A SHARP HARDY-SOBOLEV INEQUALITY WITH BOUNDARY TERM AND APPLICATIONS

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Abstract. In this paper, we state a Hardy-Sobolev type inequality with boundary terms in a borderline case. As an application, we investigate the existence of solutions for a class of zero-mass quasilinear elliptic problem of the form

$$\begin{cases} -\operatorname{div}(a(x)|\nabla u|^{N-2}\nabla u) = k(x)f(u) & \text{in } \Omega, \\ a(x)|\nabla u|^{N-2}(\nabla u\cdot\nu) + |u|^{N-2}u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega\subset\mathbb{R}^N$, $N\geq 2$, is an exterior domain, the weight functions a,k satisfy some growth conditions and the nonlinearity f has critical exponential growth.

1. Introduction and main results

As it is well-known, Hardy type inequalities have been widely used in the study of differential equations. In [10, 15], the authors have proved a Hardy-Sobolev type inequality in unbounded domains. Precisely, for any $1<p<N$ and $\Omega\subset\mathbb{R}^N$ an unbounded domain, there exists $C>0$ such that

$$\int_\Omega \frac{|u|^p}{(1+|x|)^p}dx \leq C\left(\int_\Omega |\nabla u|^p dx + \int_{\partial\Omega}\frac{|\nu\cdot x|}{(1+|x|)^p}|u|^p d\sigma\right),$$

where ν is the unit outward normal vector to $\partial\Omega$. This inequality has been extensively used in the study of quasilinear elliptic equations in unbounded domain like

$$(1.1)\qquad \begin{cases} -\operatorname{div}(a(x)|\nabla u|^{p-2}\nabla u) + b(x)|u|^{p-2}u = f(x,u) & \text{in } \Omega, \\ a(x)|\nabla u|^{p-2}(\nabla u\cdot\nu) + c(x)|u|^{p-2}u = 0 & \text{on } \partial\Omega. \end{cases}$$

We refer the interest reader to [3, 10, 12, 14, 15] for the case $b\not\equiv 0$. When $b\equiv 0$, we say that we are in the zero-mass case and the problem seems to be more difficult, since $W^{1,N}(\Omega)$ is not the natural space to look for solutions. We quote the paper [9], where the authors considered $1<p<N$, $b\equiv 0$ and a sign-changing nonlinearity f with polynomial growth.

In this paper, we aim to consider a zero-mass problem in the borderline case $p=N$. More precisely, we address the existence of solutions for the quasilinear

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elliptic problem

$$(\mathcal{P}) \qquad \begin{cases} -\mathrm{div}(a(x)|\nabla u|^{N-2}\nabla u) = k(x)f(u) & \text{in } \Omega, \\ a(x)|\nabla u|^{N-2}\,(\nabla u\cdot\nu) + |u|^{N-2}u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $N \geq 2$, $\Omega \subset \mathbb{R}^N$ is an open set satisfying the assumption

($*$) $\mathbb{R}^N \setminus \Omega$ is bounded and $0 \notin \overline{\Omega}$,

which will be assumed throughout, the nonlinearity f has critical exponential growth, and the potentials a, k verify

(a_0) $a : \overline{\Omega} \to \mathbb{R}$ is a continuous function and there exist $a_0, \gamma > 0$ such that

$$a_0|x|^\gamma \leq a(x), \quad \text{for any } x \in \Omega;$$

(k_0) $k : \Omega \to \mathbb{R}$ is a measurable function and there exist $k_0 > 0$, $\beta \geq N$ such that

$$0 < k(x) \leq \frac{k_0}{(1+|x|)^\beta}, \quad \text{for a.e. } x \in \Omega.$$

The starting point to address the existence of weak solutions for the variational borderline problem ($\mathcal{P}$) is a new Hardy-Sobolev inequality with boundary term. In order to present it, we denote by $C_\delta^\infty(\Omega)$ the space of $C_0^\infty(\mathbb{R}^N)$-functions restricted to Ω.

In [18, Theorem 3.1], the authors proved a Hardy-Sobolev type inequality with boundary term. Precisely, by assuming ($*$) and $a < (N-2)/2$ with $N \geq 3$, they proved that there exists a constant $C > 0$ (depending on Ω) such that, for any $u \in C_\delta^\infty(\Omega)$, there holds

$$\frac{(N-2-2a)^2}{4}\int_\Omega \frac{|u|^2}{|x|^{2(a+1)}}dx \leq \int_\Omega \frac{|\nabla u|^2}{|x|^{2a}}dx + C\int_{\partial\Omega}|u|^2 d\sigma. \tag{1.2}$$

Thus, a natural question is whether or not (1.2) is true in the borderline case $N = 2$. By performing a new argument we are able to prove the following result.

Theorem 1.1 (Hardy-Sobolev inequality)**.** *Suppose that $\gamma \neq 0$, $N \geq 2$ and ($*$) holds. Then there exists a constant $C_0 = C_0(\Omega) > 0$ such that*

$$\int_\Omega |x|^{\gamma-N}|u|^N\,dx \leq C_0\left(\int_\Omega |x|^\gamma|\nabla u|^N dx + \int_{\partial\Omega}|u|^N d\sigma\right), \quad \forall\, u \in C_\delta^\infty(\Omega). \tag{1.3}$$

It is worth to notice that the above conclusion can fail if $\gamma = 0$. Actually, we present in Remark 2.2 an interesting example in the case that the set Ω is the complement of a ball.

We now come back to our differential equation. Under the conditions (a_0) and (k_0), we shall look for weak solutions for ($\mathcal{P}$) in the space E_a defined as the completion of $C_\delta^\infty(\Omega)$ with respect to the norm

$$\|u\|_{E_a} := \left(\int_\Omega a(x)|\nabla u|^N dx + \int_{\partial\Omega}|u|^N d\sigma\right)^{1/N}.$$

We are going to prove that E_a embedds into the weighted Sobolev space L_k^q, for any $q \geq N$ (see Proposition 2.3), and that

$$\int_\Omega k(x)\Phi_\alpha(u)\,dx < +\infty, \quad \text{for any } \alpha > 0,\, u \in E_a,$$

where

$$\Phi_\alpha(s) := e^{\alpha|s|^{N/(N-1)}} - \sum_{j=0}^{N-2} \frac{\alpha^j}{j!}|s|^{Nj/(N-1)}, \quad \text{for all } s \in \mathbb{R}.$$

As a consequence, we are able to use Theorem 1.1 and some trick calculations to prove a Trudinger-Moser type inequality (see Lemma 4.1) in the space E_a . Hence, we may consider nonlinearities f which behave like $e^{\alpha|u|^{N/(N-1)}}$ at infinity. More specifically, we shall assume that

(f_0) $f : \mathbb{R} \to \mathbb{R}$ is continuous and there exists $\alpha_0 > 0$ such that

$$\lim_{|s|\to+\infty} \frac{|f(s)|}{e^{\alpha|s|^{N/(N-1)}}} = \begin{cases} 0 & \text{if} \quad \alpha > \alpha_0, \\ +\infty & \text{if} \quad \alpha < \alpha_0; \end{cases}$$

(f_1) $f(s) = o(|s|^{N-1})$ as $s \to 0$;

(f_2) there exists $\theta > N$ such that, for any $s \in \mathbb{R}$,

$$0 < \theta F(s) := \theta \int_0^s f(t)dt \le f(s)s;$$

(f_3) there exist $\lambda > 0$ and $\nu > N$ such that, for any $s \in \mathbb{R}$,

$$F(s) \ge \lambda |s|^\nu.$$

Our existence result for problem ($\mathcal{P}$) can be stated as follows:

Theorem 1.2. *Suppose that (a_0), (k_0), and $(f_0) - (f_2)$ hold. Then there exists $\lambda^* > 0$ such that, if (f_3) holds for $\lambda \ge \lambda^*$, then the problem ($\mathcal{P}$) has a nonzero weak solution.*

For the proof, we apply the Mountain Pass Theorem. Although the general approach is in some sense standard, it is necessary to construct all the variational setting. Actually, the abstract framework presented here can be used to deal with many other type of problems involving Robin boundary condition. Our main difficulties rely on the fact that we are dealing with the zero-mass case, the domain Ω may be not symmetric, the Hardy-Sobolev inequality generally holds for $1 < p < N$ and, as far we know, there is no appropriated Trudinger-Moser inequality for our case. So, our paper complements all the aforementioned works as well as the papers [7, 1, 4], where some related problems were considered with $N = 2$. Finally, we emphasize that our results seem to be new even in the planar (and therefore semilinear) case.

