ON A WEIGHTED TRACE EMBEDDING AND APPLICATIONS TO CRITICAL BOUNDARY PROBLEMS

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Abstract. We prove a weighted Sobolev trace embedding in the upper half-space and give its best constant. This embedding can be employed to study a number of critical boundary problems. In this direction, we obtain existence and nonexistence results for a class of semilinear elliptic equations with nonlinear boundary conditions involving critical growth. These equations are closely related to the study of self-similar solutions for nonlinear reaction-diffusion equations.

1. Introduction and main results

In this paper, we show a weighted Sobolev trace embedding and also present its best constant. As a consequence, we obtain existence and nonexistence results for semilinear elliptic equations in the upper half-space with nonlinear critical boundary conditions.

Before stating the embedding, let us set some notation. Consider $N \geq 3$ and $\mathbb{R}^N_+ := \{x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N : x_N > 0\}$. If we denote by $C_c^\infty(\overline{\mathbb{R}^N_+})$ the space of infinitely differentiable functions with compact support in $\overline{\mathbb{R}^N_+}$, we can define $\mathcal{D}^{1,2}_K(\mathbb{R}^N_+)$ as being the closure of $C_c^\infty(\overline{\mathbb{R}^N_+})$ with respect to the norm

$$\|u\| := \left(\int_{\mathbb{R}^N_+} K(x) |\nabla u|^2 dx \right)^{1/2}, \tag{1.1}$$

where $K(x) := \exp(|x|^2/4)$. For simplicity, we denote this space by X and define, for any $2 \leq s \leq 2^* := 2N/(N-2)$, the weighted Lebesgue space

$$L^s_K(\mathbb{R}^N_+) := \left\{ u \in L^s(\mathbb{R}^N_+) : \|u\|_s^s := \int_{\mathbb{R}^N_+} K(x)|u|^s dx < \infty \right\}.$$

In [12, Lemma 2.2] it was proved that the embedding $\mathcal{D}^{1,2}_K(\mathbb{R}^N_+) \hookrightarrow L^s_K(\mathbb{R}^N_+)$ is continuous for $s \in [2, 2^*]$ and compact for $s \in [2, 2^*)$.

Concerning the trace embedding, we recall that $2_* := 2(N-1)/(N-2)$ is the critical exponent of the trace Sobolev embedding $\mathcal{D}^{1,2}(\mathbb{R}^N_+) \hookrightarrow L^{2_*}(\mathbb{R}^{N-1})$. As

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proved by P.L. Lions [22], its best constant S_0 is achieved and

$$S_0 := \inf_{\varphi\in\mathcal{D}^{1,2}(\mathbb{R}^N_+)\setminus\{0\}} \frac{\displaystyle\int_{\mathbb{R}^N_+} |\nabla\varphi|^2 dx}{\left(\displaystyle\int_{\mathbb{R}^{N-1}} |\varphi|^{2_*} dx'\right)^{2/2_*}} = \frac{N-2}{2}\sigma_{N-1}^{1/(N-1)}, \tag{1.2}$$

where σ_{N-1} is the volume of the $(N-1)$ dimensional sphere. The extremal functions, found independently by Escobar [10] and Beckner [5], are given by

$$U_\varepsilon(x', x_N) := \frac{\varepsilon^{(N-2)/2}}{[|x'|^2 + (x_N+\varepsilon)^2]^{(N-2)/2}},$$

for any $\varepsilon > 0$ and $x' := (x_1, x_2, \dots, x_{N-1}) \in \mathbb{R}^{N-1}$, where we identify the boundary of the upper half-space $\partial\mathbb{R}^N_+$ with $\mathbb{R}^{N-1}$.

A natural question is whether we have trace embeddings from X into the space $L^s_K(\mathbb{R}^{N-1})$. By performing an interpolation approach in fractional Sobolev space, a partial answer was presented in [12, Lemma 2.4], where it was proved that this embedding holds for any $2 < s < 2_*$. In the first result of this paper we prove the following:

Theorem 1.1. *The Sobolev trace embedding $\mathcal{D}^{1,2}_K(\mathbb{R}^N_+) \hookrightarrow L^s_K(\mathbb{R}^{N-1})$ is continuous for $s \in [2, 2_*]$ and compact for $s \in [2, 2_*)$. Moreover, if $s = 2_*$, the best constant of this embedding is given by*

$$S_K := \inf_{\varphi\in\mathcal{D}^{1,2}_K(\mathbb{R}^N_+)\setminus\{0\}} \frac{\displaystyle\int_{\mathbb{R}^N_+} K(x)|\nabla\varphi|^2 dx}{\left(\displaystyle\int_{\mathbb{R}^{N-1}} K(x',0)|\varphi|^{2_*} dx'\right)^{2/2_*}} = S_0.$$

The above result extends that of [12], since we now cover the natural range for the trace embedding by considering also the critical case $s = 2_*$. Moreover, the proof presented here is different (and simpler) from the original one. Theorem 1.1 also complements previous weighted embedding results which can be found, for instance, in [1, 13, 14, 21, 23, 4, 17] and references therein. For example, the growth of our weight K is not of log or polynomial type neither belongs to the Muckenhoupt class A_r.

The critical trace embedding in Theorem 1.1 can be employed to study a number of semilinear equations with nonlinear boundary conditions with critical growth. In this direction, we show existence and nonexistence of solutions for the problem

$$(P) \qquad \begin{cases} -\Delta u - \frac{1}{2}(x\cdot\nabla u) &= \lambda u + |u|^{2^*-2}u, \quad \text{in } \mathbb{R}^N_+, \\ \frac{\partial u}{\partial \nu} &= |u|^{2_*-2}u, \quad \text{on } \mathbb{R}^{N-1}, \end{cases}$$

where ν is the unit outer normal on the boundary of the upper half-space.

Before presenting our results we connect the above problem with the functional space X. As observed by Escobedo and Kavian in [11], since the exponential-type weight $K(x) = \exp(|x|^2/4)$ verifies $\nabla K(x) = \frac{1}{2}xK(x)$, the first equation in (P) can be written as

$$-\mathrm{div}(K(x)\nabla u) = \lambda K(x)u + K(x)|u|^{2^*-2}u \text{ in } \mathbb{R}^N_+.$$

Thus, X is the natural space to treat problem (P). The embedding result of Theorem 1.1 shows that the energy functional

$$u \mapsto \frac{1}{2}\|u\|^2 - \frac{\lambda}{2}\|u\|_2^2 - \frac{1}{2^*}\|u\|_{2^*}^{2^*} - \frac{1}{2_*}\int_{\mathbb{R}^{N-1}} K(x',0)|u|^{2_*}dx'$$

belongs to $C^1(X,\mathbb{R})$ and its critical points are weak solutions of (P).

The starting point of our variational approach is to establish the natural range for the parameter λ in order to obtain a solution. By adapting some known arguments we are able to get a Pohozaev identity and a Hardy-type inequality for the problem (P) and, by combining these two results, we have the following:

Theorem 1.2. *If $u \in C^2(\mathbb{R}^N_+) \cap X$ is a nonzero solution of (P), then $\lambda \geq (N/4)$. Moreover, if $u \in X$ is a positive solution of (P), then $\lambda < (N/2)$.*

Closely related to problem (P) is the study of self-similar solutions for the reaction-diffusion problem

$$\text{(1.3)} \qquad \begin{cases} v_t - \Delta v &= |v|^{2^*-2}v, \quad \text{in } \mathbb{R}^N_+ \times (0,+\infty), \\ \frac{\partial v}{\partial \nu} &= |v|^{2_*-2}v, \quad \text{on } \mathbb{R}^{N-1} \times (0,+\infty). \end{cases}$$

More precisely, it is well-known that the problem (1.3) satisfies a scale invariance property and if we look for self-similar solutions of problem (1.3), namely solutions with the special form

$$v(x,t) = t^{-\bar{\lambda}}u(t^{-1/2}x), \quad x \in \mathbb{R}^N_+, \quad t > 0,$$

then the profile u satisfies the problem (P) with $\bar{\lambda} = (N-2)/4$. For further details and results about nonlinear boundary parabolic problems and self-similarity, we refer the reader to [15, 16, 25, 19, 2]. Since $\bar{\lambda} < (N/4)$, as a direct consequence of Theorem 1.2 we obtain a nonexistence result of self-similar solutions for (1.3):

Corollary 1.3. *The problem* (1.3) *does not have self-similar solutions with profile belonging to X.*

At this point it is important to emphasize the difference between the above result and its subcritical counterpart, namely problem (1.3) with the terms $|v|^{2^*-2}v$ and $|v|^{2_*-2}v$ replaced by $|v|^{q-2}v$ and $|v|^{p-2}v$, respectively, where $2 < q < 2^*$ and $2 < p < 2_*$. In this case, it was proved in [12] that the problem has infinitely many self-similar solutions whenever the parameter λ belongs to $\mathbb{R} \setminus \Sigma$, where Σ is an enumerable set which is connected with the eigenvalues of the linearized problem and can be viewed as a nonresonant set for the subcritical problem. Also in the subcritical case, the authors in [12] proved that the stationary problem (P) has a positive solution for $\lambda < (N/2)$. Again, the critical case is more delicate, since we do not have compact embeddings. However, we are able to obtain an existence result if λ is near the first eigenvalue of the linearized problem. More specifically, we shall prove the following:

Theorem 1.4. *If $N \geq 3$, then there exists $\delta > 0$ such that problem (P) has a positive solution if $(N/2) - \delta < \lambda < (N/2)$.*

For the proof we first obtain a local compactness result for the energy functional. Afterwards, in the main step, we use the first eigenfunction of the linearized problem to show that the Mountain Pass level of the functional belongs to the compactness range. At this point we use the fact that λ is near $(N/2)$.

