TWO SOLUTIONS FOR A SINGULAR ELLIPTIC EQUATION WITH CRITICAL GROWTH AT INFINITY

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Abstract. We look for positive solutions for the singular equation

$$-\Delta u - \frac{1}{2}(x \cdot \nabla u) = \mu h(x) u^{q-1} + \lambda u + u^{(N+2)/(N-2)}, \qquad \text{in } \mathbb{R}^N,$$

where $N \geq 3$, $\lambda > 0$, $\mu > 0$ is a parameter, $0 < q < 1$ and h has some summability properties. By using a perturbation method and critical point theory, we obtain two solutions when $\max\{1, N/4\} < \lambda < N/2$ and the parameter $\mu > 0$ is small.

1. Introduction

Consider the equation

$$-\Delta u - \frac{1}{2}(x \cdot \nabla u) = g(x, u), \qquad \text{in } \mathbb{R}^N,$$

with $N \geq 3$. As observed by Escobedo and Kavian in [12], if $g(x,s) = \lambda s + |s|^{p-2}s$ and $2 < p \leq 2^* := 2N/(N-2)$, this equation naturally appears when we deal with the nonlinear heat equation

$$u_t - \Delta u = |u|^{p-2}u, \qquad \text{in } (0,\infty) \times \mathbb{R}^N,$$

and look for solutions with the special form $u_\lambda(t,x) := t^{-\lambda} u(t^{-1/2}x)$, for $\lambda = 1/(p-1)$. We quote the works [17, 3, 7, 23, 9, 15, 24] and references therein for information about existence, nonexistence, decay rate and many other aspects concerning this subject. We emphasize that, in all of those works, the function $g(x,s)$ remains bounded as $s \to 0$. So, it is natural to ask what we can do in the singular case, that is, when $g(x,s) \to +\infty$ as $s \to 0^+$.

This paper aims to give a first answer for the above question. More specifically, we are concerned with positive solutions for the singular equation

$$-\Delta u - \frac{1}{2}(x \cdot \nabla u) = \mu h(x) u^{q-1} + \lambda u + u^{2^*-1}, \qquad \text{in } \mathbb{R}^N,$$

where $N \geq 3$, $\lambda > 0$, $\mu > 0$ is a parameter, $0 < q < 1$ and h has some summability properties. Before presenting the condition on h, we need to say a few words about the variational structure of the problem. We first notice that, after multiplying the equation by $K(x) := \exp(|x|^2/4)$, it can be rewritten as

$$\begin{cases} -\mathrm{div}(K(x)\nabla u) = \mu K(x) h(x) u^{q-1} + \lambda K(x) u + K(x) u^{2^*-1}, & \text{in } \mathbb{R}^N, \\ u > 0, & \text{in } \mathbb{R}^N. \end{cases} \qquad (P_\mu)$$

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It is natural to look for solutions in the space X defined as the closure of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\| := \left(\int_{\mathbb{R}^N} K(x)|\nabla u|^2 \, dx \right)^{1/2}.$$

It was proved in [12, Propositions 1.1 and 1.12] that X is a Hilbert space which is continuously embedded into the weighted Lebesgue spaces

$$L_K^p(\mathbb{R}^N) := \left\{ u \in L^p(\mathbb{R}^N) \, : \, \|u\|_p := \left(\int_{\mathbb{R}^N} K(x)|u|^p dx \right)^{1/p} < \infty \right\},$$

for any $p \in [2, 2^*]$. Moreover, the embedding is compact if $p \in [2, 2^*)$.

Due to the difficulties related to the operator and the singular nature of the nonlinearity at the origin, we do not expect to find regular solutions. Hence, as usual in the literature, we call $u \in X$ a solution for problem (P_μ) if it satisfies $u > 0$ a.e. in $\mathbb{R}^N$ and, for any $\phi \in X$, we have that $h(x)u^{q-1}\phi \in L_K^1(\mathbb{R}^N)$ and

$$\int_{\mathbb{R}^N} K(x) \left[(\nabla u \cdot \nabla \phi) - \mu h(x) u^{q-1} \phi - \lambda u \phi - u^{2^*-1} \phi \right] dx = 0. \tag{1.1}$$

In our first result, we obtain one solution with no lower limitation in the parameter $\lambda < N/2$ and for $\mu > 0$ is small. More specifically, we shall prove the following:

Theorem 1.1. *Suppose that* $\lambda < N/2$ *and* $h > 0$ *satisfies*

(h) $h \in L_K^1(\mathbb{R}^N) \cap L_K^2(\mathbb{R}^N)$.

Then there exists $\mu^* > 0$ *such that problem* (P_μ) *has a solution, whenever* $\mu \in (0, \mu^*)$.

In the proof, we apply a minimization argument for a perturbed (nonsigular) problem. We notice that condition $\lambda < N/2$ is necessary for the existence of a solution. Indeed, it is proved in [12, Proposition 2.3] that the linearized version of equation (P_μ) has the pair $(\lambda, u) = (N/2, \varphi_1)$ as a solution, where $\varphi_1(x) = \exp(-|x|^2/4) > 0$. So, if $u_0 \in X$ is a solution, we may pick $v = \varphi_1$ in the integral formulation to get

$$\left(\frac{N}{2} - \lambda \right) \int_{\mathbb{R}^N} K(x) u \varphi_1 \, dx = \int_{\mathbb{R}^N} K(x) \left[\mu h(x) u^{q-1} \varphi_1 + u^{2^*-1} \varphi_1 \right] dx > 0,$$

from which it follows that $\lambda < N/2$.

In our second result, we obtain another solution under an additional lower bound on the value of λ. More specifically, we prove the following:

Theorem 1.2. *Suppose that* $\max\{1, N/4\} < \lambda < N/2$, $h > 0$ *is continuous and satisfies* (h). *Then there exists* $0 < \mu_* < \mu^*$ *such that problem* (P_μ) *has at least two solutions, whenever* $\mu \in (0, \mu_*)$

To obtain the second solution, we apply the Mountain Pass Theorem to a perturbed functional, together with a limit process. The extra assumption on λ is related with the range of existence of positive solution for the nonsingular problem (P_0) obtained in [12, Theorem 4.10]. It is worth mentioning that the continuity of h may be replaced by the weaker condition that the infimum of h is positive in any ball.

