CRITICAL BOUNDARY VALUE PROBLEMS IN THE UPPER HALF-SPACE

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Abstract. We consider the critical problem

$$-\Delta u - \frac{1}{2}(x \cdot \nabla u) = 0 \text{ in } \mathbb{R}^N_+, \qquad \frac{\partial u}{\partial \nu} = \lambda |u|^{p-2}u + |u|^{2_*-2}u \text{ on } \partial\mathbb{R}^N_+,$$

where $\mathbb{R}^N_+ = \{(x', x_N) : x' \in \mathbb{R}^{N-1},\, x_N > 0\}$ is the upper half-space, $N \geq 4$, ν is the outward normal vector at the boundary and $2 \leq p < 2_* := 2(N-1)/(N-2)$. Using a variational approach, we obtain nonnegative nonzero solutions according to the value of the parameter $\lambda > 0$.

1. Introduction

Consider the model problem

$$-\Delta v = f(x, v, \nabla v), \text{ in } \mathbb{R}^N_+, \qquad \frac{\partial v}{\partial \nu} = g(x', v), \text{ on } \partial\mathbb{R}^N_+,$$

where $\mathbb{R}^N_+ := \{(x', x_N) : x' \in \mathbb{R}^{N-1},\, x_N > 0\}$ is the upper half-space and ν is the outward normal vector at the boundary $\partial\mathbb{R}^N_+$. Its mathematical importance arises, for instance, in the study of conformal deformation of Riemannian manifolds [12, 13, 17], problems of sharp constant in Sobolev trace inequalities [9,11] and blow-up properties of the solutions of related parabolic equations [15,18]. This kind of equations also appears in several applied contexts like glaciology [21], population genetics [2], non-Newtonian fluid mechanics [10], nonlinear elasticity [8], among others.

There are several works when the function f does not depend on the gradient (see [5–7, 19, 20, 22] and references therein). The problem turns to be more complicated if the function f also depends on ∇v. In [15], the authors considered

$$f(x, v, \nabla v) = \lambda v + \frac{1}{2}(x \cdot \nabla v), \qquad g(v) = |v|^{p-2}v,$$

with $2 < p < 2_*$ and $2_* := 2(N-1)/(N-2)$ being the critical exponent of the Sobolev trace embedding $\mathcal{D}^{1,2}(\mathbb{R}^N_+) \hookrightarrow L^{2^*}(\mathbb{R}^{N-1})$. They obtained existence and nonexistence of solutions according to the value of the parameter $\lambda > 0$. In the same paper, they present the relationship between the problem and the existence of self-similar solution

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of the nonlinear heat equation (see also [14])

$$w_t - \Delta w = 0, \text{ in } \mathbb{R}^N_+ \times (0,+\infty), \qquad \frac{\partial w}{\partial \nu} = |w|^{p-2}w, \text{ on } \mathbb{R}^{N-1} \times (0,+\infty).$$

The critical version of this problem (including the critical Sobolev trace embedding) was recently considered in [16], with

$$f(x,v,\nabla v) = \lambda v + \theta |v|^{2^*-2}v + \frac{1}{2}(x\cdot\nabla v), \qquad g(x',v) = |v|^{2^*-2}v,$$

for $2^* := 2N/(N-2)$, $\lambda > 0$ and $\theta \in \{0,1\}$.

Motivated by the aforementioned works, we deal here with the problem

$$(P_\lambda) \qquad \begin{cases} -\Delta u - \dfrac{1}{2}(x\cdot\nabla u) = 0, & \text{in } \mathbb{R}^N_+, \\ \dfrac{\partial u}{\partial \nu} = \lambda |u|^{p-2}u + |u|^{2_*-2}u, & \text{on } \in \partial\mathbb{R}^N_+ \sim \mathbb{R}^{N-1}, \end{cases}$$

where $N \geq 4$, $\lambda > 0$ is a parameter and $2 \leq p < 2_*$. In order to present our main result, we need to introduce some notation. Setting $K(x) = \exp(|x|^2/4)$ and noticing that $2\nabla K = xK$, the first equation in (P_λ) can be rewritten in the divergence form $\operatorname{div}(K(x)\nabla u) = 0$. Hence, it is natural look for solutions in the Sobolev space $\mathcal{D}^{1,2}_K(\mathbb{R}^N_+)$ defined as the closure of $C^\infty_c(\overline{\mathbb{R}^N_+})$ with respect to the norm

$$\|u\| = \left(\int_{\mathbb{R}^N_+} K(x)|\nabla u|^2 dx\right)^{1/2}.$$

This kind of space was first introduced by Escobedo and Kavian [14], who considered a problem in the whole space $\mathbb{R}^N$. In [16], it is proved that $\mathcal{D}^{1,2}_K(\mathbb{R}^N_+)$ is continuously embedded into the weighted Lebesgue space

$$L^s_K(\mathbb{R}^{N-1}) := \left\{ u \in L^s(\mathbb{R}^{N-1}) : \|u\|_s := \left(\int_{\mathbb{R}^{N-1}} K(x',0)|u|^s dx'\right)^{1/s} < \infty \right\},$$

for any $s \in [2, 2_*]$. Moreover, the embedding is compact if $s = 2$, in such way that we can define the first positive eigenvalue of the linear problem

$$(LP) \qquad -\Delta u - \frac{1}{2}(x\cdot\nabla u) = 0, \text{ in } \mathbb{R}^N_+, \qquad \frac{\partial u}{\partial \nu} = \lambda u, \text{ on } \mathbb{R}^{N-1},$$

namely

$$\lambda_1 := \inf\left\{ \int_{\mathbb{R}^N_+} K(x)|\nabla u|^2 dx \,:\, u \in \mathcal{D}^{1,2}_K(\mathbb{R}^N_+), \int_{\mathbb{R}^{N-1}} K(x',0)|u|^2 dx' = 1 \right\} > 0.$$

We state in what follows the main result of this paper:

Theorem 1.1. *Problem (P_λ) has a nonnegative nontrivial solution in each of the following cases:*

(1) $p = 2$ and $\lambda \in (0, \lambda_1)$;
(2) $p \in (2, 2_*)$ and $\lambda > 0$.