The remainder of the paper is organized as follows: in Section 2, we prove Theorem 1.1 and some useful Sobolev embeddings. In Section 3, we prove a weighted Trudinger-Moser type inequality. Finally, in Section 4, we present the proof of Theorem 1.2.

2. A Hardy-Sobolev inequality and Sobolev embeddings

We start this section by proving our Hardy-Sobolev inequality. We write $B_R(x_0)$ for the open ball of radius $R > 0$ centered at the $x_0 \in \mathbb{R}^N$. When $x_0 = 0$, we write only B_R.

We can prove our first result as follows:

Proof of Theorem 1.1. Let $\alpha \neq -N$ and $u \in C_\delta^\infty(\Omega)$ be fixed. If $\nu = (\nu_1, \nu_2, \dots, \nu_N)$ is the unit outward normal vector at $x \in \partial\Omega$, from the Divergence Theorem we obtain

$$\int_\Omega (|x|^\alpha)_{x_i} \cdot x_i |u|^N \, dx = -\int_\Omega |x|^\alpha \cdot (x_i|u|^N)_{x_i} \, dx + \int_{\partial\Omega} |x|^\alpha |u|^N x_i \nu_i \, d\sigma.$$

By summing for $i = 1, \dots, N$, we get

$$(\alpha + N) \int_\Omega |x|^\alpha |u|^N \, dx = -N \int_\Omega |x|^\alpha |u|^{N-2} u (x \cdot \nabla u) \, dx + \int_{\partial\Omega} |x|^\alpha |u|^N (x \cdot \nu) \, d\sigma,$$

and therefore

$$|\alpha + N| \int_\Omega |x|^\alpha |u|^N \, dx \leq N \int_\Omega |x|^{\alpha+1} |u|^{N-1} |\nabla u| \, dx + \int_{\partial\Omega} |x|^{\alpha+1} |u|^N \, d\sigma. \tag{2.1}$$

Given $\varepsilon > 0$, we can use Young's inequality to get

$$\begin{aligned} N \int_\Omega |x|^{\alpha+1} |u|^{N-1} |\nabla u| \, dx &= N \int_\Omega \left(|x|^{\alpha(N-1)/N} |u|^{N-1} \right) |x|^{[\alpha+1-\alpha(N-1)/N]} |\nabla u| \, dx \\ &\leq (N-1)\varepsilon \int_\Omega |x|^\alpha |u|^N \, dx + \frac{1}{\varepsilon^{N-1}} \int_\Omega |x|^{\alpha+N} |\nabla u|^N \, dx. \end{aligned}$$

If $\varepsilon \leq 1$, we can use the above inequality and (2.1) to obtain

$$[|\alpha+N| - (N-1)\varepsilon] \int_\Omega |x|^\alpha |u|^N dx \leq \frac{1}{\varepsilon^{N-1}} \left(\int_\Omega |x|^{\alpha+N} |\nabla u|^N dx + \int_{\partial\Omega} |x|^{\alpha+1} |u|^N d\sigma \right).$$

By recalling that $\alpha \neq -N$ and picking

$$0 < \varepsilon < \min \left\{ 1, \frac{|\alpha + N|}{(N-1)} \right\},$$

we get

$$\int_\Omega |x|^\alpha |u|^N \, dx \leq C_1 \left(\int_\Omega |x|^{\alpha+N} |\nabla u|^N \, dx + \int_{\partial\Omega} |x|^{\alpha+1} |u|^N \, d\sigma \right),$$

where $C_1 := [|\alpha + N| - (N-1)\varepsilon]^{-1} \varepsilon^{1-N}$. By choosing $\alpha = \gamma - N \neq -N$ in the above expression and using that $\partial\Omega$ is bounded, we obtain (1.3). The theorem is proved. □

For each $\gamma \neq 0$, we denote by $E^{1,\gamma}$ the space obtained as the completion of $C_\delta^\infty(\Omega)$ with respect to the norm

$$\|u\|_{E^{1,\gamma}} := \left(\int_\Omega \left[|x|^\gamma |\nabla u|^N + |x|^{\gamma-N} |u|^N \right] dx \right)^{1/N}.$$

As a consequence of Theorem 1.1, we obtain a second result which will play an important role in the study of the zero-mass case $b \equiv 0$ in (1.1).

Corollary 2.1. *If $\gamma \neq 0$, then the norms*

$$\|u\|_\partial := \left(\int_\Omega |x|^\gamma |\nabla u|^N dx + \int_{\partial\Omega} |u|^N d\sigma \right)^{1/N}$$

and $\|\cdot\|_{E^{1,\gamma}}$ are equivalents in $E^{1,\gamma}$.

Proof. It follows from (1.3) that, for any $u \in C_\delta^\infty(\Omega)$, one has

$$\|u\|_{E^{1,\gamma}}^N \leq \int_\Omega |x|^\gamma |\nabla u|^N dx + C_0 \left(\int_\Omega |x|^\gamma |\nabla u|^N dx + \int_{\partial\Omega} |u|^N d\sigma \right) \leq C_1 \|u\|_\partial^N,$$

for some constant $C_1 = C_1(\Omega) > 0$. On the other hand, since $\partial\Omega$ is bounded, we can choose $R > 0$ sufficiently large in such a way that the Sobolev trace embedding $W^{1,N}(\Omega \cap B_R) \hookrightarrow L^N(\partial\Omega \cup \partial B_R)$ is continuous. Therefore,

$$\begin{aligned}\int_{\partial\Omega} |u|^N d\sigma &\leq C_2 \int_{\Omega \cap B_R} \left(|\nabla u|^N + |u|^N \right) dx \\ &\leq C_3 \left(\int_{\Omega \cap B_R} \left[|x|^\gamma |\nabla u|^N + |x|^{\gamma - N} |u|^N \right] dx \right),\end{aligned}$$

with $C_3 = C_3(R, N, \gamma) > 0$ and we have used that $0 \notin \overline{\Omega}$ and $\partial\Omega$ is bounded. It follows from the above expression that

$$\|u\|_\partial^N \leq \int_\Omega |x|^\gamma |\nabla u|^N dx + C_3 \int_\Omega \left[|x|^\gamma |\nabla u|^N + |x|^{\gamma-N} |u|^N \right] dx \leq (1 + C_3) \|u\|_{E^{1,\gamma}}^N,$$

which gives the desired result. □

Remark 2.2. *If Ω is a bounded domain, then* (1.3) *holds for $\gamma = 0$, see for instance* [5, inequality (12)] *and the references* [2, 18]. *On the other hand, if $\gamma = 0$ and $\Omega = \mathbb{R}^N \setminus \overline{B_1}$, then the inequality in* (1.3) *fails in the space $E^{1,\gamma}$. Indeed, by considering the sequence of functions in $E^{1,\gamma}$ defined by*

