POSITIVE AND NODAL GROUND STATE SOLUTIONS FOR A CRITICAL SCHRÖDINGER-POISSON SYSTEM WITH INDEFINITE POTENTIALS

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Abstract. We consider the Schrödinger-Poisson system

$$\begin{cases} -\Delta u + V(x)u + K(x)\phi u = a(x)|u|^{p-2}u + |u|^4 u, & x \in \mathbb{R}^3, \\ -\Delta \phi = K(x)u^2, & x \in \mathbb{R}^3, \end{cases}$$

where $4 < p < 6$ and the potentials V, a are allowed to change their signs. Under some reasonable assumptions on V, K and a, we apply the constraint minimization argument to establish the existence of positive ground state solutions and ground state nodal solutions.

1. Introduction

Recently, more and more attention has been paid to the investigation on nodal (sign-changing) solutions for the nonlinear Schrödinger-Poisson system

$$\begin{cases} -\Delta u + V(x)u + \lambda K(x)\phi u = g(x,u), & \text{in } \mathbb{R}^3, \\ -\Delta \phi = K(x)u^2, & \text{in } \mathbb{R}^3. \end{cases} \tag{1.1}$$

Indeed, as far as this issue is concerned, Kim and Seok in [19] made the first attempt to this system for the special case that $V \equiv 1, K \equiv 1$ and $g(x,u) = |u|^{u-2}u$, with $4 < p < 6$. Explicitly, due to its own variational structure, using the Nehari manifold and gluing solution pieces together, they found a radial nodal solution with prescribed numbers of nodal domains. Since then, the constraint variational method combined with other techniques has become an effective strategy in dealing with such problems, for example, by applying variational method together with the Brouwer degree theory (see Wang and Zhou [36]), constructing invariant sets and descending flow (see Liu et al. [24]), combining constraint variational method and quantitative deformation lemma (see Wang and Shuai [30] or Chen and Tang [9]), using the approximation techniques association with the deformation lemma and Miranda's theorem (see Alves et al. [2] or Batista and Furtado [5]), and introducing filtration technique of the nodal Nehari manifold (see Sun and Wu [31]).

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All the aforementioned works dealt with the case that the nonlinearity g has subcritical or quasi-critical growth. The investigations on the existence of nodal solutions for Schrödinger-Poisson system with critical growth are more complicated and interesting, since the the embedding $H^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$ is no longer compact. Even so, there are few works in this case, with different assumptions on V, K and g. In order to more conveniently introduce our problem and make the corresponding comparison, we try our best to list the existing results. Huang et al. [18] considered the following Schrödinger-Poisson system

$$\begin{cases} -\Delta u + u + K(x)\phi u = g(x,u), & x \in \mathbb{R}^3, \\ -\Delta \phi = K(x)u^2, & x \in \mathbb{R}^3, \end{cases}$$

where $g(x,u) = \mu h(x)u + a(x)|u|^4 u$, $\mu > 0$ is a parameter, K, a and h are nonnegative functions without symmetry properties. Under reasonable assumptions on the potentials, they proved that the system possesses a pair of nodal solutions in $H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$ by using the methods in [17], which was reconsidered in [41]. Motivated by [18], both [28] and [37] were concerned with the existence of ground state nodal solutions for the case that $g(x,u) = a(x)|u|^{p-2}u + |u|^4 u$, with K and a having an appropriated exponential decay at infinity. Zhang [37] showed that the system admits one sign-changing solution with $p \in (4,6)$, which was extended in [28] to more general situation involving a potential V allowed to be indefinite.

Recently, Wang et al. [34] studied the system

$$\begin{cases} -\Delta u + V(x)u + \lambda \phi u = \mu f(u) + |u|^4 u, & x \in \mathbb{R}^3, \\ -\Delta \phi = u^2, & x \in \mathbb{R}^3, \end{cases}$$

where $\mu, \lambda > 0$. Under suitable conditions on V, which guarantee that the embedding of $H \hookrightarrow L^p(\mathbb{R}^3)$ is compact $(2 < p < 6)$, where H is energy space corresponding to the system, they established an existence result of ground state nodal solution. To achieve their conclusions, the nonlinearity $f \in C^1(\mathbb{R},\mathbb{R})$ is supposed to satisfy some class of subcritical growth hypotheses and the parameter $\mu > 0$ is required to be large enough. Therefore, it is natural to ask whether this system has ground state sign-changing solutions or not for any parameter $\mu > 0$. For this question, the authors in very recent works [10,39] gave partially affirmative answers.

Motivated by the results mentioned above especially [5,10,28,34,37,39], we are going to discuss the existence of nodal solutions for the following Schrödinger-Poisson system

$$\begin{cases} -\Delta u + V(x)u + K(x)\phi u = a(x)|u|^{p-2}u + |u|^4 u, & \text{in } \mathbb{R}^3, \\ -\Delta \phi = K(x)u^2, & \text{in } \mathbb{R}^3, \end{cases} \tag{P}$$

where $4 < p < 6$ and the potentials V, a are allowed to change their signs. Before stating explicit hypotheses on them, we give some further explanation for our motivation. In [10,28,34,37,39], except for [28], all the assumptions related to the potentials are supposed to be constant, radial or positive. However, as we pointed out, in [28] only the potential V can be indefinite. Moreover, as a perturbation to the critical term $|u|^4 u$ in [10,34,37,39], the positiveness of the potential $a(x)$ associated to subcritical term $f(x,u) =$

$a(x)|u|^{p-2}u$ $(4 < p < 6)$ is required or $f(x,u) = f(u)$ more general form than $|u|^{p-2}u$ is supposed to satisfy the Nehari type monotonicity condition:

$$\frac{f(t)}{|t|^3} \text{ is increasing in } (-\infty, 0) \cup (0, +\infty). \tag{Ne}$$

With this condition, one can show that the projection property holds true for the corresponding nodal Nehari manifold. In terms of the nodal solutions, checking this projection property is the first step and also the key point to perform the other subsequent procedures, which can be seen clearly in the sequel. At this point, we must recall the recent work due to Batista and Furtado [5], in which the authors considered problem (P) without the critical term $|u|^4u$ and the potentials V, a are allowed to be sign-changing. As far as we know, this is the only result on nodal solutions for Schrödinger-Poisson system which is concerned with indefinite potentials, especially the potential a possibly indefinite. There, they solved the problem restricted to a sequence of balls and used a limit process. However, we observe that this technique does not hold for problem (P), due to its critical term.

For any function $v : \mathbb{R}^3 \to \mathbb{R}$, we denote by $v^+(x) := \max\{v(x), 0\}$ and $v^-(x) := \min\{v(x), 0\}$, the positive and negative part of v, respectively. To state our assumptions, we define

$$S := \inf\{\|\nabla u\|^2_{L^2(\mathbb{R}^3)} : u \in \mathcal{D}^{1,2}(\mathbb{R}^3), \|u\|_{L^6(\mathbb{R}^3)} = 1\}$$

and suppose that

(V_0) $V^- \in L^{3/2}(\mathbb{R}^3)$ and $\int_{\mathbb{R}^3} |V^-(x)|^{3/2} dx < S^{3/2}$;

(V_1) there exist $\gamma > 0$ and $C_V > 0$ such that

$$V(x) \le V_\infty - C_V e^{-\gamma|x|}, \quad \text{for a.e. } x \in \mathbb{R}^3,$$

where

$$0 < V_\infty := \lim_{|x|\to+\infty} V(x);$$

(a_0) $a \in L^\infty(\mathbb{R}^3)$;

(a_1) there exist $\theta > 0$ and $C_a > 0$ such that

$$a(x) \ge a_\infty - C_a e^{-\theta|x|}, \quad \text{for a.e. } x \in \mathbb{R}^3,$$

where

$$a_\infty := \lim_{|x|\to+\infty} a(x) > 0;$$

(K_0) $K \in L^2(\mathbb{R}^3)$;

(K_1) there exist $\alpha > 0$ and $C_K > 0$ such that

$$0 \le K(x) \le C_K e^{-\alpha|x|}, \quad \text{for a.e. } x \in \mathbb{R}^3.$$