In our last result we analyze the effect of the absence of the power-type reaction term inside the domain. Actually, we remove this term and consider the following critical problem

$$(\widetilde{P}) \qquad \left\{ \begin{array}{rcll} -\Delta u - \frac{1}{2}(x \cdot \nabla u) & = & \lambda u, & \text{in } \mathbb{R}^N_+, \\ \frac{\partial u}{\partial \nu} & = & |u|^{2_*-2}u, & \text{on } \mathbb{R}^{N-1}. \end{array} \right.$$

The subcritical version of the above problem was considered in [12], where the authors obtained a positive solution for $\lambda < (N/2)$ and $N \geq 3$. Here, due to the criticality of $(\widetilde{P})$, we obtain a result in higher dimensions and with stronger restrictions on the parameter λ. More specifically, we prove the following:

Theorem 1.5. *If $N \geq 7$, then problem $(\widetilde{P})$ has a positive solution if*

$$\lambda_N^* := \frac{N}{4} + \frac{(N-4)}{8} < \lambda < \frac{N}{2}.$$

In the proof, we follow the ideas introduced by Brezis and Nirenberg in [7]. As before, the main point is to show that the Mountain Pass level of the energy functional belongs to the range where the Palais-Smale condition holds. This is done by performing some carefull estimates on the asymptotic behavior of a suitable cut of the function U_ε. The restrictions on dimension and the lower bound for λ are of technical nature. Unfortunately, we do not know what happens if $3 \leq N \leq 6$ or even if $N \geq 7$ and $(N/4) < \lambda \leq \lambda_N^*$.

In what follows we quote some papers which deal with critical nonlinear boundary conditions. The results are not comparable with ours but present some similar features. Consider the existence of solution for

$$-\Delta u = f(x,u), \text{ in } \Omega, \qquad \frac{\partial u}{\partial \nu} = g(x,u), \text{ on } \partial\Omega, \qquad u > 0, \text{ in } \Omega,$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain. In the case that Ω is a ball, the authors in [24] obtained one solution for $f(x,u) = u^{2^*-1}$, $g(x,u) = -\delta u^{2_*-1} - h(x,u)$, with $\delta \geq 0$ and the subcritical function h verifying some technical conditions on $h'(x,0)$ (see also [9] for some results in the case that $g(x,u) \leq 0$). In [8], the author supposed that $f(x,u) = \alpha(x)u^{2^*-1}$, $g(x,u) = \beta(x)u^{2_*-1}$ and obtained solution in the case that the points of maximum of the potentials are related with the set where the median curvature of the boundary is positive. He also proved that the problem has no solution if $\alpha, \beta > 0$. In [27], the authors consider $f(x,u) = u(u^{2^*-2} - 1)$, $g(x,u) = \mu u^{2_*-1}$ and use the fibering method to obtain two solutions for small $\mu > 0$. For unbounded domains, we can cite [18], where it is assumed that Ω is the complementary of a set with compact boundary, $f(x,u) = u^{2^*-1}$ and $g(x,u) = -\alpha(x)u - \beta(x)u^{2_*-1} + \mu\rho(x)u^r$, with $0 < r < 1$ and $\beta \geq 0$. The author obtained the existence of one solution for small $\mu > 0$.

The paper is structured as follows: in the next section we prove Theorem 1.1. The third section is devoted to the nonexistence results. Theorems 1.4 and 1.5 are proved in Sections 4 and 5, respectively.

2. The trace embedding

In this section we present the proof of Theorem 1.1. For this purpose, we need some notation which will be used throughout the paper. We recall that X denotes the closure of $C_c^\infty(\overline{\mathbb{R}^N_+})$ with respect to the norm

$$\|u\| := \left(\int_{\mathbb{R}^N_+} K(x)|\nabla u|^2 dx \right)^{1/2}$$

and

$$L^s_K(\mathbb{R}^N_+) := \left\{ u \in L^s(\mathbb{R}^N_+) : \int_{\mathbb{R}^N_+} K(x)|u|^s dx < \infty \right\},$$

for any $s \in [2, 2^*]$. We also use the following notation

$$|u|_s := \left(\int_{\mathbb{R}^N_+} |u|^s dx \right)^{1/s}, \quad \|u\|_s := \left(\int_{\mathbb{R}^N_+} K(x)|u|^s dx \right)^{1/s}$$

and

$$\| u \|_{2_*} := \left(\int_{\mathbb{R}^{N-1}} K(x',0)|u(x',0)|^{2_*} dx' \right)^{1/2_*}.$$

In order to be more concise, we write only $\int f$ to denote $\int_{\mathbb{R}^N_+} f(x)dx$. The points $x \in \mathbb{R}^N$ will be written as $x = (x', x_N)$, with $x' = (x_1, \ldots, x_{N-1}) \in \mathbb{R}^{N-1}$ and $x_N \in \mathbb{R}$.

Proof of Theorem 1.1. Let $\rho > 0$ and $\phi_\rho \in C^\infty(\overline{\mathbb{R}^N_+}, [0,1])$ be such that $\phi_\rho \equiv 1$ in $B_{\rho/2}(0) \cap \overline{\mathbb{R}^N_+}$ and $\phi_\rho \equiv 0$ in $\overline{\mathbb{R}^N_+} \setminus B_\rho(0)$. For any $\varepsilon > 0$, we set $\varphi_\epsilon := \phi_\rho U_\varepsilon$ and notice that, since U_ε is a minimizer of S_0, then

$$\frac{\displaystyle\int |\nabla \varphi_\varepsilon|^2}{\left(\displaystyle\int_{\mathbb{R}^{N-1}} |\varphi_\varepsilon|^{2_*} dx' \right)^{2/2_*}} = S_0 + o(\varepsilon),$$

as $\varepsilon \to 0^+$. Thus, if $K_\rho := K(\rho, 0, \cdots, 0)$, we can use $K(x) \geq 1$ and $K(x) \leq K_\rho$ in the support of φ_ε to get

$$S_K \leq \frac{\|\varphi_\varepsilon\|^2}{\| \varphi_\varepsilon \|^2_{2_*}} \leq K_\rho \frac{\displaystyle\int_{\mathbb{R}^N_+} |\nabla \varphi_\epsilon|^2 dx}{\left(\displaystyle\int_{\mathbb{R}^{N-1}} |\varphi_\varepsilon|^{2_*} dx' \right)^{2/2_*}} = K_\rho S_0 + o(\varepsilon).$$

Taking the limit as $\rho \to 0^+$ and $\varepsilon \to 0^+$, we obtain $S_K \leq S_0$.

Given $u \in C_c^\infty(\overline{\mathbb{R}^N_+})$, let $b, > 0$ be such that $u(x', x_N) = 0$ whenever $2|x'| > b$ or $2x_N > b$. If we set $\Omega := \{x' \in \mathbb{R}^{N-1} : |x'| < b\}$, we conclude that the support of u is contained in $\Omega \times [0, b]$. Moreover $K^{1/2}u \in C_c^\infty(\overline{\mathbb{R}^N_+})$ and a straightforward computation provides

$$\int |\nabla (K(x)^{1/2} u)|^2 = \int K(x)|\nabla u|^2 + B, \tag{2.1}$$

with

$$B := \int_0^b \int_\Omega \nabla \left(K(x)^{1/2} u^2 \right) \cdot \nabla \left(K(x)^{1/2} \right) dx' dx_N.$$

We claim that $B \leq 0$ for all $u \in C_c^\infty(\overline{\mathbb{R}^N_+})$. If this is true, we can (2.1), Fatou's lemma and a density argument to conclude that

$$\int |\nabla(K(x)^{1/2}u)|^2 \leq \int K(x)|\nabla u|^2, \qquad \forall\, u \in \mathcal{D}^{1,2}_K(\mathbb{R}^N_+), \tag{2.2}$$

and therefore $K^{1/2}u \in \mathcal{D}^{1,2}(\mathbb{R}^N_+)$ for any $u \in \mathcal{D}^{1,2}_K(\mathbb{R}^N_+)$. Since $K(x) \geq 1$, it follows from the definition of S_0 that

$$\begin{aligned} \int |\nabla(K(x)^{1/2}u)|^2 &\geq S_0 \left(\int_{\mathbb{R}^{N-1}} (K(x',0)^{1/2}|u|)^{2_*} dx' \right)^{2/2_*} \\ &\geq S_0 \left(\int_{\mathbb{R}^{N-1}} K(x',0)|u|^{2_*} dx' \right)^{2/2_*}. \end{aligned}$$

Hence, we can use (2.2) to obtain

$$\int K(x)|\nabla u|^2 \geq S_0 \left(\int_{\mathbb{R}^{N-1}} K(x',0)|u|^{2_*} dx' \right)^{2/2_*}, \qquad \forall\, u \in \mathcal{D}^{1,2}_K(\mathbb{R}^N_+).$$

This inequality proves the continuous trace embedding $\mathcal{D}^{1,2}_K(\mathbb{R}^N_+) \hookrightarrow L^{2_*}_K(\mathbb{R}^{N-1})$ and that $S_K \geq S_0$. Since we already know that $S_K \leq S_0$, we conclude that $S_K = S_0$.

