# ON A ZERO-MASS $(N, q)$-LAPLACIAN EQUATION IN $\mathbb{R}^N$ WITH EXPONENTIAL CRITICAL GROWTH

J. L. CARVALHO, G. M. FIGUEIREDO, M. F. FURTADO, AND E. MEDEIROS

Abstract. In this work, we investigate the existence of solution for the quasilinear elliptic equation

$$-\Delta_N u - \Delta_q u = f(u) \quad \text{in} \quad \mathbb{R}^N,$$

where $1 < q < N$ and the nonlinearity $f$ has exponential critical growth in the Trudinger-Moser sense. In order to obtain the solution, we use a variational approach based on a new Trudinger-Moser type inequality which is proved here.

*2010 Mathematics Subject Classification:* 35J60, 35A23, 35J50.
*Keywords and phrases:* Sobolev space; Trudinger-Moser inequality; Quasilinear problem.

## 1. Introduction and main results

In this paper, we aim to investigate the existence of solution for the nonlinear elliptic equation of $(N, q)$-Laplacian type

$$(\mathcal{P}) \qquad -\Delta_N u - \Delta_q u = f(u) \quad \text{in} \quad \mathbb{R}^N,$$

where $\Delta_m u := \operatorname{div}\left(|\nabla u|^{m-2}\nabla u\right)$ stands for the usual $m$-Laplacian, $1 < q < N$ and the nonlinearity $f : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying some growth conditions that we will present later.

Before stating our first main result, we need to establish some notations. For each $1 < r < N$, we denote by $D^{1,r}(\mathbb{R}^N)$ the closure of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_{D^{1,r}(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^N} |\nabla u|^r dx\right)^{1/r}.$$

In order to consider the two operators in the left-hand side of $(\mathcal{P})$, we shall look for solutions in the space $E^{N,q}$ defined as the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_{E^{N,q}} := \left(\int_{\mathbb{R}^N} |\nabla u|^N dx + \left(\int_{\mathbb{R}^N} |\nabla u|^q dx\right)^{N/q}\right)^{1/N},$$

where $1 < q < N$.

By using the Gagliardo-Nirenberg inequality and interpolation, we will verify that the embedding $E^{N,q} \hookrightarrow L^r(\mathbb{R}^N)$ is continuous, for any $r \geq q^* := Nq/(N-q)$.

The first author was supported by CAPES/Brazil.
The second author was supported by CNPq/Brazil and FAPDF/Brazil.
The third author was partially supported by CNPq/Brazil and FAPDF/Brazil.
The last author was partially supported by Grant 2019/2014 Paraíba State Research Foundation (FAPESQ) and Grant 308900/2019-7 CNPq.

This suggests that $E^{N,q}$ can be embedded into Orlicz spaces. To be more precise, we define the Young function by

$$\Phi_{\alpha,j_0}(s) := e^{\alpha|s|^{N/(N-1)}} - \sum_{j=0}^{j_0-1} \frac{\alpha^j}{j!}|s|^{Nj/(N-1)}, \quad s \in \mathbb{R},$$

where $\alpha > 0$ and

$$j_0 := \left\lfloor \frac{q^*(N-1)}{N} \right\rfloor = \inf\left\{ j \in \mathbb{N} : j \geq \frac{q^*(N-1)}{N} \right\}.$$

In our first main result, we prove the following Trudinger-Moser type result:

**Theorem 1.1.** *Suppose that $1 < q < N$. Then, for each $\alpha > 0$ and $u \in E^{N,q}$, the function $\Phi_{\alpha,j_0}(u)$ belongs to $L^1(\mathbb{R}^N)$. Moreover, if $\omega_{N-1}$ stands for the measure of the unit sphere in $\mathbb{R}^N$, then*

$$L(\alpha, N, q) := \sup_{u \in E^{N,q},\, \|u\|_{E^{N,q}} \leq 1} \int_{\mathbb{R}^N} \Phi_{\alpha,j_0}(u)dx < +\infty.$$

*for any $0 < \alpha < \alpha_N := N\omega_{N-1}^{1/(N-1)}$. Finally, if $\alpha > \alpha_N$, then $L(\alpha, N, q) = +\infty$.*

The first results concerning Trudinger-Moser type inequalities have appeared in the papers of Yudovich, Moser, Trudinger [29, 22, 28], for the bounded domain case. Similar results for unbounded domains have been established by Cao [12] and Ruf [25] in $\mathbb{R}^2$, and by do Ó [14], Adachi and Tanaka [1], Li and Ruf [20], in higher dimensions. The proof presented here follows some ideas from the papers [12, 14, 25]. We finnaly notice that, in the planar case $N = 2$, we are able to prove that $L(4\pi, 2, q) < +\infty$ (see Remark 2.2). For other related results, see for instance [2, 11, 17, 21] and references therein.

With Theorem 1.1 in hands, we can consider nonlinearities $f$ which behave like $e^{\alpha|s|^{N/(N-1)}}$ at infinity. More specifically, we shall assume that

$(f_0)$ $f : \mathbb{R} \to \mathbb{R}$ is continuous and there exists $\alpha_0 > 0$ such that

$$\lim_{|s|\to+\infty} \frac{|f(s)|}{e^{\alpha|s|^{N/(N-1)}}} = \begin{cases} 0 & \text{if} \quad \alpha > \alpha_0, \\ +\infty & \text{if} \quad \alpha < \alpha_0; \end{cases}$$

$(f_1)$ there exists $\gamma \geq \max\{N, q^*\}$ such that $f(s) = o(|s|^{\gamma-1})$, as $s \to 0$;

$(f_2)$ there exists $\mu > \gamma$ such that

$$0 < \mu F(s) := \int_0^s f(t)dt \leq sf(s), \quad \text{for all } s \in \mathbb{R};$$

$(f_3)$ there exist $\xi > 0$ and $\nu > \gamma$ such that

$$F(s) \geq \xi s^\nu, \quad \text{for all } s \in (0, 1].$$

Our existence result for equation $(\mathcal{P})$ can be stated as follows:

**Theorem 1.2.** *Assume that $1 < q < N$ and $(f_0) - (f_2)$ hold. Then there exists $\xi_0 > 0$ sufficiently large such that, if $(f_3)$ is satisfied with $\xi \geq \xi_0$, then equation $(\mathcal{P})$ has a nonzero weak solution.*

In order to put our result into perspective, we stress that a great amount of works have been done on quasilinear equations involving the $(p, q)$-laplacian problems of the form

$$-\Delta_p u - \Delta_q u + a(x)|u|^{p-2}u + b(x)|u|^{q-2}u = f(u) \quad \text{in} \quad \mathbb{R}^N. \tag{1.1}$$

They were motivated by a wide variety of problems that arises in several branches of mathematics physics and geometry. We refer to [3, 5, 7, 15, 16, 18] for recent results and more references.

As mentioned in [13], problem (1.1) comes from a general reaction-diffusion equation

$$u_t = \operatorname{div}(D(u)\nabla u) + C(x,u), \qquad D(u) = |\nabla u|^{p-2} + |\nabla u|^{q-2}$$

which has a wide spectrum of applications in physics and related sciences such as biophysics, plasma physics, solid state physics, and chemical reaction design. In such applications, $u$ represents a concentration, $\operatorname{div}(D(u)\nabla u)$ is the diffusion generated by the diffusion coefficient $D(u)$ and the reaction term $C(x,u)$ relates to source and loss processes. Usually, in chemical and biological applications, the reaction term $C(x,u)$ is a polynomial of $u$ with variable coefficients.

