# HARDY INEQUALITY FOR DOMAINS WITH A GEOMETRIC BOUNDARY CONDITION AND APPLICATIONS

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Abstract. In this paper, we state a Hardy inequality for domains with a geometric boundary condition. As a consequence, we prove a weighted Trudinger-Moser inequality. After that, we apply our results to investigate the existence of solutions for a class of quasilinear elliptic equations with Neumann boundary condition and nonlinearities with critical exponential growth.

## 1. Introduction and main results

As it is well-known, Sobolev embedding plays an important role in the study of partial differential equations. For any $1 < p < N$ and $\Omega \subset \mathbb{R}^N$ a smooth open set containing the origin, the classical $N-$dimensional Hardy inequality (see [13, 19]) assures that

$$\left(\frac{N-p}{p}\right)^p \int_\Omega \frac{|u|^p}{|x|^p} dx \leq \int_\Omega |\nabla u|^p dx, \quad u \in C_0^\infty(\Omega). \tag{1.1}$$

We refer to [2] for other results in bounded domains. The above inequality is no longer true in the borderline case $p = 2$ when $\Omega = \mathbb{R}^2$, as pointed out in the paper [17]. Although this, it is showed by Solomyak in [27] that there exists a constant $C > 0$ such that

$$\int_{\mathbb{R}^2} \frac{|u|^2}{|x|^2(1+\log^2(|x|))} dx \leq C \int_{\mathbb{R}^2} |\nabla u|^2 dx,$$

for any $u \in C_0^\infty(\mathbb{R}^2)$ satisfying the mean zero condition $\int_{\partial B_1(0)} u(x) d\sigma = 0$. For Hardy inequality in the borderline case $p = N$ and $\Omega$ the unit ball we refer the reader [14, 26] and references therein.

If $\Omega \subset \mathbb{R}^N$ is an arbitrary domain, Hardy-Sobolev inequalities and its variants have been the subject of intensive research, see [15, 16, 24, 29, 7] and references there in. For instance, Opic-Kurfner [22] provide different conditions on the weight functions $w_1$ and $w_2$ for the validity of the Hardy-Sobolev inequality

$$\int_\Omega w_1(x)|u|^p dx \leq \int_\Omega w_2(x)|\nabla u|^p dx, \quad u \in C_0^\infty(\Omega).$$

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We also emphasize that, if $p = 2$, then the Hardy-Sobolev inequality can be derived from the Caffarelli–Kohn–Nirenberg inequality (see [5])

$$\left(\int_{\mathbb{R}^N} |x|^{\beta q}|u|^q dx\right)^{1/q} \leq C\left(\int_{\mathbb{R}^N} |x|^{\alpha p}\,|\nabla u|^p\,dx\right)^{1/p}, \quad u \in C_0^\infty(\mathbb{R}^N), \quad N \geq 2, \tag{1.2}$$

for some constant $C = C(\alpha, \beta) > 0$ where the parameters $\alpha$ and $\beta$ satisfy the balanced conditions

$$\frac{\beta}{N} + \frac{1}{q} > 0, \quad \frac{\alpha}{N} + \frac{1}{p} > 0, \quad \frac{\beta - \alpha + 1}{N} = \left(\frac{1}{p} - \frac{1}{q}\right), \quad 0 \leq \beta - \alpha \leq 1.$$

In particular, if we pick $q = p = N$, we obtain $\alpha = \beta + 1 > 0$. Thus, taking $\gamma = \alpha p > 0$, it follows from (1.2) that

$$\int_{\mathbb{R}^N} |x|^{\gamma - N}|u|^N dx \leq C\int_{\mathbb{R}^N} |x|^\gamma\,|\nabla u|^N\,dx, \quad u \in C_0^\infty(\mathbb{R}^N).$$

It is worth noticing that the above inequality is no longer true for $\gamma \leq 0$. Indeed, if it holds, then we can set $\Gamma := \left\{u \in C_0^\infty(\mathbb{R}^N) \,:\, u \geq 1 \text{ in } B_1(0)\right\}$,

$$\mathrm{Cap}_{N,\gamma} := \inf_{u\in\Gamma}\int_{\mathbb{R}^N} |x|^\gamma\,|\nabla u|^N\,dx < +\infty,$$

and obtain

$$\mathrm{Cap}_{N,\gamma} \geq \frac{1}{C}\inf_{u\in\Gamma}\int_{\mathbb{R}^N} |x|^{\gamma-N}|u|^N dx \geq \frac{1}{C}\int_{B_1(0)} |x|^{\gamma-N}dx = +\infty,$$

whenever $\gamma \leq 0$. This contradiction shows that $\gamma > 0$ is a necessary condition.

In this paper, we are concerned with smooth function which can take nonzero values on the boundary of $\Omega$. More specifically, we deal with the space $C_\delta^\infty(\Omega)$ which consists of $C_0^\infty(\mathbb{R}^N)$-functions restricted to $\Omega$. We start quoting that Janssen [16] and Pfluger [24] obtained, for any $1 < p < N$, a constant $C_0 > 0$ such that

$$\int_\Omega \frac{|u|^p}{(1+|x|)^p}dx \leq C_0\left(\int_\Omega |\nabla u|^p dx + \int_{\partial\Omega} \frac{|x\cdot\nu|}{(1+|x|)^p}|u|^p d\sigma\right), \quad u \in C_\delta^\infty(\Omega), \tag{1.3}$$

where $\nu$ denotes the unit outward normal vector on $\partial\Omega$. For more results concerning Hardy inequalities in the limiting case we refer to Ioku-Ishiwata [14], Laptev [15], Sano-Sobukawa [26], Wang-Zhu [29] and its references.

Our main goal here is twofold. First, we address a version of the Hardy-Sobolev inequality (1.3) in the borderline case $p = N$. As a consequence, after imposing some geometric condition on the boundary of $\Omega$, we obtain embedding from an appropriated weighted Sobolev space into Lebesgue and Orlicz spaces. Secondly, we apply these embedding results to investigate the existence of solutions for a class of zero mass case quasilinear elliptic equation with Neumann boundary conditions involving exponential critical growth in the Trudinger-Moser sense.

1.1. **Hardy-Sobolev inequality and Sobolev embedding.** Let $N \geq 2$ and $\Omega \subset \mathbb{R}^N$ be a smooth domain. Motivated by the aforementioned results, our purpose here is to prove the following Hardy type inequality with boundary terms in the borderline case:

**Theorem 1.1** (Hardy). *Let $\gamma > 0$ and suppose that $0 \notin \partial\Omega$. Then, there exists $C > 0$ such that, for any $u \in C_\delta^\infty(\Omega)$, it holds*

$$\int_\Omega |x|^{\gamma-N}|u|^N\,dx \leq C\left(\int_\Omega |x|^\gamma|\nabla u|^N dx + \int_{\partial\Omega}|x|^{\gamma-N}|u|^N(x\cdot\nu)\,d\sigma\right). \tag{1.4}$$

Our proof is inspired by an argument presented by Mitidieri in [20], who have considered the inequality (1.1). We notice that, if we adittionaly assume that $\partial\Omega$ is bounded and $\mathbb{R}^N\backslash\Omega$ is strictly star-shaped with respect to the origin, that is, $x\cdot\nu(x) < 0$ for any $x \in \partial\Omega$, then there exists $C > 0$ such that, for any $u \in C_\delta^\infty(\Omega)$, it holds

$$\int_{\partial\Omega}|u|^N d\sigma + \int_\Omega |x|^{\gamma-N}|u|^N\,dx \leq C\int_\Omega |x|^\gamma|\nabla u|^N dx.$$

Indeed, since $\partial\Omega$ is compact, we can obtain $C_1 > 0$ such that $x\cdot\nu(x) \leq -C_1 < 0$, over $\partial\Omega$. The result follows from (1.4).

