ON A HARDY-SOBOLEV TYPE INEQUALITY AND APPLICATIONS

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ABSTRACT. In this paper, we prove a new Friedrich-type inequality. As an application, we derive some existence and nonexistence results to the quasilinear elliptic problem with Robin boundary condition

$$\begin{cases} -\mathrm{div}(|\nabla u|^{N-2}\nabla u) + h(x)|u|^{q-2}u = \lambda k(x)|u|^{p-2}u, & \text{in } \Omega, \\ |\nabla u|^{N-2}(\nabla u\cdot\nu) + |u|^{N-2}u = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is an exterior domain such that $0 \notin \overline{\Omega}$.

1. Introduction and main results

Let $\Omega \subset \mathbb{R}^N$ be an exterior domain, that is, an open set such that $\mathbb{R}^N \setminus \Omega$ is bounded, and consider the quasilinear problem

$$\begin{cases} -\mathrm{div}(|\nabla u|^{m-2}\nabla u) = f(x,u), & \text{in } \Omega \\ |\nabla u|^{m-2}(\nabla u\cdot\nu) + a(x)|u|^{m-2}u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $1 < m < N$, $N \geq 2$ and ν is the unit outward normal vector on $\partial\Omega$. Existence, non-existence and multiplicity of solutions for the above problem have been extensively investigated under different conditions on the weight a and the nonlinearity f, see for instance [3, 4, 6, 8, 10, 11, 12, 13]. This kind of problem is important because it arises in the study of nonlinear diffusion equations, in particular, in the mathematical modeling of non-Newtonian fluids. For a physical background, we refer the reader to [7, 12] and references therein.

A common aspect in most of the early papers is the use of a Friedrich type inequality proved by K. Pflüger in [12]. In order to present it, we suppose that, for contants $C_1, C_2 > 0$,

$$\frac{C_1}{(1+|x|)^{m-1}} \leq l(x) \leq \frac{C_2}{(1+|x|)^{m-1}}, \quad \text{for a.e. } x \in \Omega,$$

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and call H the completion of the $C_0^\infty(\mathbb{R}^N)-$functions restricted to Ω with respect to the norm

$$\|u\|_H = \left(\int_\Omega |\nabla u|^m dx + \int_\Omega \frac{|u|^m}{(1+|x|)^m} dx\right)^{1/m}.$$

In this setting, there holds (see [9, 12])

$$\int_\Omega \frac{|u|^m}{(1+|x|)^m} dx \le C\left(\int_\Omega |\nabla u|^m dx + \int_{\partial\Omega} \frac{|\nu\cdot x|}{(1+|x|)^m}|u|^m d\sigma\right), \tag{1.2}$$

where $C>0$ is a positive constant. Using this inequality, it can be shown that the norm $\|\cdot\|_H$ is equivalent to

$$\|u\|_{H,\partial} = \left(\int_\Omega |\nabla u|^m + \int_{\partial\Omega} a(x)|u|^m d\sigma\right)^{1/m}.$$

As an application, some results of existence, non-existence and multiplicity to problem (1.1) were obtained.

It is natural to ask if (1.2) holds in the borderline case $m=N$. In the first part of this paper, after proving an interesting inequality for compacted supported functions (see Proposition 2.1), we give a negative answer for this question. More specifically, we denote by $C_\delta^\infty(\Omega)$ the space of $C_0^\infty(\mathbb{R}^N)$-functions restricted to Ω and prove the following:

Theorem 1.1. *Suppose that $0\notin\overline{\Omega}$ and $\gamma>N$. Then, for any $u\in C_\delta^\infty(\Omega)$, there holds*

$$\int_\Omega \frac{|u|^N}{(1+|x|)^\gamma} dx \le C(\gamma,N,\Omega)\left(\int_\Omega |\nabla u|^N dx + \int_{\partial\Omega} |u|^N d\sigma\right), \tag{1.3}$$

where

$$C(\gamma,N,\Omega) := \max\left\{d_\Omega^{-\gamma+1}, d_\Omega^{-\gamma+N}\right\}\cdot\begin{cases}\left(\dfrac{N}{\gamma-N}\right)^N, & \text{if } N<\gamma<2N,\\[2mm] \dfrac{1}{\gamma-2N+1}, & \text{if } \gamma\ge 2N,\end{cases}$$

and $d_\Omega := dist(0,\partial\Omega)>0$. Moreover, if $\Omega=\{x\in\mathbb{R}^N : |x|>1\}$ and $\gamma\le N$, the inequality in (1.3) is false for any constant $C(\gamma,N)>0$, and therefore (1.2) does not hold with $m=N$.

It is worth noticing that, although the answer for the general question is negative, the abstract framework developed here permits us to consider a variation of problem (1.1) in the case $m=N$. To be more precise, in the second part of this paper, we study the quasilinear problem

$$(P_\lambda)\qquad \begin{cases} -\mathrm{div}(|\nabla u|^{N-2}\nabla u) + h(x)|u|^{q-2}u = \lambda k(x)|u|^{p-2}u, & \text{in } \Omega,\\ |\nabla u|^{N-2}(\nabla u\cdot\nu) + |u|^{N-2}u = 0, & \text{on } \partial\Omega,\end{cases}$$

where λ is a real parameter, $0\notin\overline{\Omega}$ and the weight functions k, h satisfy

(k_1) $k : \Omega \to \mathbb{R}$ is a measurable function, and there exist $k_0 > 0$, $\beta > N$, such that

$$0 < k(x) \leq \frac{k_0}{(1+|x|)^\beta}, \quad \text{for a.e. } x \in \Omega;$$

(h_1) $h : \Omega \to \mathbb{R}$ is a positive measurable function;

(h_2) there holds

$$\int_\Omega \frac{k(x)^{q/(q-p)}}{h(x)^{p/(q-p)}} dx < \infty.$$

We are going to consider problem (P_λ) in two different settings, depending on the values of q, p and $\lambda > 0$. Our results can be stated as follows:

Theorem 1.2. *Suppose that* (k_1), $(h_1) - (h_2)$ *and* $p < q$ *hold. Then,*

(i) *if* $N \leq p$, *there exists* $\lambda_* > 0$ *such that problem* (P_λ) *has only the zero solution, for any* $\lambda < \lambda_*$,

(ii) *if* $\min\{2, N\} < p$, *there exists* $\lambda^* > \lambda_*$ *such that problem* (P_λ) *has at least a non-negative non-zero weak solution, for any* $\lambda > \lambda^*$.

