ELLIPTIC PROBLEMS IN THE HALF-SPACE WITH NONLINEAR CRITICAL BOUNDARY CONDITIONS

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Abstract. We obtain multiple solutions for the nonlinear boundary value problem

$$-\Delta u - \frac{1}{2}(x \cdot \nabla u) = f(u), \text{ in } \mathbb{R}^N_+, \qquad \frac{\partial u}{\partial \eta} = \beta |u|^{2/(N-2)} u, \text{ on } \partial \mathbb{R}^N_+,$$

where $\mathbb{R}^N_+ = \{(x', x_N) \in \mathbb{R}^N_+ : x' \in \mathbb{R}^{N-1}, x_N > 0\}$, $\frac{\partial u}{\partial \eta}$ is the partial outward normal derivative, $\beta > 0$ is a parameter and f is a superlinear function with subcritical growth.

1. Introduction and main results

Let $\mathbb{R}^N_+ = \left\{(x', x_N) : x' \in \mathbb{R}^{N-1}, x_N > 0\right\}$ be the upper half-space and consider the nonlinear boundary value problem

$$-\Delta v = g(x, v), \text{ in } \mathbb{R}^N_+, \qquad \frac{\partial v}{\partial \eta} = h(x', v), \text{ on } \mathbb{R}^{N-1}, \tag{1.1}$$

where $\frac{\partial u}{\partial \eta}$ denotes the outward unit normal derivative and we have identified $\partial \mathbb{R}^N_+ \simeq \mathbb{R}^{N-1}$. It appears in several mathematical contexts, such as in the study of scalar curvature problems and conformal deformation of Riemannian manifolds [11, 18, 19], problems of sharp constant in Sobolev trace inequalities [17], nonlinear elasticity [14], glaciology [32], population genetics [3], non-Newtonian fluid mechanics [15], chemical reactions [2], among others.

There is a vast literature concerning on the solvability of problems like (1.1). Without intention of presenting a complete list of references, we quote Chipot *et al.* [13], that have used the moving plane method to obtain positive solutions when $g(v) = av^p$, $a > 0$, and $h(v) = v^q$ with $1 < p \le (N+2)/(N-2)$, $1 < q \le N/(N-2)$ and one the inequalities being strict (see also [27] to the case $a = 0$). When $g \equiv 0$ and $h(v) = (N-2)v^{N/(N-2)}$, existence of positive solution decaying as $|x|^{2-N}$ at infinity was obtained by Escobar [17] using the conformal equivalence between the unit ball in $\mathbb{R}^N$ and the half-space (see also [33]). In the same paper, it was considered the case $g(v) = N(N-2)v^{(N+2)/(N-2)}$ and $h(v) = bv^{N/(N-2)}$ (see [12] and [29]). Still in the case $g \equiv 0$, necessary and sufficient conditions on h for the existence of layer solutions (bounded solutions satisfying some monotonicity properties) and their relation with local minimizers and stable solutions for problem (1.1) in the planar case was presented by Cabré and Morales [9]. The authors also

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showed that the upper half-space naturally appears when dealing with transition profiles near to discontinuities for the same problem in bounded smooth domains through blow-up techniques. We finish quoting Wu [35], who obtained existence and multiplicity results in the double subcritical case by using Ljusternik–Schnirelmann theory.

In this paper, we consider the critical problem

$$(P)\qquad \begin{cases} -\Delta u - \dfrac{1}{2}(x\cdot\nabla u) = f(u), & \text{in } \mathbb{R}^N_+,\\ \dfrac{\partial u}{\partial \eta} = \beta |u|^{2_*-2}u, & \text{on } \mathbb{R}^{N-1}, \end{cases}$$

where $2_* := 2(N-1)/(N-2)$, $\beta > 0$ is a parameter and the conditions on f will be stated later. Setting $K(x) = \exp(|x|^2/4)$, a direct computation shows that, if u is a solution of (P), then the function $v = K^{1/2}u$ verifies (1.1) for

$$g(x,v) = -a(x)v + K(x)^{1/2} f(K(x)^{-1/2}v), \qquad h(x',v) = \beta b(x')|v|^{2_*-2}v,$$

where $a(x) = \left(\frac{N}{4} + \frac{|x|^2}{16}\right)$ and $b(x') = \exp\left(-\frac{|x'|^2}{4(N-2)}\right)$. Since g is unbounded in the spatial variable and we are dealing with an inhomogeneous problem, the techniques used in the aforementioned works do not apply and therefore we need to perform a different approach.

There is a deep connection between (P) and the equation

$$-\Delta u = g(u) + \beta|u|^{2^*-2}u, \text{ in } \Omega, \quad u \in H^1_0(\Omega),$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain and $2^* = 2N/(N-2)$. In their celebrated paper, Brezis and Nirenberg [8] have shown that the presence of the subcritical perturbation g recover compactness and the problem becomes solvable. After this, critical growth problems have been studied extensively (see [5, 16] and references therein). We highlight here the work of Silva and Xavier [34], which strongly motivate our first main result. In order to state it, we present in what follows the main assumptions on the nonlinearity f.

(f_0) $f : \mathbb{R} \to \mathbb{R}$ is continuous;

(f_1) there exist $a_1, a_2 > 0$ and $2 < p < 2^* := 2N/(N-2)$ such that

$$|f(s)| \le a_1 + a_2|s|^{p-1}, \quad \forall\, s \in \mathbb{R};$$

(f_2) there holds

$$\lim_{s\to 0} \frac{f(s)}{s} = 0;$$

(f_3) there exists $2 < \theta < 2_*$ such that

$$0 < \theta F(s) \le f(s)s, \quad \forall\, s \in \mathbb{R}\setminus\{0\},$$

where $F(s) := \int_0^s f(\tau)\,d\tau$.

By taking advantage of the symmetry properties of the problem, we prove the following:

Theorem 1.1. *Suppose that f is odd and satisfies $(f_0)-(f_3)$. Then, for any given $k \in \mathbb{N}$, there exists $\beta^* = \beta^*(k) > 0$ such that problem (P) has at least k pairs of solutions, provided $\beta \in (0, \beta^*)$.*

For the proof, we apply a version of the Symmetric Mountain Pass Theorem. The main difficult is the management of Palais-Smale sequences and we initially follow the ideas presented in [34]. However, since we are dealing with unbounded domains, the former argument does not directly apply and we need to perform a trick adaptation of Bianchi, Chabrowski and Szulkin's ideas [6,10] and the concentration compactness principle due to Lions [30].

