INDEFINITE PLANAR PROBLEM WITH EXPONENTIAL CRITICAL GROWTH

MARCELO F. FURTADO AND KARLA CAROLINA V. SOUSA

Abstract. We obtain existence of solution for the equation

$$-\Delta u + \frac{1}{2}(x \cdot \nabla u) = a(x)f(u), \quad x \in \mathbb{R}^2,$$

where a is a continuous sign-changing potential and the superlinear function f has an exponential critical growth.

1. Introduction and main results

We are concerned with the equation

$$(P) \qquad -\Delta u + \frac{1}{2}(x \cdot \nabla u) = a(x)f(u), \quad x \in \mathbb{R}^2,$$

where a is a sign-changing potential and the nonlinerity f has an exponential critical growth at infinity. The operator in (P) naturally appears when we look for self-similar solutions for homogeneous heat equations, namely solutions of the form $\omega(t,x) = t^{-1/(p-2)}u(t^{-1/2}x)$ for the evolution equation

$$\omega_t - \Delta\omega = |\omega|^{p-2}\omega, \text{ in } (0,+\infty)\times\mathbb{R}^N.$$

More specifically, ω is a solution for the above equation if, and only if, the profile u is a solution for the elliptic equation

$$-\Delta u - \frac{1}{2}(x \cdot \nabla u) = \lambda u + |u|^{p-2}u, \quad x \in \mathbb{R}^N.$$

There is a vast literature concerning the above problem with several types of nonlinearities for bounded domains, the whole space $\mathbb{R}^N$ and even the upper half-space $\mathbb{R}^N_+$. Without intention to present a complete list of references, we could cite [13, 3, 5, 16, 15, 6, 8, 10, 17] and references therein. In these works the authors find results about existence, nonexistence, multiplicity, decay rate, among other properties of solutions via ODE techniques or variational methods. As far as we know, Escobedo and Kavian [8] were the first to treat this operator in a variational way and particularly inspired works as [12, 9], that considered problem (P) with sign-changing nonlinearity having a concave-convex prototype.

In this paper, we deal with an indefinite potential a. More specifically, we follow [1] and assume that

- (a_1) $a : \mathbb{R}^2 \to \mathbb{R}$ is a bounded sign-changing continuous function;
- (a_2) if $\Omega^+ := \{x \in \mathbb{R}^2; a(x) > 0\}$ and $\Omega^- := \{x \in \mathbb{R}^2; a(x) < 0\}$, then $\text{dist}(\overline{\Omega^+}, \overline{\Omega^-}) > 0$;
- (a_3) there exists $R > 0$ such that $a(x) < 0$ for $|x| \geq R$.

The first author was partially supported by FAP-DF/Brazil and CNPq/Brazil. The second author was supported by Capes/Brazil.

We are interested in the case that f is superlinear both at the origin and at infinity, namely

(f_0) $f \in C(\mathbb{R},\mathbb{R})$ and there exists $\alpha_0 > 0$ such that
$$\lim_{s\to+\infty} \frac{f(s)}{e^{\alpha s^2}} = \begin{cases} 0 & \text{if} \quad \alpha > \alpha_0, \\ +\infty & \text{if} \quad \alpha < \alpha_0; \end{cases}$$

(f_1) $\lim\limits_{s\to 0} f(s)/s = 0.$

In order to present the other conditions on f we need to say some words about our functional space. So, we set $K(x) := \exp(|x|^2/4)$ and notice that $\operatorname{div}(K(x)\nabla u) = K(x)\left[\Delta u + (1/2)(x \cdot \nabla u)\right]$, in such way that we can use a variational approach and look for solutions in the space X defined as the closure of $C_0^\infty(\mathbb{R}^2)$ with respect to the norm
$$\|u\| := \left(\int_{\mathbb{R}^2} K(x)|\nabla u|^2\, dx \right)^{1/2}.$$
Given $s \geq 2$, it is proved in [11] that X is compactly embedded into the weighted Lebesgue space $L^s_K := L^s(\mathbb{R}^2, K(x))$. Hence, we can define the constant
$$S_2 := \inf\left\{ \int_{\mathbb{R}^2} K(x)|\nabla u|^2 dx \,:\, \int_{\mathbb{R}^2} K(x)|u|^2\, dx = 1 \right\}.$$
Since $\overline{\Omega^+}$ is far from $\overline{\Omega^-}$, we can find $\zeta \in C^\infty(\mathbb{R}^2, [0,1])$ such that
$$\zeta \equiv 1, \text{ in } \Omega^+, \qquad \zeta \equiv 0, \text{ in } \Omega^-, \qquad \mathcal{M} := \sup_{\mathbb{R}^2} |\nabla \zeta| < \infty.$$
Our technical assumptions on f can be stated as follows:

(f_2) there exist $\nu > 2$ and $0 < \theta < \nu\left[2(1+\mathcal{M}S_2^{-1/2})\right]^{-1}$ such that, for $F(s) := \int_0^s f(\tau)\, d\tau$, there holds
$$0 < \frac{\nu}{\theta} F(s) \leq f(s)s, \quad \forall\, |s| > 0;$$

(f_3) there exist K_0, $R_0 > 0$ such that
$$0 < F(s) \leq K_0|f(s)|, \quad \forall\, |s| \geq R_0;$$

(f_4) if $x_0 \in \Omega^+$ and $r > 0$ are such that $a(x_0) = \max_{\Omega^+} a$ and $a(x) \geq (\max_{\Omega^+} a)/2$ in $B_r(x_0)$, then
$$\lim_{s\to+\infty} s f(s) e^{-\alpha_0 s^2} \geq \beta_0 > \frac{8}{\alpha_0 r^2 \cdot \max_{\Omega^+} a} \exp\left(\frac{r^2}{8} + \frac{r^4}{512} \right).$$

In this paper, we prove the following existence result:

Theorem 1.1. *Suppose that $(a_1) - (a_3)$ and $(f_0) - (f_4)$ hold. Then problem (P) admits at least a weak nontrivial solution.*

In the proof we apply the Mountain Pass Theorem. Since the potential a changes it sign, it is not so easy to prove that Palais-Smale sequences are bounded. Conditions (a_2) and (f_2) are important in this issue. Condition (f_3) has first appeared in [7] and provides a compactness property for the Palais-Smale sequence. With the aim of overcome the difficulties imposed by the lack of compactness, since we are dealing with the whole space $\mathbb{R}^2$, we invoke a version of the Trudinger-Moser inequality together with assumption (f_4) and the Moser's functions to find the correct localization of the mountain pass level. We notice that (f_4) is weaker than $\lim_{s\to+\infty} f(s)se^{-\alpha_0 s^2} = +\infty$, which have been used in some former papers (see (g_5) in [1] for instance). It is not difficult to see that, if we pick $q > \nu/\theta$, then the function

$$f(s) = (q|s|^{q-2}s + 2\alpha_0|s|^q s)e^{\alpha_0|s|^q}$$

satisfies all the conditions $(f_0) - (f_4)$ above.

We finish this introduction quoting the paper [4], where the authors considered

$$-\Delta u + u = a(x)f(u), \text{ in } \Omega \qquad Bu = 0, \text{ on } \partial\Omega,$$

in a bounded domain, $Bu = \partial u/\partial\nu$ or $Bu = u$, $a \in C(\Omega, \mathbb{R})$ is a sign-changing potential and f is a power type subcritical nonlinearity. The N-laplacian case is considered in [1] for an exterior domain Ω, Dirichlet boundary conditions and f having exponential critical growth. Theorem 1.1 is a complement of these papers since we deal with the whole space case and a different operator.

The paper contains two more sections. In the first one, we present the variational framework to deal with (P) and some auxiliary results. Theorem 1.1 is proved in Section 3.

2. Variational framework and technical results

We start by quoting a Trudinger-Moser type inequality proved in [11].

Theorem 2.1 (Trudinger-Moser). *If $u \in X$, $\beta > 0$ and $p \geq 0$ then $K(x)|u|^{2+p}(e^{\beta u^2} - 1) \in L^1(\mathbb{R}^2)$. Moreover, if $\|u\| \leq M$, with $\beta M^2 < 4\pi$, then there exists a constant $C = C(\beta, M, p) > 0$ such that*

$$\int_{\mathbb{R}^2} K(x)|u|^{2+p}(e^{\beta u^2} - 1)\, dx \leq C\|u\|^{2+p}.$$

Let $\alpha > \alpha_0$ and $q \geq 1$. It follows from (f_0) that

$$\lim_{|s|\to+\infty} \frac{f(s)}{|s|^{1-q}(e^{\alpha s^2} - 1)} = 0.$$

Hence, we can use (f_1) to obtain, for any given $\varepsilon > 0$, a constant $C_\varepsilon > 0$ such that

$$\max\{|f(s)s|, |F(s)|\} \leq \varepsilon s^2 + C_\varepsilon |s|^q(e^{\alpha s^2} - 1), \tag{2.1}$$

for any $s \in \mathbb{R}$. Since $a \in L^\infty(\mathbb{R}^2)$, we can use the above estimates and Theorem 2.1 to show that the functional $I : X \to \mathbb{R}$ given by

$$I(u) := \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^2} K(x)a(x)F(u)\, dx$$

is well-defined, it belongs to $C^1(\mathbb{R}^2, \mathbb{R})$ and its critical points are weak solutions for problem (P).