Theorem 1.3. *Suppose that* (k_1), $N \leq q < p$ *and*

$(\widetilde{h_1})$ $h : \Omega \to \mathbb{R}$ *is a non-negative measurable function*

hold. Then problem (P_λ) *has a non-negative non-zero weak solution, for any* $\lambda > 0$.

Our interest in the study of problem (P_λ) comes from the works of Alama-Tarantello [1] (where the integral condition (h_2) has appeared), Filippucci-Pucci-Radulescu [8], Lyberopoulos [10], Perera [11], Pflüger [12], and others. With our abstract results at hand, we are able to perform a variational approach and prove Theorems 1.2 and 1.3. For the first one, we check that the associated energy functional is coercive and has negative energy for λ large, and therefore we can use minimization techniques. In the case $p > q$, we apply the classical Moutain Pass theorem. We want to remark that the main feature of this class of problem is that we are dealing with an indefinite nonlinearity and the weight functions k and h are not radial. Thus, we also face the difficulty to establish new Sobolev embeddings in our setting. Our results concerning problem (P_λ) generalize and/or complement the aforementioned works.

The remainder of the paper is organized as follows. In Section 2, we establish some weighted Sobolev embedding and prove Theorem 1.1. The two further sections are devoted to the proof of Theorems 1.2 and 1.3, respectively.

2. Variational framework

In this section, beside proves Theorem 1.1, we present the variational framework to deal with problem (P_λ). The basic condition (k_1) will be assumed along all the paper. For any $R > 0$, we denote by B_R the open ball $\{x \in \mathbb{R}^N : |x| < R\}$. The complement of a set $\Gamma \subset \mathbb{R}^N$ is denoted by Γ^c. Finally, we denote by $C_1, C_2, \ldots$, positive constants (possibly different).

2.1. **A Friedrich type inequality.** Our goal in this subsection is to establish the proof of our first main theorem. We recall that $C_\delta^\infty(\Omega)$ is the space of $C_0^\infty(\mathbb{R}^N)$-functions restricted to Ω. The next auxiliary result is a key point.

Proposition 2.1. *Suppose that $1 < p < \infty$ and let $\alpha \in \mathbb{R}$ be such that $\alpha \neq -N$. Then, there exists $C_0 > 0$ such that*

$$\int_\Omega |x|^\alpha |u|^p dx \leq C_0 \left(\int_\Omega |x|^{\alpha+p} |\nabla u|^p dx + \int_{\partial\Omega} |x|^{\alpha+1} |u|^p d\sigma \right), \tag{2.1}$$

for any $u \in C_\delta^\infty(\Omega)$.

Proof. Let w, v be regular functions. By applying the Divergence Theorem, we get

$$\int_\Omega w_{x_i} v \, dx = -\int_\Omega w v_{x_i} \, dx + \int_{\partial\Omega} w v \nu_i \, d\sigma.$$

Since $(|x|^\alpha)_{x_i} = \alpha |x|^{\alpha-2} x_i$, for $x \neq 0$ and $i = 1, \ldots, N$, we can choose $w = |x|^\alpha$, $v = x_i |u|^p$ and sum for $i = 1, \ldots, N$, to obtain

$$(\alpha + N) \int_\Omega |x|^\alpha |u|^p \, dx = -p \int_\Omega |x|^\alpha |u|^{p-2} u (x \cdot \nabla u) \, dx + \int_{\partial\Omega} |x|^\alpha |u|^p (x \cdot \nu) \, d\sigma,$$

which implies that

$$|\alpha + N| \int_\Omega |x|^\alpha |u|^p \, dx \leq p \int_\Omega |x|^{\alpha+1} |u|^{p-1} |\nabla u| \, dx + \int_{\partial\Omega} |x|^{\alpha+1} |u|^p \, d\sigma. \tag{2.2}$$

For any $\varepsilon > 0$, we can use Young's inequality to get

$$\begin{aligned} p \int_\Omega |x|^{\alpha+1} |u|^{p-1} |\nabla u| \, dx &= p \int_\Omega \left(|x|^{\alpha(p-1)/p} |u|^{p-1} \right) |x|^{[\alpha+1-\alpha(p-1)/p]} |\nabla u| \, dx \\ &\leq (p-1)\varepsilon \int_\Omega |x|^\alpha |u|^p \, dx + \frac{1}{\varepsilon^{p-1}} \int_\Omega |x|^{\alpha+p} |\nabla u|^p \, dx. \end{aligned}$$

If $\varepsilon < 1$, we can use the above inequality and (2.2) to obtain

$$(|\alpha + N| - (p-1)\varepsilon) \int_\Omega |x|^\alpha |u|^p dx \leq \frac{1}{\varepsilon^{p-1}} \left(\int_\Omega |x|^{\alpha+p} |\nabla u|^p dx + \int_{\partial\Omega} |x|^{\alpha+1} |u|^p d\sigma \right).$$

Recalling that $\alpha \neq -N$ and picking

$$0 < \varepsilon < \min \left\{ 1, \frac{|\alpha + N|}{(p-1)} \right\},$$

one has

$$\int_\Omega |x|^\alpha |u|^p \, dx \leq C_0 \left(\int_\Omega |x|^{\alpha+p} |\nabla u|^p \, dx + \int_{\partial\Omega} |x|^{\alpha+1} |u|^p \, d\sigma \right),$$

where

$$C_0 := \left[|\alpha + N| - (p-1)\varepsilon \right]^{-1} \varepsilon^{1-p}$$

and the lemma is proved. □

Remark 2.2. *It is worth noticing that, when considered only for $C_0^\infty(\Omega)$ functions, expression (2.1) is a Hardy type inequality (see [5, Theorem 1]).*

Our first main theorem is a consequence of this last proposition.