In our second result, we do not require symmetry for f and obtained the existence of nonnegative solution. In this case, the parameter β does not play any role and we prove the following:

Theorem 1.2. *Suppose that $N \geq 7$ and f satisfies $(f_0)-(f_3)$. Then problem (P) has a nonnegative nonzero solution provided*

$$\lim_{\varepsilon\to 0^+} \varepsilon^{N-2}\int_0^{1/\varepsilon} F\left(\frac{\varepsilon^{-(N-2)/2}}{[s^2+1]^{(N-2)/2}}\right) s^{N-1}\,ds = +\infty. \tag{1.2}$$

In the proof, we follow [8]. After obtaining a local compactness condition for the associated functional, we need to prove that its Mountain Pass level belongs to the correct range. At this point, we perform some fine estimates and use technical condition (1.2). It was inspired by a similar one which have appeared in [8, Lemma 2.1] and it holds if, for instance, $F(s) \geq \gamma|s|^p$, for some $\gamma > 0$. In order to check that, it is enough to notice that $g(s) = s^{N-1}/(1+s^2)^{p(N-2)/2}$ is increasing in the interval $[0, s_0]$, where $s_0 = [-(N-1)/(N-1-pN+2p)]^{1/2}$. Hence, for $\varepsilon > 0$ enough small,

$$\int_0^{1/\varepsilon} \frac{s^{N-1}}{(1+s^2)^{p(N-2)/2}}\,ds \geq \int_0^{s_0} \frac{s^{N-1}}{(1+s^2)^{p(N-2)/2}}\,ds > 0,$$

and therefore

$$\lim_{\varepsilon\to 0^+} \varepsilon^{N-2-p(N-2)/2}\int_0^{1/\varepsilon} \frac{s^{N-1}}{(s^2+1)^{p(N-2)/2}}\,ds = +\infty.$$

The restriction on the dimension is used only in equality (4.2), which really comes from [23]. If one could prove an analogous estimate for lower dimensions or use a different approach, maybe it would be possible to obtain the existence of a solution when $N \in \{3,4,5,6\}$. This might be an interesting open question.

It is worth noticing that the operator on the left-hand side of (P) naturally appears when we look for self-similar solutions for the following nonlinear heat equation

$$v_t - \Delta v = 0, \text{ in } \mathbb{R}^N_+ \times (0,+\infty), \qquad \frac{\partial v}{\partial \eta} = |v|^{p-2}v, \text{ on } \partial\mathbb{R}^N_+ \times (0,+\infty), \tag{1.3}$$

where $x \in \mathbb{R}^N_+$ is the spatial variable and $t > 0$ is time. This kind of solution has the special form $v(x,t) = t^{\mu}u(t^{-1/2}x)$ and, among others advantages, it preserves the PDE scaling, providing qualitative properties and giving information about large and small scale behaviors. A direct computation shows that the profile $u : \overline{\mathbb{R}^N_+} \to \mathbb{R}$ verifies

$$-\Delta u - \frac{1}{2}(x\cdot\nabla u) = g(u), \text{ in } \mathbb{R}^N_+, \qquad \frac{\partial u}{\partial \eta} = |u|^{p-2}u, \text{ on } \mathbb{R}^{N-1},$$

for $g(u) = u/(2(p-2))$.

Problem (1.3) and their variations have been studied in bounded domain, the half-space $\mathbb{R}^N_+$ and even in the whole space in the last decades; see, e.g., [4, 24–26, 28, 31] and references therein. Different types of results can be found in these works, such as existence, uniqueness of solutions, blow-up or asymptotic behavior results. To the best of our knowledge, Escobedo and Kavian [20] were the first authors to propose a variational approach to nonlinear heat problems. In their paper, they consider the whole space case and introduce the abstract Sobolev spaces appropriated to find solution with rapid decay at infinity. This abstract setting was extended to the half-space in [22], including the necessary trace embeddings. Some extensions for the critical case were recently proved in [23]. The main theorems proved here complement this last paper, since we deal with a superlinear nonlinearity in $\mathbb{R}^N_+$ and, beyond the nonnegative solution, we also obtain multiplicity results.

The rest of this paper is organized as follows. In the next section, we present the variational framework to deal with (P) and prove a compactness result. In Section 3 we prove Theorem 1.1 and, in the final section, we prove Theorem 1.2.

2. Variational framework and the Palais-Smale condition

If we define $K(x) := \exp(|x|^2/4)$, a straightforward computation shows that the first equation in (P) becomes

$$-\mathrm{div}(K(x)\nabla u) = K(x)f(u), \quad x \in \mathbb{R}^N_+.$$

Hence, it is natural looking for solutions in the space X defined as the closure of $C_0^\infty(\overline{\mathbb{R}^N_+})$ with respect to the norm

$$\|u\| := \left(\int_{\mathbb{R}^N_+} K(x)|\nabla u|^2\,dx\right)^{1/2},$$

which is induced by the inner product

$$(u,v) := \int_{\mathbb{R}^N_+} K(x)(\nabla u \cdot \nabla v)\,dx$$

From now on we identify $\partial\mathbb{R}^N_+ \sim \mathbb{R}^{N-1}$. Given $2 \le r \le 2^*$ and $2 \le s \le 2_*$ we consider the weighted Lebesgue spaces

$$L^r_K(\mathbb{R}^N_+) := \left\{ u \in L^r(\mathbb{R}^N_+) \,:\, \|u\|_r := \left(\int_{\mathbb{R}^N_+} K(x)|u|^r dx\right)^{1/r} < \infty \right\},$$

$$L^s_K(\mathbb{R}^{N-1}) := \left\{ u \in L^s(\mathbb{R}^{N-1}) : [\![u]\!]_s := \left(\int_{\mathbb{R}^{N-1}} K(x',0)|u|^s dx'\right)^{1/s} < \infty \right\}.$$

and collect in the next proposition the abstract results proved in [22, 23].