$$u_n(x) := \begin{cases} n - \log |x|, & \text{if} \quad 1 \leq |x| \leq e^n, \\ 0, & \text{if} \quad |x| \geq e^n, \end{cases}$$

we see that

$$\int_\Omega |\nabla u_n|^N \, dx = \int_{B_{e^n} \setminus \overline{B_1}} |x|^{-N} \, dx = \omega_{N-1} \int_1^{e^n} r^{-N} r^{N-1} \, dr = n\omega_{N-1},$$

where ω_{N-1} is the measure of the unit sphere in $\mathbb{R}^N$. On the other hand,

$$\begin{aligned}\int_\Omega |x|^{-N} |u_n|^N dx &= \int_{B_{e^n} \setminus \overline{B_1}} |x|^{-N} |n - \log |x||^N dx \\ &= \omega_{N-1} \int_1^{e^n} r^{-N} |n - \log r|^N r^{N-1} dr.\end{aligned}$$

By considering the change of variables $t = n - \log r$, we obtain

$$\int_\Omega |x|^{-N} |u_n|^N dx = \frac{n^{N+1}}{N+1} \omega_{N-1}.$$

Moreover,

$$\int_{\partial\Omega} |u_n|^N d\sigma = n^N \int_{\partial\Omega} d\sigma = n^N \omega_{N-1}.$$

Using the above inequalities we see that, if (1.3) *holds, then*

$$n^{N+1} \leq C_1(n + n^N),$$

for all $n \in \mathbb{N}$ and some $C_1 > 0$, which is impossible.

Given a positive function $\omega \in L^1_{loc}(\Omega)$ and $s \geq 1$, we define the weighted Lebesgue space

$$L^s_\omega := \left\{ u \in L^1_{loc}(\Omega) \,:\, \|u\|_{L^s_\omega} := \left(\int_\Omega \omega(x)|u|^s \, dx \right)^{1/s} < +\infty \right\},$$

and prove the following:

Proposition 2.3 (Sobolev inequality)**.** *Suppose that $\gamma > 0$, $\beta \geq N - \gamma$ and $q \geq N \geq 2$. Then, there exists $C = C(q, \Omega) > 0$ such that*

$$\int_\Omega \frac{|u|^q}{(1+|x|)^\beta} dx \leq C \left(\int_\Omega \left[|x|^\gamma |\nabla u|^N + |x|^{\gamma - N} |u|^N \right] dx \right)^{q/N}, \quad \forall\, u \in E^{1,\gamma}, \tag{2.2}$$

that is, the Sobolev embedding $E^{1,\gamma} \hookrightarrow L^q_{(1+|\cdot|)^{-\beta}}$ is continuous. Furthermore, this embedding is compact whenever $\beta > N - \gamma$.

Proof. Let $j_0 \in \mathbb{N}$ be such that $(\mathbb{R}^N \setminus \Omega) \subset B_{2^{j_0}}$. Setting $\Omega_{j_0} := \Omega \cap B_{2^{j_0}}$, we have that $\Omega = \Omega_{j_0} \cup (\mathbb{R}^N \setminus B_{2^{j_0}})$. Given $u \in E^{1,\gamma} \subset W^{1,N}_{loc}(\Omega)$, we can use the Sobolev embedding $W^{1,N}(\Omega_{j_0}) \hookrightarrow L^q(\Omega_{j_0})$ to obtain

$$\int_{\Omega_{j_0}} \frac{|u|^q}{(1+|x|)^\beta} dx \leq C_1 \int_{\Omega_{j_0}} |u|^q dx \leq C_2 \left(\int_{\Omega_{j_0}} \left[|\nabla u|^N + |u|^N \right] dx \right)^{q/N}.$$

By recalling that $0 \notin \overline{\Omega}$, we can write

$$\int_{\Omega_{j_0}} \frac{|u|^q}{(1+|x|)^\beta} dx \leq C_3 \left(\int_{\Omega_{j_0}} \left[|x|^\gamma |\nabla u|^N + |x|^{\gamma - N} |u|^N \right] dx \right)^{q/N}, \tag{2.3}$$

for some $C_3 = C_3(\gamma, \Omega) > 0$.

On the other hand, for any $j \in \mathbb{N} \cup \{0\}$, we have that

$$A_j := \{ z \in \Omega \,:\, 2^{j_0} \cdot 2^j < |z| < 2^{j_0} \cdot 2^{j+1} \} = B_{2^{j_0+j+1}} \setminus \overline{B}_{2^{j_0+j}}.$$

Without loss generality we may assume $\beta > 0$. The change of variables $y := 2^{-j}x$ provides

$$\int_{A_j} \frac{|u|^q}{(1+|x|)^\beta} dx \leq \frac{1}{2^{\beta j}} \int_{A_j} |u|^q dx = 2^{(N-\beta)j} \int_{A_0} |u_j(y)|^q dy,$$

where $u_j(y) := u(2^j y)$. Using the Sobolev embedding $W^{1,N}(A_0) \hookrightarrow L^q(A_0)$, we obtain $C_4 > 0$, such that

$$\begin{aligned} \int_{A_0} |u_j(y)|^q dy &\leq C_4 \left(\int_{A_0} \left[|\nabla u_j(y)|^N + |u_j(y)|^N \right] dy \right)^{q/N} \\ &= C_4 \left(\int_{A_j} \left[|\nabla u|^N + 2^{-Nj} |u|^N \right] dx \right)^{q/N}. \end{aligned}$$

We now notice that

$$\int_{A_j} |\nabla u|^N dx = \int_{A_j} |x|^{-\gamma} |x|^\gamma |\nabla u|^N dx \leq 2^{-\gamma j} \int_{A_j} |x|^\gamma |\nabla u|^N dx$$

and

$$\int_{A_j} 2^{-Nj} |u|^N dx \leq 2^{(j_0+1)N} \cdot 2^{-\gamma j} \int_{A_j} |x|^{\gamma - N} |u|^N dx.$$

Consequently, for $C_5 = C_4 \cdot 2^{(j_0+1)N}$, we have that

$$\int_{A_j} \frac{|u|^q}{(1+|x|)^\beta} dx \le C_5 2^{(N-\beta)j} \left(2^{-\gamma j} \int_{A_j} \left[|x|^\gamma |\nabla u|^N + |x|^{\gamma - N} |u|^N \right] dx \right)^{q/N}$$
$$= C_5 2^{\mu_j} \left(\int_{A_j} \left[|x|^\gamma |\nabla u|^N + |x|^{\gamma - N} |u|^N \right] dx \right)^{q/N}, \tag{2.4}$$

where

$$\mu_j := \left(N - \beta - \frac{\gamma q}{N} \right) j. \tag{2.5}$$

Since $\gamma > 0$ and $\beta \ge N - \gamma$, one has $\mu_j \le 0$, and therefore

$$\int_{A_j} \frac{|u|^q}{(1+|x|)^\beta} dx \le C_5 \left(\int_{A_j} \left[|x|^\gamma |\nabla u|^N + |x|^{\gamma - N} |u|^N \right] dx \right)^{q/N}.$$

Thus, recalling that the function $s \mapsto s^{q/N}$ is super additive for $q \ge N$, we conclude that

$$\sum_{j=0}^\infty \int_{A_j} \frac{|u|^q}{(1+|x|)^\beta} dx \le C_5 \sum_{j=0}^\infty \left(\int_{A_j} \left[|x|^\gamma |\nabla u|^N + |x|^{\gamma - N} |u|^N \right] dx \right)^{q/N}$$
$$\le C_5 \left(\int_{\mathbb{R}^N \setminus B_{2^{j_0}}} \left[|x|^\gamma |\nabla u|^N + |x|^{\gamma - N} |u|^N \right] dx \right)^{q/N}.$$

This, combined with (2.3), implies

$$\int_\Omega \frac{|u|^q}{(1+|x|)^\beta} dx \le C_6 \|u\|^q_{E^{1,\gamma}},$$

which proves (2.2).