Proof of Theorem 1.1. We are going to use the proof of Proposition 2.1 with $p = N$ and $\alpha = -\gamma$. Define the function

$$g(\varepsilon) = \frac{1}{[\gamma - N - (N-1)\varepsilon]\,\varepsilon^{N-1}}, \quad \varepsilon \in \left(0, \frac{\gamma - N}{N-1}\right).$$

It achieves its minimum value at

$$\varepsilon_0 := \frac{\gamma - N}{N} < \frac{\gamma - N}{N-1},$$

with $g(\varepsilon_0) = [N/(\gamma - N)]^N$. If $N < \gamma < 2N$, then $\varepsilon_0 < 1$. On the other hand, if $\gamma \geq 2N$, then $g(1) \leq g(\varepsilon)$, for any $0 < \varepsilon < 1$. Since $|x| \geq d_\Omega$, for any $x \in \Omega$, inequality (1.3) is now a direct consequence of Proposition 2.1 and the definition of g.

Suppose now that $\Omega = B_1^c$ and $\gamma \leq N$. Considering the sequence of functions in $C_\delta^\infty(\Omega)$ defined by

$$u_n(x) := \begin{cases} n - \log|x|, & \text{if } 1 \leq |x| \leq e^n, \\ 0, & \text{if } |x| \geq e^n, \end{cases}$$

we see that

$$\int_{B_1^c} |\nabla u_n|^N \, dx = \int_{B_{e^n} \setminus B_1} |x|^{-N} \, dx = \omega_{N-1} \int_1^{e^n} r^{-N} r^{N-1} \, dr = n\omega_{N-1},$$

where ω_{N-1} is the measure of the unit sphere in $\mathbb{R}^N$. We may assume, with no loss of generality, that $0 \leq \gamma \leq N$. Hence, since $(1 + |x|) \leq 2|x|$ in B_1^c, one has

$$\int_{B_1^c} \frac{|u_n|^N}{(1+|x|)^\gamma} dx \geq \int_{B_1^c} \frac{|u_n|^N}{2^\gamma |x|^\gamma} dx = \frac{\omega_{N-1}}{2^\gamma} \int_1^{e^n} \frac{(n - \log r)^N}{r^N} r^{N-1} dr.$$

Considering the change of variables $t = n - \log r$, we obtain

$$\int_{B_1^c} \frac{|u_n|^N}{(1+|x|)^\gamma} dx \geq \frac{\omega_{N-1}}{2^\gamma} \int_0^n t^N dt = \frac{\omega_{N-1}}{2^\gamma} n^{N+1}.$$

Moreover,

$$\int_{\partial B_1^c} |u_n|^N d\sigma = n^N \int_{\partial B_1^c} d\sigma = \omega_{N-1} n^N.$$

Using the above inequalities we see that, if (1.3) holds, then

$$n^{N+1} \leq C_1(n + n^N),$$

for all $n \in \mathbb{N}$ and some $C_1 > 0$, which is impossible. □

2.2. **Sobolev embeddings.** With Theorem 1.1 at hand we are prepared to introduce the variational framework to deal with (P_λ). Given a positive function $\omega \in L^1_{loc}(\Omega)$ and $s \geq 1$, we denote by L^s_ω the weighted Lebesgue space

$$L^s_\omega := L^s(\Omega, \omega) = \left\{ u : \Omega \to \mathbb{R} \,:\, \|u\|_{L^s_\omega} := \left(\int_\Omega \omega(x) |u|^s dx \right)^{1/s} < +\infty \right\}.$$

For each $\gamma \in \mathbb{R}$, we denote by $E^{1,\gamma}$ the space obtained as the completion of $C_\delta^\infty(\Omega)$ with respect to the norm

$$\|u\|_{E^{1,\gamma}} := \left(\int_\Omega \left[|\nabla u|^N + \frac{|u|^N}{(1+|x|)^\gamma}\right] dx\right)^{1/N}.$$

For simplicity, we write E instead of $E^{1,\gamma}$ from now on.

In our first results we establish some embedding of E into suitable weighted Lebesgue spaces.