The proof consists in applying variational methods together with some fine estimates of the minimax level of the energy functional associated to (P_λ). The approach is borrowed from the seminal paper of Brezis and Nirenberg [4], where the authors consider

$$-\Delta u = \lambda |u|^{p-2}u + |u|^{2^*-2}u, \quad u \in H_0^1(\Omega),$$

where $\Omega \subset \mathbb{R}^N$ is bounded smooth domain and $2 \le p < 2^*$. Actually, the result proved here is a version for problem (P_λ) of that presented in [4] for $N \ge 4$. It is worth recall that, if $N = 3$ and $p = 2$, Brezis and Nirenberg obtained solutions if λ is lower and close to the first positive eigenvalue of the linear problem associated to the above equation. The same existence result can be obtained here (see Remark 2.3).

We finally mention that, in the case $p = 2$, our result is different from that obtained in [14] for the problem

$$-\Delta u - \frac{1}{2}(x \cdot \nabla u) = \lambda u + |u|^{2^*-2}u \quad \text{in } \mathbb{R}^N.$$

Indeed, the range of existence of solution for the above problem is $(N/4, N/2)$. Even if we consider the half-space case, but with the linear perturbation inside the domain $\mathbb{R}^N_+$, the same kind of effect occurs. Actualy, as proved in [16, Theorem 1.5], the range of existence in this case is $(N/4 + (N-4)/8, N/2)$. Due to the L^2-integrability order of a cutoff of the *instanton* (see (2.1)) in $\mathbb{R}^{N-1}$, we are able to cover here the entire range $\lambda \in (0, \lambda_1)$. This shows that, although there is a close connection between [14, 16] and our paper, the problem studied here presents some new (and interesting) features.

We devote the next section to the proof of our main theorem. The key point is obtaining some fine estimates whose proofs are presented in Section 3.

2. Proof of Theorem 1.1

We start this section defining the energy functional associated to (P_λ), namely

$$I_\lambda(u) := \frac{1}{2}\|u\|^2 - \frac{\lambda}{p}\int_{\mathbb{R}^{N-1}} K(x',0)(u^+)^p dx' - \frac{1}{2_*}\int_{\mathbb{R}^{N-1}} K(x',0)(u^+)^{2_*} dx',$$

where $u^+(x) := \max\{u(x), 0\}$ and and $x' := (x_1, x_2, \ldots, x_{N-1}) \in \mathbb{R}^{N-1}$. It is standard to check that $I_\lambda \in C^1(\mathcal{D}^{1,2}_K(\mathbb{R}^N_+), \mathbb{R})$ and that its critical points are nonnegative solution of (P_λ).

Given $\varepsilon > 0$, set

$$U_\varepsilon(x', x_N) := \frac{\varepsilon^{(N-2)/2}}{[|x'|^2 + (x_N + \varepsilon)^2]^{(N-2)/2}}, \qquad (x', x_N) \in \mathbb{R}^N_+. \tag{2.1}$$

They are the so-called *instantons* which achieves the best constant of the Sobolev trace embedding $\mathcal{D}^{1,2}(\mathbb{R}^N_+) \hookrightarrow L^{2_*}(\mathbb{R}^{N-1})$ (see Escobar [11] and Beckner [3]) given by

$$S := \inf\left\{\int_{\mathbb{R}^N_+} |\nabla\varphi|^2 dx \,:\, u \in \mathcal{D}^{1,2}(\mathbb{R}^N_+),\ \int_{\mathbb{R}^{N-1}} |\varphi|^{2_*} dx' = 1\right\}.$$

The relation between this embedding and that used in this paper is curious. Actually, it is proved in [16] that the best constant of the Sobolev trace embedding $\mathcal{D}^{1,2}_K(\mathbb{R}^N_+) \hookrightarrow$

$L^{2_*}_K(\mathbb{R}^{N-1})$ given by

$$S_K := \inf\left\{\int_{\mathbb{R}^N_+} K(x)|\nabla\varphi|^2dx \,:\, u\in\mathcal{D}^{1,2}_K(\mathbb{R}^N_+),\ \int_{\mathbb{R}^{N-1}} K(x',0)|\varphi|^{2_*}dx' = 1\right\}$$

is also achieved and $S_K = S$.

Let $\phi \in C^\infty(\mathbb{R}^N_+,[0,1])$ be a cut-off function such that $\phi\equiv 1$ in $B_1(0)\cap\mathbb{R}^N_+$ and $\phi\equiv 0$ in $\overline{\mathbb{R}^N_+}\setminus B_2(0)$. We follow an idea of Brezis and Nirenberg [4] and define $\psi_\varepsilon:\mathbb{R}^N_+\to\mathbb{R}$ as

$$\psi_\varepsilon(x) := K(x)^{-1/2}\phi(x)U_\varepsilon(x), \quad x\in\mathbb{R}^N_+.$$

We prove in the sequel that I_λ verifies the geometric conditions of the Moutain Pass Theorem.