For the compactness, we consider a sequence $(u_n) \subset E^{1,\gamma}$ such that $u_n \rightharpoonup 0$ weakly in $E^{1,\gamma}$. Given $\varepsilon > 0$, we can use $\gamma > 0$ and the fact that $\beta > N - \gamma$ to obtain $j_1 \in \mathbb{N}$ such that $2^{\mu_j} < \varepsilon$, for all $j \ge j_1$. Thus, from (2.4), we get

$$\int_{A_j} \frac{|u_n|^q}{(1+|x|)^\beta} dx < C_5 \varepsilon \left(\int_{A_j} \left[|x|^\gamma |\nabla u_n|^N + |x|^{\gamma - N} |u_n|^N \right] dx \right)^{q/N},$$

for any $j \ge j_1$. Hence, from the embedding $E^{1,\gamma} \subset W^{1,N}_{loc}(\Omega)$ and the Rellich-Kondrachov Theorem, we obtain

$$\int_\Omega \frac{|u_n|^q}{(1+|x|)^\beta} dx \le \int_{\Omega_{j_0}} \frac{|u_n|^q}{(1+|x|)^\beta} dx + \sum_{j=0}^{j_1} \int_{A_j} \frac{|u_n|^q}{(1+|x|)^\beta} dx + C_5 \varepsilon \|u_n\|^q_{E^{1,\gamma}}$$
$$= o_n(1) + C_5 \varepsilon \|u_n\|^q_{E^{1,\gamma}},$$

where $o_n(1)$ stands for a quantity approaching zero as $n \to +\infty$. Since $\varepsilon > 0$ is arbitrary, the above expression implies that $u_n \to 0$ strongly in $L^q_{(1+|\cdot|)^{-\beta}}$ and the proposition is proved. $\square$

Remark 2.4. *The embedding $E^{1,\gamma} \hookrightarrow L^q_{(1+|\cdot|)^{-\beta}}$ is also continuous if $\gamma < 0$, $\beta \ge N - \gamma$ and $N \le q \le N(N-\beta)/\gamma$. The proof of this statement can be done as* *above since, in this case, a simple calculation shows that the number* μ_j *defined in* (2.5) *is nonpositive.*

3. Trudinger–Moser type inequality

In view of the Proposition 2.3, it is natural to look for embedding into Orlicz spaces. As we will see, this allows us to consider functions with exponential growth in problem ($\mathcal{P}$). For any $\alpha > 0$, we recall the Young function defined in the introduction

$$\Phi_\alpha(s) := e^{\alpha|s|^{N/(N-1)}} - \sum_{j=0}^{N-2} \frac{\alpha^j}{j!}|s|^{Nj/(N-1)}, \quad \text{for all } s \in \mathbb{R}.$$

If follows from the definition that

$$\Phi_\alpha(ts) = \phi_{\alpha t^{N/(N-1)}}(s), \quad s \in \mathbb{R},\, t > 0. \tag{3.1}$$

We state in the sequel the main result of this section.

Theorem 3.1 (Trudinger-Moser inequality)**.** *Suppose that* $\gamma > 0$ *and* $\beta \geq N$. *Then, for any* $\alpha > 0$ *and* $u \in E^{1,\gamma}$, *the function* $(1+|\cdot|)^{-\beta}\Phi_\alpha(u)$ *belongs to* $L^1(\Omega)$. *Moreover, there exists* $\alpha^* = \alpha^*(N) > 0$ *such that*

$$L(\alpha,\gamma,\beta) := \sup_{\{u\in E^{1,\gamma}:\|u\|_{E^{1,\gamma}}\leq 1\}} \int_\Omega \frac{\Phi_\alpha(u)}{(1+|x|)^\beta}\, dx < +\infty,$$

for any $0 < \alpha \leq \alpha^*$. *Furthermore, there exists* $\alpha^{**} > \alpha^*$ *such that*

$$L(\alpha,\gamma,\beta) = +\infty, \quad \text{for any } \alpha > \alpha^{**}. \tag{3.2}$$

For the proof of Theorem 3.1, we need two technical results.

Lemma 3.2. *Let* $x_0 \in \mathbb{R}^N$ *and* $u \in W_0^{1,N}(B_R(x_0))$ *be such that* $\int_{B_R(x_0)} |\nabla u|^N dx \leq 1$. *Then, there exists* $C = C(N) > 0$ *such that*

$$\int_{B_R(x_0)} \Phi_{\alpha_N}(u)dx \leq C(N)\cdot R^N \int_{B_R(x_0)} |\nabla u|^N dx,$$

where $\alpha_N := N\omega_{N-1}^{1/(N-1)}$ *with* ω_{N-1} *denoting the measure of the unit sphere in* $\mathbb{R}^N$.

Proof. See [19, Lemma 3.1]. □

The second auxiliary result reads as

Lemma 3.3. *Suppose that* $\gamma > 0$ *and* $\beta \geq N$. *Then, there exist* $C_N > 0$ *and* $\alpha^* = \alpha^*(N) > 0$ *such that*

$$\int_\Omega \frac{\Phi_\alpha(u)}{(1+|x|)^\beta}\, dx \leq C_N \int_\Omega \left[|x|^\gamma|\nabla u|^N + |x|^{\gamma-N}|u|^N\right] dx,$$

for any $0 < \alpha \leq \alpha^*$ *and* $u \in E^{1,\gamma}$ *verifying* $\|u\|_{E^{1,\gamma}} \leq 1$.

Proof. Let $j_0 \in \mathbb{N}$ and Ω_{j_0} as in the proof of Proposition 2.3. For each $y \in \Omega_{j_0}$, set $R_y := \operatorname{dist}(y, \partial\Omega_{j_0})$ and notice that $B_{R_y}(y) \subset \Omega_{j_0}$. Moreover, from the compactness of $\overline{\Omega_{j_0}}$, we obtain points $y_1, \ldots, y_k \in \Omega_{j_0}$ such that $\Omega_{j_0} \subset \bigcup_{i=1}^k B_{R_i/2}(y_i)$, where $R_i := R_{y_i}$. For each $i = 1, \ldots, k$, we set $B^i := B_{R_i}(y_i)$ and pick a function $\varphi_i \in C_0^\infty(B^i)$ such that $0 \leq \varphi_i \leq 1$, $\varphi_i \equiv 1$ in $B_{R_i/2}(y_i)$ and $|\nabla\varphi_i| \leq 4/R_i$ in B^i.

Since $E^{1,\gamma} \subset W^{1,N}_{loc}(\Omega)$, we have that $\varphi u \in W^{1,N}_0(B_i)$, for any $u \in E^{1,\gamma}$. So, by Poincaré's inequality, we get

$$\begin{aligned}\int_{B^i} |\nabla(\varphi_i u)|^N dx &\le C_1 \int_{B^i} |\nabla u|^N dx + C_1 R_i^{-N} \int_{B^i} |u|^N dx \\ &\le C_2 \int_{B^i} \left[|x|^\gamma |\nabla u|^N + |x|^{\gamma-N}|u|^N\right] dx,\end{aligned}$$

where $C_2 = C_2(N,\gamma) > 0$.