Proposition 2.3. *Suppose that $N < \gamma \le \beta$ and $N \le p \le N(\beta - N)/(\gamma - N)$. Then we have the continuous embedding $E \hookrightarrow L^p_{(1+|x|)^{-\beta}}$. Moreover, the embedding is compact if $N < \gamma < \beta$ and $N \le p < N(\beta - N)/(\gamma - N)$.*

Proof. For the first statement, we need to obtain $C_0 > 0$, such that

$$\int_\Omega \frac{|u|^p}{(1+|x|)^\beta} dx \le C_0 \left(\int_\Omega \left[|\nabla u|^N + \frac{|u|^N}{(1+|x|)^\gamma}\right] dx\right)^{p/N}, \quad \forall\, u \in E. \tag{2.3}$$

Let $j_0 \in \mathbb{N}$ be such that $\Omega^c \subset B_{2^{j_0}}$. Setting $\Omega_{j_0} := \Omega \cap B_{2^{j_0}}$, we have that $\Omega = \Omega_{j_0} \cup B^c_{2^{j_0}}$. Given $u \in E \subset W^{1,N}_{loc}(\Omega)$, from the Sobolev embedding $W^{1,N}(\Omega_{j_0}) \hookrightarrow L^p(\Omega_{j_0})$, we get

$$\int_{\Omega_{j_0}} \frac{|u|^p}{(1+|x|)^\beta} dx < \int_{\Omega_{j_0}} |u|^p dx \le C_1 \left(\int_{\Omega_{j_0}} \left[|\nabla u|^N + |u|^N\right] dx\right)^{p/N}.$$

Hence, since $(1+|x|)^\gamma \le (1+2^{j_0})^\gamma$ in Ω_{j_0}, there exists $C_2 = C_2(N, j_0, \gamma, p) > 0$ such that

$$\int_{\Omega_{j_0}} \frac{|u|^p}{(1+|x|)^\beta} dx \le C_2 \left(\int_{\Omega_{j_0}} \left[|\nabla u|^N + \frac{|u|^N}{(1+|x|)^\gamma}\right] dx\right)^{p/N}. \tag{2.4}$$

On the other hand, if we define $A_j := \{z \in \mathbb{R}^N : 2^{j_0} \cdot 2^j < |z| < 2^{j_0} \cdot 2^{j+1}\}$, for any given $j \in \mathbb{N} \cup \{0\}$, the change of variables $y := 2^{-j}x$ provides

$$\int_{A_j} \frac{|u|^p}{(1+|x|)^\beta} dx \le 2^{-\beta j} \int_{A_j} |u|^p dx = 2^{(N-\beta)j} \int_{A_0} |u_j(y)|^p dy,$$

where $u_j(y) := u(2^j y)$. Using the Sobolev embedding $W^{1,N}(A_0) \hookrightarrow L^p(A_0)$ we obtain $C_3 = C_3(N, j_0) > 0$, such that

$$\begin{aligned}\int_{A_0} |u_j(y)|^p dy &\le C_3 \left(\int_{A_0} \left[|\nabla u_j(y)|^N + |u_j(y)|^N\right] dy\right)^{p/N} \\ &= C_3 \left(\int_{A_j} \left[|\nabla u(x)|^N + 2^{-Nj}|u(x)|^N\right] dx\right)^{p/N}.\end{aligned}$$

Since $(1+2^{j_0}\cdot 2^{j+1}) < 2^{1+j_0}\cdot 2^{j+1}$ and $\gamma>0$, we have that

$$\int_{A_j} 2^{-Nj}|u(x)|^N dx \leq 2^{-Nj}(1+2^{j_0}\cdot 2^{j+1})^{\gamma}\int_{A_j}\frac{|u(x)|^N}{(1+|x|)^{\gamma}}dx$$
$$\leq 2^{(2+j_0)\gamma}2^{(\gamma-N)j}\int_{A_j}\frac{|u(x)|^N}{(1+|x|)^{\gamma}}dx.$$

So,

$$\int_{A_j}\frac{|u|^p}{(1+|x|)^{\beta}}dx \leq 2^{(N-\beta)j}C_3\left(\int_{A_j}\left[|\nabla u(x)|^N + C_4 2^{(\gamma-N)j}\frac{|u(x)|^N}{(1+|x|)^{\gamma}}\right]dx\right)^{p/N},$$

with $C_4 := 2^{(2+j_0)\gamma}\geq 1$. Thus,

$$\int_{A_j}\frac{|u|^p}{(1+|x|)^{\beta}}dx \leq C_5 2^{\mu_j}\left(\int_{A_j}\left[|\nabla u|^N + \frac{|u|^N}{(1+|x|)^{\gamma}}\right]dx\right)^{p/N}, \tag{2.5}$$

with $C_5 = C_3\cdot C_4^{p/N}>0$ and

$$\mu_j := \left[N-\beta+\frac{(\gamma-N)p}{N}\right]j.$$

Since $p\leq N(\beta-N)/(\gamma-N)$, one has $\mu_j\leq 0$, and therefore

$$\int_{A_j}\frac{|u|^p}{(1+|x|)^{\beta}}dx \leq C_5\left(\int_{A_j}\left[|\nabla u|^N + \frac{|u|^N}{(1+|x|)^{\gamma}}\right]dx\right)^{p/N}.$$

Thus, recalling that the function $s\mapsto s^{p/N}$ is super-additive for $p\geq N$, we conclude that

$$\sum_{j=0}^{\infty}\int_{A_j}\frac{|u|^p}{(1+|x|)^{\beta}}dx \leq C_5\sum_{j=0}^{\infty}\left(\int_{A_j}\left[|\nabla u|^N + \frac{|u|^N}{(1+|x|)^{\gamma}}\right]dx\right)^{p/N}$$
$$\leq C_5\left(\int_{B^c_{2^{j_0}}}\left[|\nabla u|^N + \frac{|u|^N}{(1+|x|)^{\gamma}}\right]dx\right)^{p/N}.$$

Combining the above estimate with (2.4), we obtain

$$\int_{\Omega}\frac{|u|^p}{(1+|x|)^{\beta}}dx \leq (C_2+C_5)\|u\|_E^p,$$

which proves (2.3).