Lemma 2.1. *Under the hypotheses of Theorem 1.1, there exist α_λ, $\rho_\lambda > 0$ and $e_\lambda\in\mathcal{D}^{1,2}_K(\mathbb{R}^N_+)$ such that*

(i) $I_\lambda(u)\ge\alpha_\lambda>0$, for any $u\in\mathcal{D}^{1,2}_K(\mathbb{R}^N_+)$ with $\|u\|=\rho_\lambda$;

(ii) $I_\lambda(e_\lambda)<0$ and $\|e_\lambda\|>\rho_\lambda$.

Proof. If $p=2$, we can use $\lambda\in(0,\lambda_1)$ to conclude that

$$I_\lambda(u)\ge\frac{1}{2}\left(\frac{\lambda-\lambda_1}{\lambda_1}\right)\|u\|^2-\frac{1}{2_*}S^{-2_*/2}\|u\|^{2_*},$$

for any $u\in\mathcal{D}^{1,2}_K(\mathbb{R}^N_+)$. Hence, we can use $p<2_*$ to conclude that, for some constants α_λ, $\rho_\lambda>0$, there holds

$$I_\lambda(u)\ge\alpha_\lambda>0, \quad \forall\, u\in B_{\rho_\lambda}(0)\cap\mathcal{D}^{1,2}_K(\mathbb{R}^N_+).$$

If $p\in(2,2_*)$, we can use the trace embedding to obtain C_1, $C_2>0$ such that

$$I_\lambda(u)\ge\frac{1}{2}\|u\|^2\left(1-\lambda C_1\|u\|^{p-2}-C_2\|u\|^{2_*-2}\right),$$

and item (i) easily follows from $2<p<2_*$.

For the second item we notice that, since the nonnegative function ψ_ε is positive in $B_1(0)\cap\mathbb{R}^{N-1}$, then $I_\lambda(t\psi_\varepsilon)\to-\infty$, as $t\to+\infty$. So, there exists $t_{*,\lambda}>0$ such that $e_\lambda:=t_{*,\lambda}\psi_\varepsilon$ is such that $I_\lambda(e_\lambda)<0$ and $\|e_\lambda\|>\rho$. The lemma is proved. □

The next lemma is an important technical result. Its proof will be postponed to the next section, since it is rather long and require some fine estimates.

Lemma 2.2. *Suppose that $N\ge 4$, $p\in[2,2_*)$ and set*

$$A_N := \int_{\mathbb{R}^N_+}|\nabla U_\varepsilon|^2dx, \qquad B_N := \left(\int_{\mathbb{R}^{N-1}} U_\varepsilon^{2_*}dx'\right)^{2/2_*},$$

$$D_{N,p} := \int_{\mathbb{R}^{N-1}}\frac{1}{[|x'|^2+1]^{p(N-2)/2}}dx'>0, \quad \gamma_N := (N+1)-\frac{p}{2}(N-2)\le 1.$$

Then $A_N/B_N = S$ *and, as* $\varepsilon \to 0^+$, *we have that*

$$\|\psi_\varepsilon\|^2 = A_N + \begin{cases} O(\varepsilon^2|\ln\varepsilon|), & \text{if } N = 4, \\ O(\varepsilon^2), & \text{if } N \geq 5, \end{cases}$$

$$|\psi_\varepsilon|_{2_*}^{2_*} = B_N^{2^*/2} + O(\varepsilon^2), \qquad |\psi_\varepsilon|_p^p = D_{N,p}\varepsilon^{\gamma_N} + O(\varepsilon^2).$$

We are ready to prove the main result of this paper.

Proof of Theorem 1.1. Using Lemma 2.1 and invoking the Moutain Pass Theorem [1], we obtain a sequence $(u_n) \subset \mathcal{D}_K^{1,2}(\mathbb{R}_+^N)$ such that

$$\lim_{n\to+\infty} I_\lambda(u_n) = c_\lambda, \quad \lim_{n\to+\infty} I'_\lambda(u_n) = 0,$$

where

$$c_\lambda := \inf_{u\in\Sigma_\lambda} \max_{t\in[0,1]} I_\lambda(\sigma(t)),$$

and $\Sigma_\lambda := \{\sigma \in C([0,1], \mathcal{D}_K^{1,2}(\mathbb{R}_+^N)) \,:\, \sigma(0) = 0,\, \sigma(1) = e_\lambda\}$.

We claim that

$$c_\lambda < \frac{1}{2(N-1)} S^{N-1}. \tag{2.2}$$

In order to prove this, we first set

$$v_\varepsilon(x) := \frac{\psi_\varepsilon(x)}{|\psi_\varepsilon|_{2_*}}, \quad x \in \mathbb{R}_+^N,$$

and notice that, in view of the definition of c_λ and e_λ (see Lemma 2.1), it is sufficient to check that

$$\max_{t\geq 0} I_\lambda(tv_\varepsilon) < \frac{1}{2(N-1)} S^{N-1}. \tag{2.3}$$

For $N \geq 5$, we can use Lemma 2.2, the mean value theorem and a straightforward computation to get

$$\|v_\varepsilon\|^2 = \frac{A_N + O(\varepsilon^2)}{\left[B_N^{2_*/2} + O(\varepsilon^2)\right]^{2/2_*}} = \frac{A_N + O(\varepsilon^2)}{B_N + O(\varepsilon^2)} = \frac{A_N}{B_N} + O(\varepsilon^2) = S + O(\varepsilon^2).$$