Proposition 2.1. *For any $r \in [2, 2^*)$ and $s \in [2, 2_*)$, the embeddings $X \hookrightarrow L^r_K(\mathbb{R}^N_+)$ and $X \hookrightarrow L^s_K(\mathbb{R}^{N-1})$ are compact. Moreover, continuous embeddings hold in the critical cases $r = 2^*$ and $s = 2_*$.*

In view of this result we can define, for $r \in [2, 2^*]$ and $s \in [2, 2_*]$, the following constants:

$$S_r := \inf_{u\in X/\{0\}} \frac{\int_{\mathbb{R}^N_+} K(x)|\nabla u|^2 dx}{\left(\int_{\mathbb{R}^N_+} K(x)|u|^r dx\right)^{2/r}},$$

and

$$S_{s,\partial} := \inf_{u\in X/\{0\}} \frac{\int_{\mathbb{R}^N_+} K(x)|\nabla u|^2 dx}{\left(\int_{\mathbb{R}^{N-1}} K(x',0)|u|^s dx'\right)^{2/s}}.$$

Setting $F(s) := \int_0^s f(\tau)\,d\tau$, it follows from $(f_0)-(f_2)$, Proposition 2.1 and standard arguments that the functional $I_\beta : X \to \mathbb{R}$ defined as

$$I_\beta(u) := \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N_+} K(x)F(u)\,dx - \frac{\beta}{2_*}\|u\|_{2_*}^{2_*},$$

is well defined. Actually, $I_\beta \in C^1(X,\mathbb{R})$ and its critical points are precisely the weak solutions of (P).

Recall that, if E is a Banach space, $I \in C^1(E,\mathbb{R})$ and $c\in\mathbb{R}$, the functional I is said to satisfy the $(PS)_c$ condition if any sequence $(u_n)\subset E$ such that

$$\lim_{n\to+\infty} I(u_n) = c, \qquad \lim_{n\to+\infty} I'(u_n) = 0,$$

has a convergent subsequence. From now on, any such sequence will be called $(PS)_c$-sequence.

The main result of this section can be stated as follows:

Proposition 2.2. *Suppose that f satisfies (f_0)-(f_3). For any given $M>0$, the functional I_β satisfies the $(PS)_c$-condition for any $0<c\le M$, provided $\beta>0$ satisfies*

$$\beta < \beta^* := \left(\frac{S_{2_*,\partial}^{N-1}}{2(N-1)M}\right)^{1/(N-2)}. \tag{2.1}$$

Proof. Let $M>0$ and $(u_n)\subset X$ be a $(PS)_c$-sequence for I_β, with $0<c\le M$. Using (f_3) and a standard argument we can prove that (u_n) is bounded. Hence, we may assume that

$$\begin{cases} u_n \rightharpoonup u, & \text{weakly in } X,\\ u_n \to u, & \text{strongly in } L^r_K(\mathbb{R}^N_+) \text{ and } L^s_K(\mathbb{R}^{N-1}),\\ u_n(x)\to u(x), & \text{for a.e. } x\in\overline{\mathbb{R}^N_+}, \end{cases} \tag{2.2}$$

for any $r\in[2,2_*)$ and $s\in[2,2_*)$. Moreover, we can easily check that $I'_\beta(u)=0$.

We claim that, if $\beta<\beta^*$, then

$$\lim_{n\to+\infty}\int_{\mathbb{R}^{N-1}} K(x',0)|u_n|^{2_*}\,dx' = \int_{\mathbb{R}^{N-1}} K(x',0)|u|^{2_*}\,dx'. \tag{2.3}$$

Since the proof of this convergence is rather long, we postpone it for the end of the section. So, assuming the claim, we can use (2.2) and Lebesgue's theorem to get

$$\begin{aligned} o(1) = I'_\beta(u_n)u_n &= \|u_n\|^2 - \int_{\mathbb{R}^N_+} K(x) f(u_n) u_n \, dx - \beta \| u_n \|_{2_*}^{2_*} \\ &= \|u_n\|^2 - \int_{\mathbb{R}^N_+} K(x) f(u) u \, dx - \beta \| u \|_{2_*}^{2_*} + o(1) \\ &= \|u_n\|^2 - \|u\|^2 + I'_\beta(u)u + o(1), \end{aligned}$$

as $n \to +\infty$. The above expression, $I'_\beta(u)u = 0$ and the weak convergence in (2.2) imply that $u_n \to u$ in X. □

We devote the rest of this section to the proof of (2.3). The first step is to apply the Lions' concentration-compactness principle (see [30, Lemma 1.2]) to obtain an at most countable family J, positive numbers $\{\mu_j\}_{j\in J}$, $\{\nu_j\}_{j\in J}$, and points $\{x_j\}_{j\in J} \subset \partial\mathbb{R}^N_+$ such that

$$\text{(2.4)} \qquad \left\{ \begin{aligned} K(x)|\nabla u_n|^2 dx &\rightharpoonup \mu \geq K(x)|\nabla u|^2 dx + \textstyle\sum_{j\in J} \mu_j \delta_{x_j}, \\ K(x',0)|u_n|^{2_*} dx' &\rightharpoonup \nu = K(x',0)|u|^{2_*} dx' + \textstyle\sum_{j\in J} \nu_j \delta_{x_j}, \\ \mu_j &\geq S_{2_*,\partial}(\nu_j)^{2/2_*}, \end{aligned} \right.$$

where $\mu \in \mathcal{M}(\overline{\mathbb{R}^N_+})$, $\nu \in \mathcal{M}(\partial\mathbb{R}^N_+)$ are Radon measures and the convergences hold in the sense of the measures.

Lemma 2.3. *If* (2.1) *holds, then J is empty.*

Proof. Suppose, by contradiction, that $\beta < \beta^*$ and there exists some $j \in J$. We first claim that

$$\text{(2.5)} \qquad \nu_j \geq \left(\frac{S_{2_*,\partial}}{\beta} \right)^{N-1}.$$

Assuming the claim, we can prove the lemma in the following way. Pick $\psi \in C_0^\infty(B_2(x_j))$ such that $0 \leq \psi \leq 1$ and $\psi \equiv 1$ in $B_1(x_j)$. Computing $I_\beta(u_n) - (1/2)I'_\beta(u_n)u_n$ and using (f_3), we obtain

$$\begin{aligned} c + o(1) &\geq \frac{\beta}{2(N-1)} \int_{\mathbb{R}^{N-1}} K(x',0)|u_n|^{2_*} \, dx' \\ &\geq \frac{\beta}{2(N-1)} \int_{\mathbb{R}^{N-1}} K(x',0)|u_n|^{2_*} \psi(x',0) \, dx', \end{aligned}$$

where $o(1)$ denotes a quantity approaching zero as $n \to +\infty$. Passing to the limit, using (2.4) and (2.5), we obtain

$$M \geq c \geq \frac{\beta}{2(N-1)} \int_{B_1(x_j)\cap\mathbb{R}^{N-1}} \psi(x',0) \, d\nu \geq \frac{\beta}{2(N-1)} \nu_j \geq \frac{\beta}{2(N-1)} \left(\frac{S_{2_*,\partial}}{\beta} \right)^{N-1},$$

which is equivalent to $\beta \geq \beta^*$, contrary to (2.1). Hence, J is empty.