For the compactness, we consider a sequence $(u_n)\subset E$ such that $u_n\rightharpoonup 0$ weakly in E. Given $\varepsilon>0$, we can use $p<N(\beta-N)/(\gamma-N)$ to obtain $j_1\in\mathbb{N}$ such that $2^{\mu_j}<\varepsilon$, for all $j>j_1$. Thus, from (2.5), we get

$$\int_{A_j}\frac{|u_n|^p}{(1+|x|)^{\beta}}dx < C_5\varepsilon\left(\int_{A_j}\left[|\nabla u_n|^N + \frac{|u_n|^N}{(1+|x|)^{\gamma}}\right]dx\right)^{p/N},$$

for any $j \geq j_1$. On the other hand, the compact embedding $W^{1,N}(\Omega_{j_0}) \hookrightarrow L^p(\Omega_{j_0})$ and $W^{1,N}(A_j) \hookrightarrow L^p(A_j)$ for $j \in \{0, 1, \ldots, j_1\}$, imply that

$$\begin{aligned}\int_\Omega \frac{|u_n|^p}{(1+|x|)^\beta} dx &\leq \int_{\Omega_{j_0}} \frac{|u_n|^p}{(1+|x|)^\beta} dx + \sum_{j=0}^{j_1} \int_{A_j} \frac{|u_n|^p}{(1+|x|)^\beta} dx + C_5 \varepsilon \|u_n\|_E^p \\ &= o_n(1) + C_5 \varepsilon \|u_n\|_E^p,\end{aligned}$$

where $o_n(1)$ stands for a quantity approaching zero as $n \to +\infty$. Since $\varepsilon > 0$ is arbitrary, the above expression implies that $u_n \to 0$ strongly in $L^p_{(1+|x|)^{-\beta}}$ and the theorem is proved. □

As a consequence of this last result together with Theorem 1.1, we get the following:

Corollary 2.4. *If $\gamma > N$, then the norms*

$$\|u\|_\partial := \left(\int_\Omega |\nabla u|^N dx + \int_{\partial\Omega} |u|^N d\sigma \right)^{1/N}$$

and $\|\cdot\|_E$ are equivalent in E.

Proof. It follows from (1.3) that

$$\|u\|_E^N \leq \int_\Omega |\nabla u|^N dx + C_1 \left(\int_\Omega |\nabla u|^N dx + \int_{\partial\Omega} |u|^N d\sigma \right) \leq C_2 \|u\|_\partial^N.$$

On the other hand, taking into account that $\partial\Omega$ is bounded, we can choose $R > 0$ sufficiently large such that the trace embedding $W^{1,N}(\Omega \cap B_R) \hookrightarrow L^N(\partial\Omega \cup \partial B_R)$ is continuous. Therefore, there exists $C_3 = C_3(R, \Omega) > 0$, such that

$$\int_{\partial\Omega} |u|^N \, d\sigma \leq C_3 \int_{\Omega \cap B_R} \left(|\nabla u|^N + |u|^N \right) dx \leq C_4 \left(\int_\Omega |\nabla u|^N dx + \int_\Omega \frac{|u|^N}{(1+|x|)^\gamma} dx \right),$$

where $C_4 = C_3(1+R)^\gamma$. Consequently,

$$\|u\|_\partial^N \leq \int_\Omega |\nabla u|^N dx + C_4 \int_\Omega \left[|\nabla u|^N + \frac{|u|^N}{(1+|x|)^\gamma} \right] dx \leq C_5 \|u\|_E^N,$$

and this yields the desired result. □

3. The case $p < q$

In this section we prove Theorem 1.2. Since $\beta > N$, we can choose γ sufficiently close to N in such way that

$$N < \gamma < \beta, \quad N \leq p < \frac{N(\beta - N)}{(\gamma - N)}. \tag{3.1}$$

We are going to look for solutions of problem (P_λ) in the subspace of E defined by

$$E^q := \left\{ u \in E : \int_\Omega h(x) |u|^q dx < \infty \right\}. \tag{3.2}$$

This is a reflexive Banach space when endowed with the norm

$$\|u\|_{E^q} := \left(\|u\|_\partial^N + \|u\|_{L_h^q}^N\right)^{1/N},$$

where $\|\cdot\|_\partial$ was defined in Corollary 2.4. Using this same corollary, (3.1) and Proposition 2.3, we conclude that the embedding $E^q \hookrightarrow L_k^p$ is compact.

Notice that one weak solution of problem (P_λ) is exactly a function $u \in E^q$ such that

$$\int_\Omega |\nabla u|^{N-2}\nabla u \cdot \nabla\varphi\, dx + \int_{\partial\Omega} |u|^{N-2}u\varphi\, d\sigma = \lambda\int_\Omega \left(\lambda k(x)|u|^{p-2}u - h(x)|u|^{q-2}u\right)\varphi\, dx,$$

for any $\varphi \in C_\delta^\infty(\Omega)$. So, the weak solutions are precisely the critical points of the functional $I_\lambda : E^q \to \mathbb{R}$ given by

$$I_\lambda(u) := \frac{1}{N}\|u\|_\partial^N + \frac{1}{q}\int_\Omega h(x)|u|^q\, dx - \frac{\lambda}{p}\int_\Omega k(x)|u|^p\, dx.$$

Using the abstract results of the previous section and standard arguments we can prove that $I_\lambda \in C^1(E^q, \mathbb{R})$.

In our first result we check that non-zero solutions do not exist if λ is close to 0.

Lemma 3.1. *Suppose that* $(h_1) - (h_2)$ *and* $N \le p < q$ *hold. Then, there exists* $\lambda_* > 0$ *such that problem* (P_λ) *has no non-zero weak solution if* $\lambda < \lambda_*$.

