FRACTIONAL ELLIPTIC SYSTEMS WITH NONCOERCIVE POTENTIALS

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ABSTRACT. We establish the existence of weak solution to the following class of fractional elliptic systems

$$\begin{cases} (-\Delta)^s u + a(x)u = F_u(x,u,v), & x \in \mathbb{R}^N, \\ (-\Delta)^s v + b(x)v = F_v(x,u,v), & x \in \mathbb{R}^N, \end{cases}$$

where $s \in (0,1)$, the potentials a, b are bounded from below and may change sign. The nonlinear term $F \in C^1(\mathbb{R}^N \times \mathbb{R}^2, \mathbb{R})$ can be asymptotically linear or superlinear at infinity. It interacts with the eigenvalues of the linearized problem. In the proofs we apply Variational Methods by considering both the resonant and non-resonant case. We notice that our results are new even in the local case $s = 1$.

1. INTRODUCTION

Recently, great attention has been paid on the study of fractional and non-local operators of elliptic type, both for the pure mathematical research and in view of concrete applications. Actually, these operators arise in a quite natural way in different contexts, such as the thin obstacle problem, optimization, finance, phase transitions, anomalous diffusion, crystal dislocation, semipermeable membranes, conservation laws, ultra-relativistic limits of quantum mechanics, multiple scattering, minimal surfaces, materials science and water waves, see for instance [6, 20] and references therein.

In this work we deal with the following class of fractional elliptic systems of gradient type

$$(P) \qquad \begin{cases} (-\Delta)^s u + a(x)u = F_u(x,u,v), & x \in \mathbb{R}^N, \\ (-\Delta)^s v + b(x)v = F_v(x,u,v), & x \in \mathbb{R}^N, \end{cases}$$

where $s \in (0,1)$, $N > 2s$, $(-\Delta)^s$ denotes the *fractional Laplacian operator* which may be defined as

$$(-\Delta)^s u(x) := C(N,s) \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{u(x)-u(y)}{|x-y|^{N+2s}} \, \mathrm{d}y,$$

where $C(N,s) > 0$ is a normalizing constant which we omit for simplicity.

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Such class of systems arise in various branches of Mathematical Physics and nonlinear optics (see for instance [2]). Solutions of System (P) are related to standing wave solutions of the following two-component system of nonlinear equations

$$\begin{cases} i\dfrac{\partial \psi}{\partial t} = (-\Delta)^s \psi + V_1(x)\psi - F_u(x,\psi,\phi), & x \in \mathbb{R}^N,\ t \geq 0, \\ i\dfrac{\partial \phi}{\partial t} = (-\Delta)^s \phi + V_2(x)\phi - F_v(x,\psi,\phi), & x \in \mathbb{R}^N,\ t \geq 0, \end{cases} \tag{1.1}$$

where i denotes the imaginary unit, $a(x) = V_1(x) - 1$ and $b(x) = V_2(x) - 1$. For System (1.1), a solution of the form $(\psi(x,t),\phi(x,t)) = (e^{-it}u(x), e^{-it}v(x))$ is called *standing wave*. Assuming also that $F_u(x, e^{i\theta}u, e^{i\theta}v) = e^{i\theta}F_u(x,u,v)$ and $F_v(x, e^{i\theta}u, e^{i\theta}v) = e^{i\theta}F_v(x,u,v)$, for $u, v \in \mathbb{R}$, it is well known that (ψ,ϕ) is a solution of (1.1) if and only if (u,v) solves System (P). For more information on the physical background we refer the readers to [2, 16, 17] and references therein.

It is worthwhile to mention that if $a(x) = b(x) = V(x)$, $u = v$, then System (P) reduces to the fractional Schrödinger equation

$$(-\Delta)^s u + V(x)u = g(x,u), \quad x \in \mathbb{R}^N,$$

where $g(x,u) = F_u(x,u,u)$. It is well known the influence of potential $V(x)$ on studying nonlinear Schrödinger equations. For the local case $s = 1$, we refer to the seminal works [1, 3, 21] and references therein. In [21], it was studied the existence of solutions for a class of nonlinear Schrödinger equations involving coercive potential, i.e, when $V(x) \to \infty$ as $|x| \to \infty$. In [3], it was introduced a less restrictive condition on the potential $V(x)$, and it was proved that the Sobolev space is compactly embedded into the Lebesgue spaces. As we point out later, our work is motivated by these classical works, since we consider potentials which may not be coercive. For the nonlocal case $s \in (0,1)$, we refer the interested reader to [5, 9, 23, 24] and references therein.

Naturally, the results have been extended to systems involving nonlinear Schrödinger equations. Regarding to System (P) in the local case $s = 1$, we refer the interesting works [10, 11, 19]. Similarly to the scalar case, the potentials play a very crucial role in the arguments. In [11, 19], the authors have considered a class of positive potentials that satisfy $\mu(\{x \in \mathbb{R}^N : a(x)b(x) < M\}) < \infty$, for every $M > 0$, where μ denotes the Lebesgue measure in $\mathbb{R}^N$. In this case, $a(x)$ and $b(x)$ may not be coercive and the Sobolev embeddings may not be compact. For the nonlocal case $s \in (0,1)$, we cite [15, 18] where it was considered fractional coupled systems.

Motivated by the above discussion, we consider potentials that may change sign, may not be coercive and we deal with a function F that can be asymptotically linear or superlinear at infinity. Before stating our assumptions, we recall the definition of the fractional Sobolev space

$$H^s(\mathbb{R}^N) := \left\{ u \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y < +\infty \right\},$$

endowed with the usual norm

$$\|u\|_s := \left([u]_s^2 + \int_{\mathbb{R}^N} u^2\,\mathrm{d}x \right)^{1/2}, \quad [u]_s := \left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y \right)^{1/2}$$

where the term $[u]_s$ is the so-called *Gagliardo semi-norm* of the function u (see [5, 20]). In view of the presence of the potential $a(x)$ in the first equation, we introduce the weighted fractional Sobolev space

$$E_a := \left\{ u \in H^s(\mathbb{R}^N) : \int_{\mathbb{R}^N} a(x)u^2 \,\mathrm{d}x < \infty \right\},$$

endowed with the inner product

$$(u,v)_{E_a} := \int_{\mathbb{R}^N} (-\Delta)^{s/2}u(-\Delta)^{s/2}v \,\mathrm{d}x + \int_{\mathbb{R}^N} a(x)uv \,\mathrm{d}x,$$

to which corresponds the induced norm $\|u\|_{E_a}^2 := (u,u)_{E_a}$. In a similar way one can define the space E_b and norm $\|\cdot\|_{E_b}$ associated to the potential $b(x)$.

We suppose that potentials $a(x)$ and $b(x)$ satisfy:

(H_1) there exist $a_0, b_0 > 0$ such that $a(x) \geq -a_0, b(x) \geq -b_0$ for all $x \in \mathbb{R}^N$. Moreover, $a(x)b(x) \geq 0$, for all $x \in \mathbb{R}^N$;

(H_2) $\mu(\{x \in \mathbb{R}^N : a(x)b(x) < M\}) < \infty$, for every $M > 0$, where μ denotes the Lebesgue measure in $\mathbb{R}^N$;

(H_3) there hold

$$\inf_{u \in E_a} \left\{ \frac{[u]_s^2 + \int_{\mathbb{R}^N} a(x)u^2 \,\mathrm{d}x}{\int_{\mathbb{R}^N} u^2 \,\mathrm{d}x} \right\} > 0 \quad \text{and} \quad \inf_{v \in E_b} \left\{ \frac{[v]_s^2 + \int_{\mathbb{R}^N} b(x)v^2 \,\mathrm{d}x}{\int_{\mathbb{R}^N} v^2 \,\mathrm{d}x} \right\} > 0.$$

An interesting prototype of potentials is given by $a(x) = (1+|x|^2)^{l_1}$, $b(x) = (1+|x|^2)^{-l_2}$, with $l_1 > l_2 > 0$. Notice that, differently from most of the aforementioned works, we have that $b(x) \to 0$, as $|x| \to \infty$. Clearly, the potentials a, b given just above satisfy hypotheses $(H_1) - (H_3)$.