We focus now on some general comments about the singular problem

$$-\Delta u = g(x,u), \quad \text{in } \Omega, \qquad u > 0, \quad \text{in } \Omega, \qquad u \in H_0^1(\Omega),$$

where $N \geq 3$, $\Omega \subset \mathbb{R}^N$ is a domain and $g(x,s) \to +\infty$, as $s \to 0$. There is a vast literature concerning this kind of problem, mainly due to its applications in boundary layer flow, fluid dynamics, non-Newtonian fluids, reaction–diffusion processes, chemical heterogeneous catalysts, in the theory of heat conduction in electrically conducting materials and in other geophysical and industrial contexts (see for instance [10, 8, 22, 27]).

Although it is impossible to give a complete reference, it seems important to quote the pioneering works of Stuart [29] and Crandall, Rabinowitz and Tartar [11], who considered a general second order operator instead of the laplacian and used some topological arguments to get solutions. Later, Lazer and McKenna [21] proved existence and regularity results for $g(x,s) = h(x)s^{q-1}$, where h is Hölder continuous. Their result was generalized in different ways by Lair and Shaker [19, 20] and Zhang and Cheng [30]. Also in the bounded domain case, we quote the paper of Boccardo and Orsina [5], where the Laplacian is replaced by the operator $u \mapsto \mathrm{div}(M(x)\nabla u)$, with M being a bounded elliptic matrix, $g(x,s) = h(x)s^{q-1}$, with $h \geq 0$ belonging to some Lebesgue space or even being a Radon measure. Some results for quasilinear operators can be found in [25, 26, 2]. For the case of the whole space, we refer the reader to [18, 20, 28], where it is supposed that $g(x,s) = h(x)s^{q-1} + f(x,s)$, h is continuous and f has some mild conditions.

We finally mention that our results are related to the pioneering work of Ambrosetti et al. [1] (see also [4]), where a concave-convex type problem was considered in a bounded domain. Here, the concave term is replaced by a singular one. We also refer to the reader versions of [1] for the same operator in (P_μ) in the whole space [14, 13] and in the upper half-space [16].

This article complements the aforementioned works, since we deal with a singular term, the problem is considered in the whole space and it has a different operator. It is organized as follows: in the next section we prove Theorem 1.1, and Section 3 is devoted to the proof of Theorem 1.2.

2. Existence of a first solution

Along all this paper we write only $\int f$ to denote $\int_{\mathbb{R}^N} f(x)dx$, where $f \in L^1(\mathbb{R}^N)$. For any $s \in \mathbb{R}$, we consider $s^+ := \max\{s, 0\}$ and $s^- := s^+ - s$. We denote by C_i, $i = 1, 2, \ldots$, positive constants depending only on the structural assumptions. The exact value of that constants will be omitted whenever it has no importance.

Before starting the proofs, we need to say a few words about the linearization of the problem (P_μ), namely

$$-\mathrm{div}(K(x)\nabla u) = \lambda K(x)u, \quad \text{in } \mathbb{R}^N.$$

Its spectrum was completely characterized in [12, Proposition 2.3], where it is proved by a Fourier approach that the first eigenvalue is given by

$$\lambda_1 = \inf\left\{\int K(x)|\nabla u|^2 \,:\, \int K(x)|u|^2 = 1\right\} = \frac{N}{2}.$$

From this, we infer the following Poincaré type inequality:

$$\lambda_1 \int_{\mathbb{R}^N} K(x)|u|^2\,dx \leq \int_{\mathbb{R}^N} K(x)|\nabla u|^2\,dx, \qquad \forall u \in X. \tag{2.1}$$

Since we are going to obtain solutions for small values of the parameter $\mu > 0$, it is important to consider the limit problem

$$-\text{div}(K(x)\nabla u) = \lambda K(x)u + K(x)|u|^{2^*-2}u, \quad \text{in } \mathbb{R}^N, \tag{P_0}$$

and its associated C^1-functional given by

$$I_0(u) := \frac{1}{2}\|u\|^2 - \frac{\lambda}{2}\|u^+\|_2^2 - \frac{1}{2^*}\|u^+\|_{2^*}^{2^*}, \quad u \in X.$$

From now on, we assume that $h \in L^1_K(\mathbb{R}^N) \cap L^2_K(\mathbb{R}^N)$. Hence, we can use interpolation to conclude that $h \in L^\theta_K(\mathbb{R}^N)$, where $\theta := 2/(2-q)$. For any $u \in X$, it follows from Hölder's inequality that

$$\frac{1}{q}\int K(x)h(x)(u^+)^q \le \frac{1}{q}\|h\|_\theta \|u^+\|_2^q \le C_1\|u\|^q, \tag{2.2}$$

Thus, we may add the singular term to I_0 and obtain the functional associated with the problem (P_μ), namely

$$I_\mu(u) := I_0(u) - \frac{\mu}{q}\int K(x)h(x)(u^+)^q, \quad u \in X.$$

It is clear that I_μ is a well-defined continuous functional in X. In our first result we study its behaviour near the origin.

Lemma 2.1. *There exists $\mu^* > 0$ such that, for any $\mu \in (0, \mu^*)$, there holds*

$$I_\mu(u) \ge \rho, \quad \forall\, u \in \partial B_R(0),$$

with ρ, $R > 0$ independent of μ.

Proof. Given $u \in X$, we can use (2.1) and the embedding $X \hookrightarrow L^{2^*}_K(\mathbb{R}^N)$ to get

$$I_0(u) \ge \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_1}\right)\|u\|^2 - C_2\|u\|^{2^*} \ge C_3\|u\|^2, \tag{2.3}$$

if $C_3 = (1 - \lambda/\lambda_1)/4$ and

$$\|u\| \le R := \left(\frac{C_3}{C_2}\right)^{1/(2^*-2)}.$$

This and (2.2) imply that

$$I_\mu(u) \ge \|u\|^q \left(C_3\|u\|^{2-q} - \mu C_1\right) \ge \rho := \frac{C_3}{2}R^q,$$

whenever $\|u\| = R$ and

$$0 < \mu < \mu^* := \frac{C_3}{2C_1}R^{2-q}.$$

The lemma is proved. □

Let μ^*, $R > 0$ as in Lemma 2.1 and $\mu \in (0, \mu^*)$. By picking a nonnegative function $\varphi \in C_0^\infty(\mathbb{R}^N) \setminus \{0\}$, we get

$$\lim_{t\to 0^+} \frac{I_\mu(t\varphi)}{t^q} = -\frac{\mu}{q}\int K(x)h(x)\varphi^q < 0,$$

and therefore there exists $t_0 > 0$ small in such a way that $\|t_0\varphi\| \le R$ and $I_\mu(t_0\varphi) < 0$. This shows that

$$m_\mu := \inf_{\|u\|\le R} I_\mu(u) < 0.$$

Since I_μ maps bounded sets onto bounded sets, we have that $m_\mu > -\infty$.