It remains to prove (2.5). For that, we consider $\phi \in C_0^\infty(B_2(0))$ such that $0 \leq \phi \leq 1$ and $\phi \equiv 1$ in $B_1(0)$ and define

$$\phi_j^\varepsilon(x) := \phi\left(\frac{x - x_j}{\varepsilon} \right), \quad x \in \mathbb{R}^N.$$

Since $I'_\beta(u_n)(u_n\phi_j^\varepsilon)=o(1)$, we obtain

$$\begin{aligned}\left[\int \phi_j^\varepsilon\,d\mu-\beta\int\phi_j^\varepsilon\,d\nu\right]+o(1) &= \int_{\mathbb{R}^N_+} K(x)f(u_n)u_n\phi_j^\varepsilon\,dx \\ &- \int_{\mathbb{R}^N_+} K(x)(\nabla u_n\cdot\nabla\phi_j^\varepsilon)u_n\,dx.\end{aligned}\tag{2.6}$$

We shall estimate each term on the right side above. First notice that, by using $(f_1)-(f_2)$, (2.2) and Lebesgue's theorem we obtain

$$\lim_{n\to\infty}\int_{\mathbb{R}^N_+} K(x)f(u_n)u_n\phi_j^\varepsilon\,dx=\lim_{n\to\infty}\int_{\mathbb{R}^N_+}K(x)f(u)u\phi_j^\varepsilon\,dx.$$

Moreover, since $\operatorname{supp}(\phi_j^\varepsilon)\subset B_{2\varepsilon}(x_j)$, we can use Lebesgue's theorem again to get

$$\lim_{\varepsilon\to0}\lim_{n\to\infty}\int_{\mathbb{R}^N_+}K(x)f(u_n)u_n\phi_j^\varepsilon\,dx=0.\tag{2.7}$$

By using Holder's inequality and that (u_n) is bounded, we obtain

$$\begin{aligned}\left|\int_{\mathbb{R}^N_+}K(x)(\nabla u_n\cdot\nabla\phi_j^\varepsilon)u_n\,dx\right| &\le \|u_n\|\left(\int_{\Omega_\varepsilon^j}K(x)(u_n)^2|\nabla\phi_j^\varepsilon|^2\,dx\right)^{1/2}\\ &= \frac{c_1}{\varepsilon}\left(\int_{\Omega_\varepsilon^j}K(x)(u_n)^2\left|\nabla\phi\left(\frac{x-x_j}{\varepsilon}\right)\right|^2\,dx\right)^{1/2},\end{aligned}$$

where $\Omega_\varepsilon^j:=B_{2\varepsilon}(x_j)\cap\mathbb{R}^N_+$ and $c_1>0$ is independent of n. If we call $\Sigma_{n,\varepsilon}$ the left-hand side of the above expression, we can use the change of variable $y=(x-x_j)/\varepsilon$ and the strong convergence $u_n\to u$ in $L^2_K(\mathbb{R}^N_+)$ to get

$$\Sigma_{n,\varepsilon}\le c_2\varepsilon^{(N-2)/2}\left(\int_{B_{2\varepsilon(0)}\cap\mathbb{R}^N_+}K(\varepsilon y+x_j)u^2(\varepsilon y+x_j)\,dy+o(1)\right)^{\frac12},$$

where $c_2=c_1\|\nabla\phi\|_\infty$. It follows that

$$\lim_{\varepsilon\to0}\lim_{n\to+\infty}\int_{\mathbb{R}^N_+}K(x)(\nabla u_n\cdot\nabla\phi_j^\varepsilon)u_n\,dx=0.$$

Passing (2.6) to the limit, using the above expression, (2.7) and (2.4), we obtain

$$\beta\nu_j=\beta\lim_{\varepsilon\to0^+}\int\phi_j^\varepsilon\,d\nu=\lim_{\varepsilon\to0^+}\int\phi_j^\varepsilon\,d\mu\ge\mu_j\phi(0)=\mu_j\ge S_{2_*,\partial}\nu_j^{2/2_*},$$

that is, $\nu_j^{1-(2/2_*)}\ge S_{2_*,\partial}/\beta$, which is equivalent to (2.5). The lemma is proved. □

In the next result we follow an argument due to Bianchi et al. [6].

Lemma 2.4. *If*

$$\nu_\infty:=\lim_{R\to+\infty}\limsup_{n\to+\infty}\int_{\{x'\in\mathbb{R}^{N-1}:|x'|\ge R\}}K(x',0)|u_n|^{2_*}\,dx',$$

then $\nu_\infty=0$ *or* $\nu_\infty\ge\left(\frac{S_{2_*,\partial}}{\beta}\right)^{N-1}$.

Proof. Set

$$\mu_\infty := \lim_{R\to+\infty} \limsup_{n\to+\infty} \int_{\mathbb{R}^N_+\setminus B_R(0)} K(x)|\nabla u_n|^2\,dx$$

and consider, for each $R > 1$, a function $\phi_R \in C^\infty(\overline{\mathbb{R}^N})$ such that $\phi_R \equiv 0$ in $B_R(0)$ and $\phi_R \equiv 1$ outside $B_{R+1}(0)$. Since $(u_n\phi_R) \subset X$, we have that $S_{2_*,\partial}\|u_n\phi_R\|_{2_*}^2 \le \|u_n\phi_R\|^2$ and we can use (2.2) to obtain

$$\begin{aligned} S_{2_*,\partial} \limsup_{n\to+\infty} \left(\int_{\mathbb{R}^{N-1}} K(x',0)|\phi_R u_n|^{2_*}\,dx'\right)^{\frac{2}{2_*}} &\le \limsup_{n\to+\infty} \int_{\mathbb{R}^N_+} K(x)|\nabla u_n|^2\phi_R^2\,dx \\ &+ \int_{\mathbb{R}^N_+} K(x)|\nabla\phi_R|^2 u^2\,dx. \end{aligned}$$