Under $(H_1) - (H_3)$, the product space $E = E_a \times E_b$ is a Hilbert space when endowed with the natural inner product

$$((u,v),(w,z))_E := (u,w)_{E_a} + (v,z)_{E_b}.$$

We denote the induced norm $\|(u,v)\|^2 := ((u,v),(u,v))_E$ for any $(u,v) \in E$. By using (H_1) and (H_3), we may argue similarly to [27, Lemma 2.1] and obtain constants $\kappa_a, \kappa_b > 0$ such that

$$[u]_s^2 + \int_{\mathbb{R}^N} a(x)u^2 \,\mathrm{d}x \geq \kappa_a \|u\|_s^2, \quad \forall\, u \in E_a \tag{1.2}$$

and

$$[v]_s^2 + \int_{\mathbb{R}^N} b(x)v^2 \,\mathrm{d}x \geq \kappa_b \|v\|_s^2, \quad \forall\, v \in E_b. \tag{1.3}$$

The above estimates easily imply that E_a and E_b are continuously embedded into $L^q(\mathbb{R}^N)$, for all $q \in [2, 2_s^*]$, where $2_s^* := 2N/(N-2s)$ is the fractional critical Sobolev exponent. We point out that hypothesis (H_2) is less restrictive than coercivity and does not imply compact embedding of the weighted fractional Sobolev spaces into the Lebesgue spaces (see [10, 12] for a similar assumption). To the best of our knowledge, this is the first work dealing with the class of potentials introduced by (H_2) in the nonlocal case.

The assumptions on the nonlinearity F are the following:

(F_1) $F \in C^1(\mathbb{R}^N \times \mathbb{R}^2, \mathbb{R})$;

(F_2) there exist $c_1, c_2 > 0$, $2 \leq \sigma \leq 2_s^*$ and $\gamma \in L^t(\mathbb{R}^N)$, for some $t \in [2N/(N+2s), 2]$ such that, for a.e. $x \in \mathbb{R}^N$ and all $z \in \mathbb{R}^2$,

$$|\nabla_z F(x,z)| \leq c_1|z|^{\sigma-1} + c_2|z| + \gamma(x),$$

where ∇_z stands for the gradient in second the variable $z \in \mathbb{R}^2$.

(F_3) there exist functions α, $\beta \in L^\infty(\mathbb{R}^N)$, $\gamma_1 \in L^1(\mathbb{R}^N) \cap L^{2N/(N+2s_1)}(\mathbb{R}^N)$ and $c_3 \geq 0$ such that, for a.e. $x \in \mathbb{R}^N$ and all $(u,v) \in \mathbb{R}^2$,

$$|F(x,u,v)| \leq \gamma_1(x)(|u|+|v|) + c_3|u||v| + \frac{\alpha(x)}{2}|u|^2 + \frac{\beta(x)}{2}|v|^2,$$

where α, β satisfy

$$\limsup_{|x|\to\infty} \alpha(x) = \alpha_\infty < \kappa_a, \quad \limsup_{|x|\to\infty} \beta(x) = \beta_\infty < \kappa_b.$$

It is important study the interaction of the nonlinearity with the spectrum of the associated linearized problem. We shall consider here asymptotic limits depending on x, as we can see from the weighted eigenvalue problem

$$(LP) \qquad \begin{cases} (-\Delta)^s u + a(x)u = \lambda A(x)v, & x \in \mathbb{R}^N, \\ (-\Delta)^s v + b(x)v = \lambda A(x)u, & x \in \mathbb{R}^N, \end{cases}$$

with $A \in L^\theta(\mathbb{R}^N)$, $\theta > N/(2s)$. We shall see in Section 2 that this problem has a sequence of eigenvalues

$$\cdots \leq \lambda_{-m}^A \leq \cdots \leq \lambda_{-1}^A < 0 < \lambda_1^A \leq \cdots \leq \lambda_m^A \leq \cdots,$$

such that $\lambda_{\pm m}^A \to \pm\infty$ as $m \to \infty$.

We also consider the following hypotheses:

(F_∞) there exists $A_\infty \in L^\theta(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $\theta > N/(2s)$, such that

$$\lim_{|(u,v)|\to+\infty} \frac{F(x,u,v) - A_\infty(x)uv}{|(u,v)|^2} = 0, \quad \text{uniformly for a.e. } x \in \mathbb{R}^N;$$

(NQ) there exists $\Gamma \in L^1(\mathbb{R}^N)$ such that

$$\begin{cases} \lim\limits_{\substack{|u|\to\infty \\ |v|\to\infty}} \nabla_z F(x,u,v)\cdot(u,v) - 2F(x,u,v) = \infty, & \text{for a.e. } x \in \mathbb{R}^N, \\ \nabla_z F(x,z)\cdot z - 2F(x,z) \geq \Gamma(x), & \forall\,(x,z) \in \mathbb{R}^N \times \mathbb{R}^2, \end{cases}$$

where $w \cdot z$ denotes the usual inner product between $w, z \in \mathbb{R}^2$.

The first result of this paper can be stated as follows:

Theorem 1.1. *Suppose that* $(H_1) - (H_3)$ *hold. If* F *satisfies* $(F_1) - (F_3)$*,* (F_∞) *and* (NQ)*, then System* (P) *has at least one solution.*

It is worthwhile to mention that under assumptions of Theorem 1.1 it is possible occurs $\nabla_z F(\cdot,0,0) \neq 0$. For this reason, $(u,v) \equiv (0,0)$ is not necessarily a weak solution for Problem (P). In the proof, we apply a version of the Saddle Point Theorem. Condition (F_∞) implies that the problem is asymptotically linear at infinity. Notice that we allow resonance, that is, it can happen $\lambda_k^{A^\infty} = 1$ for some $k \in \mathbb{N}$. The condition (NQ) is related to the boundedness of Palais-Smale type sequences. It was introduced in [8] for a scalar equation. Actually, as a byproduct of the arguments developed in Section 3, it is sufficient a local version of (NQ) where the limit holds only in a large ball (see Remark 3.3).

A class of nonlinearities verifying the hypotheses of Theorem 1.1 can be constructed in the following way: choose an odd function $\phi \in C^\infty(\mathbb{R},[0,2])$ such that $\phi(t)=t$ in $[0,1]$, $\phi\equiv 0$ in $[2,+\infty)$, $\phi(t)\le t, t\in\mathbb{R}$ and define

$$F(x,u,v)=A_\infty(x)uv+c(x)\ln(1+u^2+v^2)+\gamma_1(x)\phi(u+v),$$

where A_∞ and $\gamma_1\in L^\infty(R^N)$ have also the regularity required in (F_∞) and (F_3), respectively, $c\in L^\infty(\mathbb{R}^N)$ is negative and $c(x)\to 0$ as $|x|\to\infty$. In this case, we have that $\nabla_z F(\cdot,0,0)=(\gamma_1(\cdot),\gamma_1(\cdot))$. This example shows that, in our setting, the trivial function $(u,v)\equiv(0,0)$ may not be a solution of the system.