Even if we prove that m_μ is attained in $B_R(0)$, the singular term of the equation gives rise to a difficulty. Actually, since $0 < q < 1$, the term $\int K(x)h(x)(u^+)^q$ is continuous but not differentiable, and therefore it is not clear that minimizers are solutions of our problem. However, using a direct calculation, we may prove that this holds, as we can see in the next result.

Lemma 2.2. *If $u \in B_R(0)$ is such that $I_\mu(u) = m_\mu$, then u is a solution for problem (P_μ).*

Proof. Since $u^+ \in B_R(0)$ and $m_\mu \le I_\mu(u^+) \le I_\mu(u) = m_\mu$, we have that $I_\mu(u^+) = I_\mu(u)$. Thus, $u^- \equiv 0$ or, equivalently, $u = u^+ \ge 0$. We claim that $u > 0$, a.e. in $\mathbb{R}^N$. Indeed, suppose by contradiction that the set $\Omega_0 := \{x \in \mathbb{R}^N : u = 0\}$ has positive measure. Pick $r > 0$ such that $\Omega := \Omega_0 \cap B_r(0)$ has positive measure and a nonnegative function $\psi \in C_0^\infty(\mathbb{R}^N)$ such that $\operatorname{supp}(\psi) \subset B_{2r}(0)$ and $\psi > 0$ in $B_r(0)$. Since $\|u\| < R$, we have that $\|u + t\psi\| < R$, for any $t > 0$ small. If we divide the inequality $I_\mu(u) \le I_\mu(u + t\psi)$ by $t > 0$ we obtain

$$\begin{aligned} \frac{\mu}{q}\int_\Omega \frac{K(x)h(x)\psi^q}{t^{1-q}}dx &\le \frac{\mu}{q}\int \frac{K(x)h(x)[((u+t\psi)^+)^q - (u^+)^q]}{t} \\ &\le \frac{I_0(u+t\psi) - I_0(u)}{t}. \end{aligned}$$

Passing to the limit, using Fatou's lemma and recalling that $K(x)h(x)\psi^q > 0$ a.e. in Ω, we obtain

$$+\infty = \frac{\mu}{q}\int_\Omega \liminf_{t\to 0^+} \frac{K(x)h(x)\psi^q}{t^{1-q}}dx \le \frac{\mu}{q}\liminf_{t\to 0^+}\int_\Omega \frac{K(x)h(x)\psi^q}{t^{1-q}}dx \le I_0'(u)\psi,$$

which does not make sense. Hence, Ω_0 has zero measure and $u > 0$ a.e. in $\mathbb{R}^N$.

Now, taking an arbitrary nonnegative function $\psi \in X$, we can argue as above to get that

$$\begin{aligned} I_0'(u)\psi &\ge \frac{\mu}{q}\liminf_{t\to 0^+}\int \frac{K(x)h(x)[((u+t\psi)^+)^q - (u^+)^q]}{t} \\ &= \frac{\mu}{q}\liminf_{t\to 0^+}\int_{[\psi>0]} \frac{K(x)h(x)[((u+t\psi)^+)^q - (u^+)^q]}{t}dx. \end{aligned}$$

Since $\psi \ge 0$, we have that $(u + t\psi)^+ \ge u^+$. Then, we can use Fatou's lemma again to obtain

$$I_0'(u)\psi - \frac{\mu}{q}\int K(x)h(x)(u^+)^{q-1}\psi \ge 0, \quad \forall\, \psi \in X,\ \psi \ge 0. \tag{2.4}$$

Since $u \ne 0$, it is well-defined $t_0 := (R/\|u\|) - 1 > 0$ and a straightforward computation shows that $\|(1+t)u\| < R$, whenever $t \in (-1, t_0)$. Hence, denoting by

$$\gamma(t) := I_\mu((1+t)u), \quad t \in (-1, t_0),$$

we can use that $m_\mu = I_\mu(u)$ to get

$$m_\mu \le \inf_{t\in(-1,t_0)} \gamma(t) \le \gamma(0) = I_\mu(u) = m_\mu,$$

from which we conclude that γ attains its minimum value at $t = 0$. Since

$$\gamma(t) = \frac{(1+t)^2}{2}\left[\|u\|^2 - \lambda\|u\|_2^2\right] - \frac{(1+t)^{2^*}}{2^*}\|u\|_{2^*}^{2^*} - \frac{\mu(1+t)^q}{q}\int K(x)h(x)(u^+)^q,$$

we conclude that γ is differentiable in $(-1,t_0)$, and therefore

$$\gamma'(0)=I_0'(u)u-\mu\int K(x)h(x)(u^+)^q=0. \tag{2.5}$$

Pick $\varepsilon>0$, $\phi\in X$ and define $\Omega_\varepsilon^+:=[u^++\varepsilon\phi<0]$. By using (2.4) with $\psi=(u^++\varepsilon\phi)^+$ we get, after some computations,

$$\begin{aligned} 0 &\leq -\|u^-\|^2+I_0'(u)u-\mu\int K(x)h(x)(u^+)^q \\ &\quad +\varepsilon I_0'(u)\phi-\varepsilon\mu\int K(x)h(x)(u^+)^{q-1}\phi-\int_{\Omega_\varepsilon^+}K(x)[\nabla u\cdot\nabla(u^++\varepsilon\phi)]\,dx \\ &\quad +\int_{\Omega_\varepsilon^+}K(x)(u^++\varepsilon\phi)\left[\lambda u^++\mu h(x)(u^+)^{q-1}+(u^+)^{2^*-1}\right]dx. \end{aligned}$$

Hence, it follows from (2.5) that

$$\begin{aligned} 0 &\leq \varepsilon I_0'(u)\phi-\varepsilon\mu\int K(x)h(x)(u^+)^{q-1}\phi-\int_{\Omega_\varepsilon^+}K(x)[\nabla u\cdot\nabla(u^++\varepsilon\phi)]\,dx \\ &\leq \varepsilon\left[I_0'(u)\phi-\mu\int K(x)h(x)(u^+)^{q-1}\phi-\int_{\Omega_\varepsilon^+}K(x)[\nabla u\cdot\nabla\phi]\,dx\right] \end{aligned}$$