We now set $v := (1/C_2)^{1/N} \varphi_i u$ and suppose that

$$0 < \alpha \le \frac{\alpha_N}{C_2^{1/(N-1)}}.$$

Since $\varphi_i \equiv 1$ in $B_{R_i/2}(y_i)$ and $\Phi_\alpha \ge 0$ is monotonic in α, it follows from (3.1) and Lemma 3.2 that

$$\begin{aligned}\int_{B_{R_i/2}(y_i)} \Phi_\alpha(u) dx &= \int_{B_{R_i/2}(y_i)} \Phi_\alpha(\varphi_i u) dx \le \int_{B^i} \Phi_\alpha\left(C_2^{1/N} v\right) dx \\ &= \int_{B^i} \Phi_{\alpha C_2^{1/(N-1)}}(v) dx \le \int_{B^i} \Phi_{\alpha_N}(v) dx \\ &\le C_3 \int_{B^i} \left[|x|^\gamma |\nabla u|^N + |x|^{\gamma-N}|u|^N\right] dx,\end{aligned}$$

for some $C_3 = C_3(\gamma)$. Therefore, there exists $C_4 = C_4(\beta) > 0$ such that

$$\int_{\Omega_{j_0}} \frac{\Phi_\alpha(u)}{(1+|x|)^\beta} dx \le C_4 \sum_{i=1}^k \int_{B_{R_i/2}(y_i)} \Phi_\alpha(u) dx \le C_4 \cdot C_3 \|u\|^N_{E^{1,\gamma}}. \tag{3.3}$$

By considering again the annulus $A_j = \{z \in \Omega : 2^{j_0} \cdot 2^j < |z| < 2^{j_0} \cdot 2^{j+1}\}$, we claim that

$$\int_{A_j} \frac{\Phi_\alpha(u)}{(1+|x|)^\beta} dx \le C_5 \int_{A_j} \left[|x|^\gamma |\nabla u|^N + |x|^{\gamma-N}|u|^N\right] dx, \tag{3.4}$$

for any $j \in \mathbb{N} \cup \{0\}$ and some $C_5 > 0$. If this is true, the statement of the lemma is a direct consequence of this inequality, (3.3) and

$$\int_\Omega \frac{\Phi_\alpha(u)}{(1+|x|)^\beta} dx = \left(\int_{\Omega_{j_0}} + \sum_{j=0}^\infty \int_{A_j}\right) \frac{\Phi_\alpha(u)}{(1+|x|)^\beta} dx.$$

It remains to be proved that (3.4) holds. In order to do that, we use the change of variables $y = 2^{-j}x$ to obtain

$$\int_{A_j} \frac{\Phi_\alpha(u)}{(1+|x|)^\beta} dx \le \frac{C_6}{2^{\beta j}} \int_{A_j} \Phi_\alpha(u) dx = C_6 2^{(N-\beta)j} \int_{A_0} \Phi_\alpha(u_j)\, dy, \tag{3.5}$$

where $u_j(y) := u(2^j y)$ and $C_6 > 0$ is a constant independent of j. Arguing as before, we obtain points $y_1, \ldots, y_k \in A_0$ such that $A_0 \subset \bigcup_{i=1}^k B_{R_i/2}(y_i)$, where $R_i = \operatorname{dist}(y_i, \partial A_j)$. By setting $B^i = B_{R_i/2}(y_i)$, we pick $\varphi_i \in C_0^\infty(B^i)$ such that

$0 \le \varphi_i \le 1$, $\varphi_i \equiv 1$ in $B_{R_i/2}(y_i)$ and $|\nabla \varphi_i| \le 4/R_i$ in B^i, and compute

$$\begin{aligned}\int_{B^i} |\nabla (\varphi_i(y)u_j(y))|^N dy &\le C_7 \int_{B^i} |\nabla u_j(y)|^N dy + C_7 R_i^{-N} \int_{B^i} |u_j(y)|^N dy \\ &\le C_7 \int_{A_0} |\nabla u(2^j y)|^N 2^{jN} dy + C_7 R_i^{-N} \int_{A_0} |u(2^j y)|^N dy \\ &= C_7 \int_{A_j} |\nabla u|^N dx + \frac{C_7}{R_i^N} 2^{-Nj} \int_{A_j} |u|^N dx.\end{aligned}$$

But, as in the proof of Proposition 2.3, we have that

$$\int_{A_j} |\nabla u|^N dx \le C_8 2^{-\gamma j} \int_{A_j} |x|^\gamma |\nabla u|^N dx,$$

and

$$2^{-Nj} \int_{A_j} |u|^N dx \le C_9 2^{-\gamma j} \int_{A_j} |x|^{\gamma - N} |u|^N dx,$$

with $C_8 = C_8(\gamma) > 0$ and $C_9 = C_9(N, \gamma) > 0$. Recalling that $\gamma > 0$, one deduces

$$\begin{aligned}\int_{B^i} |\nabla (\varphi_i(y)u_j(y))|^N dy &\le C_{10} 2^{-\gamma j} \int_{A_j} \Big[|x|^\gamma |\nabla u|^N + |x|^{\gamma-N}|u|^N\Big] dx \\ &\le C_{10} \int_{A_j} \Big[|x|^\gamma |\nabla u|^N + |x|^{\gamma-N}|u|^N\Big] dx.\end{aligned}$$

Since $\|u\|_{E^{1,\gamma}} \le 1$, the above inequality shows that we can apply Lemma 3.2 with $v := (1/C_{10})^{1/N} \varphi_i u_j$ to obtain $C_{11} = C_{11}(N) > 0$ such that

$$\int_{B^i} \Phi_{\alpha_N}(v) dy \le C_{11} \int_{B^i} |\nabla v|^N dy \le C_{11} \int_{A_j} \Big[|x|^\gamma |\nabla u|^N + |x|^{\gamma-N}|u|^N\Big] dx.$$

Hence, if we define

$$\alpha^* := \min\left\{ \frac{\alpha_N}{C_2^{1/(N-1)}}, \frac{\alpha_N}{C_{10}^{1/(N-1)}} \right\},$$

we can use the definition of v and (3.1) to obtain

$$\int_{B^i} \Phi_{\alpha^*}(\varphi_i u_j) dy \le \int_{B^i} \Phi_{\alpha_N}(v) dy \le C_{11} \int_{A_j} \Big[|x|^\gamma |\nabla u|^N + |x|^{\gamma-N}|u|^N\Big] dx.$$

Thus, for any $0 < \alpha \le \alpha^*$, we can argue as in the first part of the proof to get

$$\begin{aligned}\int_{A_0} \Phi_\alpha(u_j) dy &\le \sum_{i=1}^k \int_{B_{R_i/2}(y_i)} \Phi_\alpha(u_j) dy = \sum_{i=1}^k \int_{B_{R_i/2}(y_i)} \Phi_\alpha(\varphi_i u_j) dy \\ &\le \sum_{i=1}^k \int_{B_{R_i/2}(y_i)} \Phi_{\alpha^*}(\varphi_i u_j) dy \le C_{11} \int_{A_j} \Big[|x|^\gamma |\nabla u|^N + |x|^{\gamma-N}|u|^N\Big] dx.\end{aligned}$$

This, together with (3.5) implies that

$$\int_{A_j} \frac{\Phi_\alpha(u)}{(1+|x|)^\beta} dx \le C_{12} 2^{(N-\beta)j} \int_{A_j} \Big[|x|^\gamma |\nabla u|^N + |x|^{\gamma-N}|u|^N\Big] dx.$$

The inequality in (3.4) is a consequence of the above expression and $\beta \ge N$. □

We are ready to prove the main result of this section.