We introduce, for each $\gamma > 0$, the space $E^{1,\gamma}$ obtained as the completion of $C_\delta^\infty(\Omega)$ with respect to the norm

$$\|u\|_{E^{1,\gamma}} := \left(\int_\Omega \left[|x|^\gamma|\nabla u|^N + |x|^{\gamma-N}|u|^N\right]dx\right)^{1/N}.$$

The following result is an easy consequence of our Hardy inequality which will play an important role to establish embedding results for $E^{1,\gamma}$:

**Corollary 1.2.** *If $\gamma > 0$, $0 \notin \partial\Omega$ and $x\cdot\nu(x) \leq 0$, for any $x \in \partial\Omega$, then the norms $\|\cdot\|_{E^{1,\gamma}}$ and*

$$\|u\| := \left(\int_\Omega |x|^\gamma|\nabla u|^N dx\right)^{1/N}$$

*are equivalents in $E^{1,\gamma}$.*

*Proof.* By Theorem 1.1, there exists $C > 0$ such that, for any $u \in C_\delta^\infty(\Omega)$,

$$\int_\Omega |x|^\gamma|\nabla u|^N dx \leq \int_\Omega \left[|x|^\gamma|\nabla u|^N + |x|^{\gamma-N}|u|^N\right]dx \leq (1+C)\int_\Omega |x|^\gamma|\nabla u|^N dx$$

and the result follows by density. □

From now on, we shall assume that our domain satisfies the following geometric condition:

$(*)$ $\mathbb{R}^N\setminus\Omega$ is bounded, $0 \notin \overline{\Omega}$ and $x\cdot\nu(x) \leq 0$, for any $x \in \partial\Omega$.

That is the case, for example, if $\Omega$ is the complement of an open ball centred at the origin. Given a positive function $\omega \in L^1_{loc}(\Omega)$ and $q \geq 1$, we define the weighted Lebesgue space

$$L^q_\omega := \left\{u \in L^1_{loc}(\Omega) \,:\, \|u\|_{L^q_\omega} := \left(\int_\Omega \omega(x)|u|^q\,dx\right)^{1/q} < +\infty\right\}.$$

In our next result, we prove that $E^{1,\gamma}$ embedds into the space $L^q_{(1+|\cdot|)^{-\beta}}$.

**Theorem 1.3** (Sobolev embedding). *Suppose that $\gamma > 0$, $\beta \geq N - \gamma$ and $\Omega$ satisfies $(*)$. Then, for any $q \geq N$, there exists $C > 0$ such that, for any $u \in E^{1,\gamma}$,*

$$\int_\Omega \frac{|u|^q}{(1+|x|)^\beta} dx \leq C \left( \int_\Omega |x|^\gamma |\nabla u|^N dx \right)^{q/N}, \tag{1.5}$$

*and therefore $E^{1,\gamma} \hookrightarrow L^q_{(1+|\cdot|)^{-\beta}}$ continuously. Furthermore, this embedding is compact whenever $\beta > N - \gamma$.*

1.2. **Weighted Trudinger-Moser inequality.** In view of Theorem 1.3, it is natural to look for embedding from $E^{1,\gamma}$ into Orlicz spaces. Precisely, for any $\alpha > 0$, we consider the Young function

$$\Phi_\alpha(s) := e^{\alpha |s|^{N/(N-1)}} - \sum_{j=0}^{N-2} \frac{\alpha^j}{j!} |s|^{Nj/(N-1)}, \quad \text{for all } s \in \mathbb{R}.$$

By adapting the arguments used in the proof of Theorem 1.3, we obtain the following weighted Trudinger-Moser type inequality:

**Theorem 1.4** (Trudinger-Moser). *Suppose that $\gamma > 0$, $\beta \geq N$ and $\Omega$ is a connected domain of class $C^{1,\eta}$, $\eta \in (0, 1]$, satisfying $(*)$. Then, for any $\alpha > 0$ and $u \in E^{1,\gamma}$, the function $(1 + |\cdot|)^{-\beta} \Phi_\alpha(u)$ belongs to $L^1(\Omega)$. Moreover, there exists $\alpha^* = \alpha^*(N) > 0$ such that*

$$L(\alpha, \beta, \gamma) := \sup_{\{u \in E^{1,\gamma} : \|u\| \leq 1\}} \int_\Omega \frac{\Phi_\alpha(u)}{(1+|x|)^\beta} dx < +\infty,$$

*for any $0 < \alpha \leq \alpha^*$.*

The first results concerning Trudinger-Moser type inequalities have appeared in the papers of Yudovich, Moser, Trudinger [31, 21, 28], for the bounded domain case. Similar results for unbounded domains have been established by Cao [6] and Ruf [23] in $\mathbb{R}^2$, and by do Ó [9], Adachi and Tanaka [1], Li and Ruf [18], in higher dimensions. Concerning the case of weighted Sobolev spaces, we can refer the reader to [11, 10, 12, 4] and references therein. Some of these works considered radial weight functions, in such a way that rearrangement procedures work well. Our abstract result complement and/or generalize the aforementioned papers.

1.3. **Application.** In the final part of the paper, we illustrate how the previous results can be useful to obtain existence of solutions for a class of zero-mass case quasilinear elliptic equations with Neumann boundary condition in a borderline case. More specifically, we deal with the problem

$$(\mathcal{P}_\lambda) \qquad \begin{cases} -\mathrm{div}\left(a(x)|\nabla u|^{N-2}\nabla u\right) = \lambda k(x) f(u), & \text{in } \Omega, \\ a(x)|\nabla u|^{N-2}(\nabla u \cdot \nu) = 0, & \text{on } \partial\Omega, \end{cases}$$

where $f$ is continuous and the potentials $a$ and $k$ satisfy the following assumptions:

$(a_0)$ $a : \overline{\Omega} \to \mathbb{R}$ is continuous and there exist $a_0, \gamma > 0$ such that

$$a_0 |x|^\gamma \leq a(x), \quad \text{for any } x \in \Omega;$$

$(k_0)$ $k : \Omega \to \mathbb{R}$ is measurable and there exist $k_0 > 0$ and $\beta \geq N$ such that

$$0 < k(x) \leq \frac{k_0}{(1+|x|)^\beta}, \quad \text{for a.e. } x \in \Omega.$$

We shall look for solutions of the problem in the space $E_a$ defined as the completion of $C^\infty_\delta(\Omega)$ with respect to the norm

$$\|u\|_{E_a} := \left( \int_\Omega a(x)|\nabla u|^N dx \right)^{1/N}.$$

In this case, we say that $u \in E_a$ is a weak solution for problem ($\mathcal{P}_\lambda$) if

$$\int_\Omega a(x)|\nabla u|^{N-2}(\nabla u \cdot \nabla \varphi)\, dx = \lambda \int_\Omega k(x) f(u)\varphi\, dx, \quad \text{for all } \varphi \in C^\infty_\delta(\Omega).$$