In the second part of the paper we study the superlinear case. For this purpose, we assume the following hypotheses:

(F_0) there exists $A_0\in L^\theta(\mathbb{R}^N)$, $\theta>N/(2s)$, such that

$$\lim_{|(u,v)|\to 0}\frac{F(x,u,v)-A_0(x)uv}{|(u,v)|^2}=0,\quad \text{uniformly for a.e. } x\in\mathbb{R}^N;$$

$(\widehat{F_2})$ condition (F_2) with $\gamma\equiv 0$ holds;

$(\widehat{F_3})$ there exist functions α, $\beta\in L^\infty(\mathbb{R}^N)$ and $c_4\ge 0$ such that, for a.e. $x\in\mathbb{R}^N$ and all $(u,v)\in\mathbb{R}^2$,

$$\begin{cases}|F_u(x,u,v)|\le c_4|u|^{p_1-1}|v|^{q_1}+\alpha(x)|u|+c_4|v|,\\ |F_v(x,u,v)|\le c_4|u|^{q_2}|v|^{p_2-1}+c_4|u|+\beta(x)|v|,\end{cases}$$

where $p_1>1$, $p_2>1$, $q_1>0$, $q_2>0$ and $2\le p_i+q_i\le 2_s^*$, with

$$\limsup_{|x|\to\infty}\alpha(x)=\alpha_\infty<a_0,\quad \limsup_{|x|\to\infty}\beta(x)=\beta_\infty<b_0;$$

$(\widehat{NQ})$ there exist $\theta>2,p,q,\mu,\nu>0$ and $\widehat{\Gamma}\in L^1(\mathbb{R}^N),c_5>0,c_6>0$ such that, for a.e. $x\in\mathbb{R}^N$ and all $(u,v)\in\mathbb{R}^2$,

$$\begin{cases}\nabla_z F(x,u,v)\cdot(u,v)-\theta F(x,u,v)\ge -c_5|u|^p|v|^q-\widehat{\Gamma}(x),\\ \nabla_z F(x,u,v)\cdot(u,v)-2F(x,u,v)\ge c_6|u|^\mu|v|^\nu,\end{cases}$$

and

$$\mu+\nu,\ p+q\in(2,2_s^*),\quad \mu+\nu>(p+q-2)\max\left\{\frac{N}{2s},\frac{\mu}{p},\frac{\nu}{q}\right\}.$$

In our second result, we prove the following:

Theorem 1.2. *Suppose that $(H_1)-(H_3)$ hold. If F satisfies (F_0), (F_1), $(\widehat{F_2})$, $(\widehat{F_3})$ and $(\widehat{NQ})$, then System (P) has at least one nonzero solution.*

As in our fist result, we allow resonance at the origin. For the proof we apply a version of the Generalized Mountain Pass Theorem. The condition $(\widehat{NQ})$ is a version of that introduced in [26] for the scalar case (see also [7]). It is worthwhile to mention that, in this paper, we consider a class of potentials which may change the sign and we do not assume $\lambda_1^{A_0}>1$ in the second main theorem. Hence, our main theorems are new even in the local case $s=1$.

As an application of the second theorem, we may consider

$$F(x,u,v)=A_0(x)uv+c(x)|u|^{r_1}|v|^{r_2},$$

where r_1, $r_2>1$ satisfy $r_1+r_2\le 2_s^*$, $A_0\le 0$ has the regularity required in (F_0) and $c\in L^\infty(\mathbb{R}^N)$ is bounded from below by a positive constant. Actually, the nonlinear perturbation above can be replaced by the sum $\sum_{i=1}^{l} c_i(x)|u|^{r_{1,i}}|v|^{r_{2,i}}$ with analogous conditions on c_i, $r_{1,i}$ and $r_{2,i}$.

The remainder of this paper is organized as follows: in the forthcoming section we prove some auxiliary results and we study the linear eigenvalue problem (LP). Sections 3 and 4 are devoted to the proof of Theorems 1.1 and 1.2, respectively.

2. Some technical results and the linear problem

In this section we shall consider some technical results. For $R > 0$, we set $B_R := \{x \in \mathbb{R}^N : |x| < R\}$ and $B_R^c := \mathbb{R}^N \setminus B_R$. For $u, v \in L^q(\mathbb{R}^N)$, we denote by $\|u\|_q$ the norm in $L^q(\mathbb{R}^N)$ and by $\|(u,v)\|_q$ the norm in $L^q(\mathbb{R}^N) \times L^q(\mathbb{R}^N)$. We write only $\int u$ instead of $\int_{\mathbb{R}^N} u(x)\mathrm{d}x$. Finally, $C_1, C_2, \ldots$ denote positive constants (possibly different).

The energy functional $I : E \to \mathbb{R}$ associated to System (P) is defined by

$$I(z) = \frac{1}{2}\|z\|^2 - \int F(x,z), \quad z = (u,v) \in E.$$

It is standard to check that under our assumptions the functional is well defined, $I \in C^1(E, \mathbb{R})$ and its derivative is given by

$$I'(z)w = (z,w)_E - \int \nabla_z F(x,z) \cdot w, \quad \forall\, z,\, w \in E.$$

Thus, critical points of I are precisely weak solutions for System (P) and conversely.

In the next two sections, we shall apply abstract theorems of the Critical Point Theory to prove our main results. For this purpose, we need some technical lemmas.

Lemma 2.1. *Suppose $(H_1) - (H_3)$. Then for any given $\widehat{\varepsilon} > 0$, there exists $R_1 = R_1(\widehat{\varepsilon}, a, b)$ such that*

$$\int_{B_{R_1}^c} |uv|\,\mathrm{d}x \le \widehat{\varepsilon}\left(1 + \frac{a_0 + \sqrt{a_0 b_0}}{2\kappa_a} + \frac{b_0 + \sqrt{a_0 b_0}}{2\kappa_b}\right)\|z\|^2, \quad \forall\, z = (u,v) \in E.$$

Proof. Given $\widehat{\varepsilon} > 0$, we define the set

$$C_{\widehat{\varepsilon}} := \left\{x \in \mathbb{R}^N : \sqrt{a(x)b(x)} \le \frac{1}{\widehat{\varepsilon}}\right\}.$$

Since (H_2) implies that $C_{\widehat{\varepsilon}}$ has finite Lebesgue measure, there exists $R_1 = R_1(\widehat{\varepsilon}, a, b) > 0$ such that $\mu(C_{\widehat{\varepsilon}} \cap B_{R_1}^c)^{N/(2s)} \le \widehat{\varepsilon}/(2C_1^2)$, where $C_1 > 0$ is such that $\|z\|_{2_s^*} \le C_1\|z\|$, for any $z \in E$. Thus, Hölder's inequality provide

$$\int_{C_{\widehat{\varepsilon}} \cap B_{R_1}^c} |uv|\,\mathrm{d}x \le \mu(C_{\widehat{\varepsilon}} \cap B_{R_1}^c)^{N/(2s)}\|u\|_{2_s^*}\|v\|_{2_s^*} \le \frac{\widehat{\varepsilon}}{2}\|z\|^2, \quad \forall\, z \in E. \tag{2.1}$$

In what follows we denote $\{a > 0\} := \{x \in \mathbb{R}^N : a(x) > 0\}$, with analogous notations for the sets $\{a < 0\}$, $\{b > 0\}$ and $\{b < 0\}$. We also set

$$\Theta := (\mathbb{R}^N \setminus C_{\widehat{\varepsilon}}) \cap B_{R_1}^c$$

and

$$\Omega_+ := \Theta \cap \{a > 0\}, \qquad \Omega_- := \Theta \cap \{a < 0\}.$$

Since $1 < \widehat{\varepsilon}\sqrt{a(x)b(x)}$ for any $x \in \Theta$, we obtain

$$\int_{\Omega_+} |uv|\,\mathrm{d}x \le \widehat{\varepsilon}\int_{\Omega_+} \sqrt{a(x)b(x)}|uv|\,\mathrm{d}x \le \frac{\widehat{\varepsilon}}{2}\int_{\Omega_+} \left[a(x)u^2 + b(x)v^2\right]\mathrm{d}x. \tag{2.2}$$

In view of (H_1) we have $0 \le -a(x) \le a_0$ in $\{a < 0\}$. Thus, it follows from the definition of $\|\cdot\|_s$ that

$$\begin{aligned}\int_{\Omega_+} a(x)u^2\,\mathrm{d}x &\le \int_{\{a\ge 0\}} a(x)u^2\,\mathrm{d}x\\ &= \int a(x)u^2 - \int_{\{a<0\}} a(x)u^2\,\mathrm{d}x\\ &\le \int a(x)u^2 + a_0\int_{\{a<0\}} u^2\,\mathrm{d}x\\ &\le \int a(x)u^2 + a_0\|u\|_s^2,\end{aligned}$$

which jointly with (1.2) implies that

$$\int_{\Omega_+} a(x)u^2\,\mathrm{d}x \le \int a(x)u^2 + \frac{a_0}{\kappa_a}\|u\|_{E_a}^2.$$

