}} \frac{K(x',0)^{-\tau/2}\phi(x',0)^\tau}{[|x'/\varepsilon|^2 + 1]^{\tau(N-2)/2}}\, dx' \\ &\leq C_1 \varepsilon^{-\tau(N-2)/2} \int_{B_\delta(0)\cap\partial\mathbb{R}^N_+} \frac{1}{(|x'/\varepsilon|^2 + 1)^{\tau(N-2)/2}}\, dx' \\ &\leq C_1 \varepsilon^{(N-1)-\tau(N-2)/2} \int_{\mathbb{R}^{N-1}} \frac{1}{(|y'|^2+1)^{\tau(N-2)/2}}\, dy', \end{aligned}$$

where we have used the definition of u_ε, $0 \leq \phi \leq 1$ and the change of variable $y' = x'/\varepsilon$. But

$$\begin{aligned} \int_{\mathbb{R}^{N-1}} \frac{1}{(|y'|^2+1)^{\tau(N-2)/2}}\, dy' &\leq C_2 + \int_{\partial\mathbb{R}^N_+\setminus B_1(0)} \frac{1}{|y'|^{\tau(N-2)}}\, dy' \\ &= C_2 + C_3 \int_1^{+\infty} s^{-\tau(N-2)+(N-2)} ds < +\infty, \end{aligned}$$

whenever $\tau > (N-1)/(N-2)$. Since $|u_\varepsilon|_{2_*}^\tau = B_N^{\tau/2} + o(1)$, as $\varepsilon \to 0^+$, the result follows from the above inequalities. □

We are ready to prove our first main results.

Proof of Theorems 1.1 and 1.2. According to Lemma 2.2 and Proposition 2.3, there exists $\lambda_* > 0$ such that, for any $\lambda \in (0, \lambda_*)$, the problem ($P_\lambda$) has a nonnegative solution $u_0 \in X \setminus \{0\}$ such that $I_\lambda(u_0) < 0$. The second solution will be obtained as an application of the Mountain Pass Theorem.

Recall that $\psi_\varepsilon := u_\varepsilon/\|u_\varepsilon\|_{2_*}$ and notice that

$$\begin{aligned}\int_{\mathbb{R}^{N-1}} K(x',0)b(x')(u_0+t\psi_\varepsilon)^p\,dx' &= O(1)+\int_\Omega K(x',0)b(x')(u_0+t\psi_\varepsilon)^p dx' \\ &\geq O(1)+t^p\int_\Omega K(x',0)b(x')\psi_\varepsilon^p\,dx',\end{aligned}$$

as $t\to+\infty$, where $\Omega := B_\delta(0)\cap\mathbb{R}^{N-1}$. Since a similar argument holds for the integral inside the domain, we get

$$I_\lambda(u_0+t\psi_\varepsilon)\leq O(t^2)+O(t^q)-\frac{t^p}{p}\int_\Omega K(x',0)b(x')\psi_\varepsilon^p dx',$$

as $t\to+\infty$. The function in the last integral above is positive, and therefore we can use $1<q<2<p$ to obtain

$$\lim_{t\to+\infty} I_\lambda(u_0+t\psi_\varepsilon)=-\infty. \tag{3.4}$$

Hence, there exists $t_*>0$ large such that $e:=u_0+t_*\psi_\varepsilon$ satisfies $\|e\|>\rho$ given by Lemma 2.2 and $I_\lambda(e)\leq 0$. So, it is well defined

$$c:=\inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I(\gamma(t)),$$

where $\Gamma := \{\gamma\in C([0,1],X)\,:\,\gamma(0)=0,\,\gamma(1)=e\}$. From the Mountain Pass Theorem [2] (see also [31, Theorem 1.15]), we obtain $(u_n)\subset X$ such that

$$\lim_{n\to+\infty} I_\lambda(u_n)=c,\qquad \lim_{n\to+\infty} I'_\lambda(u_n)=0.$$

If $2\leq p<2^*$, we can use Lemma 3.1 to conclude that, along a subsequence, (u_n) converges to a critical point $u_1\in X$ such that $I_\lambda(u_1)>0$. Hence, $u_1\neq u_0$ is the second solution.

The final step in the above argument is more delicate in the critical case $p=2_*$. Actually, we need to prove that, for $\varepsilon>0$ small, there holds

$$\max_{t\geq 0} I_\lambda(u_0+t\psi_\varepsilon)<\overline{c}:=I_\lambda(u_0)+\frac{1}{2(N-1)}\frac{1}{\|b\|_\infty^{N-2}}S_{2_*,\partial}^{N-1}. \tag{3.5}$$

If this is true, we can use Lemma 3.2, the Mountain Pass Theorem and a contradiction argument to obtain a nonzero solution $u_1\neq u_0$.