Proof. If $u \in E$ is a non-zero solution, we have that

$$\|u\|_\partial^N = \lambda\int_\Omega k(x)|u|^p dx - \int_\Omega h(x)|u|^q dx, \tag{3.3}$$

and therefore it is clear that $\lambda > 0$. Using Young's inequality with exponents $s = q/(q-p)$ and $s' = q/p$, we obtain

$$\lambda k(x)|s|^p = \frac{\lambda k(x)}{h(x)^{p/q}}\left(h(x)^{p/q}|s|^p\right) \le \frac{q-p}{q}\lambda^{q/(q-p)}\frac{k(x)^{q/(q-p)}}{h(x)^{p/(q-p)}} + \frac{p}{q}h(x)|s|^q,$$

for any $x \in \Omega$, $s \in \mathbb{R}$. The above inequality, (3.3) and $(h_1) - (h_2)$ provide

$$\|u\|_\partial^N \le C_1\lambda^{q/(q-p)} + \frac{p-q}{q}\int_\Omega h(x)|u|^q dx \le C_1\lambda^{q/(q-p)}, \tag{3.4}$$

with $C_1 := q/(q-p)\int_\Omega k(x)^{q/(q-p)}h(x)^{-p/(q-p)}dx$.

On the other hand, using (2.3), (3.3), (k_1) and (h_1) again, we get

$$C_2\left(\int_\Omega k(x)|u|^p dx\right)^{N/p} \le \|u\|_\partial^N \le \lambda\int_\Omega k(x)|u|^p dx,$$

with $C_2 := C_0 k_0^{-N/p} > 0$. If $p = N$, we conclude that $\lambda \ge \lambda_* := C_2$. Otherwise, this last inequality and (3.4) imply that

$$\left(C_2\lambda^{-1}\right)^{p/(p-N)} \le \int_\Omega k(x)|u|^p dx \le C_2^{-p/N}\|u\|_\partial^p \le C_2^{-p/N}C_1^{p/N}\lambda^{qp/[N(q-p)]}$$

and a straightforward computation shows that

$$\lambda \geq \lambda_* := \left[\frac{C_2^{p/(p-N)} C_2^{p/N}}{C_1^{p/N}} \right]^{qp/[N(q-p)]+p/(p-N)}.$$

So, we conclude that (P_λ) does not have non-zero solution if $\lambda < \lambda_*$. □

For the existence result, we need an elementary inequality. Let $A, B > 0$ and $q > p > 0$. A straightforward calculation shows that $f(s) := As^p - Bs^q$, for $s \geq 0$, achieves its maximum at $s_0 := [(pA)/(qB)]^{1/(q-p)}$. Hence, for any $s \in \mathbb{R}$,

$$A|s|^p - B|s|^q = f(|s|) \leq f(s_0) \leq As_0^p \leq A \left(\frac{A}{B} \right)^{p/(q-p)} = \frac{A^{q/(q-p)}}{B^{p/(q-p)}}. \tag{3.5}$$

The next lemma shows that we can deal with our problem via minimization arguments.

Lemma 3.2. *Suppose that $(h_1) - (h_2)$ and $p < q$ hold. Then I_λ is coercive and*

$$\lambda^* := \inf_{u \in E^q} \left\{ \frac{1}{N} \|u\|_\partial^N + \frac{1}{q} \int_\Omega h(x)|u|^q dx : \int_\Omega k(x)|u|^p dx = p \right\} > 0. \tag{3.6}$$

Proof. Since $p < q$, it follows from (3.5) and (h_2) that

$$\frac{\lambda}{p} \int_\Omega k(x)|u|^p dx - \frac{1}{2q} \int_\Omega h(x)|u|^q dx \leq C_1 \int_\Omega \frac{k(x)^{q/(q-p)}}{h(x)^{p/(q-p)}} dx = C_2.$$

Thus, since we can write

$$I_\lambda(u) = \frac{1}{N} \|u\|_\partial^N + \frac{1}{2q} \int_\Omega h(x)|u|^q \, dx - \frac{\lambda}{p} \int_\Omega k(x)|u|^p \, dx + \frac{1}{2q} \int_\Omega h(x)|u|^q \, dx,$$

we conclude that

$$I_\lambda(u) \geq \frac{1}{N} \|u\|_\partial^N + \frac{1}{2q} \int_\Omega h(x)|u|^q dx - C_2.$$

This and the definition of $\| \cdot \|_{E^q}$ show that $I_\lambda(u) \to +\infty$, as $\|u\|_{E^q} \to +\infty$.

We now prove that $\lambda^* > 0$. Suppose, by contradiction, that $\lambda^* = 0$. Then there exists $(u_n) \subset E^q$ such that

$$\frac{1}{N} \|u_n\|_\partial^N + \frac{1}{q} \int_\Omega h(x)|u_n|^q dx = o_n(1), \qquad \int_\Omega k(x)|u_n|^p dx = p,$$

where $o_n(1)$ stands for a quantity approaching zero as $n \to +\infty$. Hölder's inequality with exponents $s = q/p$, $s' = q/(q-p)$ and the integrability condition (h_2) provide

$$p = \int_\Omega h(x)|u_n|^p \frac{k(x)}{h(x)} dx \leq \left(\int_\Omega h(x)|u_n|^q dx \right)^{p/q} \left(\int_\Omega \frac{k(x)^{q/(q-p)}}{h(x)^{p/(q-p)}} dx \right)^{(p-q)/q} = o_n(1),$$

which is a contradiction. □

We are ready to present the proof of our first application.