Before presenting our main results, we discuss the basic framework to deal with our problem (see Section 2 for details). By using (V_0) we can prove that

$$\|u\| := \left(\int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2) dx\right)^{\frac{1}{2}}$$

defines a norm in $H^1(\mathbb{R}^3)$ which is equivalent to its standard norm. Moreover, for each $u \in H^1(\mathbb{R}^3)$, we can use (K_0) and Lax-Milgram's theorem to obtain a unique $\phi_u \in \mathcal{D}^{1,2}(\mathbb{R}^3) := \{v \in L^6(\mathbb{R}^3) : \nabla v \in L^2(\mathbb{R}^3)\}$ such that $-\triangle\phi_u = K(x)u^2$. Actually, ϕ_u is given by (see [29])

$$\phi_u(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{K(y)u^2(y)}{|x-y|}dy.$$

If we insert ϕ_u into the first equation of system (P), we obtain its following equivalent form

$$-\Delta u + V(x)u + K(x)\phi_u u = a(x)|u|^{p-2}u + |u|^4 u, \quad u \in H^1(\mathbb{R}^3). \tag{1.2}$$

Its energy functional

$$I(u) := \frac{1}{2}\|u\|^2 + \frac{1}{4}\int K(x)\phi_u u^2 - \frac{1}{p}\int a(x)|u|^p - \frac{1}{6}\int |u|^6, \quad u \in H^1(\mathbb{R}^3)$$

belongs to $C^1(H^1(\mathbb{R}^3),\mathbb{R})$ and critical points of I correspond to weak solutions of equation (1.2). Moreover, it can be proved that $u \in H^1(\mathbb{R}^3)$ is a weak solution of the equation if, and only if, the pair $(u,\phi_u) \in H^1(\mathbb{R}^3)\times\mathcal{D}^{1,2}(\mathbb{R}^3)$ is a weak solution of system (P).

Nonzero solutions of (1.2) belong to the Nehari manifold

$$\mathcal{N} := \{u \in H^1(\mathbb{R}^3)\setminus\{0\} : \langle I'(u), u\rangle = 0\},$$

and we say that $u_0 \in \mathcal{N}$ is a ground state solution if it achieves

$$m_+ := \inf_{u\in\mathcal{N}} I(u).$$

In our first result we prove the existence of such a solution:

Theorem 1.1. *Suppose that $4 < p < 6$ and (V_0)-(V_1), (K_0)-(K_1), (a_0)-(a_1) are satisfied with $\gamma < \min\{\theta,\alpha\}$. Then equation (1.2) has a positive ground state solution.*

Although the potential a changes its sign, in view of the assumption (V_1) and similar to [28, 37], we can show the existence of positive ground state solutions. For the other issues related to system (1.1), such as positive solutions, multiple solutions, ground state solutions, radial solutions, semiclassical states, we refer the reader to [8, 11, 12, 16, 21–23, 26, 27, 32, 35, 38, 40] and the references listed therein.

In our second and main result, we look for a sign-changing solution. So, we introduce the set

$$\mathcal{M} := \{u \in H^1(\mathbb{R}^3) : u^{\pm} \neq 0, \langle I'(u), u^+\rangle = 0 = \langle I'(u), u^-\rangle\},$$

which contains all the nodal solutions and consider the minimization problem

$$m_* := \inf_{u\in\mathcal{M}} I(u).$$

We prove the following:

Theorem 1.2. *Suppose that $4 < p < 6$ and (V_0)-(V_1), (K_0)-(K_1), (a_0)-(a_1) are satisfied with $\gamma < \min\{\theta,\alpha\}$. Then equation (1.2) has a solution $u \in \mathcal{M}$ such that $I(u) = m_*$.*

There are two major challenges one has to face in the proof. First, we need to check that $\mathcal{M}$ is nonempty. Due to the fact that a is indefinite in sign, the subcritical term $a(x)|u|^{p-2}u$ does not satisfy the (Ne) monotonicity condition. Hence, the usual techniques used to show that for any $u \in H^1(\mathbb{R}^3)$ with $u^\pm \neq 0$ there exists a unique pair $(s_u, t_u) \in (0,\infty) \times (0,\infty)$ such that $s_u u^+ + t_u u^- \in \mathcal{M}$ is not effective any more. To overcome this difficulty, we employ the Implicit Function Theorem in a trick way (see Lemma 4.1). Moreover, to check that the nodal level m_* belongs to the compactness range of I, it is essential to obtain an estimation of this level when compared to that of the positive solution and a limit problem associated with (P). To reach the estimate we follow [7] and use some careful calculations which involves the decay rate of the potentials. The main results of this paper extend the previous results in several way, since we deal with the critical problem and we allow the potentials change their signs.

The rest of the paper is organized as follows. In Section 2, we present the variational framework and some preliminary results. Section 3 and Section 4 are devoted to the proofs of Theorem 1.1 and Theorem 1.2, respectively.

2. Variational framework and preliminary results

In this section, we state and prove some technical results. For any $2 \leq q \leq +\infty$, we denote by $\|\cdot\|_q$ the L^q-norm for a function $u \in L^q(\mathbb{R}^3)$. To simplify notation, we write only $\int u$ instead of $\int_{\mathbb{R}^3} u(x)\,dx$.

In order to discuss the basic framework to deal with our problem, we first set

$$X := \left\{ u \in H^1(\mathbb{R}^3) : \int V(x)u^2 < +\infty \right\}$$

and recall that, in [13, Lemma 2.1], it was proved that the map

$$u \mapsto \int (|\nabla u|^2 + V^+(x)u^2)$$

defines a norm in X which is equivalent to the usual norm of $H^1(\mathbb{R}^3)$. In addition, using (V_0) and Hölder's inequality, we obtain

$$\int V^- u^2 \leq \|V^-\|_{3/2}\|u\|_6^2 \leq S^{-1}\|V^-\|_{3/2} \int |\nabla u|^2,$$

which ensures that the norm

$$\|u\| := \left(\int (|\nabla u|^2 + V(x)u^2) \right)^{\frac{1}{2}}$$

is well defined and it is also equivalent to the usual norm in $H^1(\mathbb{R}^3)$. So, $X = H^1(\mathbb{R}^3)$ and we shall use in this space the above norm from now on. Since the embedding $H^1(\mathbb{R}^3) \hookrightarrow L^q(\mathbb{R}^3)$ is continuous for $2 \leq q \leq 6$, there exists $C_q > 0$ such that

$$\|u\|_q \leq C_q \|u\|, \quad \forall\, u \in H^1(\mathbb{R}^3). \tag{2.1}$$

As quoted in the introduction, for each $u \in H^1(\mathbb{R}^3)$, the function

$$\phi_u(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{K(y)u^2(y)}{|x-y|} dy$$

belongs to $\mathcal{D}^{1,2}(\mathbb{R}^3)$ and weakly solves $-\Delta u = K(x)u^2$. As we shall see, it will be important to consider, for u, $\varphi \in H^1(\mathbb{R}^3)$, the number

$$L_{\phi_u}(\varphi) := \int K(x)\phi_u\varphi^2 = \frac{1}{4\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{K(x)K(y)\varphi^2(x)u^2(y)}{|x-y|}dxdy.$$

The following result collects the main properties of the function ϕ and actually shows that L_{ϕ_u} is well defined.