If we divide the previous expression by $\varepsilon>0$, take the limit as $\varepsilon\to0^+$ and use that $u>0$ a.e. in $\mathbb{R}^N$, we obtain

$$\lim_{\varepsilon\to0^+}\mathbf{1}_{\Omega_\varepsilon^+}(x)=0,\qquad \text{a.e. in } \mathbb{R}^N,$$

where $\mathbf{1}_{\Omega_\varepsilon^+}$ stands for the characteristic function of the set Ω_ε^+, we can use Lebesgue theorem to conclude that

$$I_0'(u)\phi-\mu\int K(x)h(x)(u^+)^{q-1}\phi\geq0,\qquad \forall\,\phi\in X.$$

Since this inequality also holds with write $-\phi$ instead of ϕ, we conclude that $u\in X$ satisfies the integral equation (1.1) and consequently, $K(x)h(x)(u^+)^{q-1}\phi\in L^1(\mathbb{R}^N)$, for all $\phi\in X$. The lemma is proved. □

We now notice that I_μ is not of class C^1, and therefore we cannot perform standard minimization arguments. So, instead of a direct approach, we are going to consider the a perturbation argument: for each $k\in\mathbb{N}$, define $\mathcal{X}_k:\mathbb{R}\to\mathbb{R}$ as

$$\mathcal{X}_k(s):=\int_0^s\left(t^++\frac{1}{k}\right)^{q-1}dt=\frac{1}{q}\left[\left(s^++\frac{1}{k}\right)^q-\left(\frac{1}{k}\right)^q\right]-\left(\frac{1}{k}\right)^{q-1}s^-, \tag{2.6}$$

and the functional

$$I_{\mu,k}(u):=I_0(u)-\mu\int K(x)h(x)\mathcal{X}_k(u),\quad u\in X.$$

Since

$$\mathcal{X}_k'(s)=\left(s^++\frac{1}{k}\right)^{q-1},\quad s\in\mathbb{R}, \tag{2.7}$$

it is clear that $I_{\mu,k}\in C^1(X,\mathbb{R})$.

We are going to show that $I_{\mu,k}$ attains its minimum at $u_k\in B_R(0)$ and the desired solution will be obtained passing to the limit as $k\to+\infty$. The details can be found in the next proposition.

Proposition 2.3. *Let μ^*, $R>0$ be given by Lemma 2.1. For any $\mu \in (0,\mu^*)$ there exists $u \in X$ such that $\|u\| < R$ and $I_\mu(u) = m_\mu$. In particular, the problem (P_μ) has a solution with negative energy.*

Proof. Since $\mathcal{X}_k(s) \le \int_0^s (t^+)^{q-1} dt$, we have that $I_{\mu,k}(u) \ge I_\mu(u)$, for any $u \in X$ and $k \in \mathbb{N}$. It follows from Lemma 2.1 that $I_{\mu,k} \ge \rho$ on $\partial B_R(0)$. Thus, since $I_{\mu,k}(0) = 0$, we can define

$$m_{\mu,k} := \inf_{\|u\| \le R} I_{\mu,k}(u),$$

and use the Ekeland Variational Principle to obtain a sequence $(u_{n,k})_{n\in\mathbb{N}} \subset B_R(0)$ such that

$$\lim_{n\to+\infty} I_{\mu,k}(u_{n,k}) = m_{\mu,k}, \qquad \lim_{n\to+\infty} I'_{\mu,k}(u_{n,k}) = 0.$$

Since $\mathcal{X}_k(s^+) \ge \mathcal{X}_k(s)$, we have that $I_{\mu,k}(u^+_{n,k}) \le I_{\mu,k}(u_{n,k})$ and we can argue as in the proof of Lemma 2.2 to suppose that $u_{n,k} \ge 0$. Up to a subsequence, as $n \to +\infty$, we have that

$$\left\{ \begin{array}{rcll} u_{n,k} & \rightharpoonup & u_k, & \text{weakly in } X, \\ u_{n,k} & \to & u_k, & \text{strongly in } L^p_K(\mathbb{R}^N), \\ u_{n,k}(x) & \to & u_k(x), & \text{a.e. in } \mathbb{R}^N, \\ |u_{n,k}(x)| & \le & g_p(x), & \text{a.e. in } \mathbb{R}^N, \end{array} \right. \tag{2.8}$$

for any $p \in [2, 2^*)$ and some $g_p \in L^p_K(\mathbb{R}^N)$. By noticing that

$$|\mathcal{X}_k(s)| \le \int_0^{|s|} k^{1-q}\, dt = k^{1-q}|s|, \qquad s \in \mathbb{R}, \tag{2.9}$$

we infer from (2.8) that

$$|K(x)h(x)\mathcal{X}_k(u_{n,k})| \le \left(\frac{1}{k}\right)^{q-1} K(x)h(x)g_2(x),$$

a.e. in $\mathbb{R}^N$. Since the right-hand side above belongs to $L^1(\mathbb{R}^N)$, we can use the Lebesgue Theorem to obtain

$$\lim_{n\to+\infty} \int K(x)h(x)\mathcal{X}_k(u_{n,k}) = \int K(x)h(x)\mathcal{X}_k(u_k). \tag{2.10}$$

Setting $v_{n,k} := u_{n,k} - u_k$, we can use the above expression, (2.8) and the Brezis-Lieb lemma [6, Theorem 2] to get

$$\begin{array}{rcl} m_{\mu,k} & = & I_{\mu,k}(u_{n,k}) + o_n(1) \\ & = & \dfrac{1}{2}\|v_{n,k}\|^2 + \dfrac{1}{2}\|u_k\|^2 - \dfrac{\lambda}{2}\|u_k^+\|_2^2 - \mu \displaystyle\int K(x)h(x)\mathcal{X}_k(u_k) \\ & & -\dfrac{1}{2^*}\displaystyle\int K(x)(v^+_{n,k})^{2^*} - \dfrac{1}{2^*}\displaystyle\int K(x)(u_k^+)^{2^*}\, dx + o_n(1), \end{array} \tag{2.11}$$

where $o_n(1)$ stands for a quantity approaching zero as $n \to +\infty$. Recalling that $\|u_{n,k}\| < R$ and using the weak convergence, we obtain

$$\begin{array}{rcl} \displaystyle\limsup_{n\to+\infty} \|v_{n,k}\|^2 & < & R^2 + \|u_k\|^2 - 2\displaystyle\lim_{n\to+\infty}\int K(x)(\nabla u_{n,k}\cdot\nabla u_k) \\ & = & R^2 - \|u_k\|^2 \le R^2. \end{array}$$