In order to prove (3.5), we first notice that, since $u_0\in B_\rho(0)$ is a local minimum of I_λ, we can use (3.4) to obtain $t_\varepsilon>0$ such that

$$m_\varepsilon := I_\lambda(u_0+t_\varepsilon\psi_\varepsilon)=\max_{t\geq 0} I_\lambda(u_0+t\psi_\varepsilon).$$

We claim that $t_\varepsilon=O(1)$, as $\varepsilon\to 0^+$. Indeed, suppose by contradiction that $t_{\varepsilon_n}\to+\infty$, for some sequence $\varepsilon_n\to 0^+$. Recalling that $a,\,b>0$ in the support of ψ_ε, we can use $I'_\lambda(u_0+t_\varepsilon\psi_\varepsilon)\psi_\varepsilon=0$ and $I'_\lambda(u_0)\psi_\varepsilon=0$ to get

$$t_\varepsilon^{2_*-1}\int_{\mathbb{R}^{N-1}} K(x',0)b(x')\psi_\varepsilon^{2_*}\,dx'\leq t_\varepsilon\|\psi_\varepsilon\|^2+\int_{\mathbb{R}^{N-1}} K(x',0)b(x')u_0^{2_*-1}\psi_\varepsilon\,dx'.$$

Thus, from (3.3), Hölder's inequality and $\|\psi_\varepsilon\|_{2_*}=1$, we obtain

$$\int_{\mathbb{R}^{N-1}} K(x',0)b(x')\psi_\varepsilon^{2_*}\,dx'\leq t_\varepsilon^{2-2_*}\left[S_{2_*,\partial}+O(1)\right]+t_\varepsilon^{1-2_*}\|b\|_\infty\|u_0\|_{2_*}^{2_*-1},$$

for all $\varepsilon>0$. In particular, we can take $\varepsilon=\varepsilon_n$ in the above inequality to conclude that

$$\lim_{n\to+\infty}\int_{\mathbb{R}^{N-1}} K(x',0)b(x')\psi_{\varepsilon_n}^{2_*}dx' = 0.$$

On the other hand, using (b_1) and $\|\psi_{\varepsilon_n}\|_{2_*}=1$ we obtain

$$o(1)=\int_{\mathbb{R}^{N-1}} K(x',0)b(x')\psi_{\varepsilon_n}^{2_*}dx' \geq \|b\|_\infty - M\int_{\mathbb{R}^{N-1}} K(x',0)|x'|^\sigma\psi_{\varepsilon_n}^{2_*}\,dx'. \tag{3.6}$$

Moreover, since $\|u_{\varepsilon_n}\|_{2_*}=B_N^{1/2}+o(1)$,

$$\begin{aligned}\int_{\mathbb{R}^{N-1}} K(x',0)|x'|^\sigma\psi_{\varepsilon_n}^{2_*}\,dx' &\leq C_1\frac{\varepsilon_n^{N-1}}{\|u_\varepsilon\|_{2_*}^{2_*}}\int_{B_\delta(0)\cap\mathbb{R}^{N-1}_+}\frac{|x'|^\sigma}{[|x'|^2+\varepsilon^2]^{N-1}}\,dx' \\ &= O(\varepsilon_n^{N-1})\int_{B_\delta(0)\cap\mathbb{R}^{N-1}_+}|x'|^{\sigma-2(N-1)}\,dx',\end{aligned}$$

as $n\to+\infty$. Since $\sigma>N-1$, the last integral above is finite and therefore

$$\int_{\mathbb{R}^{N-1}} K(x',0)|x'|^\sigma\psi_{\varepsilon_n}^{2_*}dx' = O(\varepsilon_n^{N-1}),\quad \text{as } n\to+\infty. \tag{3.7}$$

Thus, it follows from (3.6) that $\|b\|_\infty=0$, which does not make sense. This proves that (t_ε) is bounded.

Using $I'_\lambda(u_0)\psi_\varepsilon=0$, we obtain

$$m_\varepsilon = I(u_0)+\frac{t_\varepsilon^2}{2}\|\psi_\varepsilon\|^2-\frac{\lambda}{q}\Gamma_{1,\varepsilon}-\frac{1}{2_*}\Gamma_{2,\varepsilon}, \tag{3.8}$$

where

$$\Gamma_{1,\varepsilon} := \int_{\mathbb{R}^N_+} K(x)a(x)[(u_0+t_\varepsilon\psi_\varepsilon)^q-u_0^q-qt_\varepsilon u_0^{q-1}\psi_\varepsilon]\,dx$$

and

$$\Gamma_{2,\varepsilon} := \int_{\mathbb{R}^{N-1}} K(x',0)b(x')[(u_0+t_\varepsilon\psi_\varepsilon)^{2_*}-u_0^{2_*}-2_*t_\varepsilon u_0^{2_*-1}\psi_\varepsilon]\,dx'$$