Lemma 2.1. *For any $u \in H^1(\mathbb{R}^3)$, one has*

(i) $\phi_u(x) \geq 0$;
(ii) $\phi_{tu} = t^2\phi_u$, for any $t \in \mathbb{R}$;
(iii) there exist C_1, $C_2 > 0$ such that

$$\int |\nabla\phi_u|^2 = L_{\phi_u}(u) \leq C_1\|u\|_6^4 \leq C_2\|u\|^4.$$

Moreover, if $(u_n) \subset H^1(\mathbb{R}^3)$ is such that $u_n \rightharpoonup u$ weakly in $H^1(\mathbb{R}^3)$ and $u_n \to u$ a.e. in $\mathbb{R}^3$, then

$$\lim_{n\to+\infty} L_{\phi_{u_n}}(u_n) = L_{\phi_u}(u), \quad \lim_{n\to+\infty} L_{\phi_{u_n}}(u_n^{\pm}) = L_{\phi_u}(u^{\pm}), \tag{2.2}$$

and

$$\lim_{n\to+\infty}\int K(x)\phi_{u_n}u_n\varphi = \int K(x)\phi_u u\varphi, \quad \forall\, \varphi \in H^1(\mathbb{R}^3). \tag{2.3}$$

Proof. Statements (i) and (ii) easily follow from the definition of ϕ_u. Using that $-\Delta\phi_u = K(x)u^2$, Hölder's inequality and the embedding $\mathcal{D}^{1,2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$, we obtain

$$\int |\nabla\phi_u|^2 = \int K(x)u^2\phi_u \leq \|K\|_2\|u\|_6^2\|\phi_u\|_6 \leq C_1\|u\|_6^2\left(\int|\nabla\phi_u|^2\right)^{1/2},$$

from which we derive (iii). We refer to [14, Lemma 2.2] for the convergences in (2.2) and (2.3). $\square$

We now recall that, instead of considering system (P) directly, we are going to solve its equivalent form (1.2). As usual, it is important to consider also the limit problem

$$-\Delta u + V_\infty u = a_\infty|u|^{p-2}u + |u|^4u, \quad u \in H^1(\mathbb{R}^3), \tag{P_∞}$$

whose associated energy functional is given by

$$I_\infty(u) := \frac{1}{2}\int(|\nabla u|^2 + V_\infty|u|^2) - \frac{a_\infty}{p}\int|u|^p - \frac{1}{6}\int|u|^6, \quad u \in H^1(\mathbb{R}^3).$$

Ground state solutions of the limit problem can be obtained from the minimization problem

$$m_\infty := \inf_{u\in\mathcal{N}_\infty} I_\infty(u),$$

where $\mathcal{N}_\infty$ is the corresponding Nehari manifold, that is,

$$\mathcal{N}_\infty := \left\{u \in H^1(\mathbb{R}^3)\backslash\{0\} : \langle I'_\infty(u), u\rangle = 0\right\}.$$

Actually, we have the following:

Proposition 2.2. *Problem* (P_∞) *has a positive solution* $u_\infty \in L^\infty(\mathbb{R}^3) \cap C^{1,\alpha}_{loc}(\mathbb{R}^3)$, *with* $0 < \alpha < 1$ *and* $I_\infty(u_\infty) = m_\infty$. *Moreover, for any* $\bar\eta \in (0, \sqrt{V_\infty})$, *there exists* $C = C(\bar\eta) > 0$ *such that*

$$u_\infty(x) \le C e^{-(\sqrt{V_\infty}-\bar\eta)|x|}, \quad \forall\, x \in \mathbb{R}^3.$$

Proof. The existence part can be obtained as in [3, Theorem 1.7]. Moreover, arguing as in [20, Theorem 1.11], we can prove the regularity of the solution and that $\lim_{|x|\to+\infty} u_\infty(x) = 0$. In order to verify the exponential decay, we first use (V_1) to pick $R = R(\bar\eta) > 0$ such that

$$V_\infty - a_\infty |u_\infty(x)|^{p-2} - |u_\infty(x)|^4 \ge \delta^2 > 0, \quad \forall\, |x| \ge R,$$

where $\delta := (\sqrt{V_\infty} - \bar\eta)$. Hence, for $|x| \ge R$, we have

$$\delta^2 u_\infty(x) \le V_\infty u_\infty(x) - a_\infty |u_\infty(x)|^{p-2} u_\infty(x) - |u_\infty(x)|^4 u_\infty(x). \tag{2.4}$$

In addition, if we set $v(x) := \|u_\infty\|_\infty e^{-\delta(|x|-R)}$, a direct calculation gives

$$-\Delta v(x) + \delta^2 v(x) = 0, \quad x \neq 0. \tag{2.5}$$

By choosing $\varphi := (u_\infty - v)^+$ as a the test function, we can use (2.4) and (2.5) to get

$$\int_{|x|\ge R} (\nabla u_\infty \cdot \nabla\varphi + \delta^2 u_\infty \varphi)dx \le 0, \quad \int_{|x|\ge R} (\nabla v \cdot \nabla\varphi + \delta^2 v\varphi)dx = 0,$$

from which we conclude that

$$\begin{aligned} 0 &\ge \int_{|x|\ge R} (\nabla u_\infty - \nabla v)\cdot\nabla\varphi\,dx + \int_{|x|\ge R} \delta^2 (u_\infty - v)\varphi\,dx \\ &\ge \int_{\{|x|\ge R\}\cap\{u_\infty > v\}} \delta^2 (u_\infty - v)^2\,dx \ge 0. \end{aligned}$$

Since u_∞ and v are continuous, we conclude that $\{x : |x| \ge R, u_\infty > v\}$ is empty. In other words, we derive that

$$u_\infty(x) \le v(x) = \|u_\infty\|_\infty e^{-\delta(|x|-R)} = C e^{-(\sqrt{V_\infty}-\bar\eta)|x|}, \quad \forall\, |x| \ge R.$$

Since a similar inequality clearly holds for $|x| \le R$, the lemma is proved. □

We prove in the sequel that weak solutions of our problem are regular.

Proposition 2.3. *If* $u \in H^1(\mathbb{R}^3)$ *is weak solution of equation* (1.2), *then* $u, \phi_u \in C^{1,\alpha}_{loc}(\mathbb{R}^3)$, *for some* $0 < \alpha < 1$.

Proof. Define $h(x) := a(x)|u|^{p-2} + |u|^4 - V(x) - K(x)\phi_u$, $x \in \mathbb{R}^3$, and notice that $u \in H^{1,2}_{loc}(\mathbb{R}^3)$ weakly verifies $-\Delta u = g(x, u) := h(x)u$. By taking into account that $K \in L^2(\mathbb{R}^3)$, (V_0)-(V_1) and Lemma 2.1(iii), it is easy to see that $h \in L^{3/2}_{loc}(\mathbb{R}^3)$. It follows from Brezis-Kato's theorem [33, B.3 Lemma] that $u \in L^q_{loc}(\mathbb{R}^3)$, for any $2 \le q < \infty$. Meanwhile, since $Ku^2 \in L^q_{loc}(\mathbb{R}^3)$, we infer from the second equation in system (P) and the Calderón-Zygmund estimates [15, Lemma 9.9] that $\phi_u \in W^{2,q}_{loc}(\mathbb{R}^3)$, for any $1 \le q < \infty$ and ϕ_u satisfies the second equation in system (P) almost every in local sense. Therefore, we obtain that $g(\cdot, u) \in L^q_{loc}(\mathbb{R}^3)$, for any $1 \le q < \infty$. Consequently, using the Calderón-Zygmund estimates again, we show that $u \in W^{2,q}_{loc}(\mathbb{R}^3)$, for any $1 \le q < \infty$ and u satisfies (1.2) almost every in local sense. By picking $q > 3$, according to Sobolev embedding theorem (see [15, Theorem 7.26]), we conclude that $u, \phi_u \in C^{1,\alpha}_{loc}(\mathbb{R}^3)$, with $0 < \alpha := 1 - 3/q < 1$. $\square$

Once the regularity of solutions has been established, we can obtain the following decomposition property for bounded Palais-Smale sequences of I:

Lemma 2.4. *Let $(u_n) \subset H^1(\mathbb{R}^3)$ be such that*

$$I(u_n) \to c, \quad I'(u_n) \to 0$$

and $u_n \rightharpoonup u$ weakly in $H^1(\mathbb{R}^3)$. Then $I'(u) = 0$ and we have either