It remains to prove that $B \leq 0$. For this purpose, for $x' \in \mathbb{R}^{N-1}$ we set

$$A(x') = \int_0^b \left(K(x)^{1/2}u^2 \right)_{x_N} \left(K(x)^{1/2} \right)_{x_N} dx_N,$$

where $f_{x_N} := \frac{\partial f}{\partial x_N}$, for any $f \in C^\infty(\mathbb{R}^N_+)$. It follows from the Divergence Theorem that

$$\begin{aligned} B &= \int_0^b \int_\Omega \nabla_{x'} \left(K(x)^{1/2}u^2 \right) \cdot \nabla_{x'} \left(K(x)^{1/2} \right) dx' dx_N + \int_\Omega A(x') dx' \\ &= -\int_0^b \int_\Omega K(x)^{1/2}u^2 \Delta_{x'} \left(K(x)^{1/2} \right) dx' dx_N + \int_\Omega A(x') dx'. \end{aligned}$$

Since

$$\Delta_{x'} \left(K(x)^{1/2} \right) = \left(\frac{N-1}{4} + \frac{|x'|^2}{16} \right) K(x) \geq 0,$$

it is sufficient to check that $A(x') \leq 0$, for any $x' \in \mathbb{R}^{N-1}$. By integrating the equality

$$(K(x)^{1/2}u^2)_{x_N}(K(x)^{1/2})_{x_N} + \left(\frac{1}{4} + \frac{x_N^2}{16} \right) K(x)u^2 = \frac{1}{4} \left(x_N u^2 K(x) \right)_{x_N}$$

over the interval $[0,b]$, and recalling that $u(x',b) = 0$, we obtain

$$A(x') + \int_0^b \left(\frac{1}{4} + \frac{x_N^2}{16} \right) K(x)u^2 dx_N = \frac{1}{4} \int_0^b \left(x_N u^2 K(x) \right)_{x_N} dx_N = 0,$$

and therefore $A(x') \leq 0$. This finishes the proof of the claim.

We now prove the continuity of the trace embedding for $s = 2$. Given $u \in C_c^\infty(\overline{\mathbb{R}^N_+})$, we can use the Fubini Theorem to get

$$\begin{aligned} \int (K(x)u(x)^2)_{x_N} &= \int_{\mathbb{R}^{N-1}} \int_0^\infty (K(x)u^2)_{x_N} dx_N dx' \\ &= -\int_{\mathbb{R}^{N-1}} K(x',0)u(x',0)^2 dx'. \end{aligned}$$

Hence, recalling that $x_N > 0$ in $\mathbb{R}^N_+$ and $-2ab \le a^2 + b^2$, we obtain

$$\begin{aligned} \int_{\mathbb{R}^{N-1}} K(x',0)|u|^2 dx' &= -\int \frac{x_N}{2} K(x)u^2 - \int 2K(x)uu_{x_N} \\ &\le \int K(x)u^2 + \int K(x)u_{x_N}^2 \\ &\le c_1 \int K(x)|\nabla u|^2 + \int K(x)|\nabla u|^2, \end{aligned}$$

where $c_1 > 0$ comes from the embedding $\mathcal{D}_K^{1,2}(\mathbb{R}^N_+) \hookrightarrow L_K^2(\mathbb{R}^N_+)$. It follows that

$$\int_{\mathbb{R}^{N-1}} K(x',0)|u|^2 dx' \le (c_1 + 1) \int K(x)|\nabla u|^2, \qquad \forall\, u \in C_c^\infty(\overline{\mathbb{R}^N_+}).$$

The same inequality holds in $\mathcal{D}_K^{1,2}(\mathbb{R}^N_+)$ by density, and therefore we have proved the continuous embedding $\mathcal{D}_K^{1,2}(\mathbb{R}^N_+) \hookrightarrow L_K^2(\mathbb{R}^{N-1})$. The embedding for $s \in (2, 2_*)$ easily follows by interpolation. The compactness can be proved as in [12, Lemma 2.4] and we will omit the details here. □

3. The nonexistence result

In this section we prove our nonexistence result for problem (P). We first recall a basic result about the linear problem associated to (P) (see [12]). The first eigenfunction φ_1 of the linear problem

$$(3.1) \qquad -\Delta u - \frac{1}{2}(x \cdot \nabla u) = \lambda u, \quad x \in \mathbb{R}^N_+, \qquad \frac{\partial u}{\partial \nu} = 0, \quad x \in \partial\mathbb{R}^N_+,$$

can be chosen positive and the first eigenvalue λ_1 is characterized by

$$(3.2) \qquad \lambda_1 := \frac{N}{2} = \inf_{u \in X \setminus \{0\}} \frac{\displaystyle\int K(x)|\nabla u|^2}{\displaystyle\int K(x)u^2}.$$

In the sequel we use a truncation argument (see [20]) in order to obtain a Pohozaev type result for problem (P).

Proposition 3.1 (Pohozaev Identity). *If $u \in C^2(\mathbb{R}^N_+) \cap X$ is a solution of problem (P), then*

$$(3.3) \qquad |\nabla u|_2^2 - |u|_{2^*}^{2^*} - \int_{\mathbb{R}^{N-1}} |u(x',0)|^{2_*} dx' = \left(\lambda - \frac{N}{4}\right)|u|_2^2$$

and

$$(3.4) \qquad \frac{N-2}{2}\left(|\nabla u|_2^2 - |u|_{2^*}^{2^*} - \int_{\mathbb{R}^{N-1}} |u(x',0)|^{2_*} dx'\right) = \frac{1}{2}\left(\lambda N|u|_2^2 - \int (x \cdot \nabla u)^2\right).$$

Proof. Let $\phi \in C^\infty(\mathbb{R},[0,1])$ be such that $\phi \equiv 1$ in $(-\infty,1]$, $\phi \equiv 0$ in $[4,+\infty)$ and $|\phi'|_\infty \leq M$. If we set $\phi_k(x) := \phi(|x|^2/k^2)$, it follows from the Divergence Theorem that

$$\begin{aligned} -\int (\phi_k u)\Delta u &= \int u(\nabla \phi_k \cdot \nabla u) + \int \phi_k |\nabla u|^2 - \int_{\mathbb{R}^{N-1}} \phi_k u \frac{\partial u}{\partial \nu} dx' \\ &= \frac{2}{k^2} \int \phi' \left(\frac{|x|^2}{k^2} \right) (x \cdot \nabla u) u + \int \phi_k |\nabla u|^2 - \int_{\mathbb{R}^{N-1}} \phi_k u \frac{\partial u}{\partial \nu} dx'. \end{aligned}$$

By using the boundary condition and the boundedness of ϕ' we get

$$-\int (\phi_k u)\Delta u = \int \phi_k |\nabla u|^2 - \int_{\mathbb{R}^{N-1}} \phi_k |u|^{2_*} dx' + O(k^{-2}), \tag{3.5}$$

as $k \to +\infty$.

For any $s \in [2,2^*]$, it follows from the Fubini Theorem together with the Divergence Theorem and the Fundamental Theorem of Calculus that

$$\begin{aligned} \int \mathrm{div}(\phi_k |u|^s x) &= \int_0^\infty \int_{\mathbb{R}^{N-1}} \mathrm{div}_{x'}(\phi_k |u|^s x') dx' dx_N \\ &\quad + \int_{\mathbb{R}^{N-1}} \int_0^\infty (\phi_k |u|^s x_N)_{x_N} dx_N \, dx' = 0. \end{aligned}$$

Since $\mathrm{div}(\phi_k |u|^s x) = N\phi_k |u|^s + |u|^s (x \cdot \nabla \phi_k) + s\phi_k |u|^{s-2} u (x \cdot \nabla u)$, we conclude that

$$\int \phi_k |u|^{s-2} u (x \cdot \nabla u) = -\frac{N}{s} \int \phi_k |u|^s + o(1), \tag{3.6}$$

as $k \to +\infty$. For proving (3.3) it is sufficient to multiply the first equation in (P) by $\phi_k u$, integrate both sides over $\mathbb{R}^N_+$, use (3.5) and (3.6), take $k \to +\infty$ and apply the Lebesgue Theorem.

We now proceed with the proof of the second statement. A direct computation yields

$$\phi_k (x \cdot \nabla u)\Delta u = \phi_k \mathrm{div}\,(F_1 - F_2) + \frac{N-2}{2} \phi_k |\nabla u|^2,$$

with $F_1 := (x \cdot \nabla u)\nabla u$ and $F_2 := x|\nabla u|^2/2$. Notice that

$$\begin{aligned} \int \mathrm{div}(\phi_k F_1) &= \int_0^\infty \int_{\mathbb{R}^{N-1}} \mathrm{div}_{x'}(\phi_k (x \cdot \nabla u) \nabla_{x'} u) dx' dx_N \\ &\quad + \int_0^\infty \int_{\mathbb{R}^{N-1}} (\phi_k (x \cdot \nabla u) u_{x_N})_{x_N} \, dx' dx_N \\ &= -\int_{\mathbb{R}^{N-1}} \phi_k (x' \cdot \nabla_{x'} u) u_{x_N} dx' \\ &= \int_{\mathbb{R}^{N-1}} \phi_k (x' \cdot \nabla_{x'} u) |u|^{2_*-2} u \, dx'. \end{aligned}$$

The same argument used in (3.6) gives

$$\int \mathrm{div}(\phi_k F_1) = -\frac{N-1}{2_*} \int_{\mathbb{R}^{N-1}} \phi_k |u|^{2_*} dx' + o(1),$$

as $k \to +\infty$. Moreover, since $\int \mathrm{div}(\phi_k F_2) = 0$, we conclude that

$$\int \mathrm{div}\,(\phi_k (F_1 - F_2)) = \int_{\mathbb{R}^{N-1}} \phi_k (x' \cdot \nabla_{x'} u) |u|^{2_*-2} u \, dx' + o(1).$$

Recalling that $\phi_k \mathrm{div}(F_i) = \mathrm{div}(\phi_k F_i) - \nabla\phi_k \cdot F_i$, we can use the above expression, the definition of ϕ_k, the boundedness of $|\phi'|_\infty$ and the Lebesgue Theorem to obtain

$$\int \phi_k \mathrm{div}\,(F_1 - F_2) = -\frac{N-1}{2_*} \int_{\mathbb{R}^{N-1}} \phi_k |u|^{2_*} dx' + o(1),$$

as $k \to +\infty$. For proving (3.4) it is sufficient to multiply the first equation in (P) by $\phi_k(x \cdot \nabla u)$, integrate both sides over $\mathbb{R}^N_+$, use (3.6), the above equality, let $k \to +\infty$ and apply the Lebesgue Theorem. □

Remark 3.2. *If u is a solution of problem (P), we can use* (3.3) *and* (3.4) *to get*

$$\int (x \cdot \nabla u)^2 = 2\left(\lambda + \frac{N(N-2)}{8}\right)|u|_2^2.$$

In particular, problem (P) does not have solution for any $\lambda \le \bar{\lambda} := (N/4) - (N^2/8)$.