This shows that $\|v_{n,k}\| \leq R$, whenever $n \geq n_0(k)$. Hence, it follows from (2.3) that

$$\frac{1}{2}\|v_{n,k}\|^2 - \frac{1}{2^*}\int K(x)(v_{n,k}^+)^{2^*} \geq 0, \quad \forall\, n \geq n_0(k),$$

which combined with (2.11) imply that

$$m_{\mu,k} \geq I_{\mu,k}(u_k) + o_n(1).$$

Passing to the limit as $n \to +\infty$ we conclude that $m_{\mu,k} = I_{\mu,k}(u_k)$. Moreover, since $m_{\mu,k} \leq I_{\mu,k}(0) = 0$ and $I_{\mu,k} \geq \rho > 0$ on $\partial B_R(0)$, we have that $\|u_k\| < R$.

For any $\varphi \in C_0^\infty(\mathbb{R}^N)$, we have that

$$\left| K(x)h(x)\left(u_{n,k}^+(x) + \frac{1}{k}\right)^{q-1}\varphi(x)\right| \leq k^{1-q}K(x)h(x)|\varphi(x)|, \quad \text{a.e. in } \mathbb{R}^N.$$

By using the pointwise convergence and Lebesgue's theorem, we obtain

$$\lim_{n\to+\infty}\int K(x)h(x)\left(u_{n,k}^+ + \frac{1}{k}\right)^{q-1}\varphi = \int K(x)h(x)\left(u_k^+ + \frac{1}{k}\right)^{q-1}\varphi.$$

This expression, (2.8) and a standard density argument imply that $I'_{\mu,k}(u_k) = 0$.

We are going to show that

$$\lim_{k\to+\infty} I_{\mu,k}(u_k) = m_\mu. \tag{2.12}$$

Since $I_{\mu,k}(u_k) \geq I_\mu(u_k) \geq m_\mu$, it is sufficient to verify that

$$\limsup_{k\to+\infty} I_{\mu,k}(u_k) \leq m_\mu. \tag{2.13}$$

In order to do this, let $(w_n) \subset B_R(0)$ be such that $I_\mu(w_n) \to m_\mu$, as $n \to +\infty$. Notice that $w_n^+ \in B_R(0)$ and $I_\mu(w_n^+) \leq I_\mu(w_n)$. So, replacing (w_n) by (w_n^+) if necessary, we may assume that $w_n \geq 0$. Then

$$\begin{aligned} I_\mu(w_n) &= I_{\mu,k}(w_n) + \mu\int K(x)h(x)\mathcal{X}_k(w_n) - \frac{\mu}{q}\int K(x)h(x)(w_n^+)^q \\ &\geq m_{\mu,k} + \mu\int K(x)h(x)\mathcal{X}_k(w_n) - \frac{\mu}{q}\int K(x)h(x)(w_n^+)^q. \end{aligned} \tag{2.14}$$

Fixed $n \in \mathbb{N}$, we can use (2.6) and $w_n \geq 0$ to obtain

$$\begin{aligned} \int K(x)h(x)\mathcal{X}_k(w_n)\,dx &= \int K(x)h(x)\frac{(w_n^+ + \frac{1}{k})^q - (\frac{1}{k})^q}{q} \\ &= \frac{1}{q}\int K(x)h(x)(w_n^+)^q + o_k(1). \end{aligned}$$

By combining this expression with (2.14) and taking the limsup as $k \to +\infty$, we get

$$I_\mu(w_n) \geq \limsup_{k\to+\infty} m_{\mu,k} = \limsup_{k\to+\infty} I_{\mu,k}(u_k).$$

Once again, passing to the limit as $n \to +\infty$, we immediately obtain (2.13).

We are now able to prove that m_μ is attained. Since (u_k) is bounded, along a subsequence $u_k \rightharpoonup u$ weakly in X. As before, we can prove that

$$\lim_{k\to+\infty}\int K(x)h(x)\mathcal{X}_k(u_k) = \frac{1}{q}\int K(x)h(x)(u^+)^q.$$

Hence, we can use (2.12) and the same argument used to prove that $I_{\mu,k}(u_k) = m_{\mu,k}$ (but now considering the limits in the index k) to conclude that $I_\mu(u) = m_\mu$. We omit the details. □

3. The second solution

Now we have obtained a first solution, we are going to apply the Mountain Pass Theorem for the perturbed functional and obtain a second solution as a limit process. First, we present some important facts about the problem (P_0) stated in the beginning of the previous section. In order to describe some results proved in [12], we redefine the associated energy functional as

$$I_0(u) := \frac{1}{2}\|u\|^2 - \frac{\lambda}{2}\|u\|_2^2 - \frac{1}{2^*}\|u^+\|_{2^*}^{2^*}, \quad u \in X.$$

The least energy level of (P_0) is defined as

$$c_0 := \inf_{u\in\mathcal{N}_0} I_0(u),$$

where $\mathcal{N}_0 := \{u \in X \setminus \{0\} : I_0'(u)u = 0\}$ is the Nehari manifold. In [12], the authors obtained ground state solution for (P_0) using the minimization problem

$$S_\lambda(K) := \inf_{u\in X\setminus\{0\}} \frac{\|u\|^2 - \lambda\|u\|_2^2}{\|u^+\|_{2^*}^2}.$$

They proved that $0 < S_\lambda(K) < S_{2^*}$, when $\max\{1, N/4\} < \lambda < N/2$. As a consequence, the above infimum is attained by a positive function $u_0 \in X \cap C^2(\mathbb{R}^N)$. Since the problem is homogeneous, a scaling argument provides $\tau > 0$ such that the function $\omega_0 := \tau^{2^*-2}u_0$ is a solution for (P_0) with $I_0(\omega_0) = c_0$. We finally mention that, since $u_0 = \tau^{1/(2-2^*)}\omega_0$ and $\omega_0 \in \mathcal{N}_0$, we have that

$$S_{2^*} > S_\lambda(K) = \frac{\|u_0\|^2 - \lambda\|u_0\|_2^2}{\|u_0^+\|_{2^*}^2} = \frac{\|\omega_0\|^2 - \lambda\|\omega_0\|_2^2}{\|\omega_0^+\|_{2^*}^2} = \left(\|\omega_0\|^2 - \lambda\|\omega_0\|_2^2\right)^{2/N},$$

which leads to the following useful inequality

$$\frac{1}{N}S_{2^*}^{N/2} > \frac{1}{N}\left(\|\omega_0\|^2 - \lambda\|\omega_0\|_2^2\right) = I_0(\omega_0) = c_0. \tag{3.1}$$