Since (H_1) provides $\{b<0\} = \{a<0\}$, we can use the same argument to get

$$\int_{\Omega_+} b(x)v^2\,\mathrm{d}x \le \int b(x)v^2 + \frac{b_0}{\kappa_b}\|v\|_{E_b}^2.$$

Replacing the two above estimates in (2.2), we obtain

$$\int_{\Omega_+} |uv|\,\mathrm{d}x \le \frac{\widehat{\varepsilon}}{2}\left(1 + \frac{a_0}{\kappa_a} + \frac{b_0}{\kappa_b}\right)\|z\|^2, \quad \forall\, z \in E. \tag{2.3}$$

On the other hand, by using (H_1) and Young's inequality we estimate

$$\begin{aligned}\int_{\Omega_-} |uv|\,\mathrm{d}x &\le \widehat{\varepsilon}\int_{\Omega_-} \sqrt{a(x)b(x)}|uv|\,\mathrm{d}x\\ &\le \widehat{\varepsilon}\sqrt{a_0b_0}\int_{\Omega_-} |uv|\,\mathrm{d}x\\ &\le \frac{\widehat{\varepsilon}}{2}\sqrt{a_0b_0}\left(\|u\|_s^2 + \|v\|_s^2\right).\end{aligned}$$

Therefore, by (1.2)-(1.3), we conclude that

$$\int_{\Omega_-} |uv|\,\mathrm{d}x \le \frac{\widehat{\varepsilon}}{2}\sqrt{a_0b_0}\left(\frac{1}{\kappa_a} + \frac{1}{\kappa_b}\right)\|z\|^2, \quad \forall\, z \in E.$$

Since $a(x)$ and $b(x)$ does not vanish in Θ, we have that $\Theta = \Omega_+ \cup \Omega_-$. Thus, the above inequality jointly with (2.3) provide

$$\int_{(\mathbb{R}^N\setminus C_{\widehat{\varepsilon}})\cap B_{R_1}^c} |uv|\,\mathrm{d}x \le \frac{\widehat{\varepsilon}}{2}\left(1 + \frac{a_0+\sqrt{a_0b_0}}{\kappa_a} + \frac{b_0+\sqrt{a_0b_0}}{\kappa_b}\right)\|z\|^2, \quad \forall\, z \in E. \tag{2.4}$$

Therefore, (2.1) and (2.4) imply that

$$\int_{B_{R_1}^c} |uv|\,\mathrm{d}x \le \widehat{\varepsilon}\left(1 + \frac{a_0+\sqrt{a_0b_0}}{2\kappa_a} + \frac{b_0+\sqrt{a_0b_0}}{2\kappa_b}\right)\|z\|^2, \quad \forall\, z \in E,$$

and the proof is finished. □

In view of the preceding Lemma, we are able to obtain the following result:

Lemma 2.2. *Suppose* (H_1)–(H_3). *Assume also that* F *satisfies* (F_1)–(F_3). *Then, for any given* $R > 0$ *and* $\varepsilon > 0$, *there exists* $M = M(R, \varepsilon) > 0$, *such that*

$$\int_{\{|z|\le R\}} F(x,z)\,\mathrm{d}x \le M + \left(\varepsilon + \frac{\alpha_\infty}{2\kappa_a} + \frac{\beta_\infty}{2\kappa_b}\right)\|z\|^2, \quad \forall\, z \in E.$$

Proof. Let $c_3 > 0$ be given by condition (F_3) and $\widehat{\varepsilon} > 0$. Applying Lemma 2.1 we obtain $R_1 > 0$ such that

$$\int_{B^c_{R_1}} |uv|\,\mathrm{d}x \le C\widehat{\varepsilon}\|z\|^2, \quad \forall\, z \in E,$$

where $C = C(a_0, b_0, \kappa_a, \kappa_b) > 0$. In view of the limits in (F_3), we may assume that R_1 is large enough so that

$$\alpha(x) \le \alpha_\infty + \widehat{\varepsilon}, \quad \beta(x) \le \beta_\infty + \widehat{\varepsilon}, \quad \forall\, x \in B^c_{R_1}.$$

Hence, if we set $\Omega_1 := \{|z| \le R\} \cap B_{R_1}(0)$ and $\Omega_2 := \{|z| \le R\} \cap B^c_{R_1}$, then it follows from (F_1)–(F_3), the above inequalities and Sobolev embeddings that there exist $M, M_1 > 0$ such that

$$\begin{aligned}
\int\limits_{\{|z|\le R\}} F(x,z)\,\mathrm{d}x \quad &\le \int\limits_{\Omega_1} F(x,z)\,\mathrm{d}x + \int\limits_{\Omega_2} \left(c_3|uv| + \frac{\alpha_\infty + \widehat{\varepsilon}}{2}u^2 + \frac{\beta_\infty + \widehat{\varepsilon}}{2}v^2\right)\mathrm{d}x \\
&\quad + \int_{\{|z|\le R\}} \gamma_1(x)(|u| + |v|)\,\mathrm{d}x \\
&\le M_1 + \left[\left(c_3C + \frac{1}{2\kappa_a} + \frac{1}{2\kappa_b}\right)\widehat{\varepsilon} + \frac{\alpha_\infty}{2\kappa_a} + \frac{\beta_\infty}{2\kappa_b}\right]\|z\|^2 \\
&\quad + 2R\int \gamma_1(x)\,\mathrm{d}x \\
&\le M + \left[\left(c_3C + \frac{1}{2\kappa_a} + \frac{1}{2\kappa_b}\right)\widehat{\varepsilon} + \frac{\alpha_\infty}{2\kappa_a} + \frac{\beta_\infty}{2\kappa_b}\right]\|z\|^2.
\end{aligned}$$

The result follows by picking $\widehat{\varepsilon} > 0$ small. □

The next auxiliary result is a version of [8, Lemma 3.1].

Lemma 2.3. *Suppose that* F *satisfies* $(F_1), (F_\infty)$ *and* (NQ). *Then*

$$F(x,z) - A_\infty(x)uv \le -\frac{\Gamma(x)}{2}, \quad \forall\, z = (u,v) \in \mathbb{R}^2,\ a.e\ x \in \mathbb{R}^N.$$

Proof. Set $G_\infty(x,z) := F(x,z) - A_\infty(x)uv$ and notice that

$$\nabla_z G_\infty(x,z)\cdot z - 2G_\infty(x,z) = \nabla_z F(x,z)\cdot z - 2F(x,z),$$

with $\nabla_z G_\infty$ denoting the gradient of G_∞ with respect to the variable $z \in \mathbb{R}^2$. Thus, for any $s > 0$ and $\overline{z} \in \mathbb{R}^2$ such that $|\overline{z}| = 1$, by (NQ), we have

$$\frac{d}{ds}\left[\frac{G_\infty(x, s\overline{z})}{s^2}\right] = \frac{\nabla_z G_\infty(x, s\overline{z})\cdot(s\overline{z}) - 2G_\infty(x, s\overline{z})}{s^3} \ge \frac{\Gamma(x)}{s^3}.$$

Integrating over $[s,t] \subset (0,\infty)$, we get

$$\frac{G_\infty(x, s\overline{z})}{s^2} \le \frac{G_\infty(x, t\overline{z})}{t^2} - \frac{\Gamma(x)}{2}\left[\frac{1}{s^2} - \frac{1}{t^2}\right].$$

By taking the limit as t goes to infinity on the above expression and using (F_∞), we conclude that

$$G_\infty(x, s\overline{z}) \le -\frac{\Gamma(x)}{2}, \quad \forall\, s>0,\ \overline{z}\in\mathbb{R}^2 \text{ s.t. } |\overline{z}|=1, \text{ a.e. } x\in\mathbb{R}^N.$$

The argument for $s<0$ is similar. □

By similar ideas used in the preceding Lemma we get the following result:

Lemma 2.4. *Suppose that F satisfies $(F_0),(F_1)$ and $(\widehat{NQ})$. Then*

$$F(x,z) - A_0(x)uv \ge \frac{c_5}{(\mu+\nu-2)}|u|^\mu|v|^\nu, \quad \forall\, z=(u,v)\in\mathbb{R}^2,\ a.e\ x\in\mathbb{R}^N.$$

Proof. Setting $G_0(x,z) := F(x,z) - A_0(x)uv$, using $(\widehat{NQ})$ and arguing as in the proof of the above lemma we get, for any $s>0$,

$$\frac{d}{ds}\left[\frac{G_0(x,sz)}{s^2}\right] = \frac{\nabla_z G_0(x,sz)\cdot(sz) - 2G_0(x,sz)}{s^3} \ge c_5|u|^\mu|v|^\nu s^{\mu+\nu-3}.$$

Integrating over $[t,1]$, with $t>0$, we obtain

$$G_0(x,z) \ge \frac{G_0(x,tz)}{t^2} + \frac{c_5}{(\mu+\nu-2)}|u|^\mu|v|^\nu\left[1-t^{\mu+\nu-2}\right].$$

In view of (F_0) we have that $G_0(x,tz)/t^2 \to 0$, as $t\to 0^+$. Thus, by taking the limit as $t\to 0^+$ in the above expression we obtain the desired result. □

We finish this section with the study of the eigenvalue problem

$$(LP)\qquad \begin{cases} (-\Delta)^s u + a(x)u = \lambda A(x)v, & x\in\mathbb{R}^N,\\ (-\Delta)^s v + b(x)v = \lambda A(x)u, & x\in\mathbb{R}^N. \end{cases}$$

Recall that $\lambda\in\mathbb{R}$ is named an eigenvalue of (LP) if there exists a pair $(u,v)\in E$ such that

$$((u,v),(\phi,\psi))_E = \lambda\int A(x)[v\phi + u\psi], \quad \forall\,(\phi,\psi)\in E.$$

Standard calculations show that λ is an eigenvalue of (LP) if, and only if,

$$T(u,v) = \frac{1}{\lambda}(u,v),$$

where $T: E\to E$ is the self-adjoint linear operator defined by

$$\langle T(u,v),(\phi,\psi)\rangle := \int A(x)[v\phi+u\psi].$$

Since $\theta > N/(2s)$, there exists $t\in(2,2_s^*)$ such that

$$(2.5)\qquad \frac{1}{\theta}+\frac{1}{t}+\frac{1}{t} = 1.$$

Hence, by Hölder's inequality, one has

$$\left|\int A(x)v\phi\right| \le \|A\|_\theta\|v\|_t\|\phi\|_t.$$

The embedding $E\hookrightarrow L^t(\mathbb{R}^N)\times L^t(\mathbb{R}^N)$ provides $C_1>0$ such that

$$|\langle T(u,v),(\phi,\psi)\rangle| \le C_1\|A\|_\theta\|(\phi,\psi)\|\|(u,v)\|,$$

which implies that T is bounded.

We claim that T is compact. Indeed, let $(z_n) = ((u_n, v_n)) \subset E$ be a sequence such that $z_n \rightharpoonup z = (u, v)$ weakly in E (without loss of generality, we may suppose $z = (0, 0)$). Then, there exists $C_2 > 0$ such that

$$\|z_n\| \leq C_2 \quad \text{and} \quad \|Tz_n\| \leq C_2, \quad \forall\, n \in \mathbb{N}. \tag{2.6}$$

By writing $T = (T_1, T_2)$ and using the definition of T, we have

$$0 \leq \|Tz_n\|^2 = \langle Tz_n, Tz_n \rangle = \int A(x)(v_n T_1 z_n + u_n T_2 z_n).$$

Let $\varepsilon > 0$, $t \in (2, 2_s^*)$ be as in (2.5) and let $R > 0$ be such that $\|A\|_{L^\theta(\mathbb{R}^N \setminus B_R(0))} < \varepsilon$. In view of Hölder's inequality, the embedding $E \hookrightarrow L^t(\mathbb{R}^N) \times L^t(\mathbb{R}^N)$ and (2.6), we deduce

$$\left| \int_{\mathbb{R}^N \setminus B_R(0)} A(x) v_n T_1 z_n \, \mathrm{d}x \right| \leq \|A\|_{L^\theta(\mathbb{R}^N \setminus B_R(0))} \|v_n\|_t \|T_1 z_n\|_t \leq C_3 \varepsilon.$$

On the other hand, since $v_n \to 0$ strongly in $L^t(B_R(0))$, arguing as above we conclude that

$$\lim_{n \to +\infty} \int_{B_R(0)} [A(x) v_n T_1 z_n] \, \mathrm{d}x = 0.$$

Consequently, $Tz_n \to 0$, as $n \to \infty$, which proves the compactness of T.

Observing that $(u, -v)$ is an eigenfunction associated with the eigenvalue $-\lambda$ whenever (u, v) is an eigenfunction associated to λ, the spectral theory for compact operators implies that (LP) possesses a sequence $\{\lambda_m\}_{m \in \mathbb{Z}^*}$ of eigenvalues

$$\cdots \leq \lambda_{-m}^A \leq \cdots \leq \lambda_{-1}^A < 0 < \lambda_1^A \leq \cdots \leq \lambda_m^A \leq \cdots,$$

such that $\lambda_{\pm m}^A \to \pm\infty$ as $m \to \infty$. Moreover, if we denote $\{\Phi_m^A\}_{m \in \mathbb{Z}^*}$ the sequence of associated eigenfunctions and set $V_k^A = \operatorname{span}\{\Phi_1^A, \ldots, \Phi_k^A\}$, we can decompose H as $H = V_k^A \oplus (V_k^A)^\perp$ and the following variational inequalities hold

$$\frac{1}{2}\|z\|^2 \leq \lambda_k^A \int A(x) uv, \quad \forall\, z = (u, v) \in V_k^A, \tag{2.7}$$

and

$$\frac{1}{2}\|z\|^2 \geq \lambda_{k+1}^A \int A(x) uv, \quad \forall\, z = (u, v) \in (V_k^A)^\perp. \tag{2.8}$$

3. The asymptotically linear case

We devote this section to the proof of our first theorem. In order to obtain the critical point of I we shall apply a version of the Saddle Point Theorem. Before stating it, let us introduce a compactness condition as follows: let $\mathcal{E}$ be a real Hilbert space and $\mathcal{I} \in C^1(\mathcal{E}, \mathbb{R})$. A sequence $(z_n) \subset \mathcal{E}$ is said to be a *strong Cerami sequence* if $\mathcal{I}(z_n) \to c$ and $\mathcal{I}'(z_n) \to 0$ as $n \to \infty$, and $\|z_n\|_{\mathcal{E}} \|\mathcal{I}'(z_n)\|_{\mathcal{E}'}$ is bounded. We say that $\mathcal{I}$ satisfies the *strong Cerami condition for the weak topology* [(SCe)'] if any strong Cerami sequence $(z_n) \subset \mathcal{E}$ possesses a subsequence which converges weakly to a critical point of $\mathcal{I}$. For proving Theorem 1.1 we shall apply the following abstract result, which was proved in [10, Theorem 2.3] (see also [25]):

Theorem 3.1. *Let $\mathcal{E} = V \oplus W$ be a real Hilbert space with V finite dimensional and $W = V^\perp$. Suppose $\mathcal{I} \in C^1(\mathcal{E}, \mathbb{R})$ satisfies* (SCe)' *and*

($\mathcal{I}_1$) *there exists $\beta \in \mathbb{R}$ such that $\mathcal{I}(z) \leq \beta$, for all z in V;*

($\mathcal{I}_2$) *there exists* $\gamma \in \mathbb{R}$ *such that* $\mathcal{I}(z) \geq \gamma$, *for all* z *in* W.

Then $\mathcal{I}$ *possesses a critical point.*

We first prove that under the setting of Theorem 1.1 we have compactness.

Proposition 3.2. *Suppose that* F *satisfies* $(F_1)-(F_3)$, (F_∞) *and* (NQ). *Then* I *satisfies* (SCe)'.