(i) $u_n \to u$ strongly in $H^1(\mathbb{R}^3)$, or

(ii) there exists $k \in \mathbb{N}$, $(y_n^j) \subset \mathbb{R}^3$, $j = 1, \cdots, k$, and nonzero solutions $v^1, \dots, v^k \in H^1(\mathbb{R}^3)$ of problem (P_∞), such that

$$c = I(u) + \sum_{j=1}^{k} I_\infty(v^j).$$

Proof. To prove this result one can use Lemma 2.1(iii) and argue as in [8, Lemma 4.1]. We omit the details. $\square$

Corollary 2.5. *If $(u_n) \subset H^1(\mathbb{R}^3)$ is such that $I(u_n) \to c < m_\infty$ and $I'(u_n) \to 0$, then (u_n) has a convergent subsequence.*

Proof. We first notice that

$$c + o_n(1) + o_n(1)\|u_n\| \geq I(u_n) - \frac{1}{p}\langle I'(u_n), u_n\rangle \geq \left(\frac{1}{2} - \frac{1}{p}\right)\|u_n\|^2, \tag{2.6}$$

where $o_n(1)$ stands for a quantity approaching zero as $n \to +\infty$. Hence, (u_n) is bounded and there exists $u \in H^1(\mathbb{R}^3)$ such that $u_n \rightharpoonup u$ weakly in $H^1(\mathbb{R}^3)$, up to subsequence. Suppose that alternative (ii) of Lemma 2.4 holds. Since $I'(u) = 0$, the same calculation performed just above shows that $I(u) \geq 0$. Recalling that the solutions v^i of the limit problem given by Lemma 2.4 are nonzero, we get

$$c = I(u) + \sum_{i=1}^{k} I_\infty(v^i) \geq I(u) + k m_\infty \geq m_\infty,$$

which leads to a contraction. Hence, we conclude that statement (i) of Lemma 2.4 holds, i.e., $u_n \to u$ strongly in $H^1(\mathbb{R}^3)$. $\square$

3. The positive solution

We devote this section to the proof of Theorem 1.1. Since $p \in (4, 6)$, we can use Lemma 2.1(i) and (2.1) to obtain $C > 0$ such that

$$I(u) \geq \frac{1}{2}\|u\|^2 - \frac{\|a\|_\infty}{p}\|u\|_p^p - \frac{1}{6}\|u\|_6^6 \geq \left(\frac{1}{2} - C\|u\|^{p-2} - C\|u\|^4\right)\|u\|^2,$$

for any $u \in H^1(\mathbb{R}^3)$. Hence, there exists $\rho_0 > 0$ small in such a way that

$$I(u) \geq \frac{1}{8}\rho_0^2 := \alpha_0 > 0, \quad \forall\, \|u\| = \rho_0. \tag{3.1}$$

Let u_∞ be given by Proposition 2.2, $\nu = (1,0,0)$ and set

$$v_R(x) := u_\infty(x - R\nu),$$

where $R > 0$ is free for now. Since $I(tv_R) \to -\infty$, as $t \to +\infty$, there exists $t_R > 0$ large verifying $I(t_R v_R) < 0$ and $\|t_R v_R\| > \rho_0$. This and (3.1) show that we can define the Mountain Pass level

$$c_R := \inf_{\sigma\in\Sigma} \max_{t\in[0,1]} I(\sigma(t)) \geq \alpha_0 > 0,$$

where $\Sigma := \{\sigma \in C([0,1], H^1(\mathbb{R}^3)) : \sigma(0) = 0, \sigma(1) = t_R v_R\}$. Moreover, there exists a sequence $(u_n) \subset H^1(\mathbb{R}^3)$ such that

$$I(u_n) \to c_R, \quad I'(u_n) \to 0.$$

We claim that, for some $R > 0$, there holds $c_R < m_\infty$. If this is true, we may invoke Corollary 2.5 to conclude that, along a subsequence, $u_n \to u$ strongly in $H^1(\mathbb{R}^3)$, where $I'(u) = 0$ and $I(u) = c_R > 0$. Hence, problem (1.2) has a nonzero solution.

In order to prove the claim, we define the map $\psi(t) := I(tv_R)$, for $t \geq 0$. It is clear that

$$c_R \leq \max_{t\in[0,1]} I(t \cdot t_R v_R) \leq \max_{t\geq 0} \psi(t),$$

and therefore it is sufficient to verify that the last maximum above is smaller than m_∞. It follows from Lemma 2.1(iii) and the invariance of the L^s-norm that

$$\psi(t) \leq \frac{t^2}{2}\left(\|\nabla u_\infty\|_2^2 + V_\infty\|u_\infty\|_2^2\right) + C_1 t^4\|u_\infty\|_6^4 + C_2 t^p\|u_\infty\|_p^p - t^6\|u_\infty\|_6^6.$$

Thus, there exists $0 < t_* < 1 < t^*$, both independent of $R > 0$, such that

$$\psi(t) < m_\infty, \quad \forall\, t \in [0, t_*] \cup [t^*, +\infty).$$

We now consider ψ in the bounded interval $[t_*, t^*]$. By using Lemma 2.1(iii) again, we can write

$$\psi(t) = I_\infty(tv_R) + \frac{t^4}{4}\Gamma_{R,1} + \frac{t^2}{2}\Gamma_{R,2} + \frac{t^p}{p}\Gamma_{R,3}, \tag{3.2}$$

where

$$\Gamma_{R,1} := \int K(x)\phi_{v_R} v_R^2, \quad \Gamma_{R,2} := \int (V(x) - V_\infty)v_R^2$$

and

$$\Gamma_{R,3} := \int (a_\infty - a(x))v_R^p.$$

For estimating the first term above, we apply Hölder's inequality, the embedding $\mathcal{D}^{1,2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$ and Lemma 2.1(iii) to obtain $C_4 > 0$, independent of $R > 0$, such that

$$\begin{aligned}\Gamma_{R,1} &\le \|\phi_{v_R}\|_6 \left(\int K(x)^{6/5} v_R^{12/5}\right)^{5/6} \\ &\le C_3 \left(\int |\nabla \phi_{v_R}|^2\right)^{1/2} \left(\int K(x)^{6/5} v_R^{12/5}\right)^{5/6} \\ &\le C_4 \|v_R\|_6^2 \left(\int K(x)^{6/5} v_R^{12/5}\right)^{5/6} \\ &= C_4 \|u_\infty\|_6^2 \left(\int K(x+R\nu)^{6/5} u_\infty^{12/5}\right)^{5/6},\end{aligned}$$

where we also have used the translation invariance of the L^6-norm and the change of variables $x \mapsto x - R\nu$. By decreasing the numbers α, θ given in conditions (K_1) and (a_1), we may pick $\bar{\eta} > 0$ such that

$$0 < \bar{\eta} < \min\left\{\sqrt{V_\infty} - \frac{\alpha}{2}; \sqrt{V_\infty} - \frac{\theta}{p}\right\}.$$

Since $|x + R\nu| \ge R - |x|$, we can use (K_1) and the exponential decay of u_∞ given in Proposition 2.2, to get

$$\int K(x+R\nu)^{6/5} u_\infty^{12/5} \le C_5 e^{-(6/5)\alpha R} \int e^{(6/5)\left(\alpha - 2(\sqrt{V_\infty} - \bar{\eta})\right)|x|}.$$

Due to the choice of $\bar{\eta}$ the last integral above is finite and we conclude that

$$\Gamma_{R,1} \le C_6 e^{-\alpha R}. \tag{3.3}$$

By using (V_1) and changing variables again, we also obtain

$$\Gamma_{R,2} \le -C_7 \int e^{-\gamma|x+R\nu|} u_\infty^2 \le -C_7 e^{-\gamma R} \int e^{-\gamma|x|} u_\infty^2 = -C_8 e^{-\gamma R}. \tag{3.4}$$

Finally, condition (a_1) yields

$$\Gamma_{R,3} \le C_9 e^{-\theta R} \int e^{\left(\theta - p(\sqrt{V_\infty} - \bar{\eta})\right)|x|} = C_{10} e^{-\theta R}. \tag{3.5}$$

Since $I_\infty(tv_R) = I_\infty(tu_\infty)$ and the map $t \mapsto I_\infty(tu_\infty)$, $t \ge 0$, achieves its maximum at $t = 1$, we have that $I_\infty(tv_R) = I_\infty(tu_\infty) \le I_\infty(u_\infty) = m_\infty$. By replacing this, (3.3), (3.4) and (3.5) in (3.2), we obtain $C_{11} > 0$ such that

$$\psi(t) \le m_\infty + C_{11}\left(e^{-\alpha R} + e^{-\theta R} - e^{-\gamma R}\right)$$

for any $t \in [t_*, t^*]$. Recalling that $\gamma < \min\{\alpha, \theta\}$, we conclude that, for some $R > 0$ large, the inequality $c_R < m_\infty$ holds. The claim is proved.