Our aim here is to investigate the existence of weak solutions when the nonlinearity $f$ has the maximal growth for which the energy functional associated is well defined. According to Theorem 1.4 and the hypotheses $(a_0)$, $(k_0)$, we may consider nonlinearities $f$ which behave like $e^{\alpha|s|^{N/(N-1)}}$ at infinity. More specifically, we shall assume that

$(f_0)$ $f : \mathbb{R} \to \mathbb{R}$ is continuous and there exists $\alpha_0 > 0$ such that

$$\lim_{|s|\to+\infty} \frac{|f(s)|}{e^{\alpha|s|^{N/(N-1)}}} = \begin{cases} 0, & \text{if } \alpha > \alpha_0, \\ +\infty, & \text{if } \alpha < \alpha_0; \end{cases}$$

$(f_1)$ $f(s) = o(|s|^{N-1})$ as $s \to 0$;

$(f_2)$ there exists $\theta > N$ such that $0 < \theta F(s) := \theta \int_0^s f(t)dt \leq f(s)s$, for any $s \in \mathbb{R}$;

$(f_3)$ there exist $C_F > 0$ and $\nu > N$ such that $F(s) \geq C_F|s|^\nu$, for every $s \in (0, 1]$.

Our main existence result for problem ($\mathcal{P}_\lambda$) is stated in what follows.

**Theorem 1.5.** *Suppose that $\Omega$ is a connected domain of class $C^{1,\eta}$, $\eta \in (0, 1]$, satisfying $(*)$. If $(a_0)$, $(k_0)$ and $(f_0)-(f_3)$ hold, then there exists $\lambda^* > 0$ such that, for any $\lambda \geq \lambda^*$, the problem ($\mathcal{P}_\lambda$) has a nonnegative nonzero weak solution.*

For the proof, we apply the Mountain Pass Theorem. Although the general approach is in some sense standard, the idea is using all the variational setting done in the first part of the paper. Our main difficulties rely on the fact that we are dealing with the zero-mass case, the domain $\Omega$ may be not symmetric, the Hardy-Sobolev inequality generally holds for $1 < p < N$ and, as far we know, there is no appropriated Trudinger-Moser inequality for our case. So, our paper complements all the aforementioned works. We emphasize that our results seem to be new even in semilinear case $N = 2$.

The remainder of the paper is organized as follows. In Section 2, we establish the proof of Theorems 1.1 and 1.3. In section 3, we prove Theorem 1.4 and, finally, Section 4 is devoted to the proof of Theorem 1.5.

## 2. Hardy inequality and the Sobolev embedding

In this section, we prove Theorems 1.1 and 1.3 stated in the introduction. We write $B_R(x_0)$ for the open ball of radius $R>0$ centered at the $x_0 \in \mathbb{R}^N$. When $x_0 = 0$, we write only $B_R$.

*Proof of Theorem 1.1.* Let $\rho \in C^\infty(\mathbb{R})$ be such that $\rho \equiv 0$ in $[0,1]$, $\rho \equiv 1$ in $[2,+\infty)$ and $0 \le \rho \le 1$. For any $\varepsilon > 0$, we define $\rho_\varepsilon(x) := \rho(|x|/\varepsilon)$ and the vector field

$$H_\varepsilon(x) := x|x|^\alpha \rho_\varepsilon(x)|u|^N,$$

where $u \in C_\delta^\infty(\Omega)$ is fixed and $\alpha > -N$ is free for now. Since $H_\varepsilon$ vanishes in a neighbourhood of the origin, we can apply the divergence theorem to get

$$\begin{aligned}(\alpha+N)\int_\Omega |x|^\alpha \rho_\varepsilon(x)|u|^N\,dx &= -N\int_\Omega |x|^\alpha \rho_\varepsilon(x)|u|^{N-2}u(x\cdot\nabla u)\,dx \\ &\quad -\Gamma_\varepsilon + \int_{\partial\Omega} |x|^\alpha |u|^N (x\cdot\nu)\,d\sigma,\end{aligned} \tag{2.1}$$

where

$$\begin{aligned}\Gamma_\varepsilon &:= \varepsilon^{-1}\int_\Omega |x|^{\alpha+1}\rho'(|x|/\varepsilon)|u|^N\,dx \\ &\le \varepsilon^{-1}\|\rho'\|_{L^\infty(\mathbb{R})}\|u\|_{L^\infty(\Omega)}\int_{\{\varepsilon\le|x|\le 2\varepsilon\}} |x|^{\alpha+1}dx \\ &\le \varepsilon^{-1}C_1\int_0^{2\varepsilon} r^{\alpha+1}r^{N-1}dr = C_2\varepsilon^{\alpha+N},\end{aligned}$$

with $C_1 > 0$ independent of $\varepsilon$. Since $\alpha > -N$, we conclude that $\Gamma_\varepsilon \to 0$, as $\varepsilon \to 0^+$. Moreover, $\rho_\varepsilon(x) \to 1$, as $\varepsilon \to 0^+$, for any $x \in \mathbb{R}^N \setminus 0$. Hence, we can take the limit in (2.1) and use the Lebesgue Theorem to get

$$(\alpha+N)\int_\Omega |x|^\alpha |u|^N\,dx = -N\int_\Omega |x|^\alpha|u|^{N-2}u(x\cdot\nabla u)\,dx + \int_{\partial\Omega}|x|^\alpha|u|^N(x\cdot\nu)\,d\sigma,$$

for any $u \in C_\delta^\infty(\Omega)$.

We now take another $\varepsilon > 0$ and apply Young's inequality to obtain

$$\begin{aligned}-N\int_\Omega |x|^\alpha|u|^{N-2}u(x\cdot\nabla u)\,dx &\le N\int_\Omega |x|^{\alpha+1}|u|^{N-1}|\nabla u|\,dx \\ &= N\int_\Omega \left(|x|^{\alpha(N-1)/N}|u|^{N-1}\right)|x|^{[\alpha+1-\alpha(N-1)/N]}|\nabla u|\,dx \\ &\le (N-1)\varepsilon\int_\Omega |x|^\alpha|u|^N\,dx + \varepsilon^{1-N}\int_\Omega |x|^{\alpha+N}|\nabla u|^N\,dx.\end{aligned}$$

The above estimates imply that

$$(\alpha+N-(N-1)\varepsilon)\int_\Omega |x|^\alpha|u|^N dx \le \varepsilon^{1-N}\int_\Omega |x|^{\alpha+N}|\nabla u|^N dx + \int_{\partial\Omega}|x|^\alpha|u|^N(x\cdot\nu)d\sigma.$$

Picking $0 < \varepsilon < (\alpha+N)/(N-1)$ and choosing $\alpha = \gamma - N > -N$, we obtain the desired result. $\square$

The proof of the Sobolev embedding is more involved.