We now establish a Hardy-type inequality which will play an important role in the proof of Theorem 1.2.

Proposition 3.3 (Hardy Inequality)**.** *If $N \ge 3$ then, for any $u \in X$, there holds*

$$\frac{N^2}{4} \int u^2 \le \int (x \cdot \nabla u)^2. \tag{3.7}$$

Proof. Suppose first that $u \in C_c^\infty(\overline{\mathbb{R}^N_+})$. Since $\mathrm{div}(u^2 x) = 2u(x \cdot \nabla u) + N u^2$, we can use the Fubini and Divergence Theorem to get

$$0 = \frac{1}{2} \int \mathrm{div}(u^2 x) = \int u(x \cdot \nabla u) + \frac{N}{2} \int u^2.$$

Hence

$$\frac{N}{2} \int u^2 \le \int |u|\,|\,(x \cdot \nabla u)\,| \le \left(\int u^2\right)^{1/2} \left(\int (x \cdot \nabla u)^2\right)^{1/2},$$

and the inequality follows.

For the general case we consider $(u_n) \subset X$ such that $\|u_n - u\| \to 0$. Since $K(x)|\nabla u_n|^2 \to K(x)|\nabla u|^2$ in $L^1(\mathbb{R}^N_+)$, there exists $g \in L^1(\mathbb{R}^N_+)$ such that, up to a subsequence, $K(x)|\nabla u_n(x)| \le g(x)$ a.e. in $\mathbb{R}^N_+$. Thus, $|x \cdot \nabla u_n(x)|^2 \le |x|^2 g(x) K(x)^{-1}$ a.e. in $\mathbb{R}^N_+$. This last function is integrable and therefore we can use the Lebesgue Theorem to obtain the desired inequality by approximation. □

We finish this section proving our nonexistence result.

Proof of Theorem 1.2. If we multiply (3.3) by $(N-2)/2$ and add it to (3.4) we obtain

$$\int (x \cdot \nabla u)^2 = 2\left(\lambda + \frac{N(N-2)}{8}\right)|u|_2^2.$$

This, together with the Hardy inequality (3.7), gives

$$\frac{N^2}{4}|u|_2^2 \le 2\left(\lambda + \frac{N(N-2)}{8}\right)|u|_2^2.$$

Hence, if $u \not\equiv 0$, we have that $\lambda \ge (N/4)$ and this proves the first statement of Theorem 1.2.

For the second one, we first notice that the first equation in (3.1) can be written as $-\mathrm{div}(K(x)\nabla u) = \lambda K(x)u$. Thus, if $\varphi_1 > 0$ is a λ_1-eigenfunction of this linear problem, then it is a critical point of the functional

$$J(v) = \frac{1}{2}\|v\|^2 - \frac{\lambda_1}{2}\|v\|_2^2, \quad v \in X.$$

For the same reason, the solution u of the problem (P) is a critical point of

$$I_\lambda(v) = \frac{1}{2}\|v\|^2 - \frac{\lambda}{2}\|v\|_2^2 - \frac{1}{2^*}\|v\|_{2^*}^{2^*} - \frac{1}{2_*}|||v|||_{2_*}^{2_*}, \quad v \in X.$$

The equality $0 = J'(\varphi_1)u = I'_\lambda(u)\varphi_1$ can be rewritten as

$$(\lambda_1 - \lambda)\int K(x)u\varphi_1 = \int K(x)u^{2^*-1}\varphi_1 + \int_{\mathbb{R}^{N-1}} K(x',0)u^{2_*-1}\varphi_1 dx'.$$

Since $\varphi_1 > 0$ we conclude that, if $u \geq 0$ is nonzero, then $\lambda < \lambda_1 = N/2$. □

4. Proof of Theorem 1.4

In this section we prove Theorem 1.4. Besides the Sobolev constant S_0 defined in (1.2), we also consider

$$S := \inf_{\varphi \in X\setminus\{0\}} \frac{\displaystyle\int K(x)|\nabla\varphi|^2}{\left(\displaystyle\int K(x)|\varphi|^{2^*}\right)^{2/2^*}} > 0,$$

which is positive due to the embedding $X \hookrightarrow L_K^{2^*}(\mathbb{R}_+^N)$. For obtaining a nonzero solution for the problem (P) we consider the energy functional $I_\lambda : X \to \mathbb{R}$ given by

$$I_\lambda(u) := \frac{1}{2}\|u\|^2 - \frac{\lambda}{2}\|u\|_2^2 - \frac{1}{2^*}\|u^+\|_{2^*}^{2^*} - \frac{1}{2_*}|||u^+|||_{2_*}^{2_*},$$

where $u^+(x) := \max\{u(x), 0\}$. Set

$$\Sigma := \left\{(\alpha,\beta) \in [0,+\infty)^2 : \alpha+\beta > 0,\ \alpha+\beta \geq \max\{S\alpha^{2/2^*}, S_0\beta^{2/2_*}\}\right\}$$

and the minimax level

$$c^* := \min\left\{\Psi(\alpha,\beta) := \left(\frac{1}{2} - \frac{1}{2^*}\right)\alpha + \left(\frac{1}{2} - \frac{1}{2_*}\right)\beta \,:\, (\alpha,\beta) \in \Sigma\right\} > 0.$$

In order to verify that $c^* > 0$ we take $(\alpha,\beta) \in \Sigma$ arbitrary. If $\beta \leq \alpha$, then $S\alpha^{2/2^*} \leq \alpha + \beta \leq 2\alpha$ and therefore $\alpha \geq (S/2)^{N/2}$. Thus

$$\Psi(\alpha,\beta) \geq \frac{1}{2N}(\alpha+\beta) \geq \frac{\alpha}{2N} \geq \frac{1}{2N}\left(\frac{S}{2}\right)^{N/2}.$$

If $\alpha < \beta$, an analogous computation yelds $\beta \geq (S_0/2)^{N-1}$, and therefore

$$\Psi(\alpha,\beta) \geq \frac{\beta}{2N} \geq \frac{1}{2N}\left(\frac{S_0}{2}\right)^{N-1}.$$

It follows that $c^* > 0$.

The next result is a version of a convergence result due to Brezis and Lieb. Although we believe it is not essentially new, we were not able to locate a precise reference and therefore we present a proof for completeness (borrowed from [6]).

Lemma 4.1. *If $(u_n) \subset X$ is such that $u_n \rightharpoonup u$ weakly in X then, up to a subsequence,*

$$\|u_n^+\|_{2^*}^{2^*} = \|(u_n - u)^+\|_{2^*}^{2^*} + \|u^+\|_{2^*}^{2^*} + o(1)$$

and

$$|\!|\!| u_n^+ |\!|\!|_{2_*}^{2_*} = |\!|\!|(u_n - u)^+|\!|\!|_{2_*}^{2_*} + |\!|\!| u^+ |\!|\!|_{2_*}^{2_*} + o(1).$$

Proof. From the weak convergence and the compact embeddings we conclude that (u_n) is bounded in X and, up to a subsequence, $u_n(x) \to u(x)$ for a.e. $x \in \mathbb{R}^N_+$. Setting $F(s) := (s^+)^{2^*}$ and using the Mean Value Theorem, we get

$$|F(s+t) - F(s)| = |F'(s+\theta t)t| \le 2^*|s+\theta t|^{2^*-1}|t| \le c_1|s|^{2^*-1}|t| + c_2|t|^{2^*},$$

for all $s,\, t \in \mathbb{R}$, where $\theta \in [0,1]$ and $c_1,\, c_2 > 0$. Given $\varepsilon > 0$, we can use Young's inequality to obtain $A_\varepsilon > 0$ such that

$$|F(s+t) - F(s)| \le \varepsilon |s|^{2^*} + A_\varepsilon |t|^{2^*}, \quad \forall\, s,\, t \in \mathbb{R}. \tag{4.1}$$

We now consider the functions ϕ_n, $\psi_{n,\varepsilon}$ defined by

$$\phi_n(x) := |F(u_n) - F(u_n - u) - F(u)|, \quad \psi_{n,\varepsilon}(x) := \left(\phi_n - \varepsilon|u_n - u|^{2^*}\right)^+.$$

Using (4.1) with $s = u_n - u$ and $t = u$, we obtain

$$\phi_n(x) \le |F(u)| + \varepsilon|u_n - u|^{2^*} + A_\varepsilon|u|^{2^*}$$

from which it follows that

$$\psi_{n,\varepsilon}(x) \le (1 + A_\varepsilon)|u|^{2^*}.$$

Since $\psi_{n,\varepsilon}(x) \to 0$ for a.e. $x \in \mathbb{R}^N_+$, we can use Leguesgue Theorem to conclude that $\int K(x)\psi_{n,\varepsilon} \to 0$, as $n \to +\infty$. Thus,

$$\limsup_{n\to+\infty} \int K(x)\phi_n \le \limsup_{n\to+\infty} \int K(x) \left(\psi_{n,\varepsilon} + \varepsilon|u_n - u|^{2^*}\right) \le \varepsilon c_3.$$

Since $\varepsilon > 0$ is arbitrary the first statement of the lemma is proved. The argument for the second one is analogous, since we also have suitable trace embeddings and the required compactness properties. □

In the next result we present the relation between c^* and the range of compactness for the energy functional I_λ.