Passing to the limit as $R \to +\infty$, using the definition of ϕ_R and the Lebesgue's theorem, we conclude that

$$S_{2_*,\partial}\nu_\infty^{2/2_*} \le \mu_\infty. \tag{2.8}$$

Using that $I'_\beta(u_n)(u_n\phi_R) = o(1)$, together with (2.2) and Lebesgue's theorem, we get

$$\limsup_{n\to+\infty} B_n \le \int_{\mathbb{R}^N_+} K(x)f(u)u\phi_R\,dx + \beta\limsup_{n\to+\infty} C_n + \limsup_{n\to+\infty} -A_n, \tag{2.9}$$

where

$$A_n := \int_{\mathbb{R}^N_+} K(x)(\nabla u_n\cdot\nabla\phi_R)u_n\,dx, \qquad B_n := \int_{\mathbb{R}^N_+} K(x)|\nabla u_n|^2\phi_R\,dx,$$

and

$$C_n := \int_{\mathbb{R}^{N-1}} K(x',0)|u_n|^{2_*}\phi_R\,dx'.$$

From Holder's inequality, we obtain

$$-A_n \le \|u_n\|^2\left(\int_{B_{R+1}(0)\setminus B_R(0)} K(x)|\nabla\phi_R|^2 u_n^2\,dx\right)^{1/2}.$$

The above inequality, Proposition 2.1, the definition of ϕ_R and Lebesgue's theorem imply that

$$\lim_{R\to+\infty}\limsup_{n\to+\infty} -A_n \le 0. \tag{2.10}$$

Moreover, as before, it follows from the definition of ϕ_R that

$$\lim_{R\to+\infty}\limsup_{n\to+\infty} B_n = \mu_\infty, \qquad \lim_{R\to+\infty}\limsup_{n\to+\infty} C_n = \nu_\infty.$$

Passing (2.9) to the limit as $R \to +\infty$, using (2.8), (2.10), the above equalities and Lebesgue's theorem, we get $S_{2_*,\partial}\nu_\infty^{2/2_*} \le \mu_\infty \le \beta\nu_\infty$, from which the result follows. □

We are ready to prove that (2.3) holds. First notice that, in view of the pointwise convergence in (2.2) and Fatou's lemma, it is sufficient to check that

$$\limsup_{n\to+\infty}\int_{\mathbb{R}^{N-1}} K(x',0)|u_n|^{2_*}\,dx' \le \int_{\mathbb{R}^{N-1}} K(x',0)|u|^{2_*}\,dx'.$$

Since $\beta < \beta^*$, the set J is empty. Hence, the weak convergence in the sense of measure (2.4) imply that (see [21, Theorem 1, Section 1.9]), for each $R > 0$, there holds

$$\begin{aligned}\limsup_{n\to+\infty} \int_{\mathbb{R}^{N-1}} K(x',0)|u_n|^{2_*}\,dx' &= \limsup_{n\to+\infty} \int_{\{x'\in\mathbb{R}^{N-1}:|x'|>R\}} K(x',0)|u_n|^{2_*}\,dx' \\ &+ \int_{\{x'\in\mathbb{R}^{N-1}:|x'|\le R\}} K(x',0)|u|^{2_*}\,dx'.\end{aligned}$$

Passing to the limit as $R \to +\infty$, we obtain

$$\limsup_{n\to+\infty} \int_{\mathbb{R}^{N-1}} K(x',0)|u_n|^{2_*}\,dx' = \nu_\infty + \int_{\mathbb{R}^{N-1}} K(x',0)|u|^{2_*}\,dx',$$

where ν_∞ was defined in Lemma 2.4.

It remains to check that $\nu_\infty = 0$. In order to do this, notice that the same argument of the proof of Lemma 2.3 provides

$$c + o(1) \ge \frac{\beta}{2(N-1)} \int_{\{x'\in\mathbb{R}^{N-1}:|x'|\ge R\}} K(x',0)|u_n|^{2_*}\,dx',$$

for any $R > 0$. Recalling that $c \le M$, we obtain

$$M \ge \lim_{R\to+\infty} \limsup_{n\to+\infty} \frac{\beta}{2(N-1)} \int_{\{x'\in\mathbb{R}^{N-1}:|x'|\ge R\}} K(x',0)|u_n|^{2_*}\,dx' = \frac{\beta}{2(N-1)}\nu_\infty.$$

If $\nu_\infty \neq 0$, we can use the above inequality and Lemma 2.4 to obtain $\beta \ge \beta^*$, which contradicts (2.1). Hence, $\nu_\infty = 0$ and we conclude that (2.3) is verified.

3. Proof of Theorem 1.1

Our first main result will be proved as an application of the following version of the Symmetric Moutain Pass Theorem (see [1]).

Theorem 3.1. *Let $E = V \oplus W$, where E is a real Banach space and V is finite dimensional. Suppose $I \in C^1(E,R)$ is an even functional satisfying $I(0) = 0$ and*

- (I^1) *there exist constants $\rho, \alpha > 0$ such that $I\,|_{B_\rho(0)\cap W} \ge \alpha$;*
- (I^2) *there exists a subspace $\widetilde{V}$ of E such that $\dim V < \dim \tilde{V} < \infty$ and $\max_{u\in\tilde{V}} I(u) \le M$, for some constant $M > 0$;*
- (I^3) *I satisfies $(PS)_c$, for any $0 < c < M$.*

Then I has at least $\dim \tilde{V} - \dim V$ pairs of nonzero critical points.