If $N = 4$, we can use the same argument and conclude that

$$\|v_\varepsilon\|^2 = S + \begin{cases} O(\varepsilon^2|\ln\varepsilon|), & \text{if } N = 4, \\ O(\varepsilon^2), & \text{if } N \geq 5. \end{cases} \tag{2.4}$$

Moreover, for any $p \in [2, 2_*)$, there holds

$$|v_\varepsilon|_p^p = \frac{D_{N,p}\varepsilon^{\gamma_N} + O(\varepsilon^2)}{\left[B_N^{2_*/2} + O(\varepsilon^2)\right]^{p/2_*}} = D_{N,p}B_N^{-p/2}\varepsilon^{\gamma_N} + O(\varepsilon^2). \tag{2.5}$$

We now define the function

$$f_\varepsilon(t) := I_\lambda(tv_\varepsilon), \quad t > 0,$$

and call $t_\varepsilon > 0$ the point where it attains its maximum value. Then

$$I_\lambda(t_\varepsilon v_\varepsilon) \le \max_{t\ge 0}\left\{\frac{t^2}{2}\|v_\varepsilon\|^2 - \frac{t^{2_*}}{2_*}\right\} - \lambda\frac{t_\varepsilon^p}{p}|v_\varepsilon|_p^p = \frac{1}{2(N-1)}\|v_\varepsilon\|^{2(N-1)} - \lambda\frac{t_\varepsilon^p}{p}|v_\varepsilon|_p^p.$$

From $f_\varepsilon'(t_\varepsilon)t_\varepsilon = 0$ and (2.4), we obtain

$$S + O(\varepsilon^2) = \|v_\varepsilon\|^2 = \lambda t_\varepsilon^{p-2}|v_\varepsilon|_p^p + t_\varepsilon^{2_*-2}$$

and therefore there exists $\alpha_1 > 0$, independent of $\varepsilon > 0$, such that $t_\varepsilon \ge \alpha_1$, for any $\varepsilon > 0$ small.

If $N \ge 5$, the above expressions, (2.5) and the mean value theorem provide

$$\begin{aligned} I_\lambda(t_\varepsilon v_\varepsilon) &\le \frac{1}{2(N-1)}\left[S + O(\varepsilon^2)\right]^{N-1} - \lambda\frac{C_1^p}{p}|v_\varepsilon|_p^p \\ &\le \frac{1}{2(N-1)}S^{N-1} + \varepsilon^2\left[O(1) - \lambda C_2\varepsilon^{\gamma_N-2}\right], \end{aligned}$$

with $C_2 = C_2(N,p) > 0$ independent of $\varepsilon > 0$. Since $\gamma_N < 2$, the term into brackets above goes to $-\infty$ as $\varepsilon \to 0^+$. Hence,

$$\max_{t\ge 0} I_\lambda(tv_\varepsilon) = I_\lambda(t_\varepsilon v_\varepsilon) < \frac{1}{2(N-1)}S^{N-1},$$

for any $\varepsilon > 0$ small. This establishes (2.3) for $N \ge 5$.

If $N = 4$, we have that $\gamma_N = (3-p) > 0$ and we can argue as above toget

$$\begin{aligned} I_\lambda(t_\varepsilon v_\varepsilon) &\le \frac{1}{2(N-1)}S^{N-1} + O(\varepsilon^2|\ln\varepsilon|) - \lambda C_3\varepsilon^{3-p} + C_3 O(\varepsilon^2) \\ &\le \frac{1}{2(N-1)}S^{N-1} + \varepsilon^{3-p}\left[O(\varepsilon^{p-1}|\ln\varepsilon|) - \lambda C_3 + C_3 O(\varepsilon^{p-1})\right], \end{aligned}$$

and the result follows as before because $p > 1$.

After proving (2.3), we are going to show that(u_n) is bounded. Indeed, if $p = 2$ we can compute

$$c_\lambda + o(1) + o(1)\|u_n\| = I_\lambda(u_n) - \frac{1}{2_*}I_\lambda'(u_n)u_n \ge \frac{1}{2(N-1)}\left(\frac{\lambda_1-\lambda}{\lambda_1}\right)\|u_n\|^2,$$

as $n \to +\infty$. Since $\lambda \in (0,\lambda_1)$ we conclude that $(u_n) \subset \mathcal{D}_K^{1,2}(\mathbb{R}_+^N)$ is bounded. The same kind of argument can be used when $p \in (2, 2_*)$, just computing $I_\lambda(u_n) - (1/p)I_\lambda'(u_n)u_n$. In this case, there is no restriction on $\lambda > 0$.

From the boundedness, we may assume that, up to a subsequence,

$$\begin{cases} u_n \rightharpoonup u, & \text{weakly in } \mathcal{D}_K^{1,2}(\mathbb{R}_+^N), \\ u_n \to u, & \text{strongly in } L_K^p(\mathbb{R}^{N-1}), \end{cases}$$

for some $u \in \mathcal{D}_K^{1,2}(\mathbb{R}_+^N)$. Given $\varphi \in C_0^\infty(\overline{\mathbb{R}_+^N})$, we can use the above convergences and standard computations to show that

$$0 = \lim_{n\to+\infty} I_\lambda'(u_n)\varphi = I_\lambda'(u)\varphi,$$

and therefore u is a critical point of I_λ. We devote the rest of this proof to check that $u \neq 0$. If this is true, then it is the desired solution.