Proposition 4.2. *Suppose that $0 < \lambda < (N/2)$. If $(u_n) \subset X$ is such that $I_\lambda'(u_n) \to 0$ and $I_\lambda(u_n) \to c < c^*$, then (u_n) has a convergent subsequence in X.*

Proof. Since

$$c + o(1) + o(1)\|u_n\| = I_\lambda(u_n) - \frac{1}{2_*} I_\lambda'(u_n)u_n \ge \left(\frac{1}{2} - \frac{1}{2_*}\right)\left(1 - \frac{\lambda}{\lambda_1}\right)\|u_n\|^2,$$

as $n \to +\infty$, we conclude that (u_n) is bounded in X. Hence, up to a subsequence, $u_n \rightharpoonup u$ weakly in X and $u_n(x) \to u(x)$ for a.e. $x \in \mathbb{R}^N_+$. Let $s := 2^*/(2^* - 1)$ and $s' = 2^*$ its conjugated exponent. Since the sequence $g_n(x) := K(x)^{1/s}(u_n^+)^{2^*-1}$ is bounded in $L^s(\mathbb{R}^N_+)$, it follows from the pointwise convergence of (u_n) that $g_n \rightharpoonup K^{1/s}(u^+)^{2^*-1}$ weakly in $L^s(\mathbb{R}^N_+)$. Since $K^{1/s'}\varphi \in L^{s'}(\mathbb{R}^N_+)$, for any $\varphi \in C_c^\infty(\overline{\mathbb{R}^N_+})$, we get

$$\lim_{n\to+\infty} \int K(x)(u_n^+)^{2^*-1}\varphi = \int K(x)(u^+)^{2^*-1}\varphi, \quad \forall\, \varphi \in C_c^\infty(\overline{\mathbb{R}^N_+}).$$

An analogous sargument gives

$$\lim_{n\to+\infty}\int_{\mathbb{R}^{N-1}} K(x',0)(u_n^+)^{2_*-1}\varphi\, dx' = \int_{\mathbb{R}^{N-1}} K(x',0)(u^+)^{2_*-1}\varphi\, dx'.$$

The above equalities and $I'_\lambda(u_n)\to 0$ imply that $o(1)=I'_\lambda(u_n)\varphi \to I'_\lambda(u)\varphi=0$, for all $\varphi\in C_c^\infty(\overline{\mathbb{R}^N_+})$, and therefore $I'_\lambda(u)=0$. In particular, $I'_\lambda(u)u=0$ and we have that

$$I_\lambda(u)=I_\lambda(u)-\frac{1}{2}I'_\lambda(u)u=\left(\frac{1}{2}-\frac{1}{2^*}\right)\|u^+\|_{2^*}^{2^*}+\left(\frac{1}{2}-\frac{1}{2_*}\right)|u^+|_{2_*}^{2_*}\geq 0.$$

We now set $v_n:=(u_n-u)$ and notice that, by going consecutively to subsequences, we may assume that are well defined

$$\alpha:=\lim_{n\to+\infty}\|v_n\|^2,\quad \gamma:=\lim_{n\to+\infty}\|v_n^+\|_{2^*}^{2^*},\quad \beta:=\lim_{n\to+\infty}|v_n^+|_{2_*}^{2_*},$$

and that $v_n\to 0$ strongly in $L^2_K(\mathbb{R}^N_+)$. We shall prove that $\alpha=0$.

First notice that the weak convergence in the Hilbert space X implies that

$$\|u_n-u\|^2=\|u_n\|^2-2\langle u_n,u\rangle_X+\|u\|^2=\|u_n\|^2-\|u\|^2+o(1),$$

where $\langle\cdot,\cdot\rangle_X$ stands for the inner product in X. Moreover, from the classical Brezis-Lieb Lemma (see [26, Lemma 1.32]), we also have that

$$\|u_n-u\|_2^2=\|u_n\|_2^2-\|u\|_2^2+o(1).$$

Hence, we can use the above equalities, Lemma 4.1 and $I_\lambda(u)\geq 0$ to obtain

$$\begin{aligned} c+o(1) &= \frac{1}{2}\|u_n\|^2-\frac{\lambda}{2}\|u_n\|_2^2-\left(\frac{1}{2^*}\|u_n^+\|^{2^*}+\frac{1}{2_*}|u_n^+|_{2_*}^{2_*}\right)\\ &= I_\lambda(v_n)+I_\lambda(u)+o(1)\\ &\geq \frac{1}{2}\|v_n\|^2-\frac{\lambda}{2}\|v_n\|_2^2+\left(\frac{1}{2^*}\|v_n^+\|_{2^*}^{2^*}+\frac{1}{2_*}|v_n^+|_{2_*}^{2_*}\right)+o(1), \end{aligned}$$

as $n\to+\infty$. It follows that

$$c\geq \frac{1}{2}\alpha-\left(\frac{1}{2^*}\gamma+\frac{1}{2_*}\beta\right). \tag{4.2}$$

Arguing as above, we can prove that

$$\|v_n\|^2-\lambda\|v_n\|_2^2-\|v_n^+\|_{2^*}^{2^*}-|v_n^+|_{2_*}^{2_*}=I'_\lambda(v_n)v_n=I'_\lambda(u_n)u_n-I'_\lambda(u)u+o(1)=o(1),$$

and therefore, taking the limit and recalling that $\|v_n\|_2\to 0$, we obtain

$$\alpha=(\gamma+\beta). \tag{4.3}$$

Suppose, by contradiction, that $\alpha>0$. Since by definition $\alpha\geq S\gamma^{2/2^*}$ and $\alpha\geq S_0\beta^{2/2_*}$, we can use the above inequality to conclude that $(\gamma,\beta)\in\Sigma$. Thus, (4.2) and (4.3) imply that

$$c^*\leq\Psi(\gamma,\beta)=\left(\frac{1}{2}-\frac{1}{2_*}\right)\gamma+\left(\frac{1}{2}-\frac{1}{2^*}\right)\beta\leq c,$$

which contradicts $c<c^*$. Hence, $\alpha=0$ and the proof is finished. □

Proof of Theorem 1.4. Suppose that $\lambda < (N/2)$. By (3.2) and the embeddings of the space X we can obtain c_1, $c_2 > 0$ such that

$$I_\lambda(u) \geq \left[\frac{1}{2}\left(1-\frac{\lambda}{\lambda_1}\right) - c_1\|u\|^{2^*-2} - c_2\|u\|^{2_*-2}\right]\|u\|^2, \qquad \forall\, u \in X.$$

Hence, there exist η, $\rho > 0$ such that $I_\lambda(u) \geq \eta$, whenever $\|u\| = \rho$. A direct computation shows that $I_\lambda(tu) \to -\infty$, as $t \to +\infty$, for any positive function $u \in X$. So, we obtain from the Mountain Pass Theorem [3] (see also the last part of the proof of [26, Theorem 4.2]) a sequence $(u_n) \subset X$ such that $I_\lambda'(u_n) \to 0$ and

$$I_\lambda(u_n) \to \widetilde{c_\lambda} := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I_\lambda(\gamma(t)),$$

where $\Gamma := \{\gamma \in C([0,1], X) \,:\, \gamma(0) = 0,\, I_\lambda(\gamma(1)) < 0\}$.

Notice that, if $u^+ \neq 0$, then $I_\lambda(tu) \to -\infty$, as $t \to +\infty$ and therefore $I_\lambda(t_0 u) < 0$ for $t_0 > 0$ large. Using the path $\gamma_u(t) := tt_0u$, we obtain

$$\widetilde{c_\lambda} \leq \max_{t\in[0,1]} I_\lambda(\gamma_u(t)) \leq \max_{t\geq 0} I_\lambda(tu)$$

and therefore

$$\widetilde{c_\lambda} \leq c_\lambda := \inf_{u\in X,\, u^+\neq 0}\ \max_{t\geq 0} I_\lambda(tu).$$

We claim that $c_\lambda < c^*$ for any $\lambda < \lambda_1$ close to λ_1. If this is true, it follows from Proposition 4.2 that, along a subsequence, $u_n \to u$ strongly in X. Since I_λ is of class C^1 we have that $I_\lambda(u) = c_\lambda > 0$ and $I_\lambda'(u) = 0$, that is, u is a nonzero solution of (P). By using the definition of λ_1 in (3.2) and setting $u^- := u^+ - u$, we can compute

$$0 = I_\lambda'(u)u^- = -\|u^-\|^2 + \lambda\|u^-\|_2^2 \leq \left(\frac{\lambda-\lambda_1}{\lambda_1}\right)\|u^-\|^2.$$

Since $\lambda < (N/2) = \lambda_1$, we conclude that $u^- \equiv 0$, that is, $u \geq 0$ a.e. in $\mathbb{R}^N_+$. We can now use the Maximum Principle to conclude that $u > 0$ a.e. in $\mathbb{R}^N_+$.