In the recent paper [24], motivated by a first approximation of the *Born-Infeld* equation

$$-\operatorname{div}\left(\frac{\nabla u}{\sqrt{1-b^{-2}|\nabla u|^2}}\right) = f(u) \quad \text{in} \quad \mathbb{R}^N,$$

which appears in the study of electromagnetism (see [9, 10]), the authors have considered problem (1.1) in the zero mass case, namely, $a = b = 0$ for $1 < q < p < N$. We also refer to [7] for some related results.

The main contribution of this paper is the study of the zero-mass version of (1.1) in the borderline situation $p = N$. We are going to use classical Moutain Pass Theorem to obtain the desired solution. Although this variational approach is by now standard, the main difficulties here are the correct setting of the functional space as well as the proof of appropriated Trudinger-Moser type inequalities. Actually, the abstract results proved for the space $E^{N,q}$ can be used in a large spectrum of variations of equation $(\mathcal{P})$, depending on the shape of the right-hand side of the equation.

The remainder of the paper is structured as follows: In Section 2, we present and prove all the abstract setting to deal with the space $E^{N,q}$. After that, we prove Theorem 1.2 in Section 3.

## 2. A Trudinger-Moser type inequality

In this section, we prove a new Trudinger-Moser type inequality for the space $E^{N,q}$. For any $R > 0$, we denote by $B_R$ the open ball $\{x \in \mathbb{R}^N : |x| < R\}$ and we use $C, C_0, C_1, \ldots$ to denote (possibly different) positive constants. Finally $\|u\|_{L^r}$ stands for the norm of a function $u \in L^r(\mathbb{R}^N)$,

We state in the sequel a known embedding result from the space $E^{p,q}$ into Lebesgue space (see [24, Theorem 2.1]). For completeness, we provide a simplified proof.

**Proposition 2.1.** *The Sobolev embedding $E^{p,q} \hookrightarrow L^r(\mathbb{R}^N)$ is continuous for any*

$$r \in \begin{cases} [q^*, p^*], & \text{if} \quad 1 < q < p < N, \\ [q^*, \infty), & \text{if} \quad 1 < q < p = N. \end{cases}$$

*Furthermore, the embedding* $E^{p,q}_{\mathrm{rad}} := \{u \in E^{p,q} : u \text{ is radial}\} \hookrightarrow L^r(\mathbb{R}^N)$ *is compact for any*

$$r \in \begin{cases} (q^*, p^*), & \text{if} \quad 1 < q < p < N, \\ (q^*, \infty), & \text{if} \quad 1 < q < p = N. \end{cases}$$

*Proof.* If $1 < q < p < N$, the first result easily follows from the Gagliardo-Nirenberg-Sobolev inequality and interpolation. If $1 < q < p = N$, we pick $r \in [q^*, \infty)$ and choose $q < s < N$ such that $s^* = Ns/(N-s) > r$. Since $1/N < 1/s < 1/q$, there exists $\theta \in (0,1)$ such that

$$\frac{1}{s} = (1-\theta)\frac{1}{q} + \theta\frac{1}{N}.$$

By Hölder's inequality

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla u|^s dx &= \int_{\mathbb{R}^N} |\nabla u|^{(1-\theta)s} |\nabla u|^{\theta s} dx \\ &\leq \left( \int_{\mathbb{R}^N} |\nabla u|^q dx \right)^{s(1-\theta)/q} \left( \int_{\mathbb{R}^N} |\nabla u|^N dx \right)^{\theta s/N} \\ &\leq \|u\|^{s(1-\theta)}_{E^{N,q}} \|u\|^{s\theta}_{E^{N,q}} = \|u\|^s_{E^{N,q}}, \end{aligned}$$

and therefore the embedding $E^{N,q} \hookrightarrow D^{1,s}(\mathbb{R}^N) \hookrightarrow L^{s^*}(\mathbb{R}^N)$ is continuous. Since $E^{N,q} \hookrightarrow D^{1,q}(\mathbb{R}^N) \hookrightarrow L^{q^*}(\mathbb{R}^N)$ and $q^* \leq r < s^*$, we conclude by interpolation that the embedding $E^{N,q} \hookrightarrow L^r(\mathbb{R}^N)$ is continuous.

We deal now with the compactness of $E^{p,q}_{\mathrm{rad}} \hookrightarrow L^r(\mathbb{R}^N)$. Let $(u_n) \subset E^{p,q}_{\mathrm{rad}}$ be such that $u_n \rightharpoonup 0$ weakly in $E^{p,q}_{\mathrm{rad}}$. For any $R > 0$, we see that $(u_n) \subset W^{1,q}(B_R)$ and hence $u_n \to 0$ in $L^r(B_r)$ by the compact embedding $W^{1,q}(B_R) \hookrightarrow L^r(B_R)$. On the other hand, by the result proved in [27, Lemma 1], there exists $C = C(N,q) > 0$ such that,

$$|u(x)| \leq C|x|^{-(N-q)/q} \|\nabla u\|_{L^q}, \quad \text{for a.e. } x \neq 0.$$

for any $u \in D^{1,q}_{\mathrm{rad}}(\mathbb{R}^N)$. Hence, by using that $E^{p,q}_{\mathrm{rad}} \subset D^{1,q}_{\mathrm{rad}}(\mathbb{R}^N)$ and $r > q^*$, for every given $\varepsilon > 0$ we obtain $R > 0$ large in such a way that

$$\begin{aligned} \int_{\mathbb{R}^N \setminus B_R} |u_n|^r dx &= \int_{\mathbb{R}^N \setminus B_R} |u_n|^{r-q^*} |u_n|^{q^*} dx \\ &\leq C_1 R^{-(r-q^*)(N-q)/q} \|\nabla u_n\|^{r-q^*}_{L^q} \int_{\mathbb{R}^N \setminus B_R} |u_n|^{q^*} dx \\ &< C_2 \varepsilon \|\nabla u_n\|^r_{L^q}, \end{aligned}$$

where we used the embedding $D^{1,q}_{\mathrm{rad}}(\mathbb{R}^N) \hookrightarrow L^{q^*}(\mathbb{R}^N)$. The result follows from the boundedness of $(u_n)$ in $E^{p,q}$. □

We are ready to prove our first main result.