Let $x_0 \in \Omega^+$ and $r > 0$ be given by condition (f_4). We define a slight adaptation of the Green's function considered by Moser in [14], namely

$$\widetilde{M}_n(x) := \frac{1}{\sqrt{2\pi}} \cdot \begin{cases} K(r/n)^{-1/2}(\log n)^{1/2}, & \text{if } \quad |x - x_0| \leq r/n, \\ K(x)^{-1/2}\dfrac{\log(r/|x - x_0|)}{(\log n)^{1/2}}, & \text{if } \quad r/n \leq |x - x_0| < r, \\ 0, & \text{if } \quad |x - x_0| \geq r. \end{cases}$$

As we shall see, the location of $x_0 \in \mathbb{R}^2$ does not play any role in our next calculations. So, we assume with no loss of generality that $x_0 = 0$. We have that $\widetilde{M}_n \in H^1(\mathbb{R}^2)$ and $\text{supp}(\widetilde{M}_n) = \overline{B}_r(0)$. Moreover, it is proved in [11, Lemma 4.6] that there exists a sequence $(d_n) \subset \mathbb{R}$ such that

$$\|\widetilde{M}_n\|^2 = 1 + \frac{1}{\log n}\left(\frac{r^2}{8} + \frac{r^4}{512}\right) - d_n, \qquad \lim_{n\to+\infty} d_n \log n = 0. \tag{2.2}$$

In particular, $\|\widetilde{M}_n\|^2 \to 1$, as $n \to +\infty$.

Lemma 2.2. *Suppose that $(a_1) - (a_3)$, (f_2) and (f_4) hold. If $M_n := \widetilde{M}_n/\|\widetilde{M}_n\|$, then there exists $n \in \mathbb{N}$ such that*

$$\max_{s\geq 0} I(sM_n) = \max\left\{\frac{s^2}{2} - \int_{\mathbb{R}^2} K(x)a(x)F(sM_n)\,dx\right\} < \frac{2\pi}{\alpha_0}.$$

Proof. For each $n \in \mathbb{N}$, consider the function $g_n(s) := I(sM_n)$, for $s \geq 0$. From (f_2), we obtain $C_1, C_2 > 0$ such that $F(s) \geq C_1|s|^{\nu/\theta} - C_2$, for any $s \in \mathbb{R}$. Thus, since $\text{supp}(M_n) \subset \Omega^+$, we have that

$$g_n(s) \leq \frac{s^2}{2} - C_1 s^{\nu/\theta}\int_{\Omega^+} K(x)a(x)M_n^{\nu/\theta}\,dx + C_2\int_{\Omega^+} K(x)a(x)\,dx.$$

Recalling that $\nu/\theta > 2$, we obtain $g_n(s) \to -\infty$, as $s \to +\infty$. Hence, g_n attains its global maximum at $s_n > 0$ which satisfies $0 = g_n'(s_n)$ or, equivalently,

$$s_n^2 = \int_{B_r(0)} K(x)a(x)f(s_nM_n)s_nM_n\,dx. \tag{2.3}$$

Suppose, by contradiction, that the result of the lemma is false. Then $g_n(s_n) \geq (2\pi)/\alpha_0$ and we can use the definition of g_n, $\text{supp}(M_n) \subset \Omega^+$ and $F \geq 0$, to get

$$s_n^2 \geq \frac{4\pi}{\alpha_0}. \tag{2.4}$$

Let $\beta_0 > 0$ be given by (f_4). If $0 < \varepsilon < \beta_0$, there exists $R_\varepsilon > 0$ such that

$$sf(s) \geq (\beta_0 - \varepsilon)e^{\alpha_0 s^2}, \quad \forall\,|s| \geq R_\varepsilon. \tag{2.5}$$