Up to now, we have obtained a nonzero solution for (1.2) at level $c_R < m_\infty$. For obtaining a ground state solution, we proceed as follows: let $(v_n) \subset \mathcal{N}$ be such that $I(v_n) \to m_+ \le c_R < m_\infty$ and $I'(v_n)_{|\mathcal{N}} \to 0$. The same computation presented in (2.6) shows that (v_n) is bounded. Moreover,

$$\begin{aligned}\|v_n\|^2 &\le \|v_n\|^2 + \int K(x)\phi_{v_n} v_n^2 \\ &= \int a(x)|v_n|^p + \int |v_n|^6 \le C_{12}(\|v_n\|^p + \|v_n\|^6),\end{aligned}$$

and therefore $\|v_n\| \geq \Lambda$, for some $\Lambda > 0$. Thus, if we define $J(v) := \langle I'(v), v\rangle$, for $v \in H^1(\mathbb{R}^3)$, a direct calculation provides

$$\langle J'(v), v\rangle = (2-p)\|v\|^2 + (4-p)\int K(x)\phi_v v^2 + (p-6)\int |v|^6,$$

and therefore

$$\langle J'(v_n), v_n\rangle < (2-p)\|v_n\|^2 < (2-p)\Lambda^2 < 0. \tag{3.6}$$

We know that there exists $(\lambda_n) \subset \mathbb{R}$ such that $o_n(1) = I'(v_n) - \lambda_n J'(v_n)$. Since (v_n) is bounded, we see that

$$o_n(1) = \langle I'(v_n), v_n\rangle - \lambda_n\langle J'(v_n), v_n\rangle = -\lambda_n\langle J'(v_n), v_n\rangle.$$

Then it follows from (3.6) that $\lambda_n = o_n(1)$ and concludes that $I'(v_n) \to 0$, as $n \to \infty$. Recalling that $m_+ < m_\infty$, we can argue as in the first part of the proof to conclude that $v_n \to v$ strongly in $H^1(\mathbb{R}^3)$, with $v \in \mathcal{N}$ being such that $I(v) = m_+$.

Since $\phi_{v_n} = \phi_{|v_n|}$, we may replace (v_n) by $(|v_n|)$ in the last argument to assume that $v \geq 0$ a.e. in $\mathbb{R}^3$. Hence, using Proposition 2.3 and Harnack inequality (see [15, Theorem 8.20]) we guarantee that $v > 0$ a.e. in $\mathbb{R}^3$. $\square$

Remark 3.1. If u_0 is the positive solution given by Theorem 1.1, we can use (K_1) to conclude that

$$-\Delta u_0 + V(x)u_0 \leq a(x)|u_0|^{p-2}u_0 + |u_0|^4 u_0$$

in the weak sense. As in [20, Theorem 1.11], we can prove that $u_0(x) \to 0$, as $|x| \to +\infty$. Hence, the same argument of the proof of Proposition 2.2 shows that u_0 decays to zero at infinity with the same rate of u_∞. That is, for any $\eta \in (0, \sqrt{V_\infty})$, there exists $C = C(\eta) > 0$ such that

$$u_0(x) \leq Ce^{-(\sqrt{V_\infty}-\eta)|x|}, \quad \forall x \in \mathbb{R}^3.$$

4. The nodal solution

We recall the Nehari nodal set

$$\mathcal{M} := \{u \in H^1(\mathbb{R}^3) : u^\pm \neq 0, \langle I'(u), u^+\rangle = 0 = \langle I'(u), u^-\rangle\},$$

which contains all the nodal solutions and investigate the minimization problem

$$m_* := \inf_{u\in\mathcal{M}} I(u).$$

In our first step, we prove that $\mathcal{M}$ is nonempty. Actually, the following projection result holds:

Lemma 4.1. *For any $u \in H^1(\mathbb{R}^3)$ with $u^\pm \neq 0$, there exists a unique pair $(s_u, t_u) \in \mathbb{R}^2$ such that s_u, $t_u > 0$ and $s_u u^+ + t_u u^- \in \mathcal{M}$. Furthermore, we have the following relationship*

$$I(s_u u^+ + t_u u^-) = \max_{s,t\geq 0} I(su^+ + tu^-).$$

Proof. For any $u \in H^1(\mathbb{R}^3)$ with $u^\pm \neq 0$, we define $g_u : (0,+\infty) \times (0,+\infty)$ by

$$g_u(s,t) := \langle I'(su^+ + tu^-), u^+\rangle = sA + s^3B + st^2C - s^{p-1}D - s^5E,$$

where

$$A := \|u^+\|^2, \quad B := L_{\phi_{u^+}}(u^+), \quad C := L_{\phi_{u^-}}(u^+),$$

and

$$D := \int a(x)|u^+|^p, \quad E := \int |u^+|^6.$$

We now fix $t_0 > 0$ and consider the map $s \mapsto g_u(s, t_0)$. Clearly $g_u(s, t_0) > 0$, for $s > 0$ small, and $g_u(s, t_0) \to -\infty$, as $s \to +\infty$. Then, there exists $s = s(t_0) > 0$ such that $g_u(s(t_0), t_0) = 0$ and we are going to prove that this $s(t_0)$ is unique. Indeed, this is clearly true if $D \geq 0$, since $4 < p < 6$. If $D < 0$, a direct calculation gives that

$$\begin{aligned}
\frac{\partial g_u(s, t_0)}{\partial s} &= A + t_0^2 C + 3s^2 B - (p-1)s^{p-2} D - 5s^4 E, \\
\frac{\partial^2 g_u(s, t_0)}{\partial s^2} &= 6sB - (p-1)(p-2)s^{p-3} D - 20s^3 E, \\
\frac{\partial^3 g_u(s, t_0)}{\partial s^3} &= 6B - (p-1)(p-2)(p-3)s^{p-4} D - 60s^2 E, \\
\frac{\partial^4 g_u(s, t_0)}{\partial s^4} &= -(p-1)(p-2)(p-3)(p-4)s^{p-5} D - 120sE.
\end{aligned}$$

Hence, there exists a unique $s_0 > 0$ such that $\frac{\partial^4 g_u(s_0,t_0)}{\partial s^4} = 0$, $\frac{\partial^4 g_u(s,t_0)}{\partial s^4} > 0$ in $s \in (0, s_0)$ and $\frac{\partial^4 g_u(s,t_0)}{\partial s^4} < 0$ in (s_0, ∞). Using that $\lim_{s\to\infty} \frac{\partial^3 g_u(s,t_0)}{\partial s^3} = -\infty$, we reach the same conclusion to $\frac{\partial^3 g_u(s,t_0)}{\partial s^3}$. By repeating this procedure, we can prove the uniqueness of $s(t_0) > 0$. Meanwhile, for this unique $s(t_0)$, evidently it also holds that $\frac{\partial g_u(s(t_0),t_0)}{\partial s} < 0$.

By considering all the possible values of $t_0 > 0$ above, we obtain that

$$g_u(s(t), t) = 0, \quad \frac{\partial g_u(s(t), t)}{\partial s} < 0.$$

In light of the above analysis, we see that $g_u(s, t)$ satisfies the following properties:

(i) g_u has continuous partial derivatives in $(0, \infty) \times (0, +\infty)$;

(ii) $g_u(s(t), t) = 0$, for any $t > 0$;

(iii) $\frac{\partial g_u(s(t),t)}{\partial s} < 0$, for any $t > 0$.