Proof. Let $(z_n) \subset E$ be a strong Cerami sequence. Arguing as in [11, Section 4] or [10, Proposition 2.6], we see that it is sufficient to verify that (z_n) has a bounded subsequence. Suppose, by contradiction, that $\|z_n\| \to \infty$. Since $I(z_n) \to c$ and $\|z_n\|\,\|I'(z_n)\|$ is bounded, there exists $C_1 > 0$ such that

$$\liminf_{n\to+\infty} \int H(x, z_n) = \liminf_{n\to+\infty} [2I(z_n) - I'(z_n)z_n] \leq C_1, \tag{3.1}$$

where $H(x, z_n) := \nabla F_z(x, z_n) \cdot z_n - 2F(x, z_n)$.

Claim. There exists a set of positive measure $\Omega \subset \mathbb{R}^N$ such that, up to subsequences, $|u_n(x)| \to +\infty$ and $|v_n(x)| \to +\infty$ as $n \to +\infty$, for almost every $x \in \Omega$.

Assuming the claim, we can use $H(x, z_n) \geq \Gamma(x)$, Fatou's Lemma, (NQ) and the fact that $\Gamma \in L^1(\mathbb{R}^N)$ to obtain

$$\liminf_{n\to+\infty} \int H(x, z_n) \geq \int \liminf_{n\to+\infty} H(x, z_n) = \infty,$$

which contradicts (3.1).

In order to prove the claim we pick $\varepsilon > 0$ and use (F_∞) to obtain $R > 0$ such that

$$F(x, z) \leq A_\infty(x)uv + \varepsilon|z|^2, \quad \forall\, x \in \mathbb{R}^N,\ |z| > R.$$

If $S > 0$ verifies $\|z\|_2^2 \leq S\,\|z\|^2$, then for all $z \in E$, we infer that

$$\frac{1}{2}(1 - 2S\varepsilon)\,\|z_n\|^2 \leq C_2 + \|A_\infty\|_\infty \int |u_n||v_n| + \int_{\{|z_n| \leq R\}} F(x, z_n)\,\mathrm{d}x$$

and thus, in view of Lemma 2.2, we get

$$\nu_0\|z_n\|^2 \leq C_3 + \|A_\infty\|_\infty \int |u_n||v_n|, \tag{3.2}$$

where

$$\nu_0 := \frac{1}{2}\left(1 - 2S\varepsilon - 2\varepsilon - \frac{\alpha_\infty}{\kappa_a} - \frac{\beta_\infty}{\kappa_b}\right).$$

Since $\alpha_\infty < \kappa_a$ and $\beta_\infty < \kappa_b$, we can choose $\varepsilon > 0$ small in such way that $\nu_0 > 0$.

Now, by picking $\widehat{\varepsilon} > 0$ in an appropriated way, we use Lemma 2.1 to obtain $R_1 > 0$ such that

$$\|A_\infty\|_\infty \int_{B_{R_1}^c} |u_n||v_n| \leq \frac{\nu_0}{2}\|z_n\|^2.$$

Hence, we split the integral on the right-hand side of (3.2) to obtain

$$\frac{\nu_0}{2}\,\|z_n\|^2 \leq C_3 + \|A_\infty\|_\infty \int_{B_{R_1}} |u_n||v_n|\,\mathrm{d}x. \tag{3.3}$$

Defining $\widehat{z}_n = (\widehat{u}_n, \widehat{v}_n) = \dfrac{1}{\|z_n\|}(|u_n|, |v_n|)$, it follows that

$$\begin{cases} \widehat{u}_n \to \widehat{u} \text{ strongly in } L^2(B_{R_1}), \\ \widehat{v}_n \to \widehat{v} \text{ strongly in } L^2(B_{R_1}). \end{cases}$$

Hence, multiplying (3.3) by $\|z_n\|^{-2}$ and taking that expression to the limit, we get

$$\frac{\nu_0}{2} \le \|A_\infty\|_\infty \int_{B_{R_1}} \widehat{u}\widehat{v}\, \mathrm{d}x,$$

and therefore there exists $\Omega \subset B_{R_1}$, with positive measure, such that $\widehat{u}(x) \neq 0$ and $\widehat{v}(x) \neq 0$, a.e. $x \in \Omega$. The claim is now proved by observing that we are assuming that $\|z_n\| \to +\infty$ as $n \to +\infty$. □

Remark 3.3. *It is clear from the above proof that we just need that the limits in* (NQ) *hold in a ball* B_{R_1} *sufficiently large. Hence, the above compactness result holds just with a local nonquadraticity condition. On this subject, we refer the reader to the works* [13, 14] *where it was also considered a local nonquadraticity condition on unbounded and bounded domains, respectively.*

In order to check the geometric conditions $(\mathcal{I}_1) - (\mathcal{I}_2)$ we shall decompose the space E in the following way: let A_∞ be given by assumption (F_∞) and let $\{\lambda_m^{A_\infty}\}_{m\in\mathbb{Z}^*}$ be the sequence of eigenvalues of the problem (LP) with weight $A = A_\infty$. The associated eigenfunctions will be denoted by $\{\Phi_m^{A_\infty}\}_{m\in\mathbb{Z}^*}$. We consider two distinct cases: if $\lambda_1^{A_\infty} < 1$, then we fix $k \in \mathbb{N}$ such that $\lambda_k^{A_\infty} < 1 \le \lambda_{k+1}^{A_\infty}$ and set

$$V := \operatorname{span}\{\Phi_1^{A_\infty}, \ldots, \Phi_k^{A_\infty}\}, \quad W := V^\perp.$$

Otherwise, if $\lambda_1^{A_\infty} \ge 1$, then we just set $V := \{0\}$ and $W := E$.

Proposition 3.4. *Suppose* F *satisfies* (F_1)–(F_3), (F_∞) *and* (NQ). *Then the functional* I *satisfies* $(\mathcal{I}_1)$ *and* $(\mathcal{I}_2)$.

Proof. For any $z \in W$, it follows from Lemma 2.3 and (2.8) that

$$I(z) = \frac{1}{2}\|z\|^2 - \int A_\infty(x)uv - \int G_\infty(x,z) \ge \frac{1}{2}\left(1 - \frac{1}{\lambda_{k+1}^{A_\infty}}\right)\|z\|^2 + \frac{1}{2}\int \Gamma(x),$$

and therefore we infer from $\lambda_{k+1}^{A_\infty} \ge 1$ that I satisfies $(\mathcal{I}_2)$ with $\gamma = -(1/2)\|\Gamma\|_1$.

In order to verify $(\mathcal{I}_1)$, we may assume that $\lambda_1^{A^\infty} < 1$, in such way that $V = \operatorname{span}\{\Phi_1^{A^\infty}, \ldots, \Phi_k^{A^\infty}\}$, otherwise there is nothing to do. We first notice that, since V is finite dimensional, there exists $\delta > 0$ such that

$$\frac{1}{2}\|z\|^2 - \int A_\infty(x)uv \le -\delta\|z\|^2, \quad \forall\, z \in V.$$

Thus, one has

$$I(z) \le -\delta\|z\|^2 + \int \left[A_\infty(x)uv - F(x,z)\right], \quad \forall\, z \in V.$$

We shall prove that I is anticoercive on V. Suppose by contradiction that there exist $C_1 > 0$ and $(z_n) \subset E$ such that $\|z_n\| \to +\infty$ but $I(z_n) \le C_1$. Setting $\widehat{z}_n := z_n/\|z_n\|$, we infer from the above expression that

$$o_n(1) = \frac{C_1}{\|z_n\|^2} \le -\delta + \int \left[\frac{A_\infty(x)u_nv_n - F(x,z_n)}{|z_n|^2}\right]\widehat{z}_n(x)^2 \tag{3.4}$$

where $o_n(1)$ stands for a quantity which goes to zero as $n \to +\infty$. Since V is finite dimensional, there exists $\widehat{z} \in V$ such that $\widehat{z}_n(x) \to \widehat{z}(x)$ for a.e. $x \in \mathbb{R}^N$. Moreover, for any $2 \leq t < 2_s^*$, $\widehat{z}_n \to \widehat{z}$ strongly in $L^t(\mathbb{R}^N) \times L^t(\mathbb{R}^N)$, $\max\{|\widehat{u}_n(x)|, |\widehat{v}_n(x)|\} \leq \psi_t(x)$ for a.e. $x \in \mathbb{R}^N$ and some function $\psi_t \in L^t(\mathbb{R}^N)$. Now by using the embedding E into $L^{2_s^*}(\mathbb{R}^N)$ we obtain that

$$\int \frac{2\gamma_1(x)|z_n|}{\|z_n\|^2} = o_n(1), \tag{3.5}$$

where γ_1 comes from hypothesis (F_3). Hence we set

$$g_n(x) = \left[\frac{A_\infty(x)u_n v_n - F(x, z_n)}{|z_n|^2}\right] \widehat{z}_n(x)^2, \quad x \in \mathbb{R}^N.$$