It follows from the Mean Value Theorem that there exists $\theta(x)\in[0,1]$ such that

$$\begin{aligned}(u_0(x)+t_\varepsilon\psi_\varepsilon(x))^q-u_0(x)^q &= q(u_0(x)+\theta(x)t_\varepsilon\psi_\varepsilon(x))^{q-1}t_\varepsilon\psi_\varepsilon(x) \\ &\geq qt_\varepsilon u_0(x)^{q-1}\psi_\varepsilon(x),\end{aligned}$$

for a.e. $x\in\mathbb{R}^N_+$. Since $a\geq 0$ in the support of ψ_ε we conclude that $\Gamma_{1,\varepsilon}\geq 0$. For estimating $\Gamma_{2,\varepsilon}$ we notice that, given r, $s\geq 0$ and $1<\mu<2_*-1$, there holds (see [14])

$$(r+s)^{2_*}\geq r^{2_*}+s^{2_*}+2_*r^{2_*-1}s+2_*rs^{2_*-1}-A_\mu r^{2_*-\mu}s^\mu,$$

for some constant $A_\mu>0$. Picking $r=u_0(x)$ and $s=t_\varepsilon\psi_\varepsilon(x)$, we get

$$\Gamma_{2,\varepsilon}\geq\int_{\mathbb{R}^{N-1}} K(x',0)b(x')\left[t_\varepsilon^{2_*}\psi_\varepsilon^{2_*}+2_*t_\varepsilon^{2_*-1}u_0\psi_\varepsilon^{2_*-1}-A_\mu t_\varepsilon^\mu u_0^{2_*-\mu}\psi_\varepsilon^\mu\right]dx'.$$

Since $\Gamma_{1,\varepsilon}\geq 0$ and $\|\psi_\varepsilon\|_{2_*}=1$, we can use the above inequality and (3.8) to obtain

$$m_\varepsilon\leq I(u_0)+\left[\frac{t_\varepsilon^2}{2}\|\psi_\varepsilon\|^2-\frac{t_\varepsilon^{2_*}}{2_*}\|b\|_\infty\right]+\Gamma_{2,\varepsilon,1}-\Gamma_{2,\varepsilon,2}+\Gamma_{2,\varepsilon,3},$$

with

$$\Gamma_{2,\varepsilon,1} := \frac{t_\varepsilon^{2_*}}{2_*}\int_{\mathbb{R}^{N-1}} K(x',0)[\|b\|_\infty-b(x')]\psi_\varepsilon^{2_*}\,dx',$$

$$\Gamma_{2,\varepsilon,2} := t_\varepsilon^{2_*-1}\int_{\mathbb{R}^{N-1}} K(x',0)b(x')u_0\psi_\varepsilon^{2_*-1}dx'$$

and

$$\Gamma_{2,\varepsilon,3} := C_\mu \frac{t_\varepsilon^\mu}{2_*} \int_{\mathbb{R}^{N-1}} K(x',0) b(x') u_0^{2_*-\mu} \psi_\varepsilon^\mu dx'.$$

As in (3.7), the integral in $\Gamma_{2,\varepsilon,1}$ has order ε^{N-1}, as $\varepsilon \to 0^+$. So, we infer from the boundedness of (t_ε) that $\Gamma_{2,\varepsilon,1} = O(\varepsilon^{N-1})$. Moreover,

$$\max_{t \geq 0} \left\{ \frac{t^2}{2} \|\psi_\varepsilon\|^2 - \frac{t^{2_*}}{2_*} |b|_\infty \right\} = \frac{1}{2(N-1)} \frac{\|\psi_\varepsilon\|^{2(N-1)}}{|b|_\infty^{N-2}}$$

and therefore we infer from (3.3) and the above estimate for m_ε that

$$m_\varepsilon \leq \overline{c} + O(\varepsilon^2) - \Gamma_{2,\varepsilon,2} + \Gamma_{2,\varepsilon,3}. \tag{3.9}$$

In order to estimate the last two terms, we recall that $u_0 \in L^\nu_{loc}(\mathbb{R}^N_+) \cap L^\nu_{loc}(\mathbb{R}^{N-1})$ for any $\nu \geq 1$. So, if we denote $\Omega_\partial := B_\delta(0) \cap \mathbb{R}^{N-1}$, we can choose $\tau_1 > 1$ such that

$$\frac{2(N-1)}{(N+4)} < \tau_1 < \frac{2(N-1)}{N}$$

and use Hölder's inequality to get

$$\int_{\mathbb{R}^{N-1}} K(x',0) b(x') u_0 \psi_\varepsilon^{2_*-1} dx' \leq |b|_\infty \left(\int_{\Omega_\partial} K(x',0) u_0^{\tau_1'} dx' \right)^{1/\tau_1'} |\psi_\varepsilon|_{(2_*-1)\tau_1}^{2_*-1}.$$