Therefore, applying the Implicit Function Theorem, we derive that $g_u(s, t) = 0$ determines an implicit function $s(t)$ with continuous derivative in $(0, +\infty)$.

If we define

$$h_u(s, t) := \langle I'(su^+ + tu^-), u^- \rangle,$$

from a similar argument, we also deduce that there exists a unique $t(s)$ such that

$$h_u(s, t(s)) = 0 \quad \text{and} \quad \frac{\partial h_u(s, t(s))}{\partial t} < 0.$$

Furthermore, $h_u(s, t) = 0$ determines an implicit function $t(s)$ with continuous derivative in $(0, +\infty)$.

We now claim that, for some $t^* > 0$, there holds

$$s(t) < t, \quad \forall\, t > t^*. \tag{4.1}$$

Indeed, if this is not true, we can obtain $(t_n) \subset (0, +\infty)$ such that $t_n \to +\infty$ and $s(t_n) \geq t_n$. Hence, from the definition of g_u and $s(t_n)$, we get

$$0 = \frac{g_u(s(t_n), t_n)}{s(t_n)^5} = \left(\frac{1}{s(t_n)^4} A + \frac{1}{s(t_n)^2} B + \frac{t_n^2}{s(t_n)^4} C - \frac{1}{s(t_n)^{6-p}} D - E \right),$$

which implies that $E = 0$. This absurd shows that (4.1) holds. Analogously, we have

$$t(s) < s, \quad \forall\, s > s^*.$$

Hence, taking into account the above inequality, (4.1), $s(0) > 0$, $t(0) > 0$ and the continuity of s and t, we conclude that the graphs of the maps s and t must intersect at some point $(s_u, t_u) \in \mathbb{R}^+ \times \mathbb{R}^+$. In addition, noting that

$$s'(t) = -\frac{\frac{\partial}{\partial t} g_u(s,t)}{\frac{\partial}{\partial s} g_u(s,t)}(s(t), t) > 0, \quad \forall t > 0,$$

we see that $s(t)$ is strictly increasing in $(0, +\infty)$. Similarly, $t(s)$ is strictly increasing in $(0, +\infty)$. As a consequence, there is a unique pair $(s_u, t_u) \in \mathbb{R}^+ \times \mathbb{R}^+$ such that $g_u(s_u, t_u) = 0 = h_u(s_u, t_u)$, namely, $s_u u^+ + t_u u^- \in \mathcal{M}$.

To complete our proof, we finally check that (s_u, t_u) obtained above is the unique maximum point of $G_u(s,t) := I(su^+ + tu^-)$ in $[0, \infty) \times [0, +\infty)$. Indeed, we have that

$$\begin{aligned} G_u(s,t) = &\left(\frac{s^2}{2} A + \frac{s^2}{4} B - \frac{s^p}{p} D - \frac{s^6}{6} E \right) \\ &+ \left(\frac{t^2}{2} \widetilde{A} + \frac{t^2}{4} \widetilde{B} - \frac{t^p}{p} \widetilde{D} - \frac{t^6}{6} \widetilde{E} \right) + \frac{s^2 t^2}{2} C, \end{aligned}$$

where $\widetilde{A}$, $\widetilde{B}$, $\widetilde{D}$, $\widetilde{E}$ have the same meaning of A, B, D, E, with u^+ replaced by u^-, respectively. Since $G_u(s,t) > 0$ for (s,t) near $(0,0)$, it is sufficient to check that the maximum point cannot be achieved on the boundary of $\mathbb{R}^2_+$. Without loss of generality, we assume that $(0, \bar{t})$ is a maximum point of $G_u(s,t)$. Using the above expression for G_u and $p > 4$, we easily conclude that $G_u(s, \bar{t}) > G_u(0, \bar{t})$, for any $s > 0$ small, which does not make sense. Hence

$$G_u(s_u, t_u) = I(s_u u^+ + t_u u^-) = \max_{s,t \geq 0} I(su^+ + tu^-)$$

and we have done. □

Lemma 4.2. *If $(u_n) \subset \mathcal{M}$ is such that $I(u_n) \to c$, then $c > 0$ and there exists Λ_1, $\Lambda_2 > 0$ such that $\Lambda_1 \leq \|u_n^{\pm}\| \leq \Lambda_2$, for any $n \in \mathbb{N}$.*

Proof. It follows from $\langle I'(u_n), u_n^{\pm} \rangle = 0$ and (2.1) that

$$\|u_n^{\pm}\|^2 \leq \int a(x)|u_n^{\pm}|^p + \int |u_n^{\pm}|^6 \leq C(\|u_n^{\pm}\|^p + \|u_n^{\pm}\|^6).$$

Since $p > 4$, the above expression provides $\Lambda_1 > 0$ such that $\|u_n^{\pm}\| \geq \Lambda_1$. Using $\langle I'(u_n), u_n \rangle = 0$ again and (2.6), we conclude that that $c > 0$ and $\|u_n^{\pm}\| \leq \Lambda_2$. □

In order to construct a Palais-Smale sequence for I at level m_* we follow an idea introduced in [7]. So, we denote by P the cone of non-negative functions in $H^1(\mathbb{R}^3)$, $Q = [0,1]\times[0,1]$ and by Σ the set of continuous maps $\sigma \in C(Q, H^1(\mathbb{R}^3))$ such that, for any $(s,t)\in Q$ there hold

(i) $\sigma(s,0)=0$, $\sigma(0,t)\in P$ and $\sigma(1,t)\in -P$;

(ii) $(I\circ\sigma)(s,1)\le 0$ and
$$\frac{\|\sigma(s,1)\|_6^6 + \int a(x)|\sigma(s,1)|^p}{\|\sigma(s,1)\|^2 + L_{\phi_{\sigma(s,1)}}(\sigma(s,1))} \ge 2.$$

We finally define $\xi : H^1(\mathbb{R}^3)\times H^1(\mathbb{R}^3)\to\mathbb{R}$ by
$$\xi(u,v) = \begin{cases} \dfrac{\int |u|^6 + \int a(x)|u|^p}{\|u\|^2 + L_{\phi_u}(u) + L_{\phi_v}(u)}, & u\neq 0,\\ 0, & u = 0.\end{cases}$$

It is clear that $u\in\mathcal{M}$ if and only if $\xi(u^+,u^-)=\xi(u^-,u^+)=1$. Moreover, this map enables us to construct a set bigger than $\mathcal{M}$ which contains a Palais-Smale sequence for I at level m_*, as we can see from the next result.

Lemma 4.3. *If*
$$\mathcal{U} := \Big\{u\in H^1(\mathbb{R}^3) : |\xi(u^+,u^-)-1| < 1/2,\ |\xi(u^-,u^+)-1|<1/2\Big\},$$
then there exists a sequence $(u_n)\subset\mathcal{U}$ such that $I(u_n)\to m_*$ and $I'(u_n)\to 0$.