From now on, we are going to look for a second solution for problem (P_μ) as a weak limit of a sequence of positive energy critical point of

$$I_{\mu,k}(u) := I_0(u) - \mu\int_{\mathbb{R}^N} K(x)h(x)\mathcal{X}_k(u)\,dx, \quad u \in X,$$

with I_0 redefined as before. It is clear that its critical points are weak solutions for the (nonsingular) problem

$$-\mathrm{div}(K(x)\nabla u) = \mu K(x)\frac{h(x)}{(u+1/k)^{1-q}} + \lambda K(x)u + K(x)|u|^{2^*-2}u, \quad \text{in } \mathbb{R}^N. \tag{$P_{\mu,k}$}$$

In our next result we prove that such solutions are, indeed, zero or positive in $\mathbb{R}^N$.

Lemma 3.1. *If $u_k \in X$ is a nonzero critical point of $I_{\mu,k}$, then it is a positive weak solution for $(P_{\mu,k})$.*

Proof. It is clear that u_k weakly solves the problem. Moreover, computing

$$0 = I'_{\mu,k}(u_k)u_k^- = -\|u_k^-\|^2 + \lambda\|u_k^-\|_2^2 - \mu k^{1-q}\int K(x)h(x)u_k^-,$$

and recalling that $\lambda < N/2$, we conclude that $u_k \geq 0$ a.e. in $\mathbb{R}^N$. In order to prove that $u_k > 0$ a.e. in $\mathbb{R}^N$, we consider an arbitrary (but fixed) radius $r > 0$ and $\Sigma \subset B_r(0)$ a compact subset of $\mathbb{R}^N$. Given a nonnegative function $\varphi \in H_0^1(B_r(0))$, we can use $K, \lambda \geq 1$ to write

$$\int_{B_r(0)} K(x)\left(\nabla u_k \cdot \nabla\varphi\right) dx \geq \int_{B_r(0)} \left[(u_k + u_k^{2^*-1}) + \frac{\mu h(x)}{(u_k+1)^{1-q}}\right]\varphi\, dx$$

It follows from the inequality

$$(s + s^{2^*-1}) + \frac{a}{(s+1)^{1-q}} \geq \min\left\{1, \frac{a}{2^{1-q}}\right\}, \quad \forall\, a > 0,\, s \geq 0,$$

that

$$\int_{B_r(0)} K(x)\left(\nabla u_k \cdot \nabla\varphi\right) dx \geq C_r \int_{B_r(0)} \varphi, \tag{3.2}$$

for

$$C_r := \min\left\{1, \mu\frac{\min_{x\in B_r(0)} h(x)}{2^{1-q}}\right\} > 0.$$

On the other hand, using the Lax-Milgram theorem, we obtain a nonnegative $v \in H_0^1(B_r(0))$ such that

$$-\operatorname{div}(K(x)\nabla v) = C_r, \text{ in } B_r(0). \tag{3.3}$$

Following the ideas developed in [12, Theorem 3.12], we can prove that $v \in C^2(B_r(0) \cap C(\overline{B_r(0)})$ and therefore the Strong Maximum Principle ensures that $v > 0$ in $B_r(0)$. Thus, there exists a constant $C_\Sigma > 0$ such that $v(x) \geq C_\Sigma$, for any $x \in \Sigma$.

By using (3.2) and (3.3), we obtain

$$\int_{B_r(0)} K(x)(\nabla u_k \cdot \nabla\varphi)\, dx \geq \int_{B_r(0)} K(x)(\nabla v \cdot \nabla\varphi)\, dx, \quad \forall\, 0 \leq \varphi \in H_0^1(B_r(0)).$$

In particular, taking $\varphi := \max\{v - u_k, 0\} \in H_0^1(B_r(0))$ and using $K \geq 1$ again, we obtain

$$\|\varphi\|^2_{H_0^1(B_R(0))} \leq \int_{[v\geq u_k]} K(x)|\nabla\varphi|^2 dx \leq \int_{B_r(0)} K(x)(\nabla(v-u_k)\cdot\nabla\varphi)\, dx \leq 0,$$

from which we conclude that $\varphi = 0$ or, equivalently, $u_k \geq v$ a.e. in $B_r(0)$. Hence, $u_k \geq v \geq C_\Sigma > 0$ in the (arbitrary) set Σ and the lemma is proved. □

Remark 3.2. *In the above proof, we have used the continuity of h to guarantee that $C_r > 0$. So, it is clear that the same result is true if we just assume that, for any $r > 0$, there holds*

$$\inf_{x\in B_r(0)} h(x) > 0.$$

If $d \in \mathbb{R}$, we say that $(u_n) \subset X$ is a $(PS)_d$ sequence for $I_{\mu,k}$ if

$$\lim_{n\to+\infty} I_{\mu,k}(u_n) = d, \quad \lim_{n\to+\infty} I'_{\mu,k}(u_n) = 0.$$

The functional $I_{\mu,k}$ satisfies the Palais-Smale condition at level d if any such sequence has a convergent subsequence. In what follows, we prove that our functional satisfies this compactness condition in an appropriated subset of $\mathbb{R}$.