In order to prove the claim, let $\varphi_1 > 0$ be a positive λ_1-eigenfunction of the linear problem (3.1) and notice that, for some $t_\lambda > 0$, there holds

$$c_\lambda \leq I_\lambda(t_\lambda\varphi_1) = \max_{t\geq 0} I_\lambda(t\varphi_1). \tag{4.4}$$

A direct computation gives

$$D_\lambda t_\lambda^2 = t_\lambda^{2N/(N-2)}A + t_\lambda^{2(N-1)/(N-2)}B, \tag{4.5}$$

with

$$D_\lambda := \|\varphi_1\|^2 - \lambda\|\varphi_1\|_2^2, \quad A := \|\varphi_1\|_{2^*}^{2^*}, \quad B := |\!|\varphi_1|\!|_{2_*}^{2_*}.$$

Setting $s_\lambda := t_\lambda^{2/(N-2)}$ we can rewrite (4.5) as $As_\lambda^2 + Bs_\lambda - D_\lambda = 0$, and consequently

$$t_\lambda^{2/(N-2)} = \frac{-B + \sqrt{B^2 + 4AD_\lambda}}{2A}.$$

Since $\|\varphi_1\|^2 = \lambda_1\|\varphi_1\|_2^2$, we have that $\lim_{\lambda\to\lambda_1^-} D_\lambda = 0$, from which it follows that $\lim_{\lambda\to\lambda_1^-} t_\lambda = 0$. Hence, recalling that $c^* > 0$ and

$$\begin{aligned} I_\lambda(t_\lambda\varphi_1) &= I_\lambda(t_\lambda\varphi_1) - \frac{1}{2}I_\lambda'(t_\lambda\varphi_1)(t_\lambda\varphi_1) \\ &= \frac{1}{N}t_\lambda^{2^*}A + \frac{1}{2(N-1)}t_\lambda^{2_*}B, \end{aligned}$$

we infer from (4.4) that $c_\lambda < c^*$, for any λ sufficiently close to λ_1. This finishes the proof. $\square$

5. Proof of Theorem 1.5

This section is devoted to the proof of Theorem 1.5. Notice that the energy functional associated to problem $(\widetilde{P})$ is

$$I_\lambda(u) := \frac{1}{2}\|u\|^2 - \frac{\lambda}{2}\|u\|_2^2 - \frac{1}{2_*}|\!|u^+|\!|_{2_*}^{2_*}, \qquad \forall\, u \in X.$$

As in the last section, we first prove a local compactness result for the functional I_λ. The critical level is now related with the best constant S_0 of the trace embedding defined in (1.2), as we can see by the result below:

Proposition 5.1. *Suppose that* $0 < \lambda < (N/2)$*. If* $(u_n) \subset X$ *is such that* $I'_\lambda(u_n) \to 0$ *and* $I_\lambda(u_n) \to c$ *with*

$$c < c_0 := \frac{1}{2(N-1)} S_0^{N-1},$$

then (u_n) *has a convergent subsequence in* X*.*

Proof. Arguing as in the proof of Proposition 4.2 we can prove that $(u_n) \subset X$ is bounded. Also, by passing to a subquence, we may assume that $u_n \rightharpoonup u$ weakly in X, $I'_\lambda(u) = 0$ and $I_\lambda(u) \geq 0$. Moreover, if we set $v_n := u_n - u$, we can define

$$\alpha := \lim_{n\to+\infty} \|v_n\|^2, \quad \beta := \lim_{n\to+\infty} |\!|v_n^+|\!|_{2_*}^{2_*},$$

and we shall prove that $\alpha = 0$. As in the proof of Proposition 4.2, we have that

$$I_\lambda(u_n - u) = \frac{1}{2}\|v_n\|^2 - \frac{\lambda}{2}\|v_n\|_2^2 - \frac{1}{2_*}|\!|v_n^+|\!|_{2_*}^{2_*} = I_\lambda(u_n) - I_\lambda(u) + o(1),$$

as $n \to +\infty$. Since $I_\lambda(u) \geq 0$, we can take the limit to conclude that

$$\frac{\alpha}{2} - \frac{\beta}{2_*} \leq c.$$

Moreover, taking the limit in

$$\|v_n\|^2 - \lambda\|v_n\|_2^2 - |\!|v_n^+|\!|_{2_*}^{2_*} = I'_\lambda(v_n)v_n = I'_\lambda(u_n)u_n - I'_\lambda(u)u + o(1) = o(1),$$

we obtain that $\alpha = \beta$, and therefore

$$\frac{1}{2(N-1)}\alpha = \left(\frac{1}{2} - \frac{1}{2_*}\right)\alpha \leq c. \tag{5.1}$$

On the other hand, letting $n \to +\infty$ in the inequality $S_0|\!|v_n^+|\!|_{2_*}^2 \leq \|v_n\|^2$, we obtain

$$S_0\alpha^{2/2_*} \leq \alpha.$$

Suppose, by contradiction, that $\alpha > 0$. Then the above expression implies that $\alpha \geq S_0^{N-1}$ and we can use (5.1) to conclude that

$$\frac{1}{2(N-1)} S_0^{N-1} \leq \frac{1}{2(N-1)}\alpha \leq c,$$

which is a contradiction. Hence, $\alpha = 0$ and the proof is complete. $\square$

We are ready to present the proof of Theorem 1.5.

Proof of Theorem 1.5. If $0 < \lambda < (N/2)$, we can argue as in the proof of Theorem 1.4 to obtain a sequence $(u_n) \subset X$ such that $I'_\lambda(u_n) \to 0$ and $I_\lambda(u_n) \to \widetilde{c_\lambda}$, where this last number is the Mountain Pass level which, as before, satisfies

$$\widetilde{c_\lambda} \le c_\lambda := \inf_{u \in X,\, u^+ \ne 0} \max_{t \ge 0} I_\lambda(tu).$$

Let c_0, $\lambda_N^* > 0$ as in the statement of Proposition 5.1 and Theorem 1.5, respectively. We claim that

$$c_\lambda < c_0, \qquad \text{whenever } \lambda_N^* < \lambda < (N/2). \tag{5.2}$$

Since the proof of this fact is rather long, we will postpone it for the end of the paper. Assuming the claim, we can argue as in the proof of Theorem 1.4 to obtain the desired solution. □

We devote the rest of the paper to prove that (5.2) holds. Let $\phi \in C^\infty(\mathbb{R}^N_+, [0,1])$ be such that $\phi \equiv 1$ in $B_1(0) \cap \mathbb{R}^N_+$ and $\phi \equiv 0$ in $\overline{\mathbb{R}^N_+} \setminus B_2(0)$. Define the function

$$u_\varepsilon(x) := K(x)^{-1/2}\phi(x)U_\varepsilon(x),$$

with

$$U_\varepsilon(x', x_N) = \frac{\varepsilon^{(N-2)/2}}{[|x'|^2 + (x_N + \varepsilon)^2]^{(N-2)/2}}.$$

The definition of c_λ and a straightforward computation show that

$$c_\lambda \le \max_{t>0} I_\lambda(tu_\varepsilon) = \frac{1}{2(N-1)}\left(\frac{\|u_\varepsilon\|^2 - \lambda\|u_\varepsilon\|_2^2}{\rule[-0.1em]{0.15em}{0.8em}\, u_\varepsilon \,\rule[-0.1em]{0.15em}{0.8em}_{2_*}^2}\right)^{N-1}.$$

Hence, the statement (5.2) is a direct consequence of the above inequality and the following result:

Proposition 5.2. *If $N \ge 7$ and $\lambda_N^* < \lambda < (N/2)$, then*

$$\frac{\|u_\varepsilon\|^2 - \lambda\|u_\varepsilon\|_2^2}{\rule[-0.1em]{0.15em}{0.8em}\, u_\varepsilon \,\rule[-0.1em]{0.15em}{0.8em}_{2_*}^2} < S_0,$$

for any $\varepsilon > 0$ sufficiently small.

For the proof we need some preliminary results. We first set

$$A_N := \int |\nabla U_\varepsilon|^2, \qquad B_N := \left(\int_{\mathbb{R}^{N-1}} |U_\varepsilon|^{2_*} dx'\right)^{2/2_*}$$

and recall that, for any a, $b > 0$, the Beta function is defined as

$$B(a,b) := \int_0^\infty \frac{r^{a-1}}{(r+1)^{a+b}}\,dr. \tag{5.3}$$

Lemma 5.3. *Suppose that $N \ge 7$. As $\varepsilon \to 0^+$, we have that*

$$\|u_\varepsilon\|^2 = A_N + O(\varepsilon^4) + \varepsilon^2\gamma_N,$$

where

$$\gamma_N := \frac{\sigma_{N-2}(N-2)}{4(N-4)}\left[B\left(\frac{N+1}{2}, \frac{N-3}{2}\right) + \frac{1}{(N-3)}B\left(\frac{N-1}{2}, \frac{N-1}{2}\right)\right].$$

Proof. By using the definition of ϕ and U_ε, we can easily compute

$$\begin{aligned}\int K(x)|\nabla u_\varepsilon|^2 &= \int\left[|\nabla\phi|^2U_\varepsilon^2+2\phi U_\varepsilon(\nabla\phi\cdot\nabla U_\varepsilon)-\frac{1}{2}\phi U_\varepsilon^2(x\cdot\nabla\phi)\right]\\ &\quad+\int\phi^2|\nabla U_\varepsilon|^2-\frac{1}{2}\int\phi^2U_\varepsilon(x\cdot\nabla U_\varepsilon)+\frac{1}{16}\int\phi^2|x|^2U_\varepsilon^2.\end{aligned}$$

Moreover

$$\int|\nabla\phi|^2U_\varepsilon^2=\varepsilon^{N-2}\int\limits_{\{1\le|x|\le 2\}}\frac{|\nabla\phi|^2}{[|x'|^2+(x_N+\varepsilon)^2]^{N-2}}=O(\varepsilon^{N-2}),$$

as $\varepsilon\to 0^+$. Since similar computations hold for the other terms into the brackets above, we conclude that

$$\|u_\varepsilon\|^2=O(\varepsilon^{N-2})+\int\phi^2|\nabla U_\varepsilon|^2-\frac{1}{2}\int\phi^2U_\varepsilon(x\cdot\nabla U_\varepsilon)+\frac{1}{16}\int\phi^2|x|^2U_\varepsilon^2. \tag{5.4}$$