Proof of Theorem 3.1. If we consider $\alpha^* > 0$ as in Lemma 3.3, we have that

$$\sup_{\{u\in E^{1,\gamma}:\|u\|_{E^{1,\gamma}}\le 1\}} \int_\Omega \frac{\Phi_\alpha(u)}{(1+|x|)^\beta}\,dx < C_N, \tag{3.6}$$

for any $0<\alpha\le\alpha^*$. So, we need only to verify that, for each $u\in E^{1,\gamma}$, the function $(1+|\cdot|)^{-\beta}\Phi_\alpha(u)$ belongs to $L^1(\Omega)$. In order to do this, we pick $u_0\in C_\delta^\infty(\Omega)$ such that

$$\|u-u_0\|_{E^{1,\gamma}}\le\varepsilon,$$

with $\varepsilon>0$ to be chosen later.

A simple computation shows that

$$|\Phi'_\alpha(s)|\le\frac{\alpha N}{N-1}|s|^{1/(N-1)}e^{\alpha|s|^{N/(N-1)}},\quad s\ge 0.$$

Thus, for any $s,\,t\ge 0$, we can use the Mean Value Theorem to obtain $\theta\in[\min\{s,t\},\max\{s,t\}]$ such that

$$\Phi_\alpha(s)\le\Phi_\alpha(t)+\frac{\alpha N}{N-1}|\theta|^{1/(N-1)}e^{\alpha|\theta|^{N/(N-1)}}|t-s|.$$

Using this inequality with $s=|u|$ and $t=|u-u_0|$, we obtain a function $x\mapsto\theta(x)$ such that, for a.e. $x\in\Omega$,

$$\Phi_\alpha(|u|)\le\Phi_\alpha(|u-u_0|)+\frac{\alpha N}{N-1}|\theta(x)|^{1/(N-1)}\psi(x)e^{\alpha|\theta(x)|^{N/(N-1)}}, \tag{3.7}$$

where $\psi:=\Big||u-u_0|-|u|\Big|\in E^{1,\gamma}$ has its support contained in the open bounded set Θ.

We now notice that, by (3.1),

$$\int_\Omega\frac{1}{(1+|x|)^\beta}\Phi_\alpha(|u-u_0|)dx=\int_\Omega\frac{1}{(1+|x|)^\beta}\Phi_{\alpha\|u-u_0\|^{N/(N-1)}_{E^{1,\gamma}}}\left(\frac{|u-u_0|}{\|u-u_0\|_{E^{1,\gamma}}}\right)dx.$$

By choosing $\varepsilon>0$ small, we can use (3.6) to conclude that

$$\int_\Omega\frac{1}{(1+|x|)^\beta}\Phi_\alpha(|u-u_0|)dx<C_N. \tag{3.8}$$

Since u_0 is a bounded function and θ is between $|u-u_0|$ and $|u|$, it is clear that

$$|\theta(x)|\le|u-u_0|+|u|\le C_1(|u|+1),\quad\text{a.e. in }\Theta,$$

for some $C_1>0$. Thus, we can use Hölder's inequality to obtain

$$\begin{aligned}\int_\Theta\frac{1}{(1+|x|)^\beta}|\theta|^{1/(N-1)}\psi e^{\alpha|\theta|^{N/(N-1)}}dx &\le C_2\int_\Theta(|u|+1)^{1/(N-1)}\psi e^{C_3|u|^{N/(N-1)}}dx\\ &\le C_4\left(\int_\Theta e^{r_3C_3|u|^{N/(N-1)}}dx\right)^{1/r_3},\end{aligned}$$

where $C_4:=\|(|u|+1)\|^{1/(N-1)}_{L^{r_1/(N-1)}(\Theta)}\|\psi\|^{r_2}_{L^{r_2}(\Theta)}$ and r_1, r_2, r_3 are such that $1/r_1+1/r_2+1/r_3=1$, $r_1\ge N(N-1)$ and $r_2\ge N$. Since Θ is bounded, it follows from the classical Trudinger-Moser inequality in $W^{1,N}(\Theta)$, see for instance [7, 1, 4], that

$$\int_\Theta\frac{1}{(1+|x|)^\beta}|\theta|^{1/N}\psi e^{\alpha|\theta|^{N/(N-1)}}dx<+\infty.$$

Since $\Phi_\alpha(|u|)=\Phi_\alpha(u)$, we can use (3.7), (3.8) and the above expression to conclude that $(1+|\cdot|)^{-\beta}\Phi_\alpha(u)\in L^1(\Omega)$.

We now prove that (3.2) holds for some $\alpha^{**} > \alpha^*$. Indeed, let $x_0 \in \Omega$ be such that $B = B_1(x_0) \subset \Omega$ and observe that, for some constant $C_5 > 0$, one has

$$L(\alpha,\gamma,\beta) \geq C_5 \sup_{\{u \in W_0^{1,N}(B): \|u\|_{E^{1,\gamma}} \leq 1\}} \int_B \Phi_\alpha(u)dx.$$

On the other hand, for any $u \in W_0^{1,N}(B)$ with $\|u\|_{W_0^{1,N}(B)} \leq 1$, we have

$$\|u\|_{E^{1,\gamma}}^N = \int_B \left[|x|^\gamma |\nabla u|^N + |x|^{\gamma - N}|u|^N\right]dx \leq C_6 \|u\|_{W_0^{1,N}(B)}^N \leq C_6,$$

for some constant $C_6 = C_6(N,\gamma)$. As in [8], defining $v := C_6^{-1}u$, we see that $v \in W_0^{1,N}(B)$ and $\|v\|_{E^{1,\gamma}} \leq 1$. Thus, it follows from (3.1) that

$$L(\alpha,\gamma,\beta) \geq C_7 \int_B \Phi_\alpha(C_6^{-1}u)dx = C_7 \int_B \Phi_{\alpha C_6^{-N/(N-1)}}(u)dx.$$

Since $\Psi_\alpha(s) \geq e^{\alpha|s|^{N/(N-1)}}$, for any $s \in \mathbb{R}$, we obtain that

$$L(\alpha,\gamma,\beta) \geq C_7 \int_B e^{\alpha C_6^{-N/(N-1)}|u|^{N/(N-1)}}dx, \quad \text{for any } u \in W_0^{1,N}(B).$$

Consequently,

$$L(\alpha,\gamma,\beta) \geq C_7 \sup_{\{u \in W_0^{1,N}(B): \|u\|_{W_0^{1,N}} \leq 1\}} \int_B e^{\alpha C_6^{-N/(N-1)}|u|^{N/(N-1)}}dx.$$

By the classical Trudinger-Moser inequality (see for instance [11, 13, 17]) we conclude that $L(\alpha,\gamma,\beta) = +\infty$ for any $\alpha > \alpha^{**} := C_6^{N/(N-1)}\alpha_N$. This completes the proof. □

4. Proof of Theorem 1.2

By using a variational approach, we obtain in this section one weak solution for ($\mathcal{P}$). From now on, we assume that (a_0), (k_0) and $(f_0) - (f_2)$ hold. We shall look for solutions of ($\mathcal{P}$) in the space E_a defined in the introduction as the completion of $C_\delta^\infty(\Omega)$ with respect to the norm

$$\|u\|_{E_a} := \left(\int_\Omega a(x)|\nabla u|^N dx + \int_{\partial\Omega} |u|^N d\sigma\right)^{1/N}.$$