Therefore, using Young's inequality, (F_3) and Hölder's inequality we obtain

$$\begin{aligned} |g_n(x)| &\leq |A_\infty(x)|[\widehat{u}_n(x)^2 + \widehat{v}_n(x)^2]/2 + c_3[\widehat{u}_n(x)^2 + \widehat{v}_n(x)^2] \\ &\quad + \|\alpha\|_\infty \widehat{u}_n(x)^2 + \|\beta\|_\infty \widehat{v}_n(x)^2 + 2\gamma_1(x)\frac{[|\widehat{u}_n| + |\widehat{v}_n|]}{\|z_n\|} \\ &\leq |A_\infty(x)|\psi_t(x)^2 + c_4\psi_t(x)^2 + 2\gamma_1(x)\frac{[|\widehat{u}_n| + |\widehat{v}_n|]}{\|z_n\|}. \end{aligned}$$

Since $\theta > N/(2s)$, we can pick $t \in (2, 2_s^*)$ such that $1/\theta + 2/t = 1$ and use Hölder's inequality to conclude that the right-hand side just above belongs to $L^1(\mathbb{R}^N)$. Moreover, it follows from Young's inequality and (F_3) that

$$\begin{aligned} \left|\frac{A_\infty(x)u_n v_n - F(x, z_n)}{|z_n|^2}\right| \widehat{z}_n(x)^2 &\leq \frac{1}{2}(\|A_\infty\|_\infty + c_3 + \|\alpha\|_\infty + \|\beta\|_\infty)\widehat{z}_n(x)^2 \\ &\quad + 2\gamma_1(x)\frac{[|\widehat{u}_n| + |\widehat{v}_n|]}{\|z_n\|}. \end{aligned} \tag{3.6}$$

Therefore, $g_n(x) \to 0$ for a.e. $x \in \{y \in \mathbb{R}^N : \widehat{z}(y) = 0\}$. In view of (F_∞), the same occurs in the set $\{y \in \mathbb{R}^N : \widehat{z}(y) \neq 0\}$. Hence, using the Generalized Lebesgue Dominated Convergence Theorem one has

$$\lim_{n\to\infty} \int \left[\frac{A_\infty(x)u_n v_n - F(x, z_n)}{|z_n|^2}\widehat{z}_n(x)^2\right] = 0.$$

We obtain a contradiction by taking the limit in (3.4). The proof is done. □

We are ready to prove our first main result:

Proof of Theorem 1.1. Firstly, the energy functional I satisfies (SCe'), see Proposition 3.2. According to Proposition 3.4, the geometric conditions $(\mathcal{I}_1)$ and $(\mathcal{I}_2)$ are verified. Thus, we are able to apply Saddle Point Theorem 3.1 to obtain a critical point for I. As pointed out before, this critical point is a weak solution of System (P). □

4. The superlinear case

This section is devoted to the proof of Theorem 1.2. Although the energy functional is the same of the previous sections, we shall apply here a different abstract result. Let us recall another compactness condition: let $\mathcal{E}$ be a real Hilbert space and $\mathcal{I} \in C^1(\mathcal{E}, \mathbb{R})$. A sequence $(z_n) \subset \mathcal{E}$ is said to be Cerami sequence if $\mathcal{I}(z_n) \to c$ and $\|z_n\|_{\mathcal{E}}(1 + \|\mathcal{I}'(z_n)\|_{\mathcal{E}'}) \to 0$, as $n \to +\infty$. We say that $\mathcal{I}$ satisfies the *Cerami condition*, (Ce), if any Cerami sequence $(z_n) \subset \mathcal{E}$ has a convergent subsequence.

We shall use the following version of the Generalized Mountain Pass Theorem given in [1] (see also [4, 22]):

Theorem 4.1. *Let $\mathcal{E} = V \oplus W$ be a real Hilbert space with V finite dimensional and $W = V^{\perp}$. Suppose that $\mathcal{I} \in C^1(E, \mathbb{R})$ satisfies* (SCe) *and*

($\mathcal{I}_3$) *there exist $\rho > 0$, $\alpha > 0$ such that $\mathcal{I}(z) \geq \alpha$, for all $z \in W \cap \partial B_\rho(0)$;*

($\mathcal{I}_4$) *there exist $e \in W \cap \partial B_1(0)$ and $R > \rho$ such that $\mathcal{I}(z) \leq 0$, for all $z \in \partial Q$, where*
$$Q = \left(V \cap \overline{B_R(0)}\right) \oplus \{re : 0 < r < R\}$$
and ∂Q denotes the boundary relative to the subspace $V \oplus \mathbb{R}e$.

Then $\mathcal{I}$ possesses a nonzero critical point.

We first prove the following compactness condition:

Proposition 4.2. *Suppose that F satisfies $(F_1), (\widehat{F}_2), (\widehat{F}_3)$ and $(\widehat{NQ})$. Then I satisfies the (Ce) condition.*

Proof. Let $(z_n) \subset E$ be a Cerami sequence. The growth condition $(\widehat{F}_3)$ and a straightforward adaptation of [11, Section 4] show that it is sufficient to verify that (z_n) has a bounded subsequence.

Since (z_n) is a Cerami sequence we infer that $\|z_n\| \, \|I'(z_n)\|$ is bounded. Hence, we can use the first inequality in $(\widehat{NQ})$ to get
$$C_1 \geq \theta I(z_n) - I'(z_n)z_n \geq \left(\frac{\theta}{2} - 1\right) \|z_n\|^2 - c_5 \int |u_n|^p |v_n|^q - \|h\|_1.$$
Thus, since $\theta > 2$, one has
$$\|z_n\|^2 \leq C_2 + C_2 \int |u_n|^p |v_n|^q. \tag{4.1}$$
On the other hand, by using the second inequality in $(\widehat{NQ})$, we get
$$c_5 \int |u_n|^\mu |v_n|^\nu \leq 2I(z_n) - I'(z_n)z_n \leq C_3. \tag{4.2}$$

Let $\gamma := (p+q)$, $\eta := (\mu + \nu)$ and let us assume for a moment that there exists $r > 1$ such that
$$r \geq \max\left\{\frac{\mu}{p}, \frac{\nu}{q}\right\}, \qquad 2 \leq \left(\frac{\gamma r - \eta}{r}\right) r' \leq 2_s^*, \qquad 1 < r < \frac{\eta}{\gamma - 2}, \tag{4.3}$$
where $r' = r/(r-1)$ is the conjugated exponent of r. Since $\max\{|u_n(x)|, |v_n(x)|\} \leq |z_n(x)|$ for a.e. $x \in \mathbb{R}^N$, it follows from the first inequality above that
$$\int |u_n|^p |v_n|^q \leq \int \left(|u_n|^\mu |v_n|^\nu\right)^{1/r} |z_n|^{(\gamma r - \eta)/r}.$$
This, Hölder's inequality with exponents $r > 1$ and r', (4.2), the second inequality in (4.3) and the Sobolev embedding provide
$$\begin{aligned} \int |u_n|^p |v_n|^q &\leq \left(\int |u_n|^\mu |v_n|^\nu\right)^{1/r} \left(\int |z_n|^{(\gamma r - \eta)r'/r}\right)^{1/r'} \\ &\leq C_4 \|z_n\|_{(\gamma r - \eta)/(r-1)}^{(\gamma r - \eta)/r} \leq C_5 \|z_n\|^{(\gamma r - \eta)/r}. \end{aligned}$$

Thus, we infer from (4.1) that

$$\|z_n\|^2 \leq C_2 + C_6\|z_n\|^{(\gamma r-\eta)/r}.$$

The third inequality in (4.3) implies that $(\gamma r - \eta)/r < 2$. Therefore, the above expression implies that (z_n) is bounded.