*Proof of Theorem 1.1.* Let $\alpha \in (0, \alpha_N)$, $u \in E^{N,q}$ and denotes by $u^*$ the Schwarz symmetrization $u$. We know that $u^*$ is radial and non-increasing. Moreover (see [19]),

$$\int_{\mathbb{R}^N} |\nabla u^*|^N dx \leq \int_{\mathbb{R}^N} |\nabla u|^N dx, \quad \int_{\mathbb{R}^N} |\nabla u^*|^q dx \leq \int_{\mathbb{R}^N} |\nabla u|^q dx,$$

and

$$\int_{\mathbb{R}^N} \Phi_{\alpha,j_0}(u^*)dx = \int_{\mathbb{R}^N} \Phi_{\alpha,j_0}(u)dx.$$

Hence, we can assume that $u$ is radial and non-increasing. By Proposition 2.1 and interpolation, we can also guarantee that $u \in W^{1,N}(B_R)$. Thus, by using that $u$ is radial and the classical Sobolev embedding theorem, we may assume that $u$ is continuous, in such away that

$$v := u - u(R) \in W_0^{1,N}(B_R).$$

We can use the limit

$$\lim_{t\to+\infty} \frac{(1+t)^{N/(N-1)}}{t^{N/(N-1)}} = 1,$$

to obtain $t_0 = t_0(\varepsilon) > 0$ such that

$$(1+t)^{N/(N-1)} \le (1+\varepsilon)t^{N/(N-1)}, \quad \forall\, t \ge t_0,$$

where $\varepsilon > 0$ satisfies

$$\alpha(1+\varepsilon) < \alpha_N. \tag{2.1}$$

By continuity,

$$(1+t)^{N/(N-1)} \le (1+\varepsilon)t^{N/(N-1)} + C(\varepsilon), \quad \forall\, t \ge 0,$$

for some $C(\varepsilon) > 0$. By using this inequality, we deduce that

$$\begin{aligned} |u(x)|^{N/(N-1)} &\le (|v(x)| + |u(R)|)^{N/(N-1)} \\ &\le (1+\varepsilon)|v(x)|^{N/(N-1)} + C(\varepsilon)|u(R)|^{N/(N-1)}. \end{aligned} \tag{2.2}$$

Thus, for each $\alpha > 0$, it follows from the classical Trudinger-Moser inequality (see [22, 28]) that

$$\begin{aligned} \int_{B_R} \Phi_{\alpha,j_0}(u)dx &\le \int_{B_R} e^{\alpha|u|^{N/(N-1)}}dx \\ &\le e^{\alpha C(\varepsilon)|u(R)|^{N/(N-1)}} \int_{B_R} e^{\alpha(1+\varepsilon)|v|^{N/(N-1)}}dx < +\infty. \end{aligned}$$

If $\|u\|_{E^{N,q}} \le 1$, one deduce that

$$\int_{B_R} |\nabla v|^N dx = \int_{B_R} |\nabla u|^N dx \le \|u\|_{E^{N,q}}^N \le 1.$$

Hence, we can use (2.2) again, (2.1) and the classical Trudinger-Moser inequality to obtain $C_1 = C_1(R) > 0$ such that

$$\begin{aligned} \int_{B_R} \Phi_{\alpha,j_0}(u)dx &\le \int_{B_R} e^{\alpha|u|^{N/(N-1)}}dx \\ &\le e^{\alpha C(\varepsilon)|u(R)|^{N/(N-1)}} \int_{B_R} e^{(1+\varepsilon)\alpha|v|^{N/(N-1)}}dx \le C_1, \end{aligned}$$

which implies that

$$\sup_{u\in E^{N,q},\, \|u\|_{E^{N,q}}\le 1} \int_{B_R} \Phi_{\alpha,j_0}(u)dx \le C_1. \tag{2.3}$$

On the other hand,

$$\int_{\mathbb{R}^N\setminus B_R} \Phi_{\alpha,j_0}(u)dx = \sum_{j=j_0}^{\infty} \frac{\alpha^j}{j!} \int_{\mathbb{R}^N\setminus B_R} |u|^{Nj/(N-1)}dx. \tag{2.4}$$

Recalling that $u \in L^{q^*}(\mathbb{R}^N)$, we can use the result proved in [8, Lemma A.IV] to obtain $C_{N,q} > 0$ such that

$$|u(x)| \leq C_{N,q}|x|^{-N/q^*}\|u\|_{L^{q^*}}, \quad \text{for any } x \neq 0. \tag{2.5}$$

We have that

$$D_j := \sum_{j=j_0}^{\infty} \frac{\alpha^j}{j!} \int_{\mathbb{R}^N\setminus B_R} |u|^{Nj/(N-1)}dx = \sum_{j=j_0}^{\infty} \frac{\alpha^j}{j!} \int_{\mathbb{R}^N\setminus B_R} |u|^{-q^*+Nj/(N-1)}|u|^{q^*}dx.$$

Consequently, since $Nj/(N-1) \geq q^*$ for any $j \geq j_0$,

$$\begin{aligned} D_j &\leq \frac{C_2^{-q^*}}{\|u\|_{L^{q^*}}^{q^*}} \sum_{j=j_0}^{\infty} \frac{\alpha^j}{j!} \left(C_2\|u\|_{L^{q^*}}\right)^{Nj/(N-1)} \int_{\mathbb{R}^N\setminus B_R} |u|^{q^*}dx \\ &\leq C_2^{-q^*} \sum_{j=j_0}^{\infty} \frac{\left[\alpha\left(C_2\|u\|_{L^{q^*}}\right)^{N/(N-1)}\right]^j}{j!}, \end{aligned}$$

where $C_2 := R^{-N/q^*}C_{N,q}$. Thus, by using (2.4) and Proposition 2.1, we get

$$\int_{\mathbb{R}^N\setminus B_R} \Phi_{\alpha,j_0}(u)dx \leq C_2^{-q^*} e^{\alpha\left(C_2\|u\|_{L^{q^*}}\right)^{N/(N-1)}} \leq C_3 e^{\alpha\left(C_3\|u\|_{E^{N.q}}\right)^{N/(N-1)}},$$

with $C_3 = C_3(N,q,R) > 0$. This expression and (2.3) imply that

$$L(\alpha,N,q) \leq C_1 + C_3 e^{\alpha C_3^{N/(N-1)}} < +\infty.$$

It remains to be proved that $L(\alpha,N,q) = +\infty$ if $\alpha > \alpha_N$. In order to do that, we are going to use the so-called Moser's sequence (see e.g., [22]), given by

$$M_n(x) := \frac{1}{\omega_{N-1}^{1/N}} \begin{cases} (\log n)^{(N-1)/N} & \text{if} \quad |x| \leq 1/n, \\ \dfrac{\log(1/|x|)}{(\log n)^{1/N}} & \text{if} \quad 1/n \leq |x| \leq 1, \\ 0 & \text{if} \quad |x| \geq 1, \end{cases}$$

for each $n \in \mathbb{N}$. One can easily check that $M_n \in C_0^\infty(B_R) \subset E^{N,q}$ and a straightforward computation shows that

$$\int_{\mathbb{R}^N} |\nabla M_n|^N dx = \frac{1}{\log n} \int_{1/n}^1 \frac{1}{s^N} s^{N-1} ds = \frac{1}{\log n} (\log s)\Big|_{1/n}^1 = 1,$$

and

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla M_n|^q dx &= \frac{\omega_{N-1}}{\omega_{N-1}^{q/N}(\log n)^{q/N}} \int_{1/n}^1 \frac{1}{s^q} s^{N-1} ds = \frac{\omega_{N-1}^{1-q/N}}{(\log n)^{q/N}(N-q)} s^{N-q}\Big|_{1/n}^1 \\ &= \frac{\omega_{N-1}^{1-q/N}}{(\log n)^{q/N}(N-q)} \left(1 - \frac{1}{n^{N-q}}\right) =: D_n. \end{aligned}$$

Hence

$$\|M_n\|_{E^{N,q}}^{N/(N-1)} = \left(\int_{\mathbb{R}^N} |\nabla M_n|^N dx + \left(\int_{\mathbb{R}^N} |\nabla M_n|^q dx\right)^{N/q}\right)^{1/(N-1)}$$
$$= \left(1 + D_n^{N/q}\right)^{1/(N-1)} =: F_n.$$