*Proof of Theorem 1.3.* For any $j>0$, we set $\Omega_j := \Omega \cap B_{2^j}$. Let $j_0 \in \mathbb{N}$ be such that $(\mathbb{R}^N \setminus \Omega) \subset B_{2^{j_0}}$, which implies $\Omega = \Omega_{j_0} \cup (\mathbb{R}^N \setminus B_{2^{j_0}})$. Given $u \in E^{1,\gamma} \subset W^{1,N}_{loc}(\Omega)$, from the Sobolev embedding $W^{1,N}(\Omega_{j_0}) \hookrightarrow L^q(\Omega_{j_0})$, we get

$$\int_{\Omega_{j_0}} \frac{|u|^q}{(1+|x|)^\beta} dx \le C_1 \int_{\Omega_{j_0}} |u|^q dx \le C_2 \left( \int_{\Omega_{j_0}} \left[ |\nabla u|^N + |u|^N \right] dx \right)^{q/N}$$

and hence, by Corollary 1.2, one deduce

$$\begin{aligned} \int_{\Omega_{j_0}} \frac{|u|^q}{(1+|x|)^\beta} dx &\le C_3 \left( \int_{\Omega_{j_0}} \left[ |x|^\gamma |\nabla u|^N + |x|^{\gamma-N} |u|^N \right] dx \right)^{q/N} \\ &\le C_4 \left( \int_{\Omega} |x|^\gamma |\nabla u|^N dx \right)^{q/N}. \end{aligned} \tag{2.2}$$

On the other hand, if we define $A_j := \{ z \in \Omega : 2^{j_0} \cdot 2^j < |z| < 2^{j_0} \cdot 2^{j+1} \}$, for any $j \in \mathbb{N} \cup \{0\}$, the change of variables $y := 2^{-j} x$ provides

$$\int_{A_j} \frac{|u|^q}{(1+|x|)^\beta} dx \le \frac{C_5}{2^{\beta j}} \int_{A_j} |u|^q dx = C_5 2^{(N-\beta)j} \int_{A_0} |u_j(y)|^q dy,$$

where $u_j(y) := u(2^j y)$. Using the Sobolev embedding $W^{1,N}(A_0) \hookrightarrow L^q(A_0)$, we obtain $C_6 > 0$ such that

$$\begin{aligned} \int_{A_0} |u_j(y)|^q dy &\le C_6 \left( \int_{A_0} \left[ |\nabla u_j(y)|^N + |u_j(y)|^N \right] dy \right)^{q/N} \\ &= C_6 \left( \int_{A_j} \left[ |\nabla u(x)|^N + 2^{-Nj} |u(x)|^N \right] dx \right)^{q/N}. \end{aligned}$$

Now we observe that

$$\int_{A_j} |\nabla u|^N dx = \int_{A_j} |x|^{-\gamma} |x|^\gamma |\nabla u|^N dx \le 2^{-\gamma j} \int_{A_j} |x|^\gamma |\nabla u|^N dx$$

and

$$\int_{A_j} 2^{-Nj} |u|^N dx = \int_{A_j} 2^{-Nj} |x|^{-\gamma+N} |x|^{\gamma-N} |u|^N dx \le 2^{(j_0+1)N} 2^{-\gamma j} \int_{A_j} |x|^{\gamma-N} |u|^N dx.$$

Consequently, for some $C_7 = C_7(j_0, N, q) > 0$, we have

$$\begin{aligned} \int_{A_j} \frac{|u|^q}{(1+|x|)^\beta} dx &\le C_7 2^{(N-\beta)j} \left( 2^{-\gamma j} \int_{A_j} \left[ |x|^\gamma |\nabla u|^N + |x|^{\gamma-N} |u|^N \right] dx \right)^{q/N} \\ &= C_7 2^{\lambda_j} \left( \int_{A_j} \left[ |x|^\gamma |\nabla u|^N + |x|^{\gamma-N} |u|^N \right] dx \right)^{q/N}, \end{aligned} \tag{2.3}$$

where

$$\lambda_j := \left(N - \beta - \frac{\gamma q}{N}\right) j.$$

Since $\gamma > 0$ and $\beta \geq N - \gamma$, one has $\lambda_j \leq 0$, and therefore

$$\int_{A_j} \frac{|u|^q}{(1+|x|)^\beta} dx \leq C_7 \left( \int_{A_j} \left[ |x|^\gamma |\nabla u|^N + |x|^{\gamma - N} |u|^N \right] dx \right)^{q/N}.$$

Thus, recalling that the function $s \mapsto s^{q/N}$ is super additive for $q \geq N$, we conclude that

$$\begin{aligned} \sum_{j=0}^\infty \int_{A_j} \frac{|u|^q}{(1+|x|)^\beta} dx &\leq C_7 \sum_{j=0}^\infty \left( \int_{A_j} \left[ |x|^\gamma |\nabla u|^N + |x|^{\gamma - N} |u|^N \right] dx \right)^{q/N} \\ &\leq C_7 \left( \int_{\mathbb{R}^N \setminus B_{2^{j_0}}} \left[ |x|^\gamma |\nabla u|^N + |x|^{\gamma - N} |u|^N \right] dx \right)^{q/N} \\ &\leq C_8 \left( \int_\Omega |x|^\gamma |\nabla u|^N dx \right)^{q/N}, \end{aligned}$$

where we apply Corollary 1.2. This, combined with the estimate (2.2), imply

$$\int_\Omega \frac{|u|^q}{(1+|x|)^\beta} dx \leq C_9 \left( \int_\Omega |x|^\gamma |\nabla u|^N dx \right)^{q/N},$$

which proves (1.5).

For the compactness, we consider a sequence $(u_n) \subset E^{1,\gamma}$ such that $u_n \rightharpoonup 0$ weakly in $E^{1,\gamma}$. Given $\varepsilon > 0$, we can use $\gamma > 0$ and the fact that $\beta > N - \gamma$ to obtain $j_1 \in \mathbb{N}$ such that $2^{\lambda_j} < \varepsilon$, for any $j > j_1$. Thus, from (2.3), we get

$$\int_{A_j} \frac{|u_n|^q}{(1+|x|)^\beta} dx < C_7 \varepsilon \left( \int_{A_j} \left[ |x|^\gamma |\nabla u_n|^N + |x|^{\gamma - N} |u_n|^N \right] dx \right)^{q/N},$$

for any $j > j_1$. From $E^{1,\gamma} \subset W^{1,N}_{loc}(\Omega)$, Rellich–Kondrachov Theorem and Corollary 1.2, we infer

$$\begin{aligned} \int_\Omega \frac{|u_n|^q}{(1+|x|)^\beta} dx &\leq \int_{\Omega_{j_0}} \frac{|u_n|^q}{(1+|x|)^\beta} dx + \sum_{j=0}^{j_1} \int_{A_j} \frac{|u_n|^q}{(1+|x|)^\beta} dx + C_7 \varepsilon \|u_n\|^q_{E^{1,\gamma}} \\ &\leq o_n(1) + C_{10} \varepsilon \|u_n\|^q, \end{aligned}$$

where $o_n(1)$ stands for a quantity approaching zero as $n \to +\infty$. Since $\varepsilon > 0$ is arbitrary, the above expression implies that $u_n \to 0$ strongly in $L^q_{(1+|\cdot|)^{-\beta}}$, which concludes the proof. □

## 3. Trudinger-Moser type inequality

In order to prove Theorem 1.4, we need two technical lemmas. We start recalling the definition of the Young function

$$\Phi_\alpha(s) := e^{\alpha|s|^{N/(N-1)}} - \sum_{j=0}^{N-2} \frac{\alpha^j}{j!}|s|^{Nj/(N-1)}, \quad \text{for all } s \in \mathbb{R},$$

and stating a basic result which is a version of [30, Lemma 4.1] in $W^{1,N}(\Omega_0)$. Precisely, we have the following result.