We shall estimate each of the integrals on the right-hand side above. For the first one, we notice that

$$\frac{\partial U_\varepsilon}{\partial x_i}=-(N-2)\varepsilon^{(N-2)/2}\frac{x_i}{[|x'|^2+(x_N+\varepsilon)^2]^{N/2}},\qquad i=1,2,\dots,(N-1),$$

and

$$\frac{\partial U_\varepsilon}{\partial x_N}=-(N-2)\varepsilon^{(N-2)/2}\frac{(x_N+\varepsilon)}{[|x'|^2+(x_N+\varepsilon)^2]^{N/2}}.$$

Thus, using that $\phi^2|\nabla U_\varepsilon|^2=(\phi^2-1)|\nabla U_\varepsilon|^2+|\nabla U_\varepsilon|^2$, we obtain

$$\int\phi^2|\nabla U_\varepsilon|^2 = (N-2)^2\varepsilon^{N-2}\int\limits_{\{|x|\ge 1\}}\frac{(\phi^2-1)}{[|x'|^2+(x_N+\varepsilon^2)]^{N-1}}dx+\int|\nabla U_\varepsilon|^2$$

and therefore

$$\int\phi^2|\nabla U_\varepsilon|^2=O(\varepsilon^{N-2})+\int|\nabla U_\varepsilon|^2. \tag{5.5}$$

By using the same trick calculation, we get

$$\begin{aligned}\int\phi^2U_\varepsilon(x\cdot\nabla U_\varepsilon) &= O(\varepsilon^{N-2})+\int U_\varepsilon(x\cdot\nabla U_\varepsilon)\\ &= O(\varepsilon^{N-2})-(N-2)\varepsilon^{N-2}\int\frac{(x',x_N)\cdot(x',x_N+\varepsilon)}{[|x'|^2+(x_N+\varepsilon)^2]^{N-1}}\\ &= O(\varepsilon^{N-2})-(N-2)\varepsilon^{N-2}\int\frac{|x'|^2+x_N^2+\varepsilon x_N}{[|x'|^2+(x_N+\varepsilon)^2]^{N-1}}\\ &= O(\varepsilon^{N-2})-(N-2)\varepsilon^{N-2}\int\frac{\varepsilon^2\left(|x'/\varepsilon|^2+(x_N/\varepsilon)^2+(x_N/\varepsilon)\right)}{\varepsilon^{2(N-1)}\left[|x'/\varepsilon|^2+(x_N/\varepsilon+1)^2\right]^{N-1}}.\end{aligned}$$

The change of variables $x\mapsto(x/\varepsilon)$ gives

$$\int\phi^2U_\varepsilon(x\cdot\nabla U_\varepsilon)=O(\varepsilon^{N-2})-(N-2)\varepsilon^2\int\frac{|x'|^2+x_N^2+x_N}{[|x'|^2+(x_N+1)^2]^{N-1}},$$

and therefore

$$-\frac{1}{2}\int\phi^2U_\varepsilon(x\cdot\nabla U_\varepsilon)=O(\varepsilon^{N-2})+\varepsilon^2\frac{(N-2)}{2}\int\frac{|x'|^2+x_N^2+x_N}{[|x'|^2+(x_N+1)^2]^{N-1}}. \tag{5.6}$$

Arguing as above, we can compute the last term as follows

$$\int \phi^2|x|^2U_\varepsilon^2 = \varepsilon^{N-2}\int_{\{|x|\geq 1\}} \frac{\left(\phi^2-1\right)|x|^2}{[|x'|^2+(x_N+\varepsilon)^2]^{N-2}}dx + \int |x|^2U_\varepsilon^2$$

Since $N \geq 7$, the first integral on the righ-hand side above is $O(1)$ as $\varepsilon \to 0^+$. Thus,

$$\begin{aligned}\int \phi^2|x|^2U_\varepsilon^2 &= O(\varepsilon^{N-2}) + \varepsilon^{N-2}\int \frac{|x|^2}{[|x'|^2+(x_N+\varepsilon)^2]^{N-2}} \\ &= O(\varepsilon^{N-2}) + \varepsilon^{4}\int \frac{|x|^2}{[|x'|^2+(x_N+1)^2]^{N-2}},\end{aligned}$$

and we infer that

$$\frac{1}{16}\int \phi^2|x|^2U_\varepsilon^2 = O(\varepsilon^4).$$

By replacing the above expression, (5.6) and (5.5) in (5.4) and using the definition of A_N, we obtain

$$\|u_\varepsilon\|^2 = A_N + O(\varepsilon^4) + \varepsilon^2\frac{(N-2)}{2}\left[C_{1,N}+C_{2,N}\right], \tag{5.7}$$

with

$$C_{1,N} := \int \frac{|x'|^2}{[|x'|^2+(x_N+1)^2]^{N-1}}, \qquad C_{2,N} := \int \frac{x_N(x_N+1)}{[|x'|^2+(x_N+1)^2]^{N-1}}.$$

We now proceed with the computation of the two integrals above. For the first one, we use the Fubini Theorem, the change of variables $x' \mapsto \frac{x'}{x_N+1}$ and polar coordinates to get

$$\begin{aligned} C_{1,N} &= \int_0^\infty \int_{\mathbb{R}^{N-1}} \frac{|x'|^2}{[|x'|^2+(x_N+1)^2]^{N-1}}dx'dx_N \\ &= \left[\int_0^\infty (x_N+1)^{2-2(N-1)+(N-1)}dx_N\right]\left[\int_{\mathbb{R}^{N-1}} \frac{|x'|^2}{[|x'|^2+1]^{N-1}}dx'\right] \\ &= \frac{\sigma_{N-2}}{(N-4)}\int_0^\infty \frac{r^2r^{N-2}}{(r^2+1)^{N-1}}dr = \frac{\sigma_{N-2}}{(N-4)}\int_0^\infty \frac{r^N}{(r^2+1)^{N-1}}dr.\end{aligned}$$

The change of variables $r^2 \mapsto r$ in the last integral above gives

$$C_{1,N} = \frac{\sigma_{N-2}}{2(N-4)}\int_0^\infty \frac{r^{(N-1)/2}}{(r+1)^{N-1}}dr = \frac{\sigma_{N-2}}{2(N-4)}B\left(\frac{N+1}{2},\frac{N-3}{2}\right),$$

according to the definition of the function B in (5.3).

The computation of $C_{2,N}$ follows the same lines:

$$\begin{aligned} C_{2,N} &= \int_0^\infty \int_{\mathbb{R}^{N-1}} \frac{x_N(x_N+1)}{[|x'|^2+(x_N+1)^2]^{N-1}}dx'dx_N \\ &= \left[\int_0^\infty x_N(x_N+1)^{1-2(N-1)+(N-1)}dx_N\right]\left[\int_{\mathbb{R}^{N-1}} \frac{1}{[|x'|^2+1]^{N-1}}dx'\right] \\ &= \left[\int_0^\infty x_N(x_N+1)^{-N+2}dx_N\right]\sigma_{N-2}\int_0^\infty \frac{r^{N-2}}{(r^2+1)^{N-1}}dr.\end{aligned}$$

By making the changes of variables $(x_N+1)\mapsto x_N$ and $r^2\mapsto r$ in the first and second integral above, respectively, we get

$$\begin{aligned} C_{2,N} &= \frac{\sigma_{N-2}}{2(N-4)(N-3)}\int_0^\infty \frac{r^{(N-3)/2}}{(r+1)^{N-1}}dr \\ &= \frac{\sigma_{N-2}}{2(N-4)(N-3)}B\left(\frac{N-1}{2},\frac{N-1}{2}\right), \end{aligned}$$

where we have used (5.3) again. The statement of the lemma is a consequence of (5.7) and the values of $C_{1,N}$ and $C_{2,N}$ just calculated. □

Lemma 5.4. *Suppose that $N\geq 7$. As $\varepsilon\to 0^+$, we have that*

$$\|u_\varepsilon\|_2^2 = O(\varepsilon^{N-2})+\varepsilon^2\alpha_N, \qquad \| u_\varepsilon \|_{2_*}^{2_*} = B_N^{2_*/2}-\varepsilon^2 D_N + o(\varepsilon^2),$$

where

$$\alpha_N := \frac{\sigma_{N-2}}{2(N-4)}B\left(\frac{N-1}{2},\frac{N-3}{2}\right), \qquad D_N := \frac{\sigma_{N-2}}{8(N-2)}B\left(\frac{N+1}{2},\frac{N-3}{2}\right).$$

Proof. For the proof we argue as in the former lemma. Notice that

$$\int K(x)|u_\varepsilon|^2 = \int \phi^2 U_\varepsilon^2 = O(\varepsilon^{N-2})+\varepsilon^{N-2}\int \frac{1}{[|x'|^2+(x_N+\varepsilon)^2]^{N-2}},$$

and therefore

$$\|u_\varepsilon\|_2^2 = O(\varepsilon^{N-2})+\varepsilon^2\int \frac{1}{[|x'|^2+(x_N+1)^2]^{N-2}}.$$

If we call $C_{3,N}$ the integral above, we can use (5.3) to get

$$\begin{aligned} C_{3,N} &= \int_0^\infty\int_{\mathbb{R}^{N-1}} \frac{1}{[|x'|^2+(x_N+1)^2]^{N-2}}dx'dx_N \\ &= \left[\int_0^\infty (x_N+1)^{-N+3}dx_N\right]\sigma_{N-2}\int_0^\infty \frac{r^{N-2}}{(r^2+1)^{N-2}}dr \\ &= \frac{\sigma_{N-2}}{2(N-4)}\int_0^\infty \frac{r^{(N-3)/2}}{(r+1)^{N-2}}dr \\ &= \frac{\sigma_{N-2}}{2(N-4)}B\left(\frac{N-1}{2},\frac{N-3}{2}\right), \end{aligned}$$

which proves the first part of the lemma.