Since $(N-1)/(N-2) < (2_*-1)\tau_1 < 2_*$ and (t_ε) is bounded, we infer from (3.3) and the choice of τ_1 that

$$\Gamma_{2,\varepsilon,2} = O(\varepsilon^{(N-1)/\tau_1 - (N/2)}). \tag{3.10}$$

We now set $\mu := (N-1)/(N-2)$, pick $1 < \tau_2 < 2$ and apply Hölder's inequality again to obtain

$$\int_{\mathbb{R}^{N-1}} K(x',0) b(x') u_0^{2_*-\mu} \psi_\varepsilon^\mu \, dx' \leq |b|_\infty \left(\int_{\Omega_\partial} K(x',0) u_0^{(2_*-\mu)\tau_2'} \, dx' \right)^{1/\tau_2'} |\psi_\varepsilon|_{\mu\tau_2}^\mu,$$

from which we conclude that

$$\Gamma_{2,\varepsilon,3} = O(\varepsilon^{(N-1)/\tau_2 - (N-1)/2}). \tag{3.11}$$

Since

$$\lim_{\tau \to 2(N-1)/N} \left(\frac{N-1}{\tau} - \frac{N}{2} \right) = 0 < \frac{N-1}{2} = \lim_{\tau \to 1} \left(\frac{N-1}{\tau} - \frac{N-1}{2} \right),$$

we can choose the numbers τ_1, τ_2 above in such way that

$$\nu_1 := \frac{N-1}{\tau_1} - \frac{N}{2} < 2, \qquad \nu_2 := \frac{N-1}{\tau_2} - \frac{N-1}{2} > \nu_1.$$

Replacing (3.10) and (3.11) in (3.9) and using the above inequalities, we obtain

$$m_\varepsilon \leq \overline{c} + O(\varepsilon^2) - O(\varepsilon^{\nu_1}) + O(\varepsilon^{\nu_2}) = \overline{c} + \varepsilon^{\nu_1} \left[O(\varepsilon^{2-\nu_1}) - O(1) + O(\varepsilon^{\nu_2 - \nu_1}) \right],$$

as $\varepsilon \to 0^+$. We conclude that (3.5) holds, for any $\varepsilon > 0$ small. The theorem is proved. □

4. Proof of Theorems 1.3 and 1.4

We start this section presenting some definitions and abstract results which will be used to obtain infinitely many solutions for (P_λ). Let $E = V \oplus W$ be an infinite dimensional Hilbert space, with $V = \mathrm{span}\{\varphi_1^V, \varphi_2^V, \dots\}$, $W = \mathrm{span}\{\varphi_1^W, \varphi_2^W, \dots\}$ and the the basis being orthonormal. For each $n \in \mathbb{N}$, define the subspaces

$$V^n := \mathrm{span}\{\varphi_1^V, \varphi_2^V, \dots \varphi_n^V\}, \qquad V_n := \overline{\mathrm{span}\{\varphi_n^V, \varphi_{n+1}^V, \dots\}}.$$

Using the set $\left\{\varphi_i^W\right\}_{i\in\mathbb{N}}$ we define W^n and W_n in a similar way.

Given $\Phi \in C^1(E, \mathbb{R})$ and $c \in \mathbb{R}$, we say that Φ satisfies the $(PS)_c^*$-condition (with respect to $V^n \oplus W^n$) if any sequence $(u_n) \subset V^n \oplus W^n$ such that

$$\lim_{n\to+\infty} \Phi(u_n) = c, \qquad \lim_{n\to+\infty} \Phi'_{|_{V^n\oplus W^n}}(u_n) = 0,$$

has a subsequence converging to a critical point of Φ. Any such sequence will be called $(PS)_c^*$-sequence.

We are going to obtain infinitely many solutions for (P_λ) as applications of the following abstract theorems due to Tonkens [30] (see also [2, 6]):

Theorem 4.1. *Let $\Phi \in C^1(E, \mathbb{R})$ be an even functional. Suppose that, for every $n \geq n_0$, there exist $R_n > r_n > 0$ such that*

- (A_1) $\inf\{\Phi(u) \,:\, u \in V_n \oplus W, \|u\|_E = R_n\} \geq 0$;
- (A_2) $b_n := \inf\{\Phi(u) \,:\, u \in V_n \oplus W, \|u\|_E \leq R_n\} \to 0$, *as* $n \to +\infty$;
- (A_3) $d_n := \sup\{\Phi(u) \,:\, u \in V^n, \|u\|_E = r_n\} < 0$;
- (A_4) Φ *satisfies* $(PS)_c^*$*-condition for all* $c \in [b_{n_0}, 0)$.

Then Φ has a sequence of critical values $c_n \in [b_n, d_n]$ such that $c_n \to 0$, as $n \to +\infty$.

Theorem 4.2. *Let $\Phi \in C^1(E, \mathbb{R})$ be a even functional. Suppose that, for every $n \in \mathbb{N}$, there exist $R_n > r_n > 0$ such that*

- $(\widetilde{A_2})$ $b_n := \inf\{\Phi(u) \,:\, u \in V_n \oplus W, \|u\| = r_n\} \to \infty$, *as* $n \to \infty$;
- $(\widetilde{A_3})$ $a_n := \max\{\Phi(u) \,:\, u \in V^n, \|u\| = R_n\} \leq 0$;
- $(\widetilde{A_4})$ Φ *satisfies* $(PS)_c$*-condition for all* $c > 0$.