**Lemma 3.1.** *Let $\Omega_0 \subset \mathbb{R}^N$ be a bounded connected domain of class $C^{1,\beta}$, for some $\beta \in (0,1]$. If $\omega_{N-1}$ denotes the measure of the unit sphere in $\mathbb{R}^N$ then, for any $0 < \alpha < \alpha_N := (N^N\omega_{N-1}/2)^{1/(N-1)}$, there exists $C_0 = C_0(\alpha,\Omega_0) > 0$ such that*

$$\int_{\Omega_0} \Phi_\alpha(v)dx \le C_0 \int_{\Omega_0} \left[|\nabla v|^N + |v|^N\right] dx,$$

*for any $v \in W^{1,N}(\Omega_0)$ such that $\|v\|_{W^{1,N}(\Omega_0)} \le 1$.*

*Proof.* By the Trudinger-Moser inequality (see [8]) we know that, for any $\alpha < \alpha_N$,

$$L(\alpha,\Omega_0) := \sup_{\left\{u\in W^{1,N}(\Omega_0):\|u\|_{W^{1,N}(\Omega_0)}=1\right\}} \int_{\Omega_0} e^{\alpha|u|^{N/(N-1)}}dx < +\infty.$$

Thus, if $\|v\|_{W^{1,N}(\Omega_0)} \le 1$ and $\tilde{v} := v/\|v\|_{W^{1,N}(\Omega_0)}$, ones has

$$\begin{aligned} L(\alpha,\Omega_0) &\ge \int_{\Omega_0} e^{\alpha|\tilde{v}|^{N/(N-1)}}dx \ge \int_{\Omega_0}\left(\sum_{j=N-1}^{\infty}\frac{\alpha^j}{j!}\frac{|v|^{Nj/(N-1)}}{\|v\|_{W^{1,N}(\Omega_0)}^{Nj/(N-1)}}\right)dx \\ &\ge \frac{1}{\|v\|_{W^{1,N}(\Omega_0)}^N}\int_{\Omega_0}\left(\sum_{j=N-1}^{\infty}\frac{\alpha^j}{j!}|v|^{Nj/(N-1)}\right)dx, \end{aligned}$$

and the result follows. □

As a consequence of this result we can prove the next lemma.

**Lemma 3.2.** *Suppose that $\gamma > 0$ and $\beta \ge N$ hold. Then, there exists $C_N$, $\alpha^* = \alpha^*(N) > 0$ such that, for any $0 < \alpha < \alpha^*$,*

$$\int_\Omega \frac{\Phi_\alpha(u)}{(1+|x|)^\beta}\,dx \le C_N \int_\Omega |x|^\gamma|\nabla u|^N dx,$$

*whenever $u \in E^{1,\gamma}$ satisfies $\|u\| \le 1$.*

*Proof.* Let $j_0 \in \mathbb{N}$ and $\Omega_{j_0}$ be as in the proof of Theorem 1.3. If $\|u\| \le 1$, then it follows from $0 \notin \overline{\Omega}$ and Corollary 1.2 that

$$\|u\|_{W^{1,N}(\Omega_{j_0})}^N \le C_1\int_{\Omega_{j_0}}[|x|^\gamma|\nabla u|^N + |x|^{\gamma-N}|u|^N]dx \le C_2\int_{\Omega_{j_0}}|x|^\gamma|\nabla u|^N dx \le C_2.$$

From the definition of $\Phi_\alpha$, we easily conclude that

$$\Phi_\alpha(ts) = \Phi_{\alpha t^{N/(N-1)}}(s), \qquad s \in \mathbb{R},\ t > 0. \tag{3.1}$$

Thus,

$$\int_{\Omega_{j_0}} \frac{\Phi_\alpha(u)}{(1+|x|)^\beta}dx \le \int_{\Omega_{j_0}} \Phi_\alpha(u)dx = \int_{\Omega_{j_0}} \Phi_{\alpha C_2^{1/(N-1)}}\left(\frac{u}{\sqrt[N]{C_2}}\right)dx. \tag{3.2}$$

This, together with Lemma 3.1 and Corollary 1.2 imply that

$$\int_{\Omega_{j_0}} \frac{\Phi_\alpha(u)}{(1+|x|)^\beta}dx \le \frac{C_0}{C_2}\|u\|^N_{W^{1,N}(\Omega_0)} \le C_3 \int_\Omega |x|^\gamma |\nabla u|^N dx, \tag{3.3}$$

for some $C_3 = C_3(\gamma) > 0$ and any

$$0 < \alpha < \zeta_N^1 := \frac{\alpha_N}{C_2^{1/(N-1)}}.$$

Considering now the annulus $A_j := \{z \in \Omega : 2^{j_0}\cdot 2^j < |z| < 2^{j_0}\cdot 2^{j+1}\}$ we claim that, for some $C_4 > 0$ and $\alpha > 0$ small, there holds

$$\int_{A_j} \frac{\Phi_\alpha(u)}{(1+|x|)^\beta}dx \le C_4 \int_{A_j} \left[|x|^\gamma|\nabla u|^N + |x|^{\gamma-N}|u|^N\right]dx, \tag{3.4}$$

for any $j \in \mathbb{N}\cup\{0\}$. If this is true, we can apply Corollary 1.2 to get

$$\begin{aligned}\int_{B^c_{2^{j_0}}} \frac{\Phi_\alpha(u)}{(1+|x|)^\beta}dx &= \sum_{j=0}^\infty \int_{A_j} \frac{\Phi_\alpha(u)}{(1+|x|)^\beta}dx \le C_4 \int_\Omega \left[|x|^\gamma|\nabla u|^N + |x|^{\gamma-N}|u|^N\right]dx \\ &\le C_5 \int_\Omega |x|^\gamma|\nabla u|^N dx.\end{aligned}$$

So, recalling that $\Omega = \Omega_{j_0} \cup (\mathbb{R}^N \setminus B_{2^{j_0}})$, we see that the lemma is a direct consequence of this inequality and (3.3).

It remains to prove (3.4). In order to do that, we adapt our former argument. First, we fix $j \in \mathbb{N}\cup\{0\}$ and use the change of variables $y = 2^{-j}x$ to obtain

$$\int_{A_j} \frac{\Phi_\alpha(u)}{(1+|x|)^\beta}dx \le \frac{C_6}{2^{\beta j}} \int_{A_j} \Phi_\alpha(u)dx = C_6 2^{(N-\beta)j}\int_{A_0} \Phi_\alpha(u_j)\,dy, \tag{3.5}$$

where $u_j(y) := u(2^j y)$ and $C_6 > 0$ is a constant independent of $j$. Recalling that $\|u\| \le 1$, we get

$$\begin{aligned}\|u_j\|^N_{W^{1,N}(A_0)} &= \int_{A_j}\left[|\nabla u|^N + \frac{|u|^N}{2^{jN}}\right]dx = \int_{A_j}\frac{1}{|x|^\gamma}\left[|x|^\gamma|\nabla u|^N + \frac{|x|^{\gamma-N}|u|^N}{2^{jN}|x|^{-N}}\right]dx \\ &\le 2^{(j_0+1)N}\int_\Omega \left[|x|^\gamma|\nabla u|^N + |x|^{\gamma-N}|u|^N\right]dx \le C_7\end{aligned}$$

where $C_7 > 0$ does not depend on $j$. Hence, by applying Lemma 3.1 once again, we obtain

$$\int_{A_0}\Phi_\alpha(u_j)dy \le \frac{C_0}{C_7}\|u_j\|^N_{W^{1,N}(\Omega_0)} \le C_8 \int_{A_j}\left[|x|^\gamma|\nabla u|^N + |x|^{\gamma-N}|u|^N\right]dx, \tag{3.6}$$

for some $C_8 = C_8(\gamma) > 0$ and any

$$0 < \alpha < \zeta_N^2 := \frac{\alpha_N}{C_7^{1/(N-1)}}.$$

Thus, defining $\alpha^* := \min\{\zeta_N^1, \zeta_N^2\}$, we can use (3.5)-(3.6) to conclude that

$$\int_{A_j} \frac{\Phi_\alpha(u)}{(1+|x|)^\beta} dx \le C_6 2^{(N-\beta)j} \left( C_8 \int_{A_j} \left[ |x|^\gamma |\nabla u|^N + |x|^{\gamma-N}|u|^N \right] dx \right),$$

for any $0 < \alpha < \alpha^*$. Since $\beta \ge N$, it follows that inequality (3.4) holds and the proof is finished. □

We can now present the the proof of our weighted Trudinger-Moser inequality.