Setting $\tilde{M}_n := M_n/\|M_n\|_{E^{N,q}}$, we see that $\tilde{M}_n \in E^{N,q}$ and $\|\tilde{M}_n\|_{E^{N,q}} = 1$. Since

$$|\tilde{M}_n(x)|^{N/(N-1)} = \frac{|M_n(x)|^{N/(N-1)}}{\|M_n\|_{E^{N,q}}^{N/(N-1)}} = \frac{\log n}{\omega_{N-1}^{1/(N-1)} F_n}, \quad \text{for } x \in B_{1/n},$$

we have that

$$\int_{B_1} e^{\alpha|\tilde{M}_n|^{N/(N-1)}} dx \geq \int_{B_{1/n}} e^{\alpha|\tilde{M}_n|^{N/(N-1)}} dx \geq n^{F_n^{-1}\alpha/\omega_{N-1}^{1/(N-1)}} \int_{B_{1/n}} dx$$
$$= \frac{\omega_{N-1}}{N} n^{F_n^{-1}\alpha/\omega_{N-1}^{1/(N-1)} - N}. \tag{2.6}$$

But $\alpha > \alpha_N$ implies that $\alpha/\omega_{N-1}^{1/(N-1)} = N + \delta$, for some $\delta > 0$. Hence,

$$F_n^{-1}\frac{\alpha}{\omega_{N-1}^{1/(N-1)}} - N = N\left[\frac{1 - \left(1 + D_n^{N/q}\right)^{1/(N-1)}}{\left(1 + D_n^{N/q}\right)^{1/(N-1)}}\right] + \frac{\delta}{\left(1 + D_n^{N/q}\right)^{1/(N-1)}}.$$

Since $0 < q < N$ implies that $D_n \to 0$, as $n \to +\infty$, we conclude that the exponent on the last term of (2.6) is positive when $n$ is large. Thus,

$$\lim_{n\to+\infty} \int_{B_1} e^{\alpha|\tilde{M}_n|^{N/(N-1)}} dx = +\infty.$$

On the other hand,

$$\int_{B_1} \Phi_{\alpha,j_0}(\tilde{M}_n) dx + \sum_{j=0}^{j_0-1} \frac{\alpha^j}{j!} \int_{B_1} |\tilde{M}_n|^{Nj/(N-1)} dx = \int_{B_1} e^{\alpha|\tilde{M}_n|^{N/(N-1)}} dx.$$

Since the embedding $W_0^{1,N}(B_1) \hookrightarrow L^r(B_1)$ is continuous for any $r \geq 1$, it follows that

$$\sum_{j=0}^{j_0-1} \frac{\alpha^j}{j!} \int_{B_1} |\tilde{M}_n|^{Nj/(N-1)} dx \leq C_3,$$

and therefore

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} \Phi_{\alpha,j_0}(\tilde{M}_n) dx \geq \lim_{n\to+\infty} \int_{B_1} \Phi_{\alpha,j_0}(\tilde{M}_n) dx = +\infty,$$

which concludes the proof. □

**Remark 2.2.** *When $N = 2$, we car argue as in [25] to prove that*

$$L(4\pi, 2, q) = \sup_{\|u\|_{E^{2,q}} \leq 1} \int_{B_R} \Phi_{4\pi,j_0}(u) dx < +\infty.$$

*Indeed, let $u \in E^{2,q}$ and $R > 0$. As in the proof of Theorem 1.1, we can assume that $u$ is radial and nonincreasing. The function $v(x) := u(x) - u(R)$ belongs to $v \in H_0^1(B_R)$ and Young's inequality provides*

$$u^2 = v^2 + 2vu(R) + u^2(R) \leq \left[1 + u^2(R)\right] v^2 + 1 + u^2(R) = w^2 + 1 + u^2(R),$$

*in $B_R$, where $w := \left(1+u^2(R)\right)^{1/2} v \in H_0^1(B_R)$. If $\|u\|_{E^{2,q}} \leq 1$, it follows from the definition of the norm in $E^{2,q}$, (2.5) and the embedding $D^{1,q}(\mathbb{R}^2) \hookrightarrow L^{q^*}(\mathbb{R}^2)$, that*

$$\begin{aligned}\int_{B_R} |\nabla w|^2 dx = \left(1+u^2(R)\right) \int_{B_R} |\nabla u|^2 dx &\leq \left(1+u^2(R)\right)\left(1-\|\nabla u\|^2_{L^q(\mathbb{R}^2)}\right) \\ &\leq 1-\|\nabla u\|^2_{L^q(\mathbb{R}^2)} + C^2 R^{-2/q^*}\|u\|^2_{L^{q^*}(\mathbb{R}^2)} \\ &\leq 1-\|\nabla u\|^2_{L^q(\mathbb{R}^2)} + C_1 R^{-2/q^*}\|\nabla u\|^2_{L^q(\mathbb{R}^2)},\end{aligned}$$

*for some $C_1 > 0$. We now fix $R > 0$ such that $C_1 R^{-2/q^*} < 1$ to conclude that $\|\nabla w\|_{L^2(B_R)} \leq 1$. Hence, we can use the classical Trudinger-Moser inequality to get*

$$\int_{B_R} \Phi_{4\pi,j_0}(u)dx \leq \int_{B_R} e^{4\pi u^2} dx \leq e^{4\pi(1+u^2(R))} \int_{B_R} e^{4\pi w^2} dx \leq C_2,$$

*from which we conclude that*

$$\sup_{u\in E^{2,q},\, \|u\|_{E^{2,q}}\leq 1} \int_{B_R} \Phi_{4\pi,j_0}(u)dx \leq C_3.$$

*Arguing as in the proof of Theorem 1.1, we can prove that the integral on the complement of the ball $B_R$ is uniformly bounded and the result follows.*

## 3. Proof of Theorem 1.2

In this section, we obtain a weak solution for the equation $(\mathcal{P})$. The idea is consider the associated energy functional $I : E^{N,q} \to \mathbb{R}$ given by

$$I(u) = \frac{1}{N}\int_{\mathbb{R}^N} |\nabla u|^N dx + \frac{1}{q}\int_{\mathbb{R}^N} |\nabla u|^q dx - \int_{\mathbb{R}^N} F(u)dx.$$

In order to see that $I$ is well-defined, we pick $\varepsilon > 0$, $\alpha > \alpha_0$ and $r \geq 1$, and use $(f_0) - (f_1)$ to obtain $C > 0$ such that, for any $s \in \mathbb{R}$,

$$\begin{cases} |f(s)| \leq \varepsilon|s|^{\gamma-1} + C|s|^{r-1}\Phi_{\alpha,j_0}(s), \\ |F(s)| \leq \varepsilon|s|^{\gamma} + C|s|^{r}\Phi_{\alpha,j_0}(s). \end{cases} \tag{3.1}$$

For any $t \geq 1$, the following inequality holds (see [30, Lemma 2.1]):

$$(\Phi_{\alpha,j_0}(s))^t \leq \Phi_{t\alpha,j_0}(s), \quad \text{for all } s \in \mathbb{R}. \tag{3.2}$$

Hence, for any given $u \in E^{N,q}$, by Hölder's inequality, Proposition 2.1 and Theorem 1.1, we infer that

$$\int_{\mathbb{R}^N} |F(u)|dx \leq \varepsilon\|u\|^{\gamma}_{L^\gamma} + C\|u\|^r_{L^{r_1 r}}\left(\int_{\mathbb{R}^N} \Phi_{r_2\alpha,j_0}(u)dx\right)^{1/r_2} < +\infty,$$

whenever $r_1, r_2 > 1$ satisfies $1/r_1 + 1/r_2 = 1$ and $r_1 r \geq q^*$. This inequality, together with standard arguments, shows that $I \in C^1(E^{N,q}, \mathbb{R})$ and

$$I'(u)\varphi = \int_{\mathbb{R}^N} \left[|\nabla u|^{N-2} + |\nabla u|^{q-2}\right](\nabla u \cdot \nabla\varphi)dx - \int_{\mathbb{R}^N} f(u)\varphi\, dx,$$

for any $u$, $\varphi \in E^{N,q}$. Hence, the critical points of $I$ are the weak solutions of $(\mathcal{P})$.