Suppose, by contradiction, that $u = 0$. Since $u_n \to 0$ strongly in $L^p_K(\mathbb{R}^{N-1})$ and $I_\lambda(u_n) \to c_\lambda$, as $n \to +\infty$, we have that

$$\lim_{n\to+\infty}\left(\frac{1}{2}\|u_n\|^2 - \frac{1}{2_*}\int_{\mathbb{R}^{N-1}} K(x',0)(u_n^+)^{2_*}dx'\right) = c_\lambda. \tag{2.6}$$

Moreover, since $I'_\lambda(u_n)u_n \to 0$, then

$$\lim_{n\to+\infty}\|u_n\|^2 = b = \lim_{n\to+\infty}\int_{\mathbb{R}^{N-1}} K(x',0)(u_n^+)^{2_*}dx',$$

for some $b \geq 0$. Actually, since $c_\lambda > 0$, we may have $b > 0$.

Passing (2.6) to the limit we get

$$c_\lambda = \left(\frac{1}{2} - \frac{1}{2_*}\right) b = \frac{1}{2(N-1)} b. \tag{2.7}$$

On the other hand, taking the limit in the expression

$$S\left(\int_{\mathbb{R}^{N-1}} K(x',0)(u_n^+)^{2_*}dx'\right)^{2/2_*} \leq \|u_n\|^2,$$

we obtain $Sb^{2/2_*} \leq b$. Combining this inequality with (2.7) and using $b > 0$, we conclude that

$$c_\lambda \geq \frac{1}{2(N-1)} S^{N-1},$$

which contradicts (2.2). So, the weak limit u is nonzero and the theorem is proved. □

Remark 2.3. *Suppose that $N = 3$, $p = 2$ and let $\varphi_1 \in X$ be an eigenfunction associated to the first eigenvalue λ_1. If we set*

$$t_\lambda^{2_*-2} := \frac{\|\varphi_1\|^2 - \lambda|\varphi_1|_2^2}{|\varphi_1|_{2_*}^{2_*}},$$

a simple computation shows that

$$c_\lambda \leq \max_{t\geq 0} I_\lambda(t\varphi_1) = I_\lambda(t_\lambda\varphi_1) = \frac{t_\lambda^2}{2}(\lambda_1 - \lambda)|\varphi_1|_2^2 - \frac{t_\lambda^{2_*}}{2_*}|\varphi_1|_{2_*}^{2_*}.$$

Since $\|\varphi_1\|^2 = \lambda_1|\varphi_1|_2^2$, we have that $t_\lambda \to 0$, as $\lambda \to \lambda_1^+$. Hence, we conclude from the above inequality that (2.2) always holds if $\lambda < \lambda_1$ is close to λ_1. This shows that, if $p = 2$ and λ belongs to this range, our existence result also holds in the 3-dimensional case.

3. Proof of Lemma 2.2

We devote all this section to check that Lemma 2.2 really holds. Recall that

$$\psi_\varepsilon(x) := K(x)^{-1/2}\phi(x)U_\varepsilon(x), \quad x \in \mathbb{R}^N_+,$$

where the cut-off function $\phi \in C^\infty(\mathbb{R}^N_+,[0,1])$ verifies $\phi \equiv 1$ in $B_1(0)\cap\mathbb{R}^N_+$, $\phi\equiv 0$ in $\overline{\mathbb{R}^N_+}\setminus B_2(0)$ and the function U_ε is given by

$$U_\varepsilon(x',x_N) := \frac{\varepsilon^{(N-2)/2}}{[|x'|^2+(x_N+\varepsilon)^2]^{(N-2)/2}},$$

for any $\varepsilon>0$ and $(x',x_N)\in\mathbb{R}^N_+$. For convenience, we state the lemma again:

Lemma 2.2. *Suppose that $N\geq 4$, $p\in[2,2_*)$ and set*

$$A_N := \int_{\mathbb{R}^N_+}|\nabla U_\varepsilon|^2dx, \qquad B_N := \left(\int_{\mathbb{R}^{N-1}}U_\varepsilon^{2_*}dx'\right)^{2/2_*},$$

$$D_{N,p} := \int_{\mathbb{R}^{N-1}}\frac{1}{[|x'|^2+1]^{p(N-2)/2}}dx' > 0, \quad \gamma_N := (N+1)-\frac{p}{2}(N-2)\leq 1.$$

Then $A_N/B_N = S$ and, as $\varepsilon\to 0^+$, we have that

$$\|\psi_\varepsilon\|^2 = A_N + \begin{cases} O(\varepsilon^2|\ln\varepsilon|), & \text{if } N=4,\\ O(\varepsilon^2), & \text{if } N\geq 5,\end{cases}$$

$$|\psi_\varepsilon|_{2_*}^{2_*} = B_N^{2^*/2}+O(\varepsilon^2), \qquad |\psi_\varepsilon|_p^p = D_{N,p}\varepsilon^{\gamma_N}+O(\varepsilon^2).$$

Proof. The equality $A_N/B_N=S$ is not new and its proof can be found in [11]. For the second statement, we first notice that

$$\int_{\mathbb{R}^N_+}K(x)|\nabla\psi_\varepsilon|^2 = \int_{\mathbb{R}^N_+}\left[|\nabla\phi|^2U_\varepsilon^2+2\phi U_\varepsilon(\nabla\phi\cdot\nabla U_\varepsilon)-\frac{1}{2}\phi U_\varepsilon^2(x\cdot\nabla\phi)\right]dx$$
$$+\int_{\mathbb{R}^N_+}\phi^2|\nabla U_\varepsilon|^2\,dx-\frac{1}{2}\int_{\mathbb{R}^N_+}\phi^2U_\varepsilon(x\cdot\nabla U_\varepsilon)\,dx+\frac{1}{16}\int_{\mathbb{R}^N_+}\phi^2|x|^2U_\varepsilon^2\,dx.$$