*Proof of Theorem 1.4.* It follows from Lemma 3.2 that $L(\alpha, \gamma, \beta) < +\infty$, for any $0 < \alpha \le \alpha^*$. So, we need only to check that $\Phi_\alpha(u)(1 + |\cdot|)^{-\beta} \in L^1(\Omega)$, for $u \in E^{1,\gamma}$ and $\alpha > 0$. Given $\varepsilon > 0$, we pick $u_0 \in C_\delta^\infty(\Omega)$ such that

$$\|u - u_0\|_{E^{1,\gamma}} \le \varepsilon.$$

A simple computation shows that, for any $s \ge 0$,

$$0 \le \Phi'_\alpha(s) \le \alpha N' |s|^{1/(N-1)} e^{\alpha |s|^{N'}},$$

where $N' := N/(N-1)$ is the conjugated exponent of $N$. Thus, for any $s$, $t \ge 0$, we can use the Mean Value Theorem to obtain $\theta \in [\min\{s,t\}, \max\{s,t\}]$ such that

$$\Phi_\alpha(s) \le \Phi_\alpha(t) + \alpha N' |\theta|^{1/(N-1)} e^{\alpha |\theta|^{N'}} |t - s|.$$

Using this inequality with $s = |u|$ and $t = |u - u_0|$, we obtain a function $x \mapsto \theta(x)$ such that, for a.e. $x \in \Omega$,

$$\Phi_\alpha(|u|) \le \Phi_\alpha(|u-u_0|) + \alpha N' |\theta(x)|^{1/(N-1)} \psi(x) e^{\alpha |\theta(x)|^{N'}}, \tag{3.7}$$

where $\psi := \Big| |u - u_0| - |u| \Big| \in E^{1,\gamma}$ has the same support of $u_0$.

We now notice that, by (3.1),

$$\int_\Omega \frac{1}{(1+|x|)^\beta} \Phi_\alpha(|u-u_0|) dx \le \int_\Omega \frac{1}{(1+|x|)^\beta} \Phi_{\alpha \|u-u_0\|^{N'}_{E^{1,\gamma}}} \left( \frac{|u-u_0|}{\|u-u_0\|_{E^{1,\gamma}}} \right) dx,$$

and therefore we can choose $\varepsilon > 0$ small in such way that we can apply Lemma 3.2 to get

$$\int_\Omega \frac{1}{(1+|x|)^\beta} \Phi_\alpha(|u-u_0|) dx < C_N. \tag{3.8}$$

Since $u_0$ is bounded and $\theta$ is between $|u - u_0|$ and $|u|$, it is clear that

$$|\theta(x)| \le |u - u_0| + |u| \le C_1(|u| + 1), \quad \text{for a.e. } x \in \operatorname{supp} u_0,$$

and some $C_1 > 0$. Thus, we can use Hölder's inequality to obtain

$$\begin{aligned} \int_\Omega \frac{1}{(1+|x|)^\beta} |\theta|^{1/(N-1)} \psi e^{\alpha|\theta|^{N'}} dx &\le C_2 \int_\Omega (|u|+1)^{1/(N-1)} \psi e^{C_3 |u|^{N'}} dx \\ &\le C_4 \left( \int_\Omega e^{r_3 C_3 |u|^{N'}} dx \right)^{1/r_3}, \end{aligned}$$

where $C_4 := C_2 \|(|u|+1)\|^{1/(N-1)}_{L^{r_1/(N-1)}(\Omega)} \|\psi\|^{r_2}_{L^{r_2}(\Omega)}$ and $r_1$, $r_2$, $r_3$ are such that $1/r_1 + 1/r_2 + 1/r_3 = 1$, $r_1 \ge N(N-1)$ and $r_2 \ge N$. Since the first integral above can be considered only in the compact support of $u_0$, it follows from the classical Trudinger-Moser inequality that

$$\int_\Omega \frac{1}{(1+|x|)^\beta}|\theta|^{1/N}\psi e^{\alpha|\theta|^{N'}}dx < +\infty.$$

Recalling that $\Phi_\alpha(|u|) = \Phi_\alpha(u)$, we can use (3.7), (3.8) and the above expression to conclude that $(1+|\cdot|)^{-\beta}\Phi_\alpha(u) \in L^1(\Omega)$. □

## 4. Weak nonnegative solution for $(\mathcal{P}_\lambda)$

In this section, we apply our abstract result for obtaining a weak solution for problem $(\mathcal{P}_\lambda)$. In order to do this, we first use $(a_0)$, $(k_0)$ and Theorem 1.3 to get the compact embedding

$$E_a \hookrightarrow L^q_k, \quad \text{for all } q \geq N. \tag{4.1}$$

As a direct consequence of $(a_0)$ and Corollary 1.2, it follows that $E_a \subset E^{1,\gamma}$. Thus, we can use condition $(k_0)$ and Theorem 1.4 to get

$$k(\cdot)\Phi_\alpha(u) \in L^1(\Omega), \quad \text{for all } \; \alpha > 0, \;\; u \in E_a. \tag{4.2}$$

Moreover, from $(a_0)$ and Corollary 1.2, there exists $C_0 > 0$ such that

$$\|u\| \leq C_0\|u\|_{E_a}, \quad \text{for all } u \in E_a$$

and hence the following Trudinger-Moser inequality in the space $E_a$ holds:

**Lemma 4.1.** *Let $\alpha^* > 0$ be given by Theorem 1.4. Then*

$$\sup_{\{u\in E_a:\|u\|_{E_a}\leq 1\}} \int_\Omega k(x)\Phi_\alpha(u)dx < +\infty,$$

*for any $0 < \alpha \leq \overline{\alpha} := \alpha^*/C_0^{N/(N-1)}$.*

*Proof.* If $\|u\|_{E_a} \leq 1$, then $\|u/C_0\| \leq 1$. Using condition $(k_0)$, (3.1) and Theorem 1.4, we get

$$\int_\Omega k(x)\Phi_\alpha(u)dx \leq k_0 \int_\Omega \frac{1}{(1+|x|)^\beta}\Phi_{\alpha C_0^{N'}}\left(\frac{u}{C_0}\right)dx \leq C_1,$$

for some $C_1 > 0$ independent of $u$. □

For any given $\varepsilon > 0$, $\alpha > \alpha_0$ and $r \geq 1$, we can use $(f_0) - (f_1)$ to obtain $C > 0$ such that

$$|f(s)| \leq \varepsilon|s|^{N-1} + C|s|^{r-1}\Phi_\alpha(s), \quad |F(s)| \leq \varepsilon|s|^N + C|s|^r\Phi_\alpha(s), \tag{4.3}$$

for any $s \in \mathbb{R}$. Given $u \in E_a$, we can apply the above inequality to get

$$\int_\Omega k(x)F(u)dx \leq \varepsilon\int_\Omega k(x)|u|^N dx + C\int_\Omega k(x)|u|^r\Phi_\alpha(u)dx.$$