Inspired by [6, Lemma 5.1 ], we have the following version of the Principle of Symmetric Criticality due to Palais [23].

**Proposition 3.1.** *Suppose that $1<q<N$ and $(f_0)-(f_1)$ hold. If $u\in E^{N,q}_{\mathrm{rad}}$ is a critical point of $I$ restricted to $E^{N,q}_{\mathrm{rad}}$, then $u$ is a weak solution of equation $(\mathcal{P})$.*

*Proof.* For any $u\in E^{N,q}_{\mathrm{rad}}$ fixed, we consider the linear functional $T_u: E^{N,q}\to\mathbb{R}$ defined by

$$T_u(w):=\int_{\mathbb{R}^N}|\nabla u|^{N-2}\nabla u\nabla w dx+\int_{\mathbb{R}^N}|\nabla u|^{q-2}\nabla u\nabla w dx-\int_{\mathbb{R}^N}f(u)w dx.$$

We claim that $T_u$ is continuous on $E^{N,q}$. In fact, by Holder's inequality, we get

$$\left|\int_{\mathbb{R}^N}|\nabla u|^{N-2}\nabla u\nabla w dx\right|\le\|\nabla u\|^{N-1}_{L^N}\|\nabla w\|^{N}_{L^N}\le C_1\|w\|_{E^{N,q}}$$

and

$$\left|\int_{\mathbb{R}^N}|\nabla u|^{q-2}\nabla u\nabla w dx\right|\le\|\nabla u\|^{q-1}_{L^q}\|\nabla w\|^{q}_{L^q}\le C_2\left(\|w\|^N_{E^{N,q}}\right)^{1/N}=C_2\|w\|_{E^{N,q}},$$

where $C_1=C_1(u),C_2=C_2(u)>0$. If $r>\max\{3,q^*\}$, then $(r-1)r/2>q^*$. Hence, we can use Hölder's inequality, (3.1) and (3.2), to get

$$\begin{aligned}\left|\int_{\mathbb{R}^N}f(u)w dx\right|&\le\varepsilon\int_{\mathbb{R}^N}|u|^{\gamma-1}|w|dx+C\int_{\mathbb{R}^N}|u|^{r-1}\Phi_{\alpha,j_0}(u)|w|dx\\&\le\varepsilon\|u\|^{\gamma-1}_{L^\gamma}\|w\|^{\gamma}_{L^\gamma}\\&\quad+C\|u\|^{r-1}_{L^{(r-1)r/2}}\left(\int_{\mathbb{R}^N}\Phi_{r\alpha/(r-3),j_0}(u)dx\right)^{(r-3)/r}\|w\|^r_{L^r},\end{aligned}$$

where we have used $2/r+(r-3)/r+1/r=1$. Recalling that $\gamma\ge q^*$, we can apply Proposition 2.1 and Theorem 1.1, to obtain $C_3>0$ such that

$$\left|\int_{\mathbb{R}^N}f(u)w\,dx\right|\le C_3\|w\|_{E^{N,q}}.$$

All together, the above estimates show that $T_u$ is continuous on $E^{N,q}$.

By uniform convexity, there exists an unique $\overline{u}\in E^{N,q}$ such that $T_u(\overline{u})=\|T_u\|_{(E^{N,q})'}$, where $(E^{N,q})'$ denotes the dual space of $E^{N,q}$. Now, suppose that $u\in E^{N,q}_{\mathrm{rad}}$ verifies $T_u(w)=0$, for any $w\in E^{N,q}_{\mathrm{rad}}$, and denote by $\mathcal{O}(N)$ the group of orthogonal transformations in $\mathbb{R}^N$. Since $u$ is radial, we have that

$$T_u(gw)=T_u(w)\quad\text{and}\quad\|gw\|_{E^{N,q}}=\|w\|_{E^{N,q}},\quad\text{for each}\ \ g\in\mathcal{O}(N).$$

By picking $w=\overline{u}$, we infer from uniqueness that $g\overline{u}=\overline{u}$, for all $g\in\mathcal{O}(N)$, which means that $\overline{u}\in E^{N,q}_{\mathrm{rad}}$. Consequently, $\|T_u\|_{(E^{N,q})'}=T_u(\bar{u})=0$ and the proposition is proved. □

The next result shows that $I$ has the mountain pass geometry.

**Lemma 3.2.** *Suppose that $1<q<N$ and $(f_0)-(f_2)$ hold. There exist $\tau,\rho>0$ such that $I(u)\ge\tau$, if $\|u\|_{E^{N,q}}=\rho$. Moreover, there exists $e\in E^{N,q}$, with $\|e\|_{E^{N,q}}>\rho$, such that $I(e)<0$.*

*Proof.* Let $r>\max\{N,q^*\}$ and $r_1$, $r_2>1$ be such that $1/r_1+1/r_2=1$. By using Hölder's inequality, (3.2) and

$$\Phi_\alpha(ts)=\phi_{\alpha t^{N/(N-1)}}(s),\quad s\in\mathbb{R},\ t>0,\tag{3.3}$$

we get

$$\int_{\mathbb{R}^N} |u|^r \Phi_{\alpha,j_0}(u)dx \leq \|u\|^r_{L^{r_1 r}} \left( \int_{\mathbb{R}^N} \Phi_{r_2 \alpha \|u\|^{N/(N-1)}_{E^{N,q}}} \left( \frac{u}{\|u\|_{E^{N,q}}} \right) dx \right)^{1/r_2}.$$

If $\rho_0 > 0$ is such that $r_2 \alpha \rho_0^{N/(N-1)} < \alpha_N$, we can apply Theorem 1.1 and use the second inequality in (3.1) to obtain $C_1 > 0$ such that

$$\int_{\mathbb{R}^N} F(u)dx \leq \varepsilon C_1 \|u\|^\gamma_{E^{N,q}} + C_1 \|u\|^r_{E^{N,q}},$$

for any $\varepsilon > 0$ and $\|u\|_{E^{N,q}} \leq \rho_0$. Hence,

$$I(u) \geq \frac{1}{N} \int_{\mathbb{R}^N} |\nabla u|^N dx + \frac{1}{q} \int_{\mathbb{R}^N} |\nabla u|^q dx - \varepsilon C_1 \|u\|^\gamma_{E^{N,q}} - C_1 \|u\|^r_{E^{N,q}}.$$