Using the definition of M_n, (2.4) and $\|\tilde{M}_n\| \to 1$, as $n \to +\infty$, we conclude that

$$s_nM_n(x) = s_n\frac{\tilde{M}_n}{\|\tilde{M}_n\|} \geq \frac{e^{-r^2/(8n^2)}}{\|\tilde{M}_n\|}\sqrt{\frac{4\pi\log n}{\alpha_0}} \geq R_\varepsilon,$$

for any $|x| < r/n$ and n large. Hence, it follows from (2.3), (2.5), $K \geq 1$, the choice of $r > 0$ in (f_4), the previous inequality and the definition of M_n that

$$\begin{aligned}
s_n^2 &\geq \int_{B_{r/n}(0)} K(x)a(x)f(s_nM_n)s_nM_n\,dx \\
&\geq c_0(\beta_0 - \varepsilon)\int_{B_{r/n}(0)} \exp(\alpha_0(s_nM_n)^2)\,dx \\
&= c_0(\beta_0 - \varepsilon)\int_{B_{r/n}(0)} \exp\left(\alpha_0 s_n^2\frac{e^{-r^2/(4n^2)}\log n}{2\pi\|\widetilde{M}_n\|^2}\right)dx \\
&= c_0(\beta_0 - \varepsilon)\frac{\pi r^2}{n^2}\exp\left(\alpha_0 s_n^2\frac{e^{-r^2/(4n^2)}\log n}{2\pi\|\widetilde{M}_n\|^2}\right),
\end{aligned}$$

where $c_0 := (\max_{\Omega^+} a)/2$. Using that $1/n^2 = \exp(-2\log n)$, we obtain

$$s_n^2 \geq c_0(\beta_0 - \varepsilon)\pi r^2 \exp\left(2\left[\frac{e^{-r^2/(4n^2)}}{\|\widetilde{M}_n\|^2}\frac{\alpha_0}{4\pi}s_n^2 - 1\right]\log n\right), \tag{2.6}$$

and hence, recalling that $\exp(s) \geq s$, we get that

$$s_n^2 \geq 2c_0(\beta_0 - \varepsilon)\pi r^2\left[\frac{e^{-r^2/(4n^2)}}{\|\widetilde{M}_n\|^2}\frac{\alpha_0}{4\pi}s_n^2 - 1\right]\log n. \tag{2.7}$$

Since $e^{-r^2/(4n^2)}\|\widetilde{M}_n\|^{-2} \to 1$, we conclude from the above inequality that (s_n) is bounded. Hence, up to a subsequence, $s_n^2 \to \gamma \geq 4\pi/\alpha_0$. If $\gamma > 4\pi/\alpha_0$, we obtain a contradiction after passing (2.7) to the limit. Thus, $\gamma = 4\pi/\alpha_0$. Combining inequalities (2.4), (2.6) and Lemma 2.2, we obtain

$$s_n^2 \geq c_0(\beta_0 - \varepsilon)\pi r^2 \exp\left\{\frac{-2}{\|\widetilde{M}_n\|^2}(\|\widetilde{M}_n\|^2 - e^{-r^2/(4n^2)})\log n\right\}.$$

Passing to the limit in n, using (2.2) and a straightforward computation, we obtain

$$\frac{4\pi}{\alpha_0} \geq c_0(\beta_0 - \varepsilon)\pi r^2 \exp\left(-2\left(\frac{r^2}{8} + \frac{r^4}{512}\right)\right).$$

Letting $\varepsilon \to 0$ and recalling that $c_0 = (\max_{\Omega^+} a)/2$, we finally conclude that

$$\beta_0 \leq \frac{8}{\alpha_0 r^2 \cdot \max_{\Omega^+} a}\exp\left(\frac{r^2}{4} + \frac{r^4}{256}\right),$$

which contradicts assumption (f_4). The result is proved. □

We prove in the sequel that I has the Mountain Pass geometry.

Lemma 2.3. *Suppose that $(a_1) - (a_3)$ and $(f_0) - (f_2)$ hold. If $n \in \mathbb{N}$ is given by Lemma 2.2, we have that*

(i) there exist ξ, $\rho > 0$ such that $I(u) \geq \xi$, for any $u \in X$, $\|u\| = \rho$.
(ii) there exists $s_0 > 0$ such that $\|s_0 M_n\| > \rho$ and $I(s_0 M_n) < 0$.