Then Φ has a sequence of critical values $c_n \in (0, +\infty)$ such that $c_n \to +\infty$, as $n \to +\infty$.

Since we are not interested in the sign of the solutions, we redefine the energy function setting

$$I_\lambda(u) := \frac{1}{2}\|u\|^2 - \frac{\lambda}{q}\int_{\mathbb{R}^N_+} K(x)a(x)|u|^q\,dx - \frac{1}{p}\int_{\mathbb{R}^{N-1}} K(x',0)b(x')|u|^p\,dx'.$$

We are intending to apply Theorem 4.1 with $\Phi = I_\lambda$ and $E = X$. In order to define the space decomposition, we recall that $\Omega_a^+ = \left\{x \in \mathbb{R}^{\mathbb{N}}_+ \,:\, a(x) > 0\right\}$ and define

$$W := \left\{u \in X : u(x) = 0 \text{ for a.e. } x \in \mathrm{int}(\Omega_a^+)\right\}.$$

We call V the orthogonal complement of the closed subspace W, in such way that $X = V \oplus W$.

We start with the required compactness properties.

Proposition 4.3. *If $2 < p < 2_*$, then I_λ satisfies the $(PS)_c^*$ condition at any level $c \in \mathbb{R}$. The same holds if $p = 2_*$ and $b \leq 0$.*

Proof. Let $(u_n) \subset V^n \oplus W^n$ be a $(PS)^*_c$-sequence. Computing $I'_\lambda(u_n) - (1/p)I'_\lambda(u_n)u_n$, using (a_0) and Hölder's inequality, we can check that (u_n) is bounded. Then, up to a subsequence, we have that $u_n \rightharpoonup u$ weakly in X. Pick $\phi \in C_0^\infty(\overline{\mathbb{R}^N_+})$ and denote by ϕ^n its projection over the subspace $V^n \oplus W^n$. Since $(I'_\lambda(u_n)) \subset X^*$ is bounded, we have that

$$|I'_\lambda(u_n)(\phi - \phi^n)| \le \|I'_\lambda(u_n)\|_{X^*}\|\phi - \phi^n\| = o(1),$$

as $n \to +\infty$. Thus, recalling that $I'_\lambda(u_n)\phi^n = o(1)$, we obtain

$$I'_\lambda(u_n)\phi = I'_\lambda(u_n)(\phi - \phi^n) + I'_\lambda(u_n)\phi^n = o(1).$$

Arguing as in the proof of Proposition 2.3, we conclude that $I'_\lambda(u)\phi = 0$, for any $\phi \in C_0^\infty(\overline{\mathbb{R}^N_+})$. It follows from a density argument that $I'_\lambda(u) = 0$.

Using Lebesgue Theorem as in the proof of Proposition 2.3, we get

$$\lim_{n\to\infty} \int_{\mathbb{R}^N_+} K(x)a(x)|u_n|^q dx = \int_{\mathbb{R}^N_+} K(x)a(x)|u|^q dx. \tag{4.1}$$

Moreover, in the subcritical case $2 < p < 2_*$, the same kind of convergence holds for the term $\int_{\mathbb{R}^{N-1}} K(x',0)b(x')|u_n|^p dx'$, since the trace embedding is compact. These two convergences and $I'_\lambda(u)u = 0$ provide

$$o(1) = I'_\lambda(u_n)u_n - I'_\lambda(u)u = \|u_n^2\| - \|u\|^2 + o(1),$$

and we infer from the weak convergence that $u_n \to u$ strongly in X.

For the critical case $p = 2_*$, we first use assumption $b \le 0$ to guarantee that $\varphi \mapsto -\int_{\mathbb{R}^{N-1}} K(x',0)b(x')|\varphi|^p\, dx'$ is a seminorm in X. Hence, from the weak lower semicontinuity of a seminorm, we get

$$\limsup_{n\to+\infty} \int_{\mathbb{R}^{N-1}} K(x',0)b(x')|u_n|^{2_*} dx' \le \int_{\mathbb{R}^{N-1}} K(x',0)b(x')|u|^{2_*} dx'.$$

This, (4.1) and $I'_\lambda(u_n)u_n = o(1)$ imply that

$$\limsup_{n\to+\infty} \|u_n\|^2 \le \|u\|^2.$$

On the other hand, the weak convergence provides $\|u\|^2 \le \liminf_{n\to+\infty} \|u_n\|^2$, and therefore the result follows as in the former case. □

If $p = 2_*$ and b changes it sign, we need the following local compactness result.