It remains to prove that there exists a number r as in (4.3). It is equivalent to

$$1 < \max\left\{\frac{\mu}{p}, \frac{\nu}{q}, \frac{\eta-2}{\gamma-2}, \frac{2_s^*-\eta}{2_s^*-\gamma}\right\} \leq r < \frac{\eta}{\gamma-2}.$$

If $\eta < \gamma$, then we have that

$$1 < \max\left\{\frac{\eta-2}{\gamma-2}, \frac{2_s^*-\eta}{2_s^*-\gamma}\right\} = \frac{2_s^*-\eta}{2_s^*-\gamma} < \frac{\eta}{\gamma-2},$$

where we have used the inequality $\eta > N(\gamma-2)/(2s)$ from the condition $(\widehat{NQ})$. Since it also implies that $\max\{(\mu/p),(\nu/q)\} < \eta/(\gamma-2)$, it is enough to pick r sufficiently close (and small than) $\eta/(\gamma-2)$. If $\gamma < \eta$, the same argument provides

$$1 < \max\left\{\frac{\eta-2}{\gamma-2}, \frac{2_s^*-\eta}{2_s^*-\gamma}\right\} = \frac{\eta-2}{\gamma-2} < \frac{\eta}{\gamma-2},$$

and therefore the choice also can be done. The case $\gamma = \eta$ is similar and we omit the details. □

Now, we proceed with the splitting of the space E in such way that we obtain the geometric conditions of Theorem 4.1. Thus, we take the function A_0 from condition (F_0) and call $\{\lambda_m^{A_0}\}_{m\in\mathbb{Z}^*}$ the sequence of eigenvalues of the problem (LP) with weight $A = A_0$. As before, we denote by $\{\Phi_m^{A_0}\}_{m\in\mathbb{Z}^*}$ the associated eigenfunctions. We consider two distinct cases: If $\lambda_1^{A_0} \leq 1$, then we fix $k \in \mathbb{N}$ such that $\lambda_k^{A_0} \leq 1 < \lambda_{k+1}^{A_0}$ and set

$$V := \operatorname{span}\{\Phi_1^{A_0}, \ldots, \Phi_k^{A_0}\}, \quad W := V^{\perp}.$$

Otherwise, if $\lambda_1^{A_0} > 1$, then we just set $V := \{0\}$ and $W := E$. In this previous case the number k appearing in the sequel will be considered as $k = 0$.

With the above definitions, we have the following:

Proposition 4.3. *Suppose that F satisfies $(F_0), (F_1)$, $(\widehat{F_2})$ and $(\widehat{NQ})$. Then the functional I verifies the conditions $(\mathcal{I}_3)$ and $(\mathcal{I}_4)$.*

Proof. If $\varepsilon > 0$, we use $(\widehat{F_2})$ and (F_0) to obtain $C_1 > 0$ such that

$$F(x,z) \leq \varepsilon|z|^2 + A_0(x)uv + C_1|z|^{\sigma}, \quad \forall\, z = (u,v) \in \mathbb{R}^2, \text{ a.e. } x \in \mathbb{R}^N.$$

Hence, for any function $z \in W \cap \partial B_\rho(0)$, it follows from (2.8) and the embedding $E \hookrightarrow L^\sigma(\mathbb{R}^N) \times L^\sigma(\mathbb{R}^N)$ that

$$I(z) \geq \frac{1}{2}\rho^2\left[\left(1 - 2S\varepsilon - \frac{1}{\lambda_{k+1}^{A_0}}\right) - C_2\rho^{\sigma-2}\right].$$

Since $\lambda_{k+1}^{A_0} > 1$, we can choose ε, $\rho > 0$ small in such way that the terms into the brackets above are greater than $1/2$. In this way, we get $I(z) \geq \alpha := \rho^2/4$, for any $z \in W \cap \partial B_\rho(0)$, which establishes $(\mathcal{I}_3)$.

For the proof of condition $(\mathcal{I}_4)$ we first suppose that $\lambda_1^{A_0} \leq 1$, in such way that $V = \text{span}\{\Phi_1^{A_0}, \ldots, \Phi_k^{A_0}\}$. For any $z \in V$ we use (2.7) and Lemma 2.4 to get

$$I(z) = \frac{1}{2}\|z\|^2 - \int A_0(x)uv - \int G_0(x,z) \leq \frac{1}{2}\left(1 - \frac{1}{\lambda_k^{A_0}}\right)\|z\|^2 - C_1 \int |u|^\mu |v|^\nu.$$

Therefore, we infer from $\lambda_k^{A_0} \leq 1$ that $I \leq 0$ in V. We now set $e := \Phi_{k+1}^{A_0}/\|\Phi_{k+1}^{A_0}\| \in W$ and claim that

$$\lim_{\|z\|\to+\infty,\, z\in V\oplus\mathbb{R}e} I(z) \to -\infty. \tag{4.4}$$

If this is true, it easily follows that condition $(\mathcal{I}_4)$ holds if we choose $R > \rho$ sufficiently large.

In order to prove the claim we fix $j \in \mathbb{N}$, take an eigenfunction $\Phi_j^{A_0} = (\phi_j, \psi_j)$ and notice that, by Lemma 2.4,

$$I(t\Phi_j^{A_0}) \leq \frac{t^2}{2}\left(\|\Phi_j^{A_0}\|^2 - \int A_0(x)\phi_j\psi_j\right) - C_1 t^{\mu+\nu}\int |\phi_j|^\mu |\psi_j|^\nu. \tag{4.5}$$

Since

$$\begin{cases} (-\Delta)^s\phi_j + a(x)\phi_j = \lambda_j^{A_0} A_0(x)\psi_j, & x \in \mathbb{R}^N, \\ (-\Delta)^s\psi_j + b(x)\psi_j = \lambda_j^{A_0} A_0(x)\phi_j, & x \in \mathbb{R}^N, \end{cases}$$

we can multiply the first equation by ϕ_j, the second one by ψ_j and add the two equations to obtain

$$\|(\phi_j,\psi_j)\|^2 = 2\lambda_j^{A_0}\int A_0(x)\phi_j\psi_j.$$

If $\int |\phi_j|^\mu|\psi_j|^\nu = 0$, then we also have that $\int A_0(x)\phi_j\psi_j = 0$, and therefore we conclude from the above expression that $\Phi_j^{A_0} = (0,0)$, which does not make sense. Thus, $\int |\phi_j|^\mu|\psi_j|^\nu > 0$ and it follows from (4.5) that

$$\lim_{t\to+\infty} I(t\Phi_j^{A_0}) = -\infty.$$

Since $V \oplus \mathbb{R}e$ is a finite dimensional subspace spanned by $\{\Phi_1^{A_0}, \ldots, \Phi_{k+1}^{A_0}\}$, the statement (4.4) is now an easy consequence of the above limit. This finishes the proof of $(\mathcal{I}_4)$ in the case that $\lambda_1^{A_0} \leq 1$.

If $\lambda_1^{A_0} > 1$ then we set $e := \Phi_1^{A_0}/\|\Phi_1^{A_0}\|$ and notice that the above argument shows that $I(te) \to -\infty$, as $t \to +\infty$. Hence, it is sufficient to take $R > \rho$ large enough in such way that $I(Re) \leq 0$. Since $I(0) = 0$ this proves that $(\mathcal{I}_4)$ also holds in this case. □

We finish the paper by proving our second main result:

Proof of Theorem 1.2. As in the first theorem, the proof follows from Propositions 4.2, 4.3 and Theorem 4.1. □

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