If $\|u\|_{E^{N,q}} = \rho := \min\{1, \rho_0\}$, we have that $\|\nabla u\|_{L^q} \leq 1$. Hence, we infer from $q < N$ that $\|\nabla u\|^N_{L^q} \leq \|\nabla u\|^q_{L^q}$, and therefore

$$\begin{aligned} I(u) &\geq \frac{1}{N} \int_{\mathbb{R}^N} |\nabla u|^N dx + \frac{1}{N} \left( \int_{\mathbb{R}^N} |\nabla u|^q dx \right)^{N/q} - \varepsilon C_1 \|u\|^\gamma_{E^{N,q}} - C_1 \|u\|^r_{E^{N,q}} \\ &= \|u\|^N_{E^{N,q}} \left( \frac{1}{N} - \varepsilon C_1 \|u\|^{\gamma - N}_{E^{N,q}} - C_1 \|u\|^{r-N}_{E^{N,q}} \right). \end{aligned}$$

Since $\gamma \geq N$ and $r > N$, we can pick $0 < \varepsilon < 1/(NC_1)$ in the above expression to conclude that the first statement of the lemma holds if $\rho > 0$ is small enough.

For the second one, we pick a nonzero function $\varphi \in C_0^\infty(\mathbb{R}^N)$ with support contained in the ball $B_R$. From $(f_2)$, there exist $C_2, C_3 > 0$ such that $F(s) \geq C_2 |s|^\mu - C_3$, for any $s \in \mathbb{R}$. Therefore,

$$I(t\varphi) \leq C_4 \frac{t^N}{N} + C_5 \frac{t^q}{q} - C_2 t^\mu \int_{B_R} |\varphi|^\mu dx + C_3 \int_{B_R} dx.$$

Since $\mu > \gamma \geq N$, it is sufficient to take $e = t\varphi$ with $t > 0$ sufficiently large. □

In view of Lemma 3.2, the minimax level

$$c_* := \inf_{g \in \Gamma} \max_{t \in [0,1]} I(g(t)),$$

where $\Gamma := \left\{ g \in C([0,1], E^{N,q}) : g(0) = 0 \text{ and } I(g(1)) < 0 \right\}$ is well-defined and positive. Moreover, the following estimate holds:

**Lemma 3.3.** *There exists $\xi_0 > 0$ sufficiently large such that, if $(f_3)$ is satisfied with $\xi \geq \xi_0$, then the Mountain Pass level verifies*

(3.4)
$$c_* < c_0 := \min \left\{ \frac{1}{2^N} \left( \frac{\alpha_N}{\alpha_0} \right)^{N-1} \left( \frac{\mu - N}{N\mu} \right)^{N/q}, \frac{1}{2^q} \left( \frac{\alpha_N}{\alpha_0} \right)^{q(N-1)/N} \left( \frac{\mu - N}{N\mu} \right) \right\}.$$

*Proof.* Let $\varphi \in C_0^\infty(\mathbb{R}^N)$ be such that $\varphi \equiv 1$ in $B_{1/2}$, $\varphi \equiv 0$ in $\mathbb{R}^N \setminus B_1$, $0 \leq \varphi \leq 1$ and $|\nabla \varphi(x)| \leq 2$, for any $x \in \mathbb{R}^N$. A simple calculation shows that

$$\frac{1}{N} \int_{B_1} |\nabla \varphi|^N dx + \frac{1}{q} \int_{B_1} |\nabla \varphi|^q dx \leq \left( \frac{2^N}{N} + \frac{2^q}{q} \right) \frac{\omega_{N-1}}{N}.$$

This inequality and $(f_3)$ imply that

$$\begin{aligned} I(t\varphi) &\le \frac{t^N}{N}\int_{B_1}|\nabla\varphi|^N dx + \frac{t^q}{q}\int_{B_1}|\nabla\varphi|^q dx - \int_{B_{1/2}} F(t\varphi)dx \\ &\le t^q\left(\frac{2^N}{N}+\frac{2^q}{q}\right)\frac{\omega_{N-1}}{N} - t^\nu \xi \frac{\omega_{N-1}}{2^N N}, \end{aligned}$$

for any $t\in[0,1]$. We first assume that $\xi>\xi_1$, where $\xi_1>0$ is such that

$$I(\varphi) \le \left(\frac{2^N}{N}+\frac{2^q}{q}\right)\frac{\omega_{N-1}}{N} - \xi_1\frac{\omega_{N-1}}{2^N N} < 0,$$

in such a way that path $g(t)=t\varphi$ belongs to $\Gamma$. From the definition of $c_*$, we obtain

$$\begin{aligned} c_* &\le \max_{t\in[0,1]} I(t\varphi) \le \frac{\omega_{N-1}}{N}\max_{t\ge 0}\left[t^q\left(\frac{2^N}{N}+\frac{2^q}{q}\right) - t^\nu\frac{\xi}{2^N}\right] \\ &= \frac{\omega_{N-1}}{N}\left(\frac{2^N}{N}+\frac{2^q}{q}\right)^{\nu/(\nu-q)}\left(\frac{2^N}{\xi}\right)^{q/(\nu-q)}\left[\left(\frac{q}{\nu}\right)^{q/(\nu-q)} - \left(\frac{q}{\nu}\right)^{\nu/(\nu-q)}\right]. \end{aligned}$$

Since $\nu>\gamma\ge N>q$, the last term above goes to 0, as $\xi\to+\infty$. Hence, it is clear that there exists $\xi_0>\xi_1$ sufficiently large such that $c_*<c_0$, whenever $\xi>\xi_0$. $\square$

We recall that $I$ satisfies the Palais-Smale condition at level $c\in\mathbb{R}$, if any sequence $(u_n)\subset E^{N,q}$ such that

$$\lim_{n\to+\infty} I(u_n)=c \quad \text{and} \quad \lim_{n\to+\infty}\|I'(u_n)\|_{(E^{N,q})'}=0 \tag{3.5}$$

has a convergent subsequence.

**Lemma 3.4.** *Suppose that $1<q<N$ and $(f_0)-(f_2)$ hold. If $c_0>0$ is as in (3.4), then the functional $I$ restricted to $E^{N,q}_{\mathrm{rad}}$ satisfies the Palais-Smale condition at any level $0<c<c_0$.*

*Proof.* Let $(u_n)\subset E^{N,q}_{\mathrm{rad}}$ be such that (3.5) holds. By using $(f_2)$ and $1<q<N<\mu$, we obtain $C_1,C_2>0$ such that

$$\begin{aligned} C_1 + C_2\|u_n\|_{E^{N,q}} &\ge I(u_n) - \frac{1}{\mu}I'(u_n)u_n \\ &\ge \left(\frac{1}{N}-\frac{1}{\mu}\right)\left(\|\nabla u_n\|^N_{L^N} + \|\nabla u_n\|^q_{L^q}\right), \end{aligned}$$

which implies that $(u_n)$ is bounded in $E^{N,q}_{\mathrm{rad}}$. Thus, up to a consequence, we have that $u_n \rightharpoonup u$ weakly in $E^{N,q}_{\mathrm{rad}}$.