Proposition 4.4. *If $p = 2_*$, then there exists $C_0 = C_0(q, N, \|a\|_{\sigma_q}) > 0$ such that the functional I_λ satisfies the $(PS)^*_c$ condition at any level*

$$c < \frac{1}{2(N-1)} \frac{1}{\|b\|_\infty^{N-2}} S_{2_*,\partial}^{N-1} - C_0 \lambda^{2/(2-q)}.$$

Proof. Let $(u_n) \subset V^n \oplus W^n$ be a $(PS)^*_c$ sequence. As in the proof of Proposition 4.3, we may assume that $u_n \rightharpoonup u$ weakly in X, with $I'_\lambda(u) = 0$. Since $I_\lambda(u) = I_\lambda(u) - (1/2_*)I'_\lambda(u)u$, we obtain

$$I_\lambda(u) = \frac{1}{2(N-1)}\|u\|^2 - \lambda\left(\frac{2_* - q}{2_* q}\right) \int_{\mathbb{R}^N_+} K(x)a(x)|u|^q dx. \tag{4.2}$$

We now set $z_n := (u_n - u)$ and argue as in Proposition 3.2 to get

$$\lim_{n\to\infty} \|z_n\|^2 = \gamma = \lim_{n\to\infty} \int_{\mathbb{R}^{N-1}} K(x',0)b(x')|z_n|^{2_*} dx'.$$

for some $\gamma \geq 0$. If $\gamma > 0$, then

$$\gamma \geq \frac{1}{\|b\|_\infty^{\frac{N-2}{}}} S_{2_*,\partial}^{N-1}. \tag{4.3}$$

On the other, we infer from Brezis-Lieb's lemma that

$$c + o(1) = I_\lambda(u_n) = \frac{1}{2}\|z_n\|^2 - \frac{1}{2_*}\int_{\mathbb{R}^{N-1}} K(x',0)b(x')|z_n|^{2_*}dx' + I_\lambda(u) + o(1).$$

Passing to the limit, using (4.2), Hölder's inequality and (4.3), we conclude that

$$c \geq \frac{1}{2(N-1)} \frac{1}{\|b\|_\infty^{N-2}} S_{2_*,\partial}^{N-1} + g(\|u\|), \tag{4.4}$$

where

$$g(t) := \frac{t^2}{2(N-1)} - \lambda\gamma_q t^q, \qquad t > 0,$$

and $\gamma_q := \|a\|_{\sigma_q} S_{q\sigma_q'}^{-q/2}(2_* - q)/(2_* q)$. Setting

$$C_0 := \left(\frac{2-q}{2q}\right)\frac{1}{(N-1)}\left[(N-1)q\gamma_q\right]^{2/(2-q)},$$

a straightforward computation shows that $g(t) \geq -C_0\lambda^{2/(2-q)}$, for any $t > 0$. Hence, we infer from (4.4) that

$$c \geq \frac{S_{2_*,\partial}^{N-1}}{2(N-1)\|b\|_\infty^{N-2}} - C_0\lambda^{2/(2-q)},$$

which does not make sense. This contradiction proves that $\gamma = 0$ or, equivalently, $u_n \to u$ strongly in X. □

We finish this section with an important tool for the proof of Theorem 1.3.

Lemma 4.5. *Suppose that $a \geq 0$ and set*

$$\mu_n := \sup_{\{u \in V_n \oplus W \,:\, \|u\| \leq 1\}} \int_{\mathbb{R}^N_+} K(x)a(x)|u|^q\,dx.$$

Then $\mu_n \to 0$, as $n \to \infty$.

Proof. Since $(\mu_n) \subset [0, +\infty)$ is nonincreasing, we have that $\mu_n \to \mu_0 \geq 0$, as $n \to \infty$. Let $(u_n) \subset V_n \oplus W$ be such that $\|u_n\| = 1$ and

$$\frac{\mu_0}{2} \leq \frac{\mu_n}{2} \leq \int_{\mathbb{R}^N_+} K(x)a(x)|u_n|^q\,dx. \tag{4.5}$$

We may assume that $u_n \rightharpoonup u = v + w$ weakly in X, with $v \in V$ and $w \in W$. The orthogonal decomposition and the definition of V_n imply that $\langle u_n, \varphi_k^V\rangle = 0$, for any fixed $k \in \mathbb{N}$ and $n > k$. So,

$$0 = \lim_{n\to+\infty}\langle u_n, \varphi_k^V\rangle = \langle u, \varphi_k^V\rangle = \langle v, \varphi_k^V\rangle,$$

and therefore $v = 0$ or, equivalently, $u = w$. Using Lebesgue Theorem as in the proof of Proposition 2.3, we conclude that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N_+} K(x)a(x)|u_n|^q dx = \int_{\mathbb{R}^N_+} K(x)a(x)|w|^q dx = 0,$$

since $w \in W$. The above expression and (4.5) imply that $\mu_0 = 0$. □

We are ready to prove Theorem 1.3.