By (4.1), the first integral on the right-hand side above is finite. Moreover, picking $r_1$, $r_2 > 1$ such that $1/r_1 + 1/r_2 = 1$, we can use Hölder's inequality together with (4.1) and (4.2), to get

$$\int_\Omega k(x)|u|^r \Phi_\alpha(u)dx \le \|u\|^r_{L_k^{r_1 r}} \left(\int_\Omega k(x)\Phi_{r_2\alpha}(u)dx\right)^{1/r_2} < +\infty,$$

where we also have used the inequality (see [32, Lemma 2.1])

$$[\Phi_\alpha(s)]^r \le \Phi_{r\alpha}(s), \qquad s \in \mathbb{R},\ r > 1. \tag{4.4}$$

Hence, the functional $I_\lambda : E_a \to \mathbb{R}$ given by

$$I_\lambda(u) := \frac{1}{N}\|u\|^N_{E_a} - \lambda \int_\Omega k(x)F(u)dx$$

is well defined and standard arguments show that $I_\lambda \in C^1(E_a, \mathbb{R})$ with

$$I'_\lambda(u)\varphi = \int_\Omega a(x)|\nabla u|^{N-2}(\nabla u \cdot \nabla\varphi)\,dx - \lambda \int_\Omega k(x)f(u)\varphi\,dx, \quad \text{for all } u, \varphi \in E_a.$$

Consequently, critical points of $I_\lambda$ are weak solutions for problem ($\mathcal{P}_\lambda$).

The next lemma shows that the functional $I_\lambda$ satisfies the Mountain Pass geometry.

**Lemma 4.2.** *There are constants $\rho, \tau > 0$ such that $I_\lambda(u) \ge \tau$, for any $\|u\|_{E_a} = \rho$. Furthermore, there exists $e \in E_a$ such that $\|e\|_{E_a} > \rho$ and $I_\lambda(e) < 0$.*

*Proof.* From (4.1) and (4.3), we obtain $C_1 > 0$ such that

$$\int_\Omega k(x)F(u)dx \le C_1\varepsilon\|u\|^N_{E_a} + C\int_\Omega k(x)|u|^r\Phi_\alpha(u)dx.$$

By Hölder's inequality, (3.1) and (4.4), we deduce

$$\int_\Omega k(x)|u|^r\Phi_\alpha(u)dx \le \|u\|^r_{L_k^{r_1 r}} \left(\int_\Omega k(x)\Phi_{r_2\alpha\|u\|^{N'}_{E_a}}\left(\frac{u}{\|u\|_{E_a}}\right)dx\right)^{1/r_2}.$$

Hence, choosing $\rho_1 > 0$ such that $r_2\alpha\rho_1^{N'} \le \overline{\alpha}$, we can apply (4.1) and Lemma 4.1 to get

$$\int_\Omega k(x)|u|^r\Phi_\alpha(u)dx \le C_2\|u\|^r_{E_a},$$

whenever $\|u\|_{E_a} \le \rho_1$. Thus, from the above estimates, one has

$$I_\lambda(u) \ge \frac{1}{N}\|u\|^N_{E_a} - \lambda C_1\varepsilon\|u\|^N_{E_a} - \lambda C_3\|u\|^r_{E_a} = \|u\|^N_{E_a}\left(\frac{1}{N} - \lambda C_1\varepsilon - \lambda C_3\|u\|^{r-N}_{E_a}\right),$$

whenever $\|u\|_{E_a} \le \rho_1$. The first statement of the lemma follows from the above expression if we pick $0 < \varepsilon < 1/(N\lambda C_1)$ and $r > N$.

In order to prove the second one, we take $\varphi \in C^\infty_\delta(\Omega) \setminus \{0\}$. From ($f_2$) it is possible to obtain constants $C_4, C_5 > 0$ such that $F(s) \ge C_4|s|^\theta - C_5$, for any $s \in \mathbb{R}$. So, it follows that

$$I_\lambda(t\varphi) \le \frac{t^N}{N}\|\varphi\|^N_{E_a} - \lambda C_4 t^\theta \int_{\operatorname{supp}\varphi} k(x)|\varphi|^\theta dx + \lambda C_5\int_{\operatorname{supp}\varphi} k(x)dx,$$

for any $t > 0$. Since $\theta > N$, the last inequality implies that $\lim_{t\to+\infty} I_\lambda(t\varphi) = -\infty$, and so there exists $t > 0$ large in such a way that the desired result holds for $e = t\varphi$. $\square$

It follows from Lemma 4.2 that the minimax level

$$c_\lambda := \inf_{g\in\Gamma} \max_{t\in[0,1]} I_\lambda(g(t)) \geq \tau > 0,$$

where $\Gamma := \{g \in C([0,1], E_a) : g(0) = 0 \text{ and } I_\lambda(g(1)) < 0\}$ is well defined. Moreover, the following estimate holds true.

**Lemma 4.3.** *There exists $\lambda^* > 0$ such that, for any $\lambda \geq \lambda^*$, it holds*

$$c_\lambda < c_0 := \left(\frac{\overline{\alpha}}{\alpha_0}\right)^{N-1} \left(\frac{\theta - N}{N\theta}\right).$$

*Proof.* Let $R > 0$ be such that $|\Omega \cap B_{R/2}| > 0$ and pick a function $\varphi \in C^\infty(\mathbb{R}^N)$ such that $\varphi \equiv 1$ in $B_{R/2}$, $\varphi \equiv 0$ outside $B_R$ and $0 \leq \varphi \leq 1$ in $\mathbb{R}^N$. Here, $|\cdot|$ stands for the Lebesgue measure of a set. If we define $\widetilde{\varphi} := \varphi|_\Omega \in C_\delta^\infty(\Omega)$, we can easily use $\int_\Omega k(x)F(\widetilde{\varphi}) > 0$ to conclude that $I_\lambda(\widetilde{\varphi}) < 0$, for any $\lambda > \lambda_*$. Hence, for these values of $\lambda$, we have that the path $g(t) := t\widetilde{\varphi}$ belongs to $\Gamma$. Now using $(f_3)$ and recalling that $\varphi \equiv 1$ in $B_{R/2}$, we obtain

$$I_\lambda(t\widetilde{\varphi}) \leq \frac{t^N}{N} \int_{\Omega\cap B_R} a(x)|\nabla\widetilde{\varphi}|^N dx - \lambda \int_{\Omega\cap B_R} k(x)F(t\widetilde{\varphi})\, dx \leq C_1 t^N - \lambda C_2 t^\nu.$$

with

$$C_1 := \frac{1}{N} \left\|a|\nabla\widetilde{\varphi}|^N\right\|_{L^\infty(\Omega\cap B_R)} |\Omega \cap B_R|, \quad C_2 := C_F \int_{\Omega\cap B_R} k(x)|\widetilde{\varphi}|^\nu\, dx.$$