Proof. Given $\alpha > \alpha_0$ and $\varepsilon > 0$, it follows from (2.1) (with $q = 3$) that

$$\begin{aligned}\int_{\mathbb{R}^2} K(x)a(x)F(u)\,dx &\leq \int_{\Omega^+} K(x)a(x)F(u)\,dx \leq \varepsilon\|a\|_{L^\infty(\Omega^+)}\|u\|_2^2 \\ &+ \|a\|_{L^\infty(\Omega^+)}C_\varepsilon\int_{\mathbb{R}^N} K(x)|u|^3(e^{\alpha u^2} - 1)\,dx.\end{aligned}$$

If $0 < M < 1$ is such that $\alpha M^2 < 4\pi$, we can use Theorem 2.1 to obtain $C_1 = C_1(M, \alpha) > 0$ such that

$$\int_{\mathbb{R}^2} K(x)a(x)F(u)\,dx \leq \varepsilon\|a\|_{L^\infty(\Omega^+)}S_2^{-1}\|u\|_2 + C_1\|u\|^3,$$

whenever $\|u\| \leq M$. Hence, picking $\varepsilon > 0$ in such a way that $(1 - 2\varepsilon\|a\|_{L^\infty(\Omega^+)}S_2^{-1}) = C_2 > 0$, we get that

$$I(u) \geq \frac{1}{2}(1 - 2\varepsilon\|a\|_{L^\infty(\Omega^+)}S_2^{-1})\|u\|^2 - C_1\|u\|^3 = \|u\|^2\left(\frac{C_2}{2} - C_1\|u\|\right),$$

and item (i) clearly holds for $\rho := C_2/(4C_1)$ and $\xi := \rho^2 C_2/4$. The second statement is a direct consequence of the proof of the last lemma, where we have that $I(sM_n) \to -\infty$, as $s \to +\infty$. □

The above result ensures the existence of a Palais-Smale sequence at the mountain pass level [2] (see also [18, Theorem 1.15]), that is, a sequence $(u_n) \subset X$ such that

$$\lim_{n\to+\infty} I'(u_n) = 0, \qquad \lim_{n\to+\infty} I(u_n) = c_{MP}$$

where

$$c_{MP} := \inf_{\gamma\in\Gamma} \max_{s\in[0,1]} I(\gamma(s)) \in \left(0, \frac{2\pi}{\alpha_0}\right),$$

and $\Gamma := \{\gamma \in C([0,1], X); \gamma(0) = 0, \gamma(1) = e\}$, with $e := s_0 M_n \in X$ given by Lemma 2.3. Notice that the path $\gamma(s) := s s_0 M_n$ belongs to Γ and therefore we really have that $c_M < 2\pi/\alpha_0$.

Lemma 2.4. *There exists $u_0 \in X$ such that, up to a subsequence, $u_n \rightharpoonup u_0$ weakly in X.*

Proof. It is sufficient to prove that (u_n) is bounded in X. Computing $I(u_n) - (\theta/\nu)I'(u_n)(\zeta u_n)$ and using the properties of the function ζ we get that

$$\begin{aligned} c + o_n(1) + o_n(1)\|u_n\| &= \frac{1}{2}\|u_n\|^2 - \int_{\mathbb{R}^2} K(x)a(x)F(u_n)\,dx \\ &- \frac{\theta}{\nu}\int_{\mathbb{R}^2} K(x)\left[\nabla u_n \nabla(\zeta u_n) - a(x)f(u_n)\zeta u_n\right]dx \\ &\geq \left(\frac{1}{2} - \frac{\theta}{\nu}\right)\|u_n\|^2 - \frac{\theta\mathcal{M}}{\nu}\int_{\mathbb{R}^2} K(x)|\nabla u_n||u_n|\,dx \\ &+ \int_{\Omega^+} K(x)a(x)\left[\frac{\theta}{\nu}f(u_n)u_n - F(u_n)\right]dx \end{aligned}$$

and therefore we can use (f_2) to obtain

$$c + o_n(1) + o_n(1)\|u_n\| \geq \left(\frac{1}{2} - \frac{\theta}{\nu}\right)\|u_n\|^2 - \frac{\theta\mathcal{M}}{\nu}\int_{\mathbb{R}^2} K(x)|\nabla u_n||u_n|\,dx. \tag{2.8}$$

It follows from Hölder's inequality and the continuous embedding that

$$\frac{\theta\mathcal{M}}{\nu}\int_{\mathbb{R}^2} K(x)|\nabla u_n||u_n|\,dx \leq \frac{\theta\mathcal{M}S_2^{-1/2}}{\nu}\|u_n\|^2,$$

which together with (2.8) lead to

$$c + o_n(1) + o_n(1)\|u_n\| \geq \left(\frac{1}{2} - \frac{\theta}{\nu} - \frac{\theta\mathcal{M}S_2^{-1/2}}{\nu}\right)\|u_n\|^2.$$

By (f_2), the term into parenthesis above is positive, which implies that (u_n) is bounded in X.