From (a_0), (k_0), Corollary 2.1 and Proposition 2.3, we obtain the compact embedding

$$E_a \hookrightarrow L_k^q, \quad \text{for any } q \geq N. \tag{4.1}$$

On the other hand, as direct consequence of (a_0) and Corollary 2.1, it follows that $E_a \subset E^{1,\gamma}$ and therefore we can use condition (k_0) and Theorem 3.1 to obtain

$$\int_\Omega k(x)\Phi_\alpha(u)\,dx < +\infty, \quad \text{for any } \alpha > 0,\ u \in E_a. \tag{4.2}$$

By (a_0) and Corollary 2.1 again, we can assure the existence of $C_0 > 0$ such that

$$\|u\|_{E^{1,\gamma}} \leq C_0\|u\|_{E_a}, \quad \text{for all } u \in E_a$$

and hence the following Trudinger-Moser inequality in the space E_a holds:

Lemma 4.1. *Let* α^* *be given by Theorem 3.1. Then,*

$$\sup_{\{u\in E_a:\|u\|_{E_a}\le 1\}}\int_\Omega k(x)\Phi_\alpha(u)\,dx<+\infty.$$

for any $0<\alpha\le\overline{\alpha}:=\alpha^*/(C_0)^{N/(N-1)}$.

Proof. If $\|u\|_{E_a}\le 1$, then $\|u/C_0\|_{E^{1,\gamma}}\le 1$. So, by using (k_0) and (3.1), we obtain

$$\int_\Omega k(x)\Phi_\alpha(u)dx\le k_0\int_\Omega\frac{1}{(1+|x|)^\beta}\Phi_{\alpha C_0^{N/(N-1)}}\left(\frac{u}{C_0}\right)dx,$$

and the result follows from Theorem 3.1. □

In order to define the energy functional associated to ($\mathcal{P}$), we pick $\alpha>\alpha_0$ and $q\ge 1$. For any given $\varepsilon>0$, we obtain from $(f_0)-(f_1)$ a constant $C>0$ such that

$$|f(s)|\le\varepsilon|s|^{N-1}+C|s|^{q-1}\Phi_\alpha(s),\quad |F(s)|\le\varepsilon|s|^N+C|s|^q\Phi_\alpha(s),\tag{4.3}$$

for any $s\in\mathbb{R}$. Hence, if $u\in E_a$ and $r_1,r_2>1$ are such that $1/r_1+1/r_2=1$, $r_1\ge N$, we can use (4.3), Hölder's inequality, (4.1) and (4.2), to get

$$\int_\Omega k(x)F(u)dx\le\varepsilon\|u\|^N_{L^N_k}+C_1\|u\|^q_{L^{r_1q}_k}\left(\int_\Omega k(x)\Phi_{r_2\alpha}(u)dx\right)^{1/r_2}<+\infty,$$

where we also have used the inequality (see [20, Lemma 2.1])

$$[\Phi_\alpha(s)]^t\le\Phi_{t\alpha,N}(s),\quad s\in\mathbb{R},\ t\ge 1.\tag{4.4}$$

All the above considerations show that the functional $I:E_a\to\mathbb{R}$ given by

$$I(u)=\frac{1}{N}\|u\|^N_{E_a}-\int_\Omega k(x)F(u)dx$$

is well defined. Moreover, we can use standard arguments to check that $I\in C^1(E_a,\mathbb{R})$ with

$$I'(u)\varphi=\int_\Omega a(x)|\nabla u|^{N-2}(\nabla u\cdot\nabla\varphi)\,dx+\int_{\partial\Omega}|u|^{N-2}u\varphi\,d\sigma-\int_\Omega k(x)f(u)\varphi\,dx,$$

for any u, $\varphi\in E_a$. Hence, the critical points of I are precisely the weak solutions of the problem ($\mathcal{P}$).

We recall that $(u_n)\subset E_a$ is a Palais-Smale sequence for I at level $c\in\mathbb{R}$ ($(PS)_c$ for short), if

$$\lim_{n\to+\infty}I(u_n)=c\quad\text{and}\quad\lim_{n\to+\infty}\|I'(u_n)\|_{E_a^*}=0.$$

We prove in the sequel a local compactness result for the functional I.

Lemma 4.2. *Suppose that* $(u_n)\subset E_a$ *is a* $(PS)_c$ *sequence with*

$$c<\left(\frac{\overline{\alpha}}{\alpha_0}\right)^{N-1}\left(\frac{\theta-N}{N\theta}\right).$$

Then (u_n) *has a convergent subsequence.*

Proof. By computing $I(u_n)-(1/\theta)I'(u_n)u_n$ and using (f_2) and (k_0), we obtain $C_1,C_2>0$ such that

$$\begin{aligned}C_1+C_2\|u_n\|_{E_a}&\ge\left(\frac{1}{N}-\frac{1}{\theta}\right)\|u_n\|^N_{E_a}+\int_\Omega k(x)\left(\frac{1}{\theta}f(u_n)u_n-F(u_n)\right)dx\\&\ge\left(\frac{1}{N}-\frac{1}{\theta}\right)\|u_n\|^N_{E_a}.\end{aligned}$$

Recalling that $\theta > N$, we conclude that (u_n) is bounded in E_a and therefore, up to a subsequence, we have that $u_n \rightharpoonup u$ weakly in E_a.

We claim that

$$\int_\Omega k(x)f(u_n)(u_n - u)dx = o_n(1). \tag{4.5}$$

Indeed, by using (4.3), we get

$$\left|\int_\Omega k(x)f(u_n)(u_n - u)dx\right| \le \varepsilon A_n + C_3 D_n,$$

where

$$A_n := \int_\Omega k(x)|u_n|^{N-1}|u_n - u|\, dx, \quad D_n := \int_\Omega k(x)|u_n|^{q-1}\Phi_\alpha(u_n)|u_n - u|\, dx.$$

Hölder's inequality and the Sobolev embedding (4.1) ensure that

$$A_n \le \|u_n\|^{N-1}_{L^N_k}\|u_n - u\|_{L^N_k} \le C_4\|u_n\|^{N-1}_{E_a}\|u_n - u\|_{E_a}.$$

Hence, $(A_n) \subset \mathbb{R}$ is bounded and, since $\varepsilon > 0$ is arbitrary, we see that (4.5) will be proved if we can guarantee that $D_n = o_n(1)$.

It follows from (f_2) that

$$c = \lim_{n\to+\infty}\left(I(u_n) - \frac{1}{\theta}I'(u_n)u_n\right) \ge \left(\frac{1}{N} - \frac{1}{\theta}\right)\lim_{n\to+\infty}\|u_n\|^N_{E_a},$$

and therefore we can use $\theta > N$ and the hypothesis on c to get

$$\lim_{n\to+\infty}\|u_n\|^{N/(N-1)}_{E_a} \le \left(\frac{N\theta}{\theta - N}\right)^{1/(N-1)} c^{1/(N-1)} < \frac{\overline{\alpha}}{\alpha_0}.$$

We now pick $r_1 > 1$ and $\alpha > \alpha_0$ in such way that $r_1\alpha\|u_n\|^{N/(N-1)}_{E_a} < \overline{\alpha}$, for any $n \in \mathbb{N}$ large enough. So, Hölder's inequality, (k_0), (4.1), Lemma 4.1, (4.4) and (3.1) imply that

$$\begin{aligned} D_n &\le \|u_n\|^{q-1}_{L^{r_2(q-1)}_k}\|u_n - u\|_{L^{r_3}_k}\left(\int_\Omega k(x)\Phi_{r_1\alpha\|u_n\|^{N/(N-1)}_{E_a}}\left(\frac{u_n}{\|u_n\|_{E_a}}\right)dx\right)^{1/r_1} \\ &\le C_5\|u_n\|^{q-1}_{L^{r_2(q-1)}_k}\|u_n - u\|_{L^{r_3}_k} = o_n(1), \end{aligned}$$

where $1/r_1 + 1/r_2 + 1/r_3 = 1$, $r_3 \ge N$ and $q > 1$ is such that $r_2(q-1) \ge N$. This concludes the proof of (4.5).