Proof of Theorem 1.3. It follows from Lemma 4.5 and Proposition 2.1 that, for any $u \in V_n \oplus W$, there holds

$$I_\lambda(u) \geq \frac{1}{2}\|u\|^2 - \frac{\lambda}{q}\mu_n\|u\|^q - \frac{1}{p}\|b\|_\infty S_{p,\partial}^{-p/2}\|u\|^p.$$

Hence,

$$I_\lambda(u) \geq \frac{1}{4}\|u\|^2 - \frac{\lambda}{q}\mu_n\|u\|^q, \qquad \forall\, u \in \overline{B_{\rho_1}(0)} \cap (V_n \oplus W), \tag{4.6}$$

where $\rho_1 := \left[pS_{p,\partial}^{p/2}/(4\|b\|_\infty)\right]^{1/(p-2)}$. Since $\mu_n \to 0$, there exists $n_1 \in \mathbb{N}$ such that

$$R_n := \left(\frac{4\lambda\mu_n}{q}\right)^{1/(2-q)} < \rho_1, \qquad \forall\, n \geq n_1.$$

Using (4.6) we can check that $I_\lambda(u) \geq 0$, for any $u \in V_n \oplus W$ such that $\|u\| = R_n$. This proves that (A_1) holds.

In order to verify (A_2) we notice that, for $n \geq n_1$,

$$I_\lambda(u) \geq -\frac{\lambda}{q}\mu_n R_n^q \qquad \forall\, u \in \overline{B_{R_n}(0)} \cap (V_n \oplus W).$$

Thus,

$$0 \geq b_n = \inf\left\{I_\lambda(u) \,:\, u \in \overline{B_{R_n}(0)} \cap (V_n \oplus W)\right\} \geq -\frac{\lambda}{q}\mu_n R_n^q.$$

Since the right-hand side above goes to 0, as $n \to +\infty$, we conclude that (A_2) holds.

Given $u \in V^n$, we have that $\int_{\mathbb{R}^N_+} K(x)a(x)|u|^q dx = 0$ if, and only if, $u = 0$. Hence, this integral defines a norm in the finite dimensional subspace V^n. The equivalence of norms provides $0 < \beta_n < (8\mu_n)/q$, such that

$$\beta_n\|u\|^q \leq \int_{\mathbb{R}^N_+} K(x)a(x)|u|^q\, dx, \qquad \forall\, u \in V^n.$$

Hence, we can argue as above to get

$$I_\lambda(u) \leq \|u\|^2 - \frac{\lambda}{q}\beta_n\|u\|^q, \qquad \forall\, u \in \overline{B_{\rho_2}(0)} \cap V^n, \tag{4.7}$$

where $\rho_2 := \left[pS_{p,\partial}^{p/2}/(2\|b\|_\infty)\right]^{1/(p-2)}$. Since $\beta_n \to 0$, there exists $n_2 \in \mathbb{N}$ such that

$$r_n := \left(\frac{\lambda\beta_n}{2}\right)^{1/(2-q)} < \rho_2, \qquad \forall\, n \geq n_2.$$

A straightforward computation shows that the function $g(t) := t^2 - (\lambda/q)\beta_n t^q$, for $t > 0$, attains its minimum value at $t = r_n$ and

$$d_n := g(r_n) = -\frac{(2-q)}{2q}\lambda\beta_n\left(\frac{\lambda\beta_m}{2}\right)^{q/(2-q)} < 0.$$

Hence, we infer from (4.7) that $I_\lambda(u) \leq d_n$, for any $u \in \partial B_{r_n}(0) \cap V^n$ and $n \geq n_2$.

We now define $n_0 := \max\{n_1, n_2\}$. According to the above considerations, I_λ verifies (A_1) and (A_2). Moreover, since $\beta_n < (8\mu_n)/q$, we have that $r_n < R_n$ and therefore (A_3) also holds. It remains to check (A_4). If $2 < p < 2_*$ or $p = 2_*$ and $b \leq 0$, condition (A_4) is a direct consequence of Proposition 4.3. If $p = 2_*$ but we have no information about the sign of b, we have compactness at any negative level provided $\lambda > 0$ is small in such way that

$$C_0\lambda^{2/(2-q)} < \frac{1}{2(N-1)}\frac{1}{\|b\|_\infty^{N-2}} S_{2_*,\partial}^{N-1},$$

where $C_0 > 0$ comes from Proposition 4.4. In any case, we may invoke Theorem 4.1 to obtain infinitely many critical points for I_λ. □

Remark 4.6. *Suppose that $b \leq 0$ and let $(u_n) \subset X$ be a sequence of solutions given by Theorem 1.3. If we denote by $c_n = I_\lambda(u_n) \in [b_n, 0]$ the energy of the solutions, we can use $I'_\lambda(u_n)u_n = 0$ and an easy computation to get*

$$c_n = \lambda\left(\frac{1}{2}-\frac{1}{q}\right)\int_{\mathbb{R}^N_+} K(x)a(x)|u_n|^q\,dx + \left(\frac{1}{2}-\frac{1}{p}\right)\int_{\mathbb{R}^{N-1}} K(x',0)b(x')|u_n|^p\,dx'.$$