We claim that

$$\lim_{n\to+\infty}\|u_n\|^{N/(N-1)}_{E^{N,q}} < \frac{\alpha_N}{\alpha_0}. \tag{3.6}$$

If this is true, we can pick $\alpha>\alpha_0$ and $r_1>1$ such that $r_1\alpha\|u_n\|^{N/(N-1)}_{E^{N,q}}<\alpha_N$, for large $n\in\mathbb{N}$. From (3.1) with $r\ge q^*+1$, Hölder's inequality, (3.2) and (3.3), we infer that

$$\left|\int_{\mathbb{R}^N} f(u_n)(u_n-u)dx\right| \le \varepsilon\|u_n\|^{\gamma-1}_{L^\gamma}\|u_n-u\|_{L^\gamma}$$

$$+C\,\|u_n\|^{r-1}_{L^{r_2(r-1)}}\left(\int_{\mathbb{R}^N}\Phi_{r_1\alpha\|u_n\|^{N/(N-1)}_{E^{N,q}},j_0}\left(\frac{u_n}{\|u_n\|_{E^{N,q}}}\right)dx\right)^{1/r_1}\|u_n-u\|_{L^{r_3}},$$

where $1/r_1+1/r_2+1/r_3=1$ and $r_3>q^*$. It follows from the boundedness of $(u_n)$, Theorem 1.1 and the compactness in Proposition 2.1 that

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N} f(u_n)(u_n-u)dx=0. \tag{3.7}$$

We proceed now with the proof of (3.6). To this purpose, take into account that $(u_n)$ is bounded and using $(f_2)$ together with the fact $1<N<q<\mu$, we get

$$c=\lim_{n\to+\infty}\left(I(u_n)-\frac{1}{\mu}I'(u_n)u_n\right)\geq \lim_{n\to+\infty}\left(\frac{1}{N}-\frac{1}{\mu}\right)\left(\|\nabla u_n\|_{L^N}^N+\|\nabla u_n\|_{L^q}^q\right),$$

from which it follows that

$$\begin{cases}\displaystyle\lim_{n\to+\infty}\|\nabla u_n\|_{L^N}^N\leq c\left(\frac{N\mu}{\mu-N}\right)\\[2ex] \displaystyle\lim_{n\to+\infty}\|\nabla u_n\|_{L^q}^N\leq c^{N/q}\left(\frac{N\mu}{\mu-N}\right)^{N/q}.\end{cases}$$

Hence,

$$\begin{aligned}\lim_{n\to+\infty}\|u_n\|_{E^{N,q}}^{N/(N-1)}\leq{}&(2c)^{1/(N-1)}\left(\frac{N\mu}{\mu-N}\right)^{1/(N-1)}\\&+(2c^{N/q})^{1/(N-1)}\left(\frac{N\mu}{\mu-N}\right)^{N/[q(N-1)]}.\end{aligned}$$

Since $N\mu/(\mu-N)>1$ and $1<q<N$, we get

$$\left(\frac{N\mu}{\mu-N}\right)^{1/(N-1)}<\left(\frac{N\mu}{\mu-N}\right)^{N/[q(N-1)]}.$$

Therefore,

$$\lim_{n\to+\infty}\|u_n\|_{E^{N,q}}^{N/(N-1)}\leq\left[2\left(\frac{N\mu}{\mu-N}\right)^{N/q}\right]^{1/(N-1)}\left(c^{1/(N-1)}+c^{N/[q(N-1)]}\right).$$

If $c<1$, then $c^{1/(N-1)}+c^{N/[q(N-1)]}\leq 2c^{1/(N-1)}$ and therefore

$$\lim_{n\to+\infty}\|u_n\|_{E^{N,q}}^{N/(N-1)}\leq\left(2^N\left(\frac{N\mu}{\mu-N}\right)^{N/q}c\right)^{1/(N-1)}.$$

Otherwise, if $c>1$, then $c^{1/(N-1)}+c^{N/[q(N-1)]}\leq 2c^{N/[q(N-1)]}$ and in this case

$$\lim_{n\to+\infty}\|u_n\|_{E^{N,q}}^{N/(N-1)}\leq\left(2^N\left(\frac{N\mu}{\mu-N}\right)^{N/q}c^{N/q}\right)^{1/(N-1)}.$$

All together, the above estimates show that (3.6) is a consequence of $c<c_0$.

We now notice that, since $\lim_{n\to+\infty}I'(u_n)(u_n-u)=0$, we can use (3.7) to get

$$\int_{\mathbb{R}^N}|\nabla u_n|^{N-2}\nabla u_n\nabla(u_n-u)dx+\int_{\mathbb{R}^N}|\nabla u_n|^{q-2}\nabla u_n\nabla(u_n-u)dx=o_n(1), \tag{3.8}$$

where $o_n(1)$ stands for a quantity approaching zero as $n\to+\infty$. On the other hand, from the weak convergence $u_n\rightharpoonup u$ in $E^{N,q}_{\text{rad}}$, we obtain

$$\int_{\mathbb{R}^N}|\nabla u|^{N-2}\nabla u\nabla(u_n-u)dx+\int_{\mathbb{R}^N}|\nabla u|^{q-2}\nabla u\nabla(u_n-u)dx=o_n(1). \tag{3.9}$$

If we set, for $r \geq 1$,

$$T_{N,r}(y_1, y_2) := \left(|y_1|^{r-2}y_1 - |y_2|^{r-2}y_2\right), \quad y_1, y_2 \in \mathbb{R}^N,$$

we can use [26, equation (2.2)] to write

$$T_{N,r}(y_1, y_2) \cdot (y_1 - y_2) \geq C(N, r) \cdot \begin{cases} |y_1 - y_2|^r, & \text{if } r \geq 2, \\ \dfrac{|y_1 - y_2|^2}{(|y_1| + |y_2|)^{2-r}}, & \text{if } 1 < r < 2, \end{cases} \tag{3.10}$$

for any $y_1$, $y_2 \in \mathbb{R}^N$. It follows from (3.8)-(3.9) that

$$\int_{\mathbb{R}^N} T_{N,r}(\nabla u_n, \nabla u) \cdot (\nabla u_n - \nabla u) dx = o_n(1), \quad r \in \{q, N\}. \tag{3.11}$$

If $2 \leq q < N$, we infer from the above expression and (3.10) that

$$C(N, N)\|\nabla(u_n - u)\|_{L^N}^N + C(N, q)\|\nabla(u_n - u)\|_{L^q}^q \leq o_n(1),$$

from which we conclude that $u_n \to u$ strongly in $E_{\mathrm{rad}}^{N,q}$. If $1 < q < 2 \leq N$, we also have $\|\nabla(u_n - u)\|_{L^N}^N = o_n(1)$. Moreover, if $C_3 := 2^{q-1}C(N, q)^{-q/2}$, we can use Hölder's inequality with exponents $2/q$ and $2/(2-q)$, and the boundedness of $(u_n)$, to get

$$\begin{aligned} \|\nabla(u_n - u)\|_{L^q}^q &\leq C_3 \int_{\mathbb{R}^N} \left[T_{N,q}(\nabla u_n, \nabla u)\nabla(u_n - u)\right]^{q/2} \left[|\nabla u_n|^q + |\nabla u|^q\right]^{(2-q)/2} dx \\ &\leq C_3 \left(\int_{\mathbb{R}^N} T_{N,q}(\nabla u_n, \nabla u)\nabla(u_n - u) dx\right)^{q/2}. \end{aligned}$$

Thus, by using (3.11) with $r = q$, we obtain $\|\nabla(u_n - u)\|_{L^q} = o_n(1)$. This finished the proof. □

We are ready to prove second main result.