For the second statement, we notice that

$$\begin{aligned} \int_{\mathbb{R}^{N-1}} K(x',0)|u_\varepsilon|^{2_*}dx' &= \varepsilon^{N-1}\int_{\mathbb{R}^{N-1}} \frac{K(x',0)^{1/(2-N)}\phi^{2_*}}{[|x'|^2+\varepsilon^2]^{N-1}}dx' \\ &= O(\varepsilon^{N-1})+\varepsilon^{N-1}\int_{\mathbb{R}^{N-1}} \frac{K(x',0)^{1/(2-N)}}{[|x'|^2+\varepsilon^2]^{N-1}}dx' \\ &= O(\varepsilon^{N-1})+\int_{\mathbb{R}^{N-1}} \frac{K(\varepsilon x',0)^{1/(2-N)}}{[|x'|^2+1]^{N-1}}dx'. \end{aligned}$$

Since $K(x)=\exp(|x|^2/4)$, we get

$$\int_{\mathbb{R}^{N-1}} \frac{K(\varepsilon x',0)^{1/(2-N)}}{[|x'|^2+1]^{N-1}}dx' = \int_{\mathbb{R}^{N-1}} |U_\varepsilon|^{2_*}dx' + \int_{\mathbb{R}^{N-1}} \frac{\left[\exp\left(-\varepsilon^2\frac{|x'|^2}{4(N-2)}\right)-1\right]}{[|x'|^2+1]^{N-1}}dx'$$

and therefore we can use $N \geq 7$, the definition of B_N and Taylor's formula to get

$$\|u_\varepsilon\|_{2_*}^{2_*} = B_N^{2_*/2} - \varepsilon^2 \frac{C_{4,N}}{4(N-2)} + O(\varepsilon^4),$$

where

$$\begin{aligned} C_{4,N} \quad &:= \quad \int_{\mathbb{R}^{N-1}} \frac{|x'|^2}{[|x'|^2+1]^{N-1}} dx' = \sigma_{N-2} \int_0^\infty \frac{r^2 r^{N-2}}{(r^2+1)^{N-1}} dr \\ &= \quad \frac{\sigma_{N-2}}{2} \int \frac{r^{(N-1)/2}}{(r+1)^{N-1}} dr = \frac{\sigma_{N-2}}{2} B\left(\frac{N+1}{2}, \frac{N-3}{2}\right). \end{aligned}$$

The result follows from the above expressions. □

Now we turn to prove Proposition 5.2.

Proof of Proposition 5.2. We start by using the second statement of Lemma 5.4 and the Mean Value Theorem to get

$$\begin{aligned} \|u_\varepsilon\|_{2_*}^2 &= \left(B_N^{2_*/2} - \varepsilon^2 D_N + o(\varepsilon^2)\right)^{2/2_*} \\ &= B_N + \frac{2}{2_*}\left[B_N^{2_*/2} + O(\varepsilon^2)\right]^{-1+(2/2_*)} \left[-\varepsilon^2 D_N + o(\varepsilon^2)\right] \\ &= B_N - \varepsilon^2 \frac{2}{2_*} B_N^{(2-2_*)/2} D_N + o(\varepsilon^2), \end{aligned}$$

as $\varepsilon \to 0^+$, from which it follows that

$$\frac{1}{\|u_\varepsilon\|_{2_*}^2} = \frac{1}{B_N} + \varepsilon^2 \frac{2}{2_*} B_N^{-2+(2-2_*)/2} D_N + o(\varepsilon^2).$$

Thus, if we set

$$\beta_N := \frac{2}{2_*} A_N B_N^{-2/2_*} D_N,$$

we can use the expression of $\|u_\varepsilon\|_{2_*}^{-2}$, Lemma 5.3 and some computations to get, for $N \geq 7$,

$$\frac{\|u_\varepsilon\|^2 - \lambda \|u_\varepsilon\|_2^2}{\|u_\varepsilon\|_{2_*}^2} = \frac{A_N}{B_N} + \varepsilon^2 \left(\frac{\gamma_N - \lambda \alpha_N + \beta_N}{B_N} + o(1)\right).$$

Since $A_N/B_N = S_0$ (see for instance [10]), we conclude that the left-hand side above is smaller than S_0 if $\varepsilon > 0$ is sufficiently small and λ satisfies

$$\lambda > \Lambda_N := \frac{\gamma_N}{\alpha_N} + \frac{\beta_N}{\alpha_N}.$$

In view of the definition of λ_N^* in the statement of Theorem 1.5, it remains to show that

$$\Lambda_N = \frac{N}{4} + \frac{(N-4)}{8}. \tag{5.8}$$

In order to do this, we first recall that the Beta function can be written as

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, \tag{5.9}$$

where

$$\Gamma(a) := \int_0^{+\infty} r^{a-1} e^{-r} dr, \quad a > 0,$$

is the Gamma function. For simplicity, we denote

$$\Gamma_0 := \Gamma\left(\frac{N-1}{2}\right).$$

Since $\Gamma(a) = (a-1)\Gamma(a-1)$, we have that

$$\Gamma\left(\frac{N+1}{2}\right) = \frac{(N-1)}{2}\Gamma_0, \qquad \Gamma\left(\frac{N-3}{2}\right) = \frac{2}{(N-3)}\Gamma_0. \tag{5.10}$$

This, the definition of γ_N in Lemma 5.3 and (5.9), give

$$\begin{aligned} \gamma_N &= \frac{\sigma_{N-2}(N-2)}{4(N-4)}\left[\frac{\Gamma(\frac{N+1}{2})\Gamma(\frac{N-3}{2})}{\Gamma(N-1)} + \frac{1}{(N-3)}\frac{\Gamma(\frac{N-1}{2})\Gamma(\frac{N-1}{2})}{\Gamma(N-1)}\right]. \\ &= \frac{\sigma_{N-2}(N-2)}{4(N-4)\Gamma(N-1)}\left[\frac{(N-1)}{2}\frac{2}{(N-3)}\Gamma_0^2 + \frac{1}{(N-3)}\Gamma_0^2\right] \end{aligned}$$

and therefore

$$\gamma_N = \frac{\sigma_{N-2}N(N-2)}{4(N-3)(N-4)}\frac{\Gamma_0^2}{\Gamma(N-1)}.$$

On the other hand, from Lemma 5.4, (5.9), (5.10) and $\Gamma(N-1) = (N-2)\Gamma(N-2)$, we get

$$\alpha_N = \frac{\sigma_{N-2}}{2(N-4)}\frac{\Gamma(\frac{N-1}{2})\Gamma(\frac{N-3}{2})}{\Gamma(N-2)} = \frac{\sigma_{N-2}(N-2)}{(N-3)(N-4)}\frac{\Gamma_0^2}{\Gamma(N-1)}. \tag{5.11}$$

Thus,

$$\frac{\gamma_N}{\alpha_N} = \frac{N}{4}. \tag{5.12}$$

We now compute the other term of Λ_N, namely β_N/α_N. First notice that, arguing as in Lemma 5.3, we get

$$\begin{aligned} A_N &= \int |\nabla U_\varepsilon|^2 = (N-2)^2 \int \frac{1}{[|x'|^2 + (x_N+1)^2]^{N-1}} \\ &= (N-2)^2 \left[\int_0^{+\infty} (x_N+1)^{-N+1} dx_N\right]\left[\int_{\mathbb{R}^{N-1}} \frac{1}{[|x'|^2+1]^{N-1}} dx'\right] \\ &= \frac{\sigma_{N-2}(N-2)}{2}\int_0^{\infty} \frac{r^{(N-3)/2}}{(r+1)^{N-1}} dr \end{aligned}$$

and therefore

$$A_N = \frac{\sigma_{N-2}(N-2)}{2} B\left(\frac{N-1}{2}, \frac{N-1}{2}\right).$$

The same argument gives

$$\begin{aligned} B_N^{-2_*/2} &= \left(\int_{\mathbb{R}^{N-1}} |U_\varepsilon|^{2_*} dx'\right)^{-1} = \left(\int_{\mathbb{R}^{N-1}} \frac{1}{[|x'|^2+1]^{N-1}} dx'\right)^{-1} \\ &= \frac{2}{\sigma_{N-2}}\frac{1}{B\left(\frac{N-1}{2}, \frac{N-1}{2}\right)}. \end{aligned}$$

Now, using the value of D_N given in Lemma 5.4, (5.9) and (5.10), we obtain

$$D_N = \frac{\sigma_{N-2}}{8(N-2)} B\left(\frac{N+1}{2}, \frac{N-3}{2}\right) = \frac{\sigma_{N-2}(N-1)}{8(N-2)(N-3)} \frac{\Gamma_0^2}{\Gamma(N-1)}.$$

Hence,

$$\frac{2}{2_*} A_N B_N^{-2_*/2} D_N = \frac{\sigma_{N-2}(N-2)}{8(N-3)} \frac{\Gamma_0^2}{\Gamma(N-1)},$$

and therefore we can recall the value of α_N in (5.11) to get

$$\frac{\beta_N}{\alpha_N} = \frac{\sigma_{N-2}(N-2)}{8(N-3)} \frac{\Gamma_0^2}{\Gamma(N-1)} \frac{(N-3)(N-4)}{\sigma_{N-2}(N-2)} \frac{\Gamma(N-1)}{\Gamma_0^2},$$

that is,

$$\frac{\beta_N}{\alpha_N} = \frac{(N-4)}{8}.$$

The equation (5.8) is a consequence of the above equality and (5.12). The theorem is proved. □

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