Hence,

$$0 \leq \lambda\int_{\mathbb{R}^N_+} K(x)a(x)|u_n|^q dx \leq -\frac{2q}{(2-q)}c_n$$

and

$$\frac{2p}{(p-2)}c_n \leq \int_{\mathbb{R}^{N-1}} K(x',0)b(x')|u_n|^p dx' \leq 0.$$

Recalling that $b_n \to 0$, the above inequalities and $I'_\lambda(u_n)u_n = 0$ imply that $\|u_n\| \to 0$.

In order to prove Theorem 1.4 we recall that $\Omega_b^+ = \{x' \in \mathbb{R}^{N-1} : b(x') > 0\}$ and redefine the subspace W in the following way:

$$W := \{u \in X : u(x') = 0 \text{ for a.e. } x' \in \operatorname{int}(\Omega_b^+)\}.$$

As before, V is the orthogonal complement of W in X, in such way that $X = V \oplus W$.

Proof of Theorem 1.4. Setting

$$\mu_n := \sup_{\{u\in V_n\oplus W:\ \|u\|\leq 1\}} \int_{\mathbb{R}^{N-1}} K(x',0)b(x')|u|^p dx',$$

we can use $2 < p < 2_*$ and the same argument of Lemma 4.5 to conclude that $\mu_n \to 0$, as $n \to +\infty$.

If $u \in V_n \oplus W$, then

$$I_\lambda(u) \geq \frac{1}{2}\|u\|^2 - \frac{\lambda}{q}S_{q\sigma'_q}^{-q/2}\|a\|_{\sigma_q}\|u\|^q - \frac{\mu_n}{p}\|u\|^p,$$

and therefore

$$I_\lambda(u) \geq \frac{1}{4}\|u\|^2 - \frac{\mu_n}{p}\|u\|^p, \qquad \forall\, u \in V_n \oplus W, \quad \|u\| \geq \rho_1, \tag{4.8}$$

where $\rho_1 := \left[4\lambda\|a\|_{\sigma_q}S_{q\sigma'_q}^{-q/2}/q\right]^{1/(2-q)}$. Since $\mu_n \to 0$, there exists $n_1 \in \mathbb{N}$ such that

$$r_n := \left(\frac{p}{8\mu_n}\right)^{1/(p-2)} > \rho_1, \qquad \forall\ n \geq n_1.$$

So, using (4.8) we conclude that

$$b_n = \inf\{I_\lambda(u) :\ u \in V_n \oplus W;\ \|u\| = r_n\} \geq \frac{1}{8}r_n^2.$$

It follows from $\mu_n \to 0$ and the definition of r_n that $(\widetilde{A_2})$ holds.

Arguing as in Theorem 1.3, we have that $\int_{\mathbb{R}^{N-1}} K(x',0)b(x')|u|^p dx'$ defines a norm in the finite dimensional subspace V^n. Then, there exists $0 < \beta_n < 8\mu_n$ such that

$$\beta_n \|u\|^p \le \int_{\mathbb{R}^{N-1}} K(x',0)b(x')|u|^p dx', \qquad \forall\, u \in V_n.$$

Hence,

$$I_\lambda(u) \le \|u\|^2 - \frac{\beta_n}{p}\|u\|^p \qquad u \in V^n, \ \|u\| \ge \rho_2,$$

where $\rho_2 := \left(2\lambda S_{q\sigma_q'}^{-q/2}\|a\|_{\sigma_q}/q\right)^{1/(2-q)}$. Setting

$$R_n := \max\left\{2\rho_2, \left(\frac{p}{\beta_n}\right)^{1/(p-2)}\right\},$$

a straightforward computation shows that $I_\lambda(u) \le 0$, whenever $u \in V^n$ satisfies $\|u\| = R_n$. Since $R_n > r_n$, we conclude that requirement $(\widetilde{A_3})$ if fulfilled.

Since $(PS)_c^*$ implies $(PS)_c$ condition, the proof of $(\widetilde{A_4})$ is analogous to that of Proposition 4.3. So, we may invoke Theorem 4.2 to obtain a sequence of solutions $(u_n) \subset X$ such that $I_\lambda(u_n) = c_n \to +\infty$, as $n \to +\infty$. Since

$$c_n = I_\lambda(u_n) \le \frac{1}{2}\|u_n\|^2 + \frac{\lambda}{q}\|a\|_{\sigma_q} S_{q\sigma_q'}^{-q/2}\|u_n\|^q + \frac{1}{p}\|b\|_\infty S_{p,\partial}^{-p/2}\|u_n\|^p,$$

we conclude that $\|u_n\| \to +\infty$. The theorem is proved. □

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