□

Since X is compactly embedded in $L^s_K(\mathbb{R}^2)$, it follows from the above lemma that

$$\begin{cases} u_n \to u_0 \text{ strongly in } L^s(\mathbb{R}^2), \\ u_n(x) \to u_0(x) \text{ a.e. in } \mathbb{R}^2, \\ |u_n(x)| \leq h_s(x) \text{ a.e. in } \mathbb{R}^2, \end{cases} \tag{2.9}$$

for any $s \geq 2$ and some $h_s \in L^s_K(\mathbb{R}^2)$.

Lemma 2.5. *Suppose that* $(a_1) - (a_3)$ *and* $(f_0) - (f_4)$ *hold. If* $a^{\pm}(x) := \max\{\pm a(x), 0\}$ *and* $u_0 \in X$ *is given by Lemma 2.4, then* $K(x)a^{\pm}(x)f(u_n) \to K(x)a^{\pm}(x)f(u_0)$ *in* $L^1_{loc}(\mathbb{R}^2)$.

Proof. Fixed $\sigma > 0$, we can compute $I(u_n) - (\sigma/\nu)I'(u_n)(\zeta u_n)$ and argue as in Lemma 2.4 to obtain

$$\begin{aligned} c + o_n(1) + o_n(1)\|u_n\| &\geq \left(\frac{1}{2} - \frac{\sigma}{\nu} - \frac{\sigma \mathcal{M} S_2^{-1/2}}{\nu}\right)\|u_n\|^2 \\ &+ \left(\frac{\sigma}{\nu} - \frac{\theta}{\nu}\right)\int_{\Omega^+} K(x)a(x)f(u_n)u_n\,dx. \end{aligned}$$

Choosing $\sigma > \nu\left[2(1+\mathcal{M}S_2^{-1/2})\right]^{-1} > \theta$ and recalling that (u_n) is bounded, we obtain

$$\int_{\Omega^+} K(x)a(x)f(u_n)u_n\,dx \leq C_1.$$

Moreover, since $I'(u_n)u_n = 0$, we have that

$$\int_{\Omega^-} K(x)a(x)f(u_n)u_n\,dx \leq \int_{\mathbb{R}^N} K(x)a(x)f(u_n)u_n\,dx = \|u_n\| + o_n(1) \leq C_2.$$

Let $\Omega \subset \mathbb{R}^2$ be a bounded set. Given $\varepsilon > 0$, is is clear that

$$|f(s)| \leq \varepsilon f(s)s, \quad \forall\,|s| \geq R_\varepsilon := 1/\varepsilon.$$

Consequently,

$$\int_{[|u_n|\geq R_\varepsilon]\cap\Omega} K(x)a^{\pm}(x)|f(u_n)|\,dx \leq \varepsilon \int_{[|u_n|\geq R_\varepsilon]\cap\Omega} K(x)a^{\pm}(x)f(u_n)u_n\,dx \leq \varepsilon C_3, \tag{2.10}$$

with $C_3 := (C_1 + C_2)$. Thus, from the pointwise convergence and Fatou's lemma, we obtain

$$\int_{[|u_0|\geq R_\varepsilon]\cap\Omega} K(x)a^{\pm}(x)|f(u_0)|\,dx \leq \varepsilon C_3. \tag{2.11}$$

On the other hand,

$$\begin{aligned} \int_\Omega K(x)a^{\pm}(x)|f(u_n) - f(u_0)|\,dx &\leq \int_{[|u_n|\geq R_\varepsilon]\cap\Omega} K(x)a^{\pm}(x)|f(u_0)|\,dx \\ &+ \int_{[|u_n|\geq R_\varepsilon]\cap\Omega} K(x)a^{\pm}(x)|f(u_n)|\,dx \\ &+ \int_{[|u_n|< R_\varepsilon]\cap\Omega} K(x)a^{\pm}(x)|f(u_n) - f(u_0)|\,dx. \end{aligned}$$

Thus, we infer from (2.10) and (2.11) that

$$\begin{aligned} \int_\Omega K(x)a^{\pm}(x)|f(u_n) - f(u_0)|\,dx &\leq 2\varepsilon C_3 + \int_{\Sigma_{n,\varepsilon}\cap\Omega} K(x)a^{\pm}(x)|f(u_0)|\,dx \\ &+ \int_{[|u_n|< R_\varepsilon]\cap\Omega} K(x)a^{\pm}(x)|f(u_n) - f(u_0)|\,dx, \end{aligned}$$

with $\Sigma_{n,\varepsilon} := [|u_0| < R_\varepsilon] \cap [|u_n| \geq R_\varepsilon]$. Passing the above inequality to the limit as $n \to +\infty$, using that Ω is bounded, Lebesgue's theorem and the arbitrariness of $\varepsilon > 0$, we obtain