We have that

$$\int_{\mathbb{R}^N_+}|\nabla\phi|^2U_\varepsilon^2\,dx = \varepsilon^{N-2}\int_{B_2^+\setminus B_1^+}\frac{|\nabla\phi|^2}{[|x'|^2+(x_N+\varepsilon)^2]^{N-2}}dx = O(\varepsilon^{N-2}),$$

as $\varepsilon\to 0^+$. Similiar arguments for the other terms into the brackets above provide

$$\|\psi_\varepsilon\|^2 = O(\varepsilon^{N-2})+\int_{\mathbb{R}^N_+}\phi^2|\nabla U_\varepsilon|^2\,dx-\frac{1}{2}\int_{\mathbb{R}^N_+}\phi^2U_\varepsilon(x\cdot\nabla U_\varepsilon)\,dx + \frac{1}{16}\int_{\mathbb{R}^N_+}\phi^2|x|^2U_\varepsilon^2\,dx. \tag{3.1}$$

In order to estimate each of the integrals on the right-hand side above, we first compute

$$\nabla U_\varepsilon = -\frac{(N-2)\varepsilon^{(N-2)/2}}{[|x'|^2+(x_N+\varepsilon)^2]^{N/2}}\left(x_1,\ldots,x_{N-1},x_N+\varepsilon\right).$$

For any $r > 0$, we set $B_r^+ := \mathbb{R}_+^N \cap B_r(0)$. Since $\phi^2|\nabla U_\varepsilon|^2 = |\nabla U_\varepsilon|^2 + (\phi^2 - 1)|\nabla U_\varepsilon|^2$, we obtain

$$\int_{\mathbb{R}_+^N} \phi^2|\nabla U_\varepsilon|^2 dx = A_N + (N-2)^2\varepsilon^{N-2} \int_{\mathbb{R}_+^N \setminus B_1^+} \frac{(\phi^2 - 1)}{[|x'|^2 + (x_N + \varepsilon^2)]^{N-1}} dx,$$

and therefore

$$\int_{\mathbb{R}_+^N} \phi^2|\nabla U_\varepsilon|^2 = A_N + O(\varepsilon^{N-2}). \tag{3.2}$$

Using the same argument and the definition of ϕ, we obtain

$$\int_{\mathbb{R}_+^N} \phi^2 U_\varepsilon(x \cdot \nabla U_\varepsilon)\, dx = O(\varepsilon^{N-2}) + \int_{B_2^+} U_\varepsilon(x \cdot \nabla U_\varepsilon)\, dx$$

$$= O(\varepsilon^{N-2}) - (N-2)\varepsilon^{N-2} \int_{B_2^+} \frac{|x'|^2 + x_N(x_N + \varepsilon)}{[|x'|^2 + (x_N + \varepsilon)^2]^{N-1}}\, dx.$$

So, we infer from the change of variables $y = x/\varepsilon$ that

$$\int_{\mathbb{R}_+^N} \phi^2 U_\varepsilon(x \cdot \nabla U_\varepsilon)\, dx = O(\varepsilon^{N-2}) - (N-2)\varepsilon^2 \left(\Gamma_{1,N,\varepsilon} + \Gamma_{2,N,\varepsilon}\right) \tag{3.3}$$

where

$$\Gamma_{1,N,\varepsilon} := \int_{B_{2/\varepsilon}^+} \frac{|y'|^2}{[|y'|^2 + (y_N + 1)^2]^{N-1}} dy$$

and

$$\Gamma_{2,N,\varepsilon} := \int_{B_{2/\varepsilon}^+} \frac{y_N(y_N + 1)}{[|y'|^2 + (y_N + 1)^2]^{N-1}} dy.$$

If $N \geq 5$, we have that

$$\begin{aligned} \Gamma_{1,N,\varepsilon} &\leq \int_{\mathbb{R}_+^N} \frac{|y'|^2}{[|y'|^2 + (y_N + 1)^2]^{N-1}} dy \\ &= \int_{B_1^+} \frac{|y'|^2}{[|y'|^2 + (y_N + 1)^2]^{N-1}} dy + \int_{\mathbb{R}_+^N \setminus B_1^+} |y|^{2-2(N-1)} dy < +\infty. \end{aligned}$$

Since a similar estimate holds for $\Gamma_{2,N,\varepsilon}$, we conclude from (3.3) that

$$\int_{\mathbb{R}_+^N} \phi^2 U_\varepsilon(x \cdot \nabla U_\varepsilon) = O(\varepsilon^2), \qquad \text{if } N \geq 5. \tag{3.4}$$

The case $N = 4$ is more involved, since the function $|y|^{2-2(N-1)}$ is not integrable at infinity. We first observe that

$$\Gamma_{1,4,\varepsilon} \leq \int_{\mathbb{R}^3 \times [0,2]} \frac{|y'|^2}{[|y'|^2 + (y_4 + 1)^2]^3} dy + \int_{\mathbb{R}^3 \times [2,2/\varepsilon]} \frac{|y'|^2}{[|y'|^2 + (y_4 + 1)^2]^3} dy.$$

If we call Σ_1 and $\Sigma_{1,\varepsilon}$ the two last terms above, respectively, we can use Fubini's Theorem and the change of variable $x' = y'/(y_4+1)$, to get

$$\begin{aligned} \Sigma_1 &= \int_0^2 \int_{\mathbb{R}^3} \frac{|y'|^2}{[|y'|^2 + (y_4+1)^2]^3} dy' dy_4 \\ &= \left(\int_0^2 \frac{1}{y_4+1} dy_4 \right) \left(\int_{\mathbb{R}^3} \frac{|x'|^2}{[|x'|^2+1]^3} dx' \right) = O(1). \end{aligned}$$