Since $I'(u_n)(u_n - u) = o_n(1)$, we can use (4.5) to get

$$\int_\Omega a(x)|\nabla u_n|^{N-2}\nabla u_n \cdot \nabla(u_n - u)dx + \int_{\partial\Omega}|u_n|^{N-2}u_n(u_n - u)dx = o_n(1).$$

Moreover, from the weak convergence, we have that

$$\int_\Omega a(x)|\nabla u|^{N-2}\nabla u \cdot \nabla(u_n - u)dx + \int_{\partial\Omega}|u|^{N-2}u(u_n - u)dx = o_n(1).$$

Hence,

$$\int_\Omega T_N(\nabla u_n, \nabla u)\cdot\nabla(u_n - u)dx + \int_{\partial\Omega}T_1(u_n, u)(u_n - u)dx = o_n(1), \tag{4.6}$$

where

$$T_k(y_1, y_2) := \left(|y_1|^{N-2}y_1 - |y_2|^{N-2}y_2\right), \quad y_1, y_2 \in \mathbb{R}^k,$$

for $k \in \{1, N\}$. But we know that (see [16, inequality (2.2)])

$$T_k(y_1, y_2) \cdot (y_1 - y_2) \geq C(k, N)|y_1 - y_2|^N, \quad \forall\, y_1,\, y_2 \in \mathbb{R}^k.$$

From this inequality and (4.6) we obtain $C_6 > 0$ such that

$$C_6 \|u_n - u\|_{E_a}^N \leq o_n(1),$$

and therefore $u_n \to u$ strongly in E_a. The lemma is proved. □

In what follows we prove that I has the Mountain Pass geometry.

Lemma 4.3. *There exist $\tau, \rho > 0$ such that $I(u) \geq \tau$, if $\|u\|_{E_a} = \rho$. Moreover, there exists $e \in E_a$, with $\|e\|_{E_a} > \rho$, such that $I(e) < 0$.*

Proof. Let $q > N$ and r_1, $r_2 > 1$ be such that $1/r_1 + 1/r_2 = 1$. By using Hölder's inequality, (4.4) and (3.1), we get

$$\int_\Omega k(x)|u|^q \Phi_\alpha(u)dx \leq \|u\|_{L_k^{r_1 q}}^q \left(\int_\Omega k(x) \Phi_{r_2 \alpha \|u\|_{E_a}^{N/(N-1)}} \left(\frac{u}{\|u\|_{E_a}} \right) dx \right)^{1/r_2}.$$

If $\rho_1 > 0$ is such that $r_2 \alpha \rho_1^{N/(N-1)} \leq \overline{\alpha}$, we can apply Lemma 4.1 and use the second inequality in (4.3) to obtain $C_1 > 0$ such that

$$\int_\Omega k(x)F(u)dx \leq \varepsilon \|u\|_{L_k^N}^N + C_1 \|u\|_{L_k^{r_1 q}}^q,$$

for any $\varepsilon > 0$ and $\|u\|_{E_a} \leq \rho_1$. Hence, according to (4.1), there exists $C_2 > 0$ with

$$I(u) \geq \frac{1}{N}\|u\|_{E_a}^N - \varepsilon C_2 \|u\|_{E_a}^N - C_2 \|u\|_{E_a}^q = \|u\|_{E_a}^N \left(\frac{1}{N} - \varepsilon C_2 - C_2 \|u\|_{E_a}^{q-N} \right).$$

Picking $0 < \varepsilon < 1/(NC_2)$ and recalling that $q > N$, we can easily use the above expression to obtain the first statement of the lemma for $\rho > 0$ small enough.

For the second one, we consider a function $\varphi \in C_\delta^\infty(\Omega) \setminus \{0\}$ with support contained in the open bounded set Θ. From (f_2), we obtain constants C_3, $C_4 > 0$ such that $F(s) \geq C_3|s|^\theta - C_4$, for all $s \in \mathbb{R}$. Thus, for all $t > 0$, it holds that

$$I(t\varphi) \leq \frac{t^N}{N}\|\varphi\|_{E_a}^N - C_3 t^\theta \int_\Theta k(x)|\varphi|^\theta dx + C_4 \int_\Theta k(x)dx.$$

Since $\theta > N$, the second statement holds for $e := t\varphi$, with $t > 0$ sufficiently large. □

We finish the paper by presenting the proof of our last theorem.

Proof of Theorem 1.2. By using Lemma 4.3, we can define the minimax level

$$c_{MP} := \inf_{g \in \Gamma} \max_{t \in [0,1]} I(g(t)) \geq \tau > 0,$$

with $\Gamma := \{g \in C([0,1], E_a) : g(0) = 0 \text{ and } I(g(1)) < 0\}$. From the Mountain Pass Theorem without the Palais-Smale condition (see e.g. [6]), we obtain a sequence $(u_n) \subset E_a$ such that

$$\lim_{n \to +\infty} I(u_n) = c_{MP} \quad \text{and} \quad \lim_{n \to +\infty} \|I'(u_n)\|_{E_a^*} = 0.$$

We claim that there are $D(\lambda) > 0$ and $\lambda_* > 0$ such that

$$c_{MP} \leq D(\lambda) < \left(\frac{\overline{\alpha}}{\alpha_0} \right)^{N-1} \left(\frac{\theta - N}{N\theta} \right), \tag{4.7}$$

whenever f satisfies (f_3) with $\lambda \geq \lambda_*$. If this is true, we infer from Lemma 4.2 that, along a subsequence, $u_n \to u$ strongly in E_a. From the regularity of I we conclude that $I'(u) = 0$ and $I(u) = c_{MP} > 0$, and therefore u is the desired solution of ($\mathcal{P}$).

It remains to be proved that (4.7) holds. We first use the compact embedding (4.1) to obtain $\omega \in E_a$ such that

$$\|\omega\|_{E_a}^N = S_\nu := \inf \left\{ \|u\|_{E_a}^N : \int_\Omega k(x)|u|^\nu dx = 1 \right\}.$$

From (f_3), we know that $F(s) \geq \lambda |s|^\nu$, for all $s \in \mathbb{R}$. Thus,

$$I(\omega) \leq \frac{1}{N} S_\nu - \lambda \int_\Omega k(x)|\omega|^\nu dx = \frac{1}{N} S_\nu - \lambda < 0$$

for any $\lambda > (S_\nu / N)$. Thus, for such values of λ, the path $g(t) := t\omega$ belongs to the set of paths Γ which appears in the definition of c_{MP}, and therefore we obtain

$$c_{MP} \leq \max_{t \geq 0} I(t\omega) \leq \left(\frac{1}{N} S_\nu t^N - \lambda t^\nu \right) = \frac{S_\nu^{\nu/(\nu - N)}}{(\lambda \nu)^{N/(\nu - N)}} \left(\frac{\nu - N}{N \nu} \right) =: D(\lambda).$$

Since $D(\lambda) \to 0$ as $\lambda \to +\infty$, it is clear that (4.7) holds for any $\lambda \geq \lambda_*$ large enough. This concludes the proof. □

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