*Proof of Theorem 1.2.* Let $\xi_0 > 0$ be given by the last lemma and suppose that $(f_3)$ holds with $\xi > \xi_0$. If we consider the functional $I$ restrict to $E_{\mathrm{rad}}^{N,q}$, it follows from all the above lemmas and the Mountain Pass Theorem [4] that $I$ has a nonzero critical point $u \in E_{\mathrm{rad}}^{N,q}$. By Proposition 3.1, this function is a weak solution for equation $(\mathcal{P})$. □

**ACKNOWLEDGMENTS:** The authors would like to express their sincere gratitude to the anonymous referee for his/her valuable comments and mainly to point out some available references.

## References

[1] S. Adachi; K. Tanaka.: *Trudinger type inequalities in $\mathbb{R}^N$ and their best exponents.* Proc. Amer. Math. Soc. **128**, 2051–2057 (2000).

[2] Adimurthi; K. Sandeep.: *A singular Moser-Trudinger embedding and its applications.* NoDEA Nonlinear Differential Equations Appl. **13**, 585–603 (2007).

[3] M. J. Alves; R. B. Assunção; O. H. Miyagaki.: *Existence result for a class of quasilinear elliptic equations with $(p, q)$-Laplacian and vanishing potentials.* Illinois J. Math. **59**, 545–575 (2015).

[4] A. Ambrosetti; P. H. Rabinowitz.: *Dual variational methods in critical point theory and applications.* J. Funct. Anal. **14**, 349–381 (1973).

[5] V. Ambrosio; V. Rădulescu.: *Fractional double-phase patterns: concentration and multiplicity of solutions.* J. Math. Pures Appl. **142**, 101–145 (2020).

[6] M. Badiale; L. Pisani; S. Rolando.: *Sum of weighted Lebesgue spaces and nonlinear elliptic equations*. Nonlinear Differential Equations Appl. **18**, 369–405 (2011).

[7] L. Baldelli; Y. Brizi; R. Filippucci.: *Multiplicity results for $(p,q)$-Laplacian equations with critical exponent in $\mathbb{R}^N$ and negative energy*. Calc. Var. Partial Differential Equations **60**, (2021) Paper No. 8, 30 pp.

[8] H. Berestycki; P. L. Lions.: *Nonlinear scalar field equations. I. Existence of a ground state.* Arch. Rational Mech. Anal. **82**, 313–345 (1983).

[9] D. Bonheure; P. d'Avenia; A. Pomponio.: *On the electrostatic Born-Infeld equation with extended charges*. Comm. Math. Phys. **346**, 877–906 (2016).

[10] D. Bonheure; A. Iacopetti.: *On the regularity of the minimizers ofr the electrostatic Born-Infeld energy*. Arch. Rational Mech. Anal. **232**, 697–725 (2019).

[11] S. Deng; T. Hu; C.-L. Tang.: *$N-$Laplacian problems with critical double exponential nonlinearities*. Discrete Contin. Dyn. Syst. **41**, 987–1003 (2021).

[12] D. M. Cao.: *Nontrivial solution of semilinear elliptic equation with critical exponent in $\mathbb{R}^2$*. Comm. Partial Differential Equations **17**, 407–435 (1992).

[13] L. Cherfils; Y. Il'yasov.: *On the stationary solutions of generalized reaction diffusion equations with $(p,q)$-Laplacian*. Comm. Pure Appl. Anal. **4**, 9–22 (2005).

[14] J. M. do Ó.: *$N$-Laplacian equations in $\mathbb{R}^N$ with critical growth*. Abstr. Appl. Anal. **2**, 301–315 (1997).

[15] G. M. Figueiredo.: *Existence of positive solutions for a class of $p,q$ elliptic problems with critical growth on $\mathbb{R}^N$*. J. Math. Anal. Appl. **378**, 507–518 (2011).

[16] A. Fiscella; P. Pucci.: *$(p,N)$ equations with critical exponential nonlinearities in $\mathbb{R}^N$*. J. Math. Anal. Appl., to appear.

[17] J. Giacomoni; P.K. Mishra; K. Sreenadh, *Critical growth problems for $1/2-$Laplacian in $\mathbb{R}$*. Differ. Equ. Appl. **8**, 295–317 (2016).

[18] C. He; G. Li.: *The regularity of weak solutions to nonlinear scalar field elliptic equations containing $(p,q)$-Laplacians*. Ann. Acad. Sci. Fenn. Math. **33**, 337–371 (2008).

[19] B. Kawohl.: *Rearrangements and Convexity of Level Sets in PDE*. Lecture Notes in Mathematics, vol. 1150, Springer-Verlag, Berlin, 1985.

[20] Y. Li; B. Ruf.: *A sharp Trudinger-Moser type inequality for unbounded domains in $\mathbb{R}^n$*. Indiana Univ. Math. J. **57**, 451–480 (2008).

[21] G. Lu; H. Tang; M. Zhu.: *Best constants for Adams' inequalities with the exact growth condition in $\mathbb{R}^n$*. Adv. Nonlinear Stud. **15**, 763–788 (2015).

[22] J. Moser.: *A sharp form of an inequality by N. Trudinger*. Indiana Univ. Math. J. **20**, 1077–1092 (1971).

[23] R. S. Palais.: *The principle of symmetric criticality*. Comm. Math. Phys. **69**, 19–30 (1979).

[24] A. Pomponio; T. Watanabe.: *Some quasilinear elliptic equations involving multiple p-Laplacians*. Indiana Univ. Math. J. **67**, 2199–2224 (2018).

[25] B. Ruf.: *A sharp Trudinger-Moser type inequality for unbounded domains in $\mathbb{R}^2$*. J. Funct. Anal. **219**, 340–367 (2005).

[26] J. Simon.: *Regularité de la Solution D'Une Equation Non Lineaire Dans $\mathbb{R}^N$*. Lecture Notes in Math, vol. 665, Springer, Heidelberg, (1978).

[27] J. Su; Z. Q. Wang; M. Willem.: *Weighted Sobolev embedding with unbounded and decaying radial potentials*. J. Differential Equations **238**, 201–219 (2007).

[28] N. S. Trudinger.: *On the embedding into Orlicz spaces and some applications*. J. Math. Mech. **17**, 473–484 (1967).

[29] V. I. Yudovich.: *Some estimates connected with integral operators and with solutions of elliptic equations* Dok. Akad. Nauk SSSR **138** (1961), 804–808, [English translation in Soviet Math. Doklady **2** (1961), 746–749.]

[30] Y. Yunyan.: *Existence of positive solutions to quasi-linear elliptic equations with exponential growth in the whole Euclidean space*. J. Funct. Anal. **262**, 1679–1704 (2012).

Universidade Federal da Paraíba, Departamento de Matemática, 58051-900 João Pessoa-PB, Brazil
*Email address*: jonison.mat@gmail.com

Universidade de Brasília, Departamento de Matemática, 70910-900 Brasília-DF, Brazil
*Email address*: giovany@unb.br

Universidade de Brasília, Departamento de Matemática, 70910-900 Brasília-DF, Brazil
*Email address*: mfurtado@unb.br

Universidade Federal da Paraíba, Departamento de Matemática, 58051-900 João Pessoa-PB, Brazil
*Email address*: everaldo@mat.ufpb.br