Analagoulsy,

$$\Sigma_{1,\varepsilon} \leq \left(\int_2^{2/\varepsilon} \frac{1}{y_4} dy_4 \right) \left(\int_{\mathbb{R}^3} \frac{|x'|^2}{[|x'|^2+1]^3} dx' \right) = O(|\ln \varepsilon|)$$

and therefore we conclude that

$$\Gamma_{1,4,\varepsilon} = O(|\ln \varepsilon|). \tag{3.5}$$

Using the same ideas for $\Gamma_{2,4,\varepsilon}$ we obtain

$$\begin{aligned} \Gamma_{2,4,\varepsilon} &\leq O(1) + \left(\int_2^{2/\varepsilon} \frac{y_4}{(y_4+1)^2} dy_4 \right) \left(\int_{\mathbb{R}^3} \frac{1}{[|x'|^2+1]^{N-1}} dx' \right) \\ &\leq O(1) + \left(\int_2^{2/\varepsilon} \frac{1}{y_4} dy_4 \right) \left(\int_{\mathbb{R}^3} \frac{|x'|^2}{[|x'|^2+1]^{N-1}} dx' \right) = O(|\ln \varepsilon|). \end{aligned}$$

This inequality, (3.5), (3.4) and (3.3) imply that

$$\int_{\mathbb{R}^N_+} \phi^2 U_\varepsilon (x \cdot \nabla U_\varepsilon)\, dx = \begin{cases} O(\varepsilon^2), & \text{if } N \geq 5, \\ O(\varepsilon^2 |\ln \varepsilon|), & \text{if } N = 4. \end{cases} \tag{3.6}$$

It remains to estimate the last term in (3.1). If $N \geq 7$, we can write $\phi^2 = (\phi^2 - 1) + 1$ and use the change of variables $y = x/\varepsilon$ to compute

$$\int_{\mathbb{R}^N_+} \phi^2 |x|^2 U_\varepsilon^2\, dx = O(\varepsilon^{N-2}) + \varepsilon^4 \int_{\mathbb{R}^N_+} \frac{|y|^2}{[|y'|^2 + (y_N+1)^2]^{N-2}} dy.$$

Since the last integral above is finite for $N \geq 7$, we conclude that

$$\int_{\mathbb{R}^N_+} \phi^2 |x|^2 U_\varepsilon^2\, dx = O(\varepsilon^4), \qquad \text{if } N \geq 7. \tag{3.7}$$

For the other cases, we notice that

$$\begin{aligned} \int_{\mathbb{R}^N_+} \phi^2 |x|^2 U_\varepsilon^2\, dx &= \varepsilon^4 \int_{B^+_{2/\varepsilon}} \frac{|y|^2}{[|y'|^2 + (y_N+1)^2]^{N-2}} dy \\ &= O(\varepsilon^4) + \varepsilon^4 \int_{B^+_{2/\varepsilon} \setminus B^+_2} \frac{|y|^2}{[|y'|^2 + (y_N+1)^2]^{N-2}} dy. \end{aligned}$$

But

$$\int_{B^+_{2/\varepsilon}\setminus B^+_2} \frac{|y|^2}{[|y'|^2+(y_N+1)^2]^{N-2}}dy \le \int_{B^+_{2/\varepsilon}\setminus B^+_2} |y|^{6-2N}dy = C_1\int_2^{2/\varepsilon} r^{5-N}dr,$$

for some $C_1 = C_1(N) > 0$. Hence,

$$\int_{B^+_{2/\varepsilon}\setminus B^+_2} \frac{|y|^2}{[|y'|^2+(y_N+1)^2]^{N-2}}dy \le C_2 \begin{cases} \varepsilon^{N-6}-1, & \text{if } N\in\{4,5\},\\ -\ln\varepsilon, & \text{if } N=6,\end{cases}$$

with $C_2 = C_2(N) > 0$. All together, the above inequalities and (3.7) imply that

$$\int_{\mathbb{R}^N_+} \phi^2|x|^2U_\varepsilon^2\,dx = \begin{cases} O(\varepsilon^{N-2}), & \text{if } N\in\{4,5\},\\ O(\varepsilon^4|\ln\varepsilon|), & \text{if } N=6,\\ O(\varepsilon^4), & \text{if } N\ge 7.\end{cases}$$

The equality for $\|\psi_\varepsilon\|^2$ in the lemma follows from the above inequality, (3.6), (3.2) and (3.1).

We perform now the calculations on the boundary. For saving notation, we write only K to denote $K(x',0)$, for any $x'\in\mathbb{R}^{N-1}$. We also set $B_r^\partial := \{x'\in\mathbb{R}^{N-1} : |x'|<r\}$, for any $r>0$. Since $\phi\equiv 1$ in B_1^∂ and $\phi\equiv 0$ outside B_2^∂, we have that

$$\begin{aligned}|\!|\psi_\varepsilon|\!|_{2_*}^{2_*} &= \varepsilon^{N-1}\int_{B_1^\partial}\frac{K^{-1/(N-2)}}{[|x'|^2+\varepsilon^2]^{N-1}}dx' + O(\varepsilon^{N-1})\\ &= \varepsilon^{N-1}\Gamma_{1,\varepsilon}+\varepsilon^{N-1}\Gamma_{2,\varepsilon}+O(\varepsilon^{N-1}),\end{aligned} \tag{3.8}$$

with

$$\Gamma_{1,\varepsilon} := \int_{B_1^\partial}\frac{1}{[|x'|^2+\varepsilon^2]^{N-1}}dx', \qquad \Gamma_{2,\varepsilon} := \int_{B_1^\partial}\frac{K^{-1/(N-2)}-1}{[|x'|^2+\varepsilon^2]^{N-1}}dx'.$$