Proof. Given $v\in\mathcal{M}$, we define $\sigma_v(s,t) := \gamma_v t(1-s)v^+ + \gamma_v tsv^- \in \Sigma$. A simple computation shows that $\sigma_v\in\Sigma$, for some $\gamma_v>0$ large. Hence, the set Σ is non-empty and we can define
$$c_\Sigma := \inf_{\sigma\in\Sigma}\sup_{u\in\sigma(Q)} I(u).$$

We claim that $c_\Sigma = m_*$. Actually, considering the map σ_v defined above and using Lemma 4.1, we get
$$I(v) = \max_{s,t\ge 0} I(sv^+ + tv^-) \ge \sup_{u\in\sigma_v(Q)} I(u) \ge c_\Sigma.$$

Since $v\in\mathcal{M}$ is arbitrary, we conclude that $m_*\ge c_\Sigma$. On the other hand, for each $\sigma\in\Sigma$, we can use item (i) of the definition of Σ to get
$$\begin{aligned} 0 &\le \xi(\sigma^+(0,t),\sigma^-(0,t))\\ &= \xi(\sigma^+(0,t),\sigma^-(0,t)) - \xi(\sigma^-(0,t),\sigma^+(0,t))\end{aligned} \tag{4.2}$$
and
$$\begin{aligned} 0 &\ge -\xi(\sigma^-(1,t),\sigma^+(1,t))\\ &= \xi(\sigma^+(1,t),\sigma^-(1,t)) - \xi(\sigma^-(1,t),\sigma^+(1,t)).\end{aligned} \tag{4.3}$$
Again from the definition of Σ, we have
$$\begin{aligned} 2 &\le \frac{\int |\sigma(s,1)|^6 + \int a(x)|\sigma(s,1)|^p}{\|\sigma(s,1)\|^2 + L_{\phi_{\sigma(s,1)}}(\sigma(s,1))}\\ &\le \xi(\sigma^+(s,1),\sigma^-(s,1)) + \xi(\sigma^-(s,1),\sigma^+(s,1))\end{aligned}$$
for any $s\in[0,1]$. Consequently,
$$\xi(\sigma^+(s,1),\sigma^-(s,1)) + \xi(\sigma^-(s,1),\sigma^+(s,1)) - 2 \ge 0$$
and
$$\xi(\sigma^+(s,0),\sigma^-(s,0)) + \xi(\sigma^-(s,0),\sigma^+(s,0)) - 2 = -2 < 0.$$

Taking into account the two above expressions, (4.2) and (4.3), we can apply Miranda's theorem [25] to obtain $(s_\sigma, t_\sigma) \in Q$ such that

$$\begin{aligned} 0 &= \xi(\sigma^+(s_\sigma,t_\sigma),\sigma^-(s_\sigma,t_\sigma)) - \xi(\sigma^-(s_\sigma,t_\sigma),\sigma^+(s_\sigma,t_\sigma)) \\ &= \xi(\sigma^+(s_\sigma,t_\sigma),\sigma^-(s_\sigma,t_\sigma)) + \xi(\sigma^-(s_\sigma,t_\sigma),\sigma^+(s_\sigma,t_\sigma)) - 2. \end{aligned}$$

As a result, it must be

$$\xi(\sigma^+(s_\sigma,t_\sigma),\sigma^-(s_\sigma,t_\sigma)) = \xi(\sigma^-(s_\sigma,t_\sigma),\sigma^+(s_\sigma,t_\sigma)) = 1,$$

which implies that $u_\sigma = \sigma(s_\sigma,t_\sigma) \in \sigma(Q) \cap \mathcal{M}$. Hence,

$$m_* = \inf_{u\in\mathcal{M}} I(u) \le I(u_\sigma) \le \sup_{u\in\sigma(Q)} I(u).$$

Since $\sigma \in \Sigma$ is arbitrary we conclude that $c_\Sigma \ge m_*$ and the claim is proved.

Let $(w_n) \subset \mathcal{M}$ be such that $I(w_n) \to m_*$ and define $\sigma_n(s,t) := \gamma_n t(1-x)w_n^+ + \gamma_n t s w_n^-$, where $\gamma_n > 0$ is such that $\sigma_n \in \Sigma$. From the first part of the proof, we obtain

$$m_* \le \max_{u\in\sigma_n(Q)} I(u) \le I(w_n)$$

and therefore

$$\lim_{n\to+\infty} \max_{u\in\sigma_n(Q)} I(u) = m_*.$$

We can now use a deformation argument as in the proof of Theorem A in [7] to obtain a sequence $(u_n) \subset H^1(\mathbb{R}^3)$ such that

$$I(u_n) \to m_*, \quad I'(u_n) \to 0, \quad \operatorname{dist}(u_n, \sigma_n(Q)) \to 0. \tag{4.4}$$

We need only to verify that, for $n \ge n_0$, there holds $u_n \in \mathcal{U}$. As in the proof of (2.6), the sequence (u_n) is bounded and $\langle I'(u_n), u_n^\pm\rangle = o_n(1)$. Therefore, it is enough to show that $u_n^\pm \ne 0$, which means that $\xi(u_n^+,u_n^-) \to 1$, $\xi(u_n^-,u_n^+) \to 1$, and then $u_n \in \mathcal{U}$, for all n large enough.

From (4.4), there exist sequences (s_n), $(t_n) \subset [0,+\infty)$ and $(v_n) \subset H^1(\mathbb{R}^3)$ such that

$$v_n := s_n w_n^+ + t_n w_n^- \in \sigma_n(Q), \quad \|v_n - u_n\| \to 0. \tag{4.5}$$

We are going to prove that $s_n w_n^+ \not\to 0$ and $t_n w_n^- \not\to 0$, which guarantee that $u_n^\pm \ne 0$. Suppose, by contradiction, that $s_n w_n^+ \to 0$. Then, from Lemma 4.2, we conclude that $s_n \to 0$ and therefore, from the continuity of I and (4.5), we get

$$m_* = \lim_{n\to\infty} I(v_n) = \lim_{n\to\infty} I(s_n w_n^+ + t_n w_n^-) = \lim_{n\to\infty} I(t_n w_n^-).$$

By using this expression and Lemma 4.2 again, we obtain C_1, $C_2 > 0$ such that

$$\begin{aligned} m_* = I(w_n) + o_n(1) &= \max_{s,t\ge0} I(sw_n^+ + tw_n^-) + o_n(1) \\ &\ge \max_{s\ge0} I(sw_n^+ + t_n w_n^-) + o_n(1) \\ &\ge \max_{s\ge0} I(sw_n^+) + o_n(1) + I(t_n w_n^-) \\ &\ge \max_{s\ge0}\Big[\frac{s^2}{2}\|w_n^+\|^2 - \frac{s^p}{p}\int a(x)|w_n^+|^p - \frac{s^6}{6}\|w_n^+\|_6^6\Big] + m_* + o_n(1) \\ &\ge \max_{s\ge0}\Big(\frac{\Lambda_1^2}{2}s^2 - \Lambda_2^p C_1 s^p - \Lambda_2^6 C_2 s^6\Big) + m_* + o_n(1), \end{aligned}$$

where we also have used Lemma 4.1 and the Sobolev embeddings. Taking the limit as $n \to +\infty$ we obtain a contradiction, since the last maximum above is positive. Arguing along the same lines, we can prove that $t_n w_n^- \not\to 0$ and the lemma is proved. $\square$

The next result shows that the positive and the negative part of the sequence given by last lemma is far away from zero.

Lemma 4.4. *There exists $\varrho > 0$ such that*

$$\|u^\pm\| \geq \varrho > 0, \;\; \forall\, u \in \mathcal{U}.$$

Proof. For any $u \in \mathcal{U}$, by the definition of ξ, one has

$$\frac{1}{2}\|u^\pm\|^2 \leq \int a(x)|u^\pm|^p + \int |u^\pm|^6 \leq \varepsilon \int |u^\pm|^2 + C(\varepsilon)\int |u^\pm|^6.$$

Here, we use the fact that, for any $\varepsilon > 0$, there exists $C(\varepsilon) > 0$ such that $s^p + t^6 \leq \varepsilon s^2 + C(\varepsilon)t^6$, for any $s,\, t \geq 0$. Thus, taking $\varepsilon > 0$ small and using (2.1), we see that

$$\|u^\pm\|^2 \leq C_1 \int |u^\pm|^6 \leq C_2\|u^\pm\|^6,$$

and the result follows. $\square$

Lemma 4.5. *If $(u_n) \subset \mathcal{U}$ is such that $I(u_n) \to c \in (0, m_+ + m_\infty)$ and $I'(u_n) \to 0$, then (u_n) has a convergent subsequence.*