$$\lim_{n\to+\infty} \int_\Omega K(x)a^\pm(x)f(u_n)\,dx = \int_\Omega K(x)a^\pm(x)f(u_0)\,dx,$$

and the lemma is proved. □

3. Proof of Theorem 1.1

We prove in this section our main theorem. The idea is proving that the weak limit u_0 given by Lemma 2.4 is a nonzero solution of (P). First notice that, since $I'(u_n) \to 0$, as $n \to +\infty$, we can use Lemmas 2.4 and 2.5 to conclude that $I'(u_0)\varphi = 0$, for all $\varphi \in C_0^\infty(\mathbb{R}^2)$. A density argument shows that u_0 is a critical point of I.

Suppose, by contradiction, that $u_0 = 0$. Using condition (f_3), the continuity of f and that Ω^+ is bounded, we obtain $C_1 > 0$ such that

$$K(x)a(x)F(u_n) \leq C_1 + K_0K(x)a(x)|f(u_n)|, \quad \text{for a.e. } x \in \Omega^+.$$

As a byproduct of the proof of Lemma 2.5, we see that the right hand side above goes to zero. So, we can use the pointwise convergence and Lebesgue's theorem to conclude that $\int_{\Omega^+} K(x)a(x)F(u_n)\,dx \to 0$. Hence,

$$\begin{aligned} c_{MP} + o_n(1) = I(u_n) &= \frac{1}{2}\|u_n\|^2 - \int_{\mathbb{R}^2} K(x)a(x)F(u_n)\,dx \\ &\geq \frac{1}{2}\|u_n\|^2 - \int_{\Omega^+} K(x)a(x)F(u_n)\,dx = \frac{1}{2}\|u_n\|^2 + o_n(1), \end{aligned}$$

from which we conclude that $\limsup_{n\to+\infty} \|u_n\|^2 \leq 2c_{MP} < 4\pi/\alpha_0$. This provides m, $n_0 > 0$ be such that

$$\|u_n\|^2 < m < \frac{4\pi}{\alpha_0}, \quad \forall\, n \geq n_0.$$

We now claim that $\int_{\mathbb{R}^2} K(x)a(x)f(u_n)u_n = o_n(1)$. If this is true, we can use $I'(u_n)u_n = o_n(1)$ and (2.1) to get

$$\|u_n\|^2 = \int_{\mathbb{R}^2} K(x)a(x)f(u_n)u_n\,dx + o_n(1) = o_n(1),$$

which implies that $I(u_n) \to 0$. But this is impossible because $I(u_n) \to c_{MP} > 0$. Then, $u_0 \neq 0$ is the desired solution.

In order to prove the claim, we pick $\alpha > \alpha_0$, $q > 2$ and $s > 1$ to be chosen later, and apply (2.1) together with Hölder's inequality to write

$$\begin{aligned} \int_{\mathbb{R}^2} K(x)a(x)f(u_n)u_n\,dx &\leq C_2\|u_n\|^2_{L^2_K} + C_3\int_{\mathbb{R}^2} K(x)|u_n|^{2q}(e^{\alpha u_n^2} - 1)dx \\ &\leq C_2\|u_n\|^2_{L^2_K} \\ &+ C_3\|u_n\|^q_{L_K^{qs'}} \left[\int_{\mathbb{R}^2} K(x)|u_n|^{qs}\left(e^{\alpha u_n^2} - 1\right)^s\right]^{1/s}, \end{aligned}$$

Using the inequality $(1+a)^s \geq 1 + a^s$ with $a = e^t - 1$, we get $(e^t - 1)^s \leq e^{ts} - 1$. So, setting $v_n := u_n/\|u_n\|$ and noticing that $\alpha s u_n^2 = \alpha s\|u_n\|^2|v_n|^2 \leq \alpha s m|v_n|^2$, for $n \geq n_0$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^2} K(x)a(x)f(u_n)u_n\,dx &\leq C_2\|u_n\|^2_{L^2_K}\\ &+ C_4\|u_n\|^q_{L^{qs'}_K}\left[\int_{\mathbb{R}^2} K(x)|v_n|^{qs}\left(e^{\alpha sm|v_n|^2}-1\right)\right]^{1/s}.\end{aligned}$$

Since $\alpha sm \to \alpha_0 m < 4\pi$, as $\alpha \to \alpha_0$ and $s \to 1^+$, we can choose α, s, q close to the numbers α_0, 1, 2, respectively, and use Theorem 2.1 to guarantee that the term into brackets above is uniformly bounded. It is sufficient now to recall that $u_n \to 0$ strongly in the weighted Lebesgue spaces to obtain

$$\int_{\mathbb{R}^2} K(x)a(x)f(u_n)u_n \leq C_1\|u_n\|^2_{L^2_K} + C_5\|u_n\|^q_{L^{qs'}_K} = o_n(1),$$

and we have done.