We are intending to apply this abstract result with $E = X$ and $I = I_\beta$. For the required decomposition of the space X we consider the linearized problem

$$(LP) \qquad \begin{cases} -\mathrm{div}(K(x)\nabla u) = \lambda K(x)u, & \text{in } \mathbb{R}^N_+, \\ \frac{\partial u}{\partial \eta} = 0, & \text{on } \mathbb{R}^{N-1}. \end{cases}$$

Thanks to the compact embedding $X \hookrightarrow L^2_K(\mathbb{R}^N_+)$, we can use standard spectral theory to obtain a sequence of eigenvalues $(\lambda_j)_{j\in\mathbb{N}}$ such that

$$0 < \lambda_1 < \lambda_2 \le \cdots \le \lambda_j \le \cdots$$

with $\lim_{j\to\infty} \lambda_j = +\infty$. Moreover, the first eigenvalue is given by

$$\lambda_1 = \inf \left\{ \int_{\mathbb{R}^N_+} K(x)|\nabla u|^2 \, dx : \int_{\mathbb{R}^N_+} K(x) u^2 \, dx = 1 \right\}.$$

From this, we obtain the following Poincaré inequality

$$\lambda_1 \int_{\mathbb{R}^N_+} K(x) u^2 \, dx \leq \int_{\mathbb{R}^N_+} K(x)|\nabla u|^2 \, dx, \qquad \forall u \in X. \tag{3.1}$$

We are ready to prove our multiplicity result.

Proof of Theorem 1.1. In order to apply Theorem 3.1, we consider $V = \{0\}$ and $W = X$. Given $\varepsilon > 0$, we can use (f_1) and (f_2) to obtain $c_1 = c_1(\varepsilon) > 0$ such that

$$|F(s)| \leq \frac{\varepsilon}{2} s^2 + c_1 |s|^p, \qquad \forall\, s \in \mathbb{R}.$$

Picking $\varepsilon > 0$ such that $\varepsilon < \lambda_1$, we can use (3.1) and Proposition 2.1 to get

$$\begin{aligned} I_\beta(w) &\geq \frac{1}{2}\|w\|^2 - \frac{1}{2}\varepsilon \|w\|_2^2 - c_1 \|w\|_p^p - \frac{\beta}{2_*} \| w \|_{2_*} \\ &\geq \frac{1}{2}\left[\frac{\lambda_1 - \varepsilon}{\lambda_1}\right] \|w\|^2 - c_1 S_p^{-p/2} \|w\|^p - S_{2_*,\partial}^{-2_*/2} \|w\|^{2_*}, \end{aligned}$$

for any $w \in W$. Since $2 < p < 2^*$, we conclude that

$$I_\beta(w) \geq \frac{\|w\|^2}{2} \left[c_2 + o(\|w\|^2) \right], \quad \text{as } \|w\| \to 0,\ w \in W,$$

with $c_2 = (\lambda_1 - \varepsilon)/(\lambda_1) > 0$. This proves that (I^1) holds.

Given $k \in \mathbb{N}$, we consider $\{\psi_i\}_{i=1}^k \subset C_0^\infty(\overline{\mathbb{R}^N_+})$ smooth functions with disjoint supports and denote

$$\widetilde{V} := \operatorname{span}\{\psi_1, \ldots, \psi_k\}.$$

Then, $\dim \widetilde{V} = k$ and there exists a large ball $B_R(0) \subset \overline{\mathbb{R}^N_+}$ containing the support of all the functions $\psi_1, \ldots, \psi_k$.

Notice that (f_3) provides c_3, $c_4 > 0$ such that

$$F(s) \geq c_3 s^\theta - c_4, \qquad \forall s \in \mathbb{R}, \tag{3.2}$$

with $\theta > 2$. Hence, for any $v \in \widetilde{V}$, the equivalence of norms in $\tilde{V}$ implies that

$$I_\beta(v) \leq \frac{1}{2}\|v\| - c_5 \|v\|^\theta - c_4 \text{meas}(B_R(0)) \to -\infty, \qquad \text{as } \|v\| \to +\infty.$$

Since I_β maps bounded sets into bounded sets, it follows from the above expression that $\max_{v \in \widetilde{V}} I_\beta(v) \leq M$, for some constant $M > 0$. This proves (I^2).

We now consider $\beta^* > 0$ as in Proposition 2.2 and invoke Theorem 3.1 to obtain k pairs of nonzero solution whenever $\beta \in (0, \beta*)$. The theorem is proved. □

4. Nonnegative Solution

We prove in this section our second main result. Since we are looking for positive solutions, we shall assume that $f(s) = 0$, for any $s \le 0$. Moreover, since the parameter $\beta > 0$ does not play any rule in Theorem 1.2, we assume from now on that $\beta = 1$ and consider the functional

$$I(u) := \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N_+} K(x)F(u)\,dx - \frac{1}{2_*}\int_{\mathbb{R}^{N-1}} K(x',0)(u^+)^{2_*}\,dx',$$

where $u^+(x) := \max\{u(x),0\}$. It is clear that $I \in C^1(X,\mathbb{R})$. Moreover, if u is such that $I'(u) = 0$ and $u^- := u^+ - u$, then $0 = I'(u)u^- = -\|u^-\|^2$. Hence, the critical points of I are nonnegative solutions of our problem.

We start with a local compactness result.

Lemma 4.1. *The functional I satisfies the $(PS)_c$-condition for any*

$$c < c^* := \frac{S_{2_*,\partial}^{N-1}}{2(N-1)}.$$

Proof. Let $(u_n) \subset X$ be such that $I(u_n) \to c$ and $I'(u_n) \to 0$. As before, (u_n) is bounded and therefore there exists $u \in X$ such that (2.2) holds. Moreover, from (f_1), (f_2) and the Lebesgue's Theorem, we conclude that $I'(u) = 0$ and

$$\int_{\mathbb{R}^N_+} K(x)F(u_n)\,dx = \int_{\mathbb{R}^N_+} K(x)F(u)\,dx + o(1)$$

and

$$\int_{\mathbb{R}^N_+} K(x)f(u_n)u_n\,dx = \int_{\mathbb{R}^N_+} K(x)f(u)u\,dx + o(1),$$

as $n \to +\infty$.