Proof of Theorem 1.2. We focus on item (ii), since the non-existence part is exactly Lemma 3.1. Let $\lambda^* > 0$ be given by (3.6) and $\lambda > \lambda^*$. From the embedding $E^q \hookrightarrow L^p_k$, we conclude that I_λ maps bouded sets into bounded sets. So, recalling that I_λ is coercive, we conclude that

$$c_\lambda := \inf_{u \in E^q} I_\lambda(u) > -\infty.$$

We claim that, if $\lambda > \lambda^*$, then $c_\lambda < 0$. Indeed, using the definition of λ^* we obtain $u_\lambda \in E^q$ such that $\int_\Omega k(x)|u_\lambda|^p dx = p$ and

$$\lambda > \frac{1}{N}\|u_\lambda\|_\partial^N + \frac{1}{q}\int_\Omega h(x)|u_\lambda|^q dx.$$

This implies that $I_\lambda(u_\lambda) < 0$, and therefore $c_\lambda < 0$. Once we have proved that

$$J(u) := \int_\Omega F(x,u)dx = \int_\Omega \left[\frac{\lambda k(x)|u|^p}{p} - \frac{h(x)|u|^q}{q}\right] dx,$$

is weakly continuous, the direct method of calculus of variations (cf. [15, Theorem 1.2]) shows that I_λ has a global minimum u_λ. Since $I_\lambda(u_\lambda) = c_\lambda < 0$, we have that $u_\lambda \neq 0$. Noticing that $I_\lambda(u_\lambda) = I_\lambda(|u_\lambda|)$, we may assume that $u_\lambda \geq 0$ and the theorem is proved.

It remains to prove the claimed regularity for J. For simplicity, we will assume $\lambda = 1$. Let $(u_n) \subset E^q$ be such that $u_n \rightharpoonup u_0$ weakly in E^q. Using the Fundamental Theorem of Calculus, we obtain

$$\begin{aligned} F(x,u_n) - F(x,u_0) = &\int_0^1 \int_0^t F_{ss}(x, u_0 + \tau(u_n - u_0))(u_n - u_0)^2 d\tau dt \\ &+ F_s(x,u_0)(u_n - u_0). \end{aligned}$$

A standard computation shows that $F_s(x,s) = k(x)|s|^{p-2}s - h(x)|s|^{q-2}s$ and $F_{ss}(x,s) = (p-1)k(x)|s|^{p-2} - (q-1)h(x)|s|^{q-2}$. This, the above equality and (3.5) (with $p-2$ instead of p and $q-2$ instead of q) imply that

$$\left|F_{ss}(x, u_0 + \tau(u_n - u_0))(u_n - u_0)^2\right| \leq C_1 k(x) \left(\frac{k(x)}{h(x)}\right)^{(p-2)/(q-p)} (u_n - u_0)^2.$$

Therefore,

$$\left|\int_\Omega [F(x,u_n) - F(x,u_0)]\, dx\right| \leq C_1 I_n^1 + I_n^2,$$

with

$$I_n^1 := \int_\Omega k(x)\left(\frac{k(x)}{h(x)}\right)^{(p-2)/(q-p)} (u_n - u_0)^2 dx, \quad I_n^2 := \left|\int_\Omega F_u(x,u_0)(u_n - u_0) dx\right|,$$

and therefore it sufficent to show that $I_n^1 = o_n(1)$ and $I_n^2 = o_n(1)$.

For the first one, we use Hölder's inequality with exponents $s = p/(p-2)$, $s' = p/2$, hypothesis (k_2) and the compactness of $E \hookrightarrow L^p_k$ to get

$$I_n^1 = \left(\int_\Omega \frac{k(x)^{q/(q-p)}}{h(x)^{p/(q-p)}} dx\right)^{(p-2)/p} \|u_n - u_0\|^2_{L^p_k} = o_n(1).$$

We now estimate I_n^2 observing that the linear functional

$$\Psi(v) := \int_\Omega F_s(x,u_0)v dx = \int_\Omega \left(k(x)|u_0|^{p-2}u_0 - h(x)|u_0|^{q-2}u_0\right) v dx, \quad v \in E^q,$$

is such that

$$|\Psi(v)| \le \|u_0\|_{L_k^p}^{p-1}\|v\|_{L_k^p} + \|u_0\|_{L_h^q}^{q-1}\|v\|_{L_h^q} \le C_2\|v\|_{E^q}.$$

Hence, Ψ is continuous and the weak convergence of (u_n) implies that $I_n^2 = |\Psi(u_n - u_0)| = o_n(1)$, finishing the proof. □

4. The case $p > q$

This section is devoted to the proof of Theorem 1.3. As in the last section, we pick $\gamma > N$ such that

$$N < \gamma < \beta, \quad N \le q < p < \frac{N(\beta - N)}{(\gamma - N)}$$

and consider E^q defined in (3.2). In order to find non-negative solutions for (P_λ), we consider now the energy functional given by

$$I_\lambda(u) := \frac{1}{N}\|u\|_\partial^N + \frac{1}{q}\int_\Omega h(x)|u|^q\, dx - \frac{\lambda}{p}\int_\Omega k(x)(u^+)^p\, dx,$$

where $u^+(x) := \max\{u(x), 0\}$.

We prove in the sequel that the functional I_λ satisfies the hypotheses of the Mountain Pass Theorem.