Lemma 3.3. *There exists $M_1 = M_1(q, \lambda, N, \|h\|_\theta) > 0$ and $M_2 = M_2(q, \|h\|_1) > 0$ such that, for any $\mu > 0$ and $k \in \mathbb{N}$, the functional $I_{\mu,k}$ satisfies the Palais-Smale condition at any level*

$$d < \frac{1}{N} S_{2^*}^{N/2} - M_1 \mu^\theta - \frac{M_2}{k^q} \mu.$$

Proof. Let $(u_n) \subset X$ be a $(PS)_d$ sequence for $I_{\mu,k}$. In order to verify that it is a bounded sequence, we set

$$\alpha_0 := \frac{1}{2} \left(\frac{1}{2} - \frac{1}{2^*} \right) \left(\frac{\lambda_1 - \lambda}{\lambda_1} \right) \tag{3.4}$$

and use (2.1) to get

$$\begin{aligned} d + o_n(1)(1 + \|u_n\|) &= I_{\mu,k}(u_n) - \frac{1}{2^*} I'_{\mu,k}(u_n) u_n \\ &\geq 2\alpha_0 \|u_n\|^2 - \mu \int K(x) h(x) \mathcal{X}_k(u_n) \\ &\quad + \frac{\mu}{2^*} \int K(x) h(x) \mathcal{X}'_k(u_n) u_n. \end{aligned}$$

It follows from the above expression and (2.6)-(2.7) that

$$d + o_n(1)(1 + \|u_n\|) \geq 2\alpha_0 \|u_n\|^2 - \frac{\mu}{q} \int K(x) h(x) \left(u_n^+ + \frac{1}{k} \right)^q. \tag{3.5}$$

Since $(a + b)^q \leq C_q(a^q + b^q)$, for some $C_q > 0$ and any a, $b \geq 0$, we can use Young's inequality to obtain, for each $\varepsilon > 0$, a constant $C_{\varepsilon,q} > 0$ such that

$$\begin{aligned} \frac{\mu}{q} K(x) h(x) \left(u_n^+(x) + \frac{1}{k} \right)^q &\leq C_q \frac{\mu}{q} K(x) h(x) \left[(u_n^+)^q(x) + k^{-q} \right] \\ &\leq \varepsilon K(x) u_n^+(x)^2 + C_{\varepsilon,q} \mu^\theta K(x) h(x)^\theta \\ &\quad + C_q \frac{\mu}{q} K(x) h(x) k^{-q}, \end{aligned}$$

for a.e. $x \in \mathbb{R}^N$ and where $1 < \theta := (2/q)' = 2/(2 - q) < 2$. Picking $\varepsilon = \alpha_0 \lambda_1$, we can use the above expression, (3.5) and (2.1), to obtain

$$d + o_n(1)(1 + \|u_n\|) \geq \alpha_0 \|u_n\|^2 - M_1 \mu^\theta - M_2 \frac{\mu}{k^q},$$

where

$$M_1 := C_{\varepsilon,q} \|h\|_\theta^\theta, \quad M_2 := \frac{C_q}{q} \|h\|_1.$$

Thus, $(u_n) \subset X$ is bounded.

Up to a subsequence, we may assume that $u_n \rightharpoonup u$ weakly in X and an analogous of (2.8) holds. Arguing as in the proof of Proposition 2.3, we can prove that $I'_{\mu,k}(u) = 0$ and

$$\lim_{n \to +\infty} \int K(x) h(x) \mathcal{X}'_k(u_n) u_n = \int K(x) h(x) \mathcal{X}'_k(u) u.$$

Moreover, the former computations provide

$$I_{\mu,k}(u) = I_{\mu,k}(u) - \frac{1}{2^*} I'_{\mu,k}(u)u \geq \alpha_0 \|u\|^2 - M_1\mu^\theta - M_2\frac{\mu}{k^q}. \tag{3.6}$$

Hence, if we set $v_n := (u_n - u)$, we can use (2.8) and the Brezis-Lieb lemma to get

$$o_n(1) = I'_{\mu,k}(u_n)u_n = \|v_n\|^2 - \|v_n\|_{2^*}^{2^*} + I'_{\mu,k}(u)u + o_n(1),$$

from which it follows that

$$\lim_{n\to+\infty} \|v_n\|^2 = l = \lim_{n\to+\infty} \|v_n\|_{2^*}^{2^*},$$

for some $l \geq 0$.

Suppose, by contradiction, that $l > 0$. Then we can use the definition of S_{2^*} to conclude that $l \geq S_{2^*}^{N/2}$. On the other hand, using Brezis-Lieb lemma again, (2.10) and (3.6) we obtain

$$\begin{aligned} d + o_n(1) &= I_{\mu,k}(u_n) = \frac{1}{2}\|v_n\|^2 - \frac{1}{2^*}\|v_n\|_{2^*}^{2^*} + I_{\mu,k}(u) + o_n(1) \\ &\geq \frac{1}{2}\|v_n\|^2 - \frac{1}{2^*}\|v_n\|_{2^*}^{2^*} + \alpha_0\|u\|^2 - M_1\mu^\theta - M_2\frac{\mu}{k^q} + o_n(1). \end{aligned}$$

Taking the limit as $n \to +\infty$ and recalling that $l \geq S_{2^*}^{N/2}$, we obtain

$$d \geq \frac{1}{N} S_{2^*}^{N/2} - M_1\mu^\theta - M_2\frac{\mu}{k^q},$$

which contradicts the hypotheses. Hence, $l = 0$ or, equivalently, $u_n \to u$ strongly in X. □

We solve in the sequel the modified problem.

Proposition 3.4. *Let μ^*, $\rho > 0$ be given by Lemma 2.1. Then, there exists $k_* = k_*(q,h) > 0$ and $\mu_* = \mu_*(q,N,h) < \mu^*$ such that, for any $k \geq k_*$ and $\mu \in (0,\mu_*)$, the functional $I_{\mu,k}$ has a positive critical point $u_k \in X$ verifying $I_{\mu,k}(u_k) \geq \rho > 0$.*

Proof. Let M_1, M_2 be given by Lemma 3.3. Recalling that the function ω_0 obtained in the beginning of the section is positive, we obtain $I_{\mu,k}(t\omega_0) \leq I_0(t\omega_0)$, for any $t \geq 0$. Since $I_0(t\omega_0) \to 0$, as $t \to 0^+$, we can find $t_* > 0$, independent of μ and k, such that

$$\max_{0\leq t\leq t_*} I_{\mu,k}(t\omega_0) < \frac{c_0}{2} < c_0 - M_1\mu^\theta - M_2\frac{\mu}{k^q}, \tag{3.7}$$

whenever

$$\mu < \min\left\{1, \frac{c_0}{2(M_1+M_2)}\right\}.$$

Moreover, since the function $t \mapsto t\omega_0(x)\,[t\omega_0(x)+1]^{q-1}$ is increasing in $[0,+\infty)$, a change of variables provides

$$\begin{aligned} \mathcal{X}_k(t\omega_0(x)) = \int_{1/k}^{t\omega_0(x)+1/k} \frac{1}{\tau^{1-q}}\,d\tau &\geq \frac{t\omega_0(x)}{[t\omega_0(x)+1/k]^{1-q}} \\ &\geq \frac{t\omega_0(x)}{[t\omega_0(x)+1]^{1-q}} \\ &\geq \frac{t_*\omega_0(x)}{[t_*\omega_0(x)+1]^{1-q}}, \end{aligned} \tag{3.8}$$

for any $x \in \mathbb{R}^N$ and $t \geq t_*$. Hence, if we define

$$C_{h,q} := t_* \int K(x)h(x)\frac{\omega_0}{(t_*\omega_0+1)^{1-q}},$$

we can use (3.8) and that $I_0(\omega_0) = \max_{t\geq 0} I_0(t\omega_0)$ to obtain

$$I_{\mu,k}(t\omega_0) = I_0(t\omega_0) - \mu\int K(x)h(x)\mathcal{X}_k(t\omega_0) \leq c_0 - C_{h,q}\mu, \quad t \geq t_*. \tag{3.9}$$