If $z_n := (u_n - u)$, we can use the above expressions, $I'(u_n)u_n = o(1)$ and Brezis-Lieb's lemma [7] to get

$$\begin{aligned} o(1) &= \|u_n\|^2 - \int_{\mathbb{R}^N_+} K(x)f(u_n)u_n\,dx - \int_{\mathbb{R}^{N-1}} K(x',0)(u_n^+)^{2_*}\,dx' \\ &= I'(u)u + \|z_n\|^2 - \int_{\mathbb{R}^{N-1}} K(x',0)(z_n^+)^{2_*}\,dx' + o(1). \end{aligned}$$

Passing to the limit and using $I'(u) = 0$, we obtain $b \ge 0$ such that

$$\lim_{n\to+\infty} \|z_n\|^2 = b = \lim_{n\to+\infty} \int_{\mathbb{R}^{N-1}} K(x',0)(z_n^+)^{2_*}\,dx'.$$

We claim that $b = 0$. In order to prove this, we first pass to the limit the inequality

$$\int_{\mathbb{R}^{N-1}} K(x',0)(z_n^+)^{2_*}\,dx' \le S_{2_*,\partial}^{-2_*/2}\left(\int_{\mathbb{R}^N_+} K(x)|\nabla z_n|^2 dx\right)^{2_*/2},$$

to obtain $b \le S_{2_*,\partial}^{-2_*/2} b^{2_*/2}$. Hence, if $b > 0$, we get

$$b \ge S_{2_*,\partial}^{N-1}. \tag{4.1}$$

On the other hand, using Brezis-Lieb again, we obtain

$$c + o(1) = I(u_n) = I(u) + \frac{1}{2}\|z_n\|^2 - \frac{1}{2_*}\int_{\mathbb{R}^{N-1}} K(x',0)(z_n^+)^{2_*}\,dx' + o(1).$$

Taking the limit and using (4.1), we get that

$$c = I(u) + \frac{\lambda}{2(N-1)} \geq I(u) + \frac{S_{2_*,\partial}^{N-1}}{2(N-1)} = I(u) + c^*.$$

Using (f_3) we obtain $I(u) = I(u) - (1/\theta)I'(u)u \geq 0$, and therefore the above expression implies that $c \geq c_*$, which does not make sense. ☐

Let us take $\phi \in C^\infty(\overline{\mathbb{R}^N_+}, [0,1])$ such that $\phi \equiv 1$ in $\overline{\mathbb{R}^N_+} \cap B_1(0)$ and $\phi \equiv 0$ in $\overline{\mathbb{R}^N_+} \backslash B_2(0)$. Set, for each $\varepsilon > 0$,

$$u_\varepsilon(x) := K(x)^{-1/2}\phi(x)U_\varepsilon(x), \quad x \in \mathbb{R}^N_+,$$

where

$$U_\varepsilon(x', x_N) := \frac{\varepsilon^{(N-2)/2}}{[|x'|^2 + (x_N + \varepsilon)^2]^{(N-2)/2}}.$$

When $N \geq 7$, it is proved in [23] that, as $\varepsilon \to 0^+$,

$$\|u_\varepsilon\|^2 = A_N + O(\varepsilon^2), \qquad |u_\varepsilon|_{2_*}^{2_*} = B_N^{2_*/2} + O(\varepsilon^2), \tag{4.2}$$

with the constants above being such that $A_N/B_N = S_{2_*,\partial}$. We shall need the following estimates:

Lemma 4.2. *If $\psi_\varepsilon := u_\varepsilon/|u_\varepsilon|_{2_*}$ and $N/(N-2) < q < 2N/(N-2)$, then*

$$\|\psi_\varepsilon\|^{2(N-1)} = S_{2_*,\partial}^{N-1} + O(\varepsilon^2), \qquad \|\psi_\varepsilon\|_q^q = O(\varepsilon^{N-q(N-2)/2}), \tag{4.3}$$

as $\varepsilon \to 0^+$.

Proof. Using the Mean Value theorem for $g(r) = r^s$ and a simple computation, we can check that

$$\left[A + O(\varepsilon^t)\right]^s = A^s + O(\varepsilon^t),$$

for any A, s, $t > 0$. Hence, we infer from (4.2) and the defintion of 2_* that

$$\|\psi_\varepsilon\|^{2(N-1)} = \frac{\left[A_N + O(\varepsilon^2)\right]^{N-1}}{\left[B_N^{2_*/2} + O(\varepsilon^2)\right]^{N-2}} = \frac{A_N^{N-1} + O(\varepsilon^2)}{B_N^{2_*(N-2)/2} + O(\varepsilon^2)} = \left(\frac{A_N}{B_N}\right)^{N-1} + O(\varepsilon^2).$$

The first statement in (4.3) follows from the above inequality and $A_N/B_N = S_{2_*,\partial}$.

For the second one, we first notice that

$$\begin{aligned}
\|u_\varepsilon\|_q^q &= \varepsilon^{-q(N-2)/2}\int_{\mathbb{R}^N_+} \frac{K(x)^{-q/2}\phi(x)^q}{[|x'/\varepsilon|^2 + (x_N/\varepsilon + 1)^2]^{q(N-2)/2}}\,dx \\
&\leq C_1\varepsilon^{-q(N-2)/2}\int_{B_2(0)\cap\mathbb{R}^N_+} \frac{1}{[|x/\varepsilon|^2 + 1]^{q(N-2)/2}}\,dx \\
&\leq C_1\varepsilon^{-q(N-2)/2+N}\int_{\mathbb{R}^N_+} \frac{1}{[|y|^2 + 1]^{q(N-2)/2}}\,dy,
\end{aligned}$$

where we have used the definition of u_ε, $0 \leq \phi \leq 1$ and the change of variable $y = x/\varepsilon$. But

$$\begin{aligned}
\int_{\mathbb{R}^N_+} \frac{1}{[|y|^2 + 1]^{q(N-2)/2}}\,dy &\leq C_2 + \int_{\mathbb{R}^N_+\backslash B_1(0)} \frac{1}{|y|^{q(N-2)}}\,dy \\
&= C_2 + C_3\int_1^{+\infty} s^{-q(N-2)+(N-1)}\,ds < +\infty,
\end{aligned}$$

whenever $q > N/(N-2)$. Since $|u_\varepsilon|_{2_*}^q = B_N^{q/2} + o(1)$, as $\varepsilon \to 0^+$, the result follows from the above inequalities. □

We are ready to prove our second main result.