The first term can be computed as follows:

$$\Gamma_{1,\varepsilon} = \int_{\mathbb{R}^{N-1}}\frac{1}{[|x'|^2+\varepsilon^2]^{N-1}}dx' - \int_{\mathbb{R}^{N-1}\setminus B_1^\partial}\frac{1}{[|x'|^2+\varepsilon^2]^{N-1}}dx'.$$

Since $|x'|^{-2(N-1)}\in L^1(\mathbb{R}^{N-1}\setminus B_1^\partial)$ for any $N\ge 2$, the last integral above is bounded as $\varepsilon\to 0^+$, and we infer from (3.8) and the definition of U_ε that

$$|\!|\psi_\varepsilon|\!|_{2_*}^{2_*} = B_N^{2_*/2}+\varepsilon^{N-1}\Gamma_{2,\varepsilon}+O(\varepsilon^{N-1}). \tag{3.9}$$

The change of variables $y'=x'/\varepsilon$ and the definition of K provide

$$\varepsilon^{N-1}\Gamma_{2,\varepsilon} = \int_{B_{1/\varepsilon}^\partial}\frac{K(\varepsilon y',0)^{-1/(N-2)}-1}{[|y'|^2+1]^{N-1}}dy' = -\int_{B_{1/\varepsilon}^\partial}\frac{1-\exp(-\alpha_0\varepsilon^2|y'|^2)}{[|y'|^2+1]^{N-1}}dy',$$

where $\alpha_0 := [4(N-2)]^{-1} > 0$. Using (3.9), we see that the estimate for $|\!|\psi_\varepsilon|\!|_{2_*}^{2_*}$ is equivalent to show that

$$\lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon^2}\int_{\mathbb{R}^{N-1}}\frac{1-\exp(-\gamma\varepsilon^2|y'|^2)}{[|y'|^2+1]^{N-1}}dy' = \alpha_0\int_{\mathbb{R}^{N-1}}\frac{|y'|^2}{[|y'|^2+1]^{N-1}}dy', \tag{3.10}$$

since the last integral above is finite whenever $N \geq 4$. In order to check the above limit, we suppose that $\varepsilon \in (0,1)$ and set $f_\varepsilon : \mathbb{R}^{N-1} \to \mathbb{R}$ by

$$f_\varepsilon(y') := \frac{1 - \exp(-\alpha_0 \varepsilon^2 |y'|^2)}{\varepsilon^2}, \qquad y' \in \mathbb{R}^{N-1}.$$

By L'Hospital's rule, we obtain

$$\lim_{\varepsilon \to 0^+} f_\varepsilon(y') = \alpha_0 |y'|^2, \quad \forall\, y' \in \mathbb{R}^{N-1}. \tag{3.11}$$

Moreover, the inequality $e^t \geq 1 + t$, for any $t \in \mathbb{R}$, imply that

$$\frac{d}{d\varepsilon} f_\varepsilon(y') = \frac{2}{\varepsilon^3} \left[\frac{\alpha_0 \varepsilon^2 |y'|^2 + 1}{\exp(\alpha_0 \varepsilon^2 |y'|^2)} - 1 \right] \leq 0.$$

Hence, the sequence of nonnegative functions $(f_\varepsilon)_{\varepsilon \in (0,1)}$ is nonincreasing, and we can use (3.11) together with the Monotone Convergence Theorem to conclude that (3.10) holds. This proves the equality for $\| \psi_\varepsilon \|_{2_*}^{2_*}$

For the norm involving the power $p \in [2, 2_*)$, we argue as above to get

$$\| \psi_\varepsilon \|_p^p = \varepsilon^{\gamma_N} \int_{\mathbb{R}^{N-1}} \frac{1}{[|x'|^2 + 1]^{p(N-2)/2}} dx' - \varepsilon^{\gamma_N} \Gamma_{3,\varepsilon} + O(\varepsilon^{p(N-2)/2}), \tag{3.12}$$

where $\gamma_N = (N-1) - p(N-2)/2$,

$$\Gamma_{3,\varepsilon} := \int_{B^\partial_{1/\varepsilon}} \frac{1 - \exp(-\alpha_1 \varepsilon^2 |y'|^2)}{[|y'|^2 + 1]^{p(N-2)/2}} dy',$$

and $\alpha_1 := (p-2)/8 \geq 0$. As in the proof of (3.10), we can check that

$$\frac{1}{\varepsilon^2} \Gamma_{3,\varepsilon} = \alpha_1 \int_{\mathbb{R}^{N-1}} \frac{|y'|^2}{[|y'|^2 + 1]^{p(N-2)/2}} dy' + o(1),$$

and therefore

$$\varepsilon^{\gamma_N} \Gamma_{3,\varepsilon} = O(\varepsilon^{\gamma_N + 2}) = O(\varepsilon^2),$$

since the inequality $p < 2_*$ implies that $\gamma_N > 0$. Using $N \geq 4$, we conclude that $p(N-2)/2 \geq 2$ and therefore the last term in (3.12) also has order $O(\varepsilon^2)$. The lemma is proved. □

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