Thus, using the definition of $c_\lambda$, we get

$$\begin{aligned} c_\lambda \leq \max_{t\geq 0} I_\lambda(t\widetilde{\varphi}) &\leq \max_{t\geq 0} \left[C_1 t^N - \lambda C_2 t^\nu\right] \\ &= \frac{1}{(\lambda C_2)^{N/(\nu-N)}} \left(\frac{NC_1}{\nu}\right)^{\nu/(\nu-N)} \left(\frac{\nu - N}{N}\right). \end{aligned}$$

Since $\nu > N$, the right-hand side above goes to zero, as $\lambda \to +\infty$. Hence, there exists $\lambda^* > \lambda_*$ such that $c_\lambda < c_0$, for any $\lambda > \lambda^*$. $\square$

**Lemma 4.4.** *The functional $I_\lambda$ satisfies the $(PS)_c$ condition for all $c < c_0$, that is, every sequence $(u_n) \subset E_a$ such that*

$$\lim_{n\to+\infty} I_\lambda(u_n) = c \quad \text{and} \quad \lim_{n\to+\infty} \|I'_\lambda(u_n)\|_{E_a^*} = 0 \tag{4.5}$$

*has a convergent subsequence.*

*Proof.* From $(k_0)$, $(f_2)$ and (4.5), one deduces

$$\begin{aligned} C_1 + C_2\|u_n\|_{E_a} \geq I_\lambda(u_n) - \frac{1}{\theta} I'_\lambda(u_n)u_n &= \left(\frac{1}{N} - \frac{1}{\theta}\right)\|u_n\|^N_{E_a} \\ &\quad + \lambda \int_\Omega k(x)\left(\frac{1}{\theta} f(u_n)u_n - F(u_n)\right)dx \\ &\geq \left(\frac{1}{N} - \frac{1}{\theta}\right)\|u_n\|^N_{E_a}. \end{aligned}$$

It follows from $\theta > N$ that $(u_n)$ is bounded in $E_a$ and hence, up to a subsequence, we may assume that $u_n \rightharpoonup u$ weakly in $E_a$.

We claim that

$$\int_\Omega k(x) f(u_n)(u_n - u)dx = o_n(1). \tag{4.6}$$

If this is true, we can use $I'_\lambda(u_n)(u_n - u) = o_n(1)$ to conclude that

$$\int_\Omega a(x)|\nabla u_n|^{N-2}\nabla u_n \cdot \nabla(u_n - u)dx = o_n(1).$$

On the other hand, from the weak convergence, we have

$$\int_\Omega a(x)|\nabla u|^{N-2}\nabla u \cdot \nabla(u_n - u)dx = o_n(1).$$

Consequently, we get

$$\int_\Omega a(x)[|\nabla u_n|^{N-2}\nabla u_n - |\nabla u|^{N-2}\nabla u] \cdot \nabla(u_n - u)dx = o_n(1).$$

This, together with the inequality (see [25, inequality (2.2)])

$$\left(|y_1|^{N-2}y_1 - |y_2|^{N-2}y_2\right) \cdot (y_1 - y_2) \geq C(N)|y_1 - y_2|^N, \quad \forall\, y_1,\, y_2 \in \mathbb{R}^N,$$

implies that

$$C(N)\|u_n - u\|^N_{E_a} = C(N)\int_\Omega a(x)|\nabla(u_n - u)|^N dx \leq o_n(1),$$

and therefore $u_n \to u$ strongly in $E_a$.

Now we will verify that (4.6) holds. To see this, we first use (4.3) to compute

$$\begin{aligned} \left|\int_\Omega k(x) f(u_n)(u_n - u)dx\right| &\leq \varepsilon \int_\Omega k(x)|u_n|^{N-1}|u_n - u|dx \\ &\quad + C_3 \int_\Omega k(x)|u_n|^{r-1}\Phi_\alpha(u_n)|u_n - u|dx. \end{aligned}$$

Using Hölder's inequality, (4.1) and the fact that $(u_n)$ is bounded in $E_a$, one has

$$\begin{aligned} \int_\Omega k(x)|u_n|^{N-1}|u_n - u|dx &\leq \|u_n\|^{N-1}_{L^N_k}\|u_n - u\|_{L^N_k} \\ &\leq C_4\left(\|u_n\|^N_{E_a} + \|u_n\|^{N-1}_{E_a}\|u\|_{E_a}\right) \leq C_5, \end{aligned}$$

for any $n \in \mathbb{N}$. Since $\varepsilon > 0$ is arbitrary, it remains to be proved that

$$\int_\Omega k(x)|u_n|^{r-1}\Phi_\alpha(u_n)|u_n - u|dx = o_n(1). \tag{4.7}$$

For this purpose, from $(f_2)$ we obtain

$$c = \lim_{n\to+\infty}\left(I_\lambda(u_n) - \frac{1}{\theta}I'_\lambda(u_n)u_n\right) \geq \left(\frac{1}{N} - \frac{1}{\theta}\right)\lim_{n\to+\infty}\|u_n\|^N_{E_a},$$

and hence the hypothesis on $c$ implies that

$$\lim_{n\to+\infty}\|u_n\|^{N'}_{E_a} \leq \left(\frac{N\theta}{\theta - N}\right)^{1/(N-1)} c^{1/(N-1)} < \frac{\overline{\alpha}}{\alpha_0}.$$

Thus, we can choose $\alpha > \alpha_0$ and $r_1 > 1$ such that $r_1\alpha\|u_n\|^{N'}_{E_a} \leq \overline{\alpha}$, for all $n \in \mathbb{N}$ large enough. Applying Hölder's inequality, (3.1), (4.4) and Theorem 1.4, we conclude that

$$\begin{aligned}\int_\Omega k(x)|u_n|^{r-1}\Phi_\alpha(u_n)|u_n - u|dx &\leq \left(\int_\Omega k(x)\Phi_{r_1\alpha\|u_n\|^{N'}_{E_a}}\left(\frac{u_n}{\|u_n\|_{E_a}}\right)dx\right)^{1/r_1} \\ &\quad\times \|u_n\|^{r-1}_{L_k^{r_2(r-1)}}\|u_n - u\|_{L_k^{r_3}} \\ &\leq C_6\|u_n\|^{r-1}_{L_k^{r_2(r-1)}}\|u_n - u\|_{L_k^{r_3}},\end{aligned}$$

where $1/r_1 + 1/r_2 + 1/r_3 = 1$, $r_3 \geq N$ and $r_2(r-1) \geq N$. The convergence in (4.7) is now a consequence of the above expression and (4.1). The lemma is proved. ☐

We are ready to prove our existence result.

*Proof of Theorem 1.5.* Let $\lambda^* > 0$ be given by Lemma 4.3 and suppose that $\lambda > \lambda^*$. According to all the previous lemmas we may invoke the Mountain Pass Theorem [3] to obtain $u_\lambda \in E_a$ such that $I'_\lambda(u_\lambda) = 0$ and $I_\lambda(u_\lambda) = c_\lambda > 0$. In order to prove that this critical point can be taken nonnegative, we notice that we may assume $f(s) = 0$, for $s \leq 0$, and repeat all the previous calculations. So, if we set $u_\lambda^-(x) := \max\{-u_\lambda(x), 0\}$, we obtain $0 = I'_\lambda(u_\lambda)u_\lambda^- = -\|u_\lambda^-\|^N_{E_a}$, and therefore $u_\lambda \geq 0$ a.e. in $\Omega$. The theorem is proved. ☐

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