We now notice that, if

$$k \geq k_* := \left(\frac{2M_2}{C_{h,q}}\right)^{1/q}, \quad \mu^{\theta-1} < \frac{C_{h,q}}{2M_1},$$

then

$$M_1\mu^\theta + M_2\frac{\mu}{k^q} < \frac{C_{h,q}}{2}\mu + \frac{C_{h,q}}{2}\mu = \mu C_{h,q}.$$

This inequality, together with (3.9) and (3.7), imply that

$$\sup_{t\geq 0} I_{\mu,k}(t\omega_0) < c_0 - M_1\mu^\theta - M_2\frac{\mu}{k^q}, \tag{3.10}$$

whenever $k \geq k_*$ and

$$0 < \mu < \mu_* := \min\left\{1, \frac{c_0}{2(M_1+M_2)}, \left(\frac{C_{h,q}}{2M_1}\right)^{1/(\theta-1)}\right\}.$$

Since

$$\lim_{t\to+\infty}\frac{I_{\mu,k}(t\omega_0)}{t^{2^*}} \leq -\frac{1}{2^*}\|\omega_0\|_{2^*}^{2^*} < 0,$$

there exists $T > 0$, independent of μ and k, such that $\|T\omega_0\| > \rho$ and $I_{\mu,k}(t\omega_0) < 0$, for any $t \geq T$. Thus, we can use Lemma 2.1 to define the Moutain Pass level

$$c_{\mu,k} := \inf_{\gamma\in\Gamma}\sup_{0\leq t\leq 1} I_{\mu,k}(\gamma(t)),$$

where $\Gamma := \{\gamma \in C([0,1], X) : \ \gamma(0) = 0, \gamma(1) = T\omega_0\}$.

The definition of Γ and (3.10) imply that

$$c_{\mu,k} < c_0 - M_1\mu^\theta - M_2\frac{\mu}{k^q}, \tag{3.11}$$

whenever $k \geq k_*$ and $\mu \in (0, \mu_*)$. Using Lemma 3.3 and the Mountain Pass Theorem we obtain a critical point $u_k \in X$ such that $I_{\mu,k}(u_k) \geq \rho$. By Lemma 3.1 this solution is positive and the proposition is proved. □

We are ready to prove our final main result.

Proof of Theorem 1.2. Let $\mu^* > 0$ be given by Lemma 2.1 and $0 < \mu < \mu^*$. Using Proposition 2.3, we obtain a first positive solution with negative energy. In order to obtain the second one, we denote by $(u_k)_{k\geq k^*} \subset X$ the sequence of positive solutions given by Proposition 3.4. As in Lemma 3.3, we can prove that

$$c_{\mu,k} = I_{\mu,k}(u_k) - \frac{1}{2^*}I'_{\mu,k}(u_k)u_k \geq \alpha_0\|u_k\|^2 - M_1\mu^\theta - M_2\frac{\mu}{k^q},$$

where $\alpha_0 > 0$ was defined in (3.4) and M_1, M_2 come from Lemma 3.3. The above inequality and (3.11) imply that $(u_k) \subset X$ is bounded.

Up to a subsequence, we may assume that $u_k \rightharpoonup u$ weakly in X, as $k \to +\infty$, and an analogous of (2.8) holds. Arguing as in the proof of Proposition 2.3, we can prove that $I'_\mu(u) = 0$. Moreover, for each compact set $\Sigma \subset \mathbb{R}^N$, it follows from the proof of Lemma 3.1 that $u_k(x) \geq C_\Sigma$, for some $C_\Sigma > 0$ independent of k. Thus, we infer from the pointwise convergence of (u_k) that $u \geq C_\Sigma > 0$ a.e. in the (arbitrary) set Σ, and therefore u is a solution for (P_μ).

In order to guarantee that u is different from the first solution, we shall prove that $I_\mu(u) > 0$. We first notice that, arguing as in Lemma 3.3 and using $u \neq 0$, we get

$$I_\mu(u) \geq \alpha_0 \|u\|^2 - M_1 \mu^\theta > -M_1 \mu^\theta. \tag{3.12}$$

By setting $v_k := u_k - u$, using Brezies-Lieb lemma, $I_\mu(u_k)u_k = 0$ and repeating the calculations of Lemma 3.3, we obtain

$$o_k(1) = \|v_k\|^2 - \|v_k\|_{2^*}^{2^*} + I'_\mu(u)u + o_k(1),$$

and therefore, for some $l \geq 0$, there holds

$$\lim_{k \to +\infty} \|v_k\|^2 = l = \lim_{k \to +\infty} \|v_k\|_{2^*}^{2^*}.$$

Thus, we can use (3.11) and the same argument employed in the proof of Lemma 3.3 to obtain

$$c_0 - M_1 \mu^\theta - M_2 \frac{\mu}{k^q} > I_{\mu,k}(u_k) = \frac{1}{N} l + I_\mu(u) + o_k(1).$$

If $l > 0$, then $l \geq S_{2^*}^{N/2}$ and we can pass to the limit as $k \to +\infty$, use (3.1) and (3.12) to obtain

$$c_0 - M_1 \mu^\theta \geq \frac{1}{N} S_{2^*}^{2/N} + I_\mu(u) > c_0 - M_1 \mu^\theta,$$

which does not make sense. Hence, $l = 0$ and therefore $u_k \to u$ strongly in X. This implies that

$$\rho \leq I_{\mu,k}(u_k) = I_\mu(u) + o_k(1),$$

and therefore $I_\mu(u) \geq \rho > 0$. The theorem is proved. □

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