References

[1] C.O. Alves, L.R. de Freitas and S.H.M. Soares, *Indefinite quasilinear elliptic equations in exterior domains with exponential critical growth,* Differential Integral Equations **24** (2011) 1047–1062.

[2] A. Ambrosetti and P.H. Rabinowitz, *Dual variational methods in critical point theory and applications,* J. Functional Analysis **14** (1973) 349–381.

[3] F.V. Atkinson and L.A. Peletier, *Sur les solutions radiales de l'équation* $\Delta u + \frac{1}{2}x\cdot\nabla u + \frac{1}{2}\lambda u + |u|^{p-1}u = 0$, (French) [On the radial solutions of the equation $\Delta u + \frac{1}{2}x\cdot\nabla u + \frac{1}{2}\lambda u + |u|^{p-1}u = 0$] C. R. Acad. Sci. Paris Sér. I Math. **302** (1986) 99–101.

[4] H. Berestycki, I. Capuzzo-Dolcetta, and L. Nirenberg, *Variational methods for indefinite superlinear homogeneous elliptic problems*. NoDEA Nonlinear Differential Equations Appl., **2** (1995), 553–572.

[5] H. Brezis, L.A. Peletier, D. Terman, *A very singular solution of the heat equation with absorption,* Arch. Rational Mech. Anal. **95** (1986) 185–209.

[6] F. Catrina, M. Montenegro and M.F. Furtado, *Positive solutions for nonlinear elliptic equations with fast increasing weights.* Proc. Roy. Soc. Edinburgh Sect. A **137** (2007), 1157–1178.

[7] D.G. de Figueiredo, O.H. Miyagaki and B. Ruf, *Elliptic equations in* $\mathbf{R}^2$ *with nonlinearities in the critical growth range*, Calc. Var. Partial Differential Equations **3** (1995) 139-153.

[8] M. Escobedo and O. Kavian, *Variational problems related to self-similar solutions of the heat equation,* Nonlinear Anal. **11** (1987) 1103—1133.

[9] M.F. Furtado, *Two solutions for a planar equation with combined nonlinearities and critical growth,* Proc. Amer. Math. Soc. **147** (2019) 4397-4408.

[10] M.F. Furtado, J.P.P. da Silva and M.S. Xavier, *Multiplicity of self-similar solutions for a critical equation.* J. Differential Equations **254** (2013), 2732–2743.

[11] M.F. Furtado, E.S. Medeiros and U.B. Severo *A Trudinger-Moser inequality in a weighted Sobolev space and applications,* Math. Nachr. **287** (2014) 1255–1273.

[12] M.F. Furtado, R. Ruviaro and J.P.P. Silva, *Two solutions for an elliptic equation with fast increasing weight and concave-convex nonlinearities*, J. Math. Anal. Appl. **416** (2014) 698–709.

[13] A. Haraux and F. Weissler, *Nonuniqueness for a semilinear initial value problem*, Indiana Univ. Math. J. **31** (1982) 167–189.

[14] J. Moser, *A sharp form of an inequality by N. Trudinger,* Indiana Univ. Math. J. **20** (1971) 1077–1092.

[15] Y. Naito, *Self-similar solutions for a semilinear heat equation with critical Sobolev exponent*, Indiana Univ. Math. J. **57** (2008) 1283–1315.

[16] Y. Naito and T. Suzuki, *Radial symmetry of self-similar solutions for semilinear heat equations*, J. Differential Equations **163** (2000) 407–428.

[17] L. A. Peletier, D. Terman and F.B. Weissler, *On the equation* $\Delta u + \frac{1}{2}x\cdot\nabla u + f(u) = 0$. Arch. Rational Mech. Anal. **94** (1986), 83–99.

[18] M. Willem, *Minimax theorems,* Birkhäuser, Boston, 1996.

(M.F. Furtado) Department of Mathematics, University of Brasília
70910-900, Brasília-DF, Brazil
Email address: mfurtado@unb.br

(K.C.V. de Sousa) Department of Mathematics, University of Brasília
70910-900,Brasília-DF, Brazil
Email address: karlakcvs@gmail.com