Proof of Theorem 1.2. Arguing as in the proof of Theorem 1.1 we obtain $\rho, \alpha > 0$ such that $I(u) \geq \alpha$, whenever $\|u\| \geq \rho$. Moreover, it follows from (3.2) that

$$\frac{I(t\psi_\varepsilon)}{t^{2_*}} \leq \frac{1}{2t^{2_*-2}}\|\psi_\varepsilon\|^2 - \frac{c_3}{t^{2_*-\theta}}\|\psi_\varepsilon\|_\theta^\theta + \frac{c_4}{t^{2_*}}\text{meas}(\text{supp}\psi_\varepsilon) - \frac{1}{2_*}|\psi_\varepsilon|_{2_*}^{2_*},$$

for $t > 0$. Thus, there exists $\bar{t} > 0$ such that $e = \bar{t}\psi_\varepsilon$ satisfies $I(e) < 0$ and $\|e\| > \rho$. If we set

$$c_\varepsilon := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)) \geq \alpha,$$

where $\Gamma := \{\gamma \in C([0,1], X) : \ \gamma(0) = 0, \ \gamma(1) = e\}$, we obtain from the Mountain Pass Theorem [1] a sequence $(u_n) \subset X$ such that $I(u_n) \to c_\varepsilon$ and $I'(u_n) \to 0$. If $c_\varepsilon < c^*$, it follows from Lemma 4.1 that, along a subsequence, (u_n) strongly converges to a critical point $u \in X$ such that $I(u) = c_\varepsilon \geq \alpha > 0$. Thus, $u \geq 0$ is a nonzero solution of the problem.

It remains to check that, for some $\varepsilon > 0$ small, there holds $c_\varepsilon < c^*$. In order to do that, we set

$$m_\varepsilon := \max_{t \geq 0} I(t\psi_\varepsilon)$$

and notice that it is sufficient to prove that $m_\varepsilon < c^*$. Let $t_\varepsilon > 0$ be such that $m_\varepsilon = I(t_\varepsilon\psi_\varepsilon)$. Since $I'(t_\varepsilon\psi_\varepsilon)\psi_\varepsilon = 0$ and $|\psi_\varepsilon|_{2_*} = 1$, we get

$$t_\varepsilon^{2_*-1} = t_\varepsilon\|\psi_\varepsilon\|^2 - \int_{\mathbb{R}_+^N} K(x)f(t_\varepsilon\psi_\varepsilon)\psi_\varepsilon\,dx, \tag{4.4}$$

The above identity and (f_3) imply that

$$t_\varepsilon \leq \|\psi_\varepsilon\|^{2/(2_*-2)}.$$

Since the function $g : [0, +\infty) \to \mathbb{R}$ defined by $g(t) := (t^2/2)\|\psi_\varepsilon\|^2 - t^{2_*}/2_*$ is increasing in the interval $[0, \|\psi_\varepsilon\|^{2/(2_*-2)}]$, we can use the above inequality and (4.3) to get

$$\begin{aligned} m_\varepsilon &= g(t_\varepsilon) - \int_{\mathbb{R}_+^n} K(x)F(t_\varepsilon\psi_\varepsilon)\,dx \\ &\leq \frac{\|\psi_\varepsilon\|^{2(N-1)}}{2(N-1)} - \int_{\mathbb{R}_+^n} K(x)F(t_\varepsilon\psi_\varepsilon)\,dx \\ &= \frac{S_{2_*,\partial}^{N-1}}{2(N-1)} + O(\varepsilon^2) - \int_{\mathbb{R}_+^n} K(x)F(t_\varepsilon\psi_\varepsilon)\,dx. \end{aligned}$$

So, it is sufficient to prove that

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^2}\int_{\mathbb{R}_+^n} K(x)F(t_\varepsilon\psi_\varepsilon)\,dx = +\infty. \tag{4.5}$$

First notice that, by (f_1), (f_2), (4.3) and $p < 2^*$, it follows that

$$\left|\int_{\mathbb{R}_+^N} K(x)f(t_\varepsilon\psi_\varepsilon)\psi_\varepsilon\,dx\right| \leq O(\varepsilon^2) + O(\varepsilon^{N-p(N-2)/2}) = o(1),$$

as $\varepsilon \to 0^+$. This, together with (4.3) and (4.4), implies that $t_\varepsilon \to S_{2_*,\partial}^{(N-2)/2} > 0$, as $\varepsilon \to 0^+$. Thus, since (f_3) implies that F is increasing in $[0, +\infty)$, we can use (4.2), $K \geq 1$ and the definition of ϕ to obtain $C_1 > 0$ such that

$$\int_{\mathbb{R}^N_+} K(x)F(t_\varepsilon \psi_\varepsilon)\,dx \geq \int_{B_1(0)\cap\mathbb{R}^N_+} F\left(C_1 \frac{\varepsilon^{(N-2)/2}}{[|x'|^2 + (x_N+\varepsilon)^2]^{(N-2)/2}}\right) dx, \tag{4.6}$$

for any $\varepsilon > 0$ small. If we call Γ_ε the right-hand side above, the change of variables $y = x/\varepsilon$ gives

$$\Gamma_\varepsilon = \varepsilon^N \int_0^{1/\varepsilon} \int_{\partial B_s(0)\cap\mathbb{R}^N_+} F\left(C_1 \frac{\varepsilon^{-(N-2)/2}}{[|y'|^2 + (y_N+1)^2]^{(N-2)/2}}\right) d\sigma_y\,ds.$$

Now, using the change of variable $y = sx$, with $x \in \partial B_1(0)$, the monotonicity of F and the inequality $s^2|x'|^2 + (sx_N+1)^2 \leq 4(s^2+1)$, for $x \in \partial B_1(0)$, we obtain

$$\begin{aligned}\Gamma_\varepsilon &\geq \varepsilon^N \int_0^{1/\varepsilon} \int_{\partial B_1(0)\cap\mathbb{R}^N_+} F\left(C_2 \frac{\varepsilon^{-(N-2)/2}}{[s^2+1]^{(N-2)/2}}\right) s^{N-1}\,d\sigma_x\,ds \\ &= C_3\varepsilon^N \int_0^{1/\varepsilon} F\left(C_2 \frac{\varepsilon^{-(N-2)/2}}{[s^2+1]^{(N-2)/2}}\right) s^{N-1}\,ds\end{aligned}$$

with $C_2 = 4^{-(N-2)/2}C_1 > 0$ and $C_3 = C_3(N)$. After rescaling, we obtain

$$\frac{1}{\varepsilon^2}\Gamma_\varepsilon \geq C_4 \varepsilon^{N-2} \int_0^{C_2^{-2/(N-2)}/\varepsilon} F\left(\frac{\varepsilon^{-(N-2)/2}}{[s^2+1]^{(N-2)/2}}\right) s^{N-1}\,ds.$$

with $C_4 := C_3 C_2^{2N/(N-2)}$. It is easy to see that (4.5) is a consequence of the above expression, (4.6) and hypothesis (1.2). The theorem is proved. □

5. Data availability statement

Data sharing not applicable

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