Lemma 4.1. *Suppose that $(\widetilde{h_1})$ holds. Then, for each $\lambda > 0$,*

(i) *there exist $\xi, \rho > 0$ such that $I_\lambda(u) \ge \xi$, for any $u \in E^q$, $\|u\|_{E^q} = \rho$;*
(ii) *there exists $e \in E^q$ such that $\|e\|_{E^q} > \rho$ and $I_\lambda(e) < 0$.*

Proof. If $u \in E^q$ is such that $\|u\|_{E^q} \le 1$, we can use $N \le q$ to obtain $\|u\|_\partial^q \le \|u\|_\partial^N$. Hence,

$$I_\lambda(u) \ge \frac{1}{q}\left(\|u\|_\partial^q + \|u\|_{L_h^q}^q\right) - \frac{\lambda}{p}\int_\Omega k(x)|u|^p\, dx.$$

So, using the inequality $(a^N + b^N)^{q/N} \le 2^{(q-N)/N}(a^q + b^q)$, Proposition 2.3 and Corollary 2.4, one has

$$I_\lambda(u) \ge C_1\|u\|_{E^q}^q - \lambda C_2\|u\|_{E^q}^p = \|u\|_{E^q}^q(C_1 - \lambda C_2\|u\|_{E^q}^{p-q}),$$

for constants C_1, $C_2 > 0$. Then, item (i) holds for $\rho := \min\left\{1, [C_1/(2\lambda C_2)]^{1/(p-q)}\right\}$ and $\xi := C_1\rho^q/2$.

For proving (ii), we pick $u \in E^q \setminus \{0\}$ such that $u \ge 0$ a.e. in $\mathbb{R}^N$. Using $N \le q < p$ and $k > 0$, we conclude that

$$I_\lambda(su) = \frac{s^N}{N}\|u\|_\partial^N + \frac{s^q}{q}\int_\Omega h(x)|u|^q\, dx - \frac{\lambda s^p}{p}\int_\Omega k(x)(u^+)^p\, dx \to -\infty,$$

as $s \to +\infty$. So, the result follows for $e = s_0 u$, with $s_0 > 0$ sufficiently large. □

We say that I_λ satisfies the Palais-Smale condition if any sequence $(u_n) \subset E^q$ such that

$$I_\lambda(u_n) \to c \in \mathbb{R}, \quad I'_\lambda(u_n) \to 0. \tag{4.1}$$

has a convergent subsequence. In the next result we see that, in the setting of Theorem 1.3, this property is verified.

Lemma 4.2. *Suppose that* $(\widetilde{h_1})$ *holds. Then, for each* $\lambda > 0$*, the functional* I_λ *satisfies the Palais-Smale condition.*

Proof. Let $(u_n) \subset E^q$ be such that (4.1) holds. Computing $I_\lambda(u_n) - (1/p)I'_\lambda(u_n)u_n$, we obtain $C_1, C_2 > 0$ such that

$$\left(\frac{1}{N} - \frac{1}{p}\right) \|u_n\|_\partial^N + \left(\frac{1}{q} - \frac{1}{p}\right) \|u_n\|_{L_h^q}^q \le C_1 + C_2 \|u_n\|_{E^q}.$$

From $N \le q < p$, we conclude that (u_n) is bounded in E^q. Hence, up to a subsequence, we have that $u_n \rightharpoonup u$ weakly in E^q. As in the last section, we may also assume that $u_n \to u$ strongly in L_k^p. Using Hölder's inequality with exponents $s = p/(p-1)$ and $s' = p$, we get

$$\left| \int_\Omega k(x)(u_n^+)^{p-1}(u_n - u)dx \right| \le \|u_n^+\|_{L_k^p}^{p-1} \|u_n - u\|_{L_k^p}^p = o_n(1).$$

This and $I'_\lambda(u_n)(u_n - u) = o_n(1)$ imply that

$$\int_\Omega |\nabla u_n|^{N-2} \left[\nabla u_n \cdot \nabla(u_n - u)\right] dx + \int_\Omega h(x)|u_n|^{q-2}u_n(u_n - u)\, dx + \int_{\partial\Omega} |u_n|^{N-2}u_n(u_n - u)d\sigma = o_n(1).$$

On the other hand, using that $u_n \rightharpoonup u$ weakly in E^q and arguing as in the final part of the proof of Theorem 1.2, ones has

$$\int_\Omega |\nabla u|^{N-2} \left[\nabla u \cdot \nabla(u_n - u)\right] dx + \int_\Omega h(x)|u|^{q-2}u(u_n - u)\, dx + \int_{\partial\Omega} |u|^{N-2}u(u_n - u)\, d\sigma = o_n(1).$$

For any $k \in \mathbb{N}$ and $r \ge 2$, we set

$$T_{k,r}(y_1, y_2) := (|y_1|^{r-2}y_1 - |y_2|^{r-2}y_2), \quad y_1,\, y_2 \in \mathbb{R}^k.$$

We deduce from the two above convergences that

$$\int_\Omega T_{N,N}(\nabla u_n, \nabla u) \cdot \nabla(u_n - u)\, dx + \int_\Omega h(x) T_{1,q}(u_n, u)(u_n - u)\, dx + \int_{\partial\Omega} T_{1,N}(u_n, u)(u_n - u)\, d\sigma = o_n(1).$$

But we know that, for any $k \in \mathbb{N}$ and $r \ge 2$, there holds (see [14, inequality (2.2)])

$$T_{k,r}(y_1, y_2) \cdot (y_1 - y_2) \ge C(k, r)|y_1 - y_2|^k, \quad \forall\, y_1,\, y_2 \in \mathbb{R}^k,$$

and therefore we infer from the last convergence that

$$\|u_n - u\|_{\partial}^{N} + \|u_n - u\|_{L_h^q}^{q} = o_n(1),$$

which implies that $u_n \to u$ strongly in E^q and completes the proof. □

We can now finish the paper proving our second existence result.

Proof of Theorem 1.3. Using Lemmas 4.1 and 4.2 together with the Mountain Pass Theorem [2], we obtain a non-zero critical point of I_λ. If $u^- := u^+ - u$, a straightforward computation shows that $0 = I_\lambda(u)u^- = -\|u^-\|_{E^q}^N$, and therefore $u_\lambda \geq 0$ a.e. in $\mathbb{R}^N$. Hence $u_\lambda \neq 0$ is a non-negative solution of (P_λ) and the theorem is proved. □

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