Proof. As before, the sequence is bounded and therefore there exists $u \in H^1(\mathbb{R}^3)$ such that, up to a subsequence, $u_n \rightharpoonup u$ weakly in $H^1(\mathbb{R}^3)$, $u_n \to u$ a.e. in $\mathbb{R}^3$ and $I'(u) = 0$. Letting $v_n := u_n - u$, since $a(x) \to a_\infty$, as $|x| \to +\infty$, and $v_n \rightharpoonup 0$ weakly in $H^1(\mathbb{R}^3)$, we conclude that

$$\int a(x)|v_n|^p = \int a_\infty |v_n|^p + \int (a(x) - a_\infty)|v_n|^p = \int a_\infty|v_n|^p + o_n(1). \tag{4.6}$$

Analogously, from (V_2),

$$\|v_n\|^2 = \int \left(|\nabla v_n|^2 + V_\infty v_n^2\right) + o_n(1).$$

The two above expressions, the weak convergence of (u_n), Brezis-Lieb's lemma [6] and (2.2) imply that

$$\begin{aligned}
I(u_n) =& \frac{1}{2}\|u\|^2 + \frac{1}{4}L_{\phi_{u_n}}(u_n) - \int a(x)|u|^p - \int |u|^6 \\
&+ \frac{1}{2}\|v_n\|^2 - \int a(x)|v_n|^p - \int |v_n|^6 + o_n(1) \\
=& I(u) + \frac{1}{2}\int \left(|\nabla v_n|^2 + V_\infty v_n^2\right) - \int a_\infty|v_n|^p - \int |v_n|^6 + o_n(1)
\end{aligned}$$

and therefore

$$c = I(u) + I_\infty(v_n) + o_n(1). \tag{4.7}$$

We are going to prove that $v_n^\pm \to 0$, by excluding the three other possibilities.

Case 1: $v_n^- \to 0$ but $v_n^+ \not\to 0$ strongly in $H^1(\mathbb{R}^3)$.

If $u = 0$, then $v_n^- = u_n^- \to 0$, which contradicts Lemma 4.4. Hence $u \neq 0$ and therefore $I(u) \geq m_+$. By using $I'(u_n) \to 0$ and the argument of the first part of the proof, with (2.3) instead of (2.2), we get

$$\begin{aligned} o_n(1) = \langle I'(u_n), u_n^+\rangle &= \|u_n^+\|^2 + L_{\phi_{u_n}}(u_n^+) - \int a(x)|u_n^+|^p - \int |u_n^+|^6 \\ &= \langle I'(u), u^+\rangle + \|v_n^+\|^2 - \int a(x)|v_n^+|^p - \int |v_n^+|^6 + o_n(1) \end{aligned}$$

and therefore from $\langle I'(u), u^+\rangle = 0$ we obtain

$$\|v_n^+\|^2 - \int a(x)|v_n^+|^p - \int |v_n^+|^6 = o_n(1). \tag{4.8}$$

So, we can argue as in the proof of Lemma 4.4 to conclude that

$$\|v_n^+\|^2 \leq C_1 \int |v_n^+|^6 + o_n(1).$$

On the other hand, since $v_n^+ \not\to 0$, up to a subsequence we have that

$$\int |v_n^+|^6 \geq C_2 > 0. \tag{4.9}$$

Thus, we can easily obtain $s_n > 0$ such that $s_n v_n^+ \in \mathcal{N}_\infty$, namely

$$s_n^2\|v_n^+\|^2 = s_n^p \int a_\infty |v_n^+|^p + s_n^6 \int |v_n^+|^6. \tag{4.10}$$

This equality and (4.9) imply that $(s_n) \subset (0, +\infty)$ is bounded, and therefore we may assume that $s_n \to s \geq 0$.

Arguing as in (4.6) and recalling that $v_n^+ \to 0$, we obtain

$$\int a(x)|v_n|^p = \int a_\infty |v_n|^p + o_n(1).$$

This, (4.10) and (4.8), imply that

$$(s_n^{p-2} - 1)\int a_\infty |v_n^+|^p + (s_n^4 - 1)\int |v_n^+|^6 = o_n(1),$$

and therefore it follows from (4.9) that $s = 1$. So, we can use (4.7), $v_n^- \to 0$ and $I(u) \geq m_+$, to obtain

$$\begin{aligned} c + o_n(1) &= I(u) + I_\infty(v_n^+) + I_\infty(v_n^-) \\ &= m_+ + I_\infty(s_n v_n^+) + o_n(1) \\ &\geq m_+ + m_\infty + o_n(1), \end{aligned}$$

where we also have used the trivial decomposition $I_\infty(v) = I_\infty(v^+) + I_\infty(v^-)$. The above expression provides $c \geq m_+ + m_\infty$, which does not make sense. Thus, Case 1 cannot occurs.

Case 2: $v_n^- \not\to 0$ but $v_n^+ \to 0$ strongly in $H^1(\mathbb{R}^3)$.

This case can be discarded arguing as in the first one.

Case 3: $v_n^- \not\to 0$ and $v_n^+ \not\to 0$ strongly in $H^1(\mathbb{R}^3)$.

In this case, we obtain sequences (s_n), $(t_n) \subset (0,+\infty)$ such that both $s_n u_n^+$ and $t_n u_n^-$ belong to $\mathcal{N}_\infty$ and s_n, $t_n \to 1$. Since $I(u) \geq 0$ and $m_+ < m_\infty$, we can use (4.7) again to get

$$\begin{aligned} c + o_n(1) &\geq I_\infty(v_n^+) + I_\infty(v_n^-) \\ &= I_\infty(s_n v_n^+) + I_\infty(t_n v_n^-) + o_n(1) \\ &\geq m_\infty + m_\infty + o_n(1) \\ &> m_+ + m_\infty + o_n(1), \end{aligned}$$

which is absurd. This finishes the proof. □

We are able to prove our main theorem.

Proof of Theorem 1.2. We claim that

$$m_* < m_+ + m_\infty. \tag{4.11}$$

If this is true, we may invoke Lemma 4.3 to obtain $(u_n) \subset \mathcal{U}$ such that

$$I(u_n) \to m_* < m_+ + m_\infty, \quad I'(u_n) \to 0.$$

According to Lemma 4.5, up to a subsequence $u_n \to u$ strongly in $H^1(\mathbb{R}^3)$. Hence, $I'(u) = 0$ and $I(u) = m_*$, that is, u is a minimal nodal solution.

We verify now that (4.11) really holds. Let u_∞, v_R as in the proof of Theorem 1.1 and $u_0 \in H^1(\mathbb{R}^3)$ be the positive solution given by this same theorem. Let $D := \left[\frac{1}{2}, \frac{3}{2}\right] \times \left[\frac{1}{2}, \frac{3}{2}\right]$ and

$$\Psi(s,t) := \left(\Psi_+(s,t), \Psi_-(s,t)\right),$$

where $\Psi_\pm(s,t) := \langle I'(su_0 - tv_R), (su_0 - tv_R)^\pm \rangle$. Since $\langle I'(u_0), u_0 \rangle = 0$ and $p > 4$, we can easily compute

$$\langle I'(1/2u_0), 1/2u_0 \rangle > 0, \quad \langle I'(3/2u_0), 3/2u_0 \rangle < 0.$$

Moreover, since $v_R \rightharpoonup 0$ in $H^1(\mathbb{R}^3)$, it follows from (2.2) that $\int K(x)\phi_{v_R} v_R^2 = o_R(1)$, as $R \to +\infty$. Hence, we can use (a_1) and (V_1) to get

$$\langle I'(1/2v_R), 1/2v_R \rangle = \langle I'_\infty(1/2u_\infty), 1/2u_\infty \rangle + o_R(1)$$

and

$$\langle I'(3/2v_R), 3/2v_R \rangle = \langle I'_\infty(3/2u_\infty), 3/2u_\infty \rangle + o_R(1).$$

From $\langle I'_\infty(u_\infty), u_\infty \rangle = 0$, we obtain

$$\langle I'_\infty(1/2u_\infty), 1/2u_\infty \rangle > 0, \quad \langle I'_\infty(3/2u_\infty), 3/2u_\infty \rangle < 0.$$

By using all the above expressions, together with $v_R \rightharpoonup 0$, we obtain $R_0 > 0$ such that

$$\Psi_+(1/2, t) > 0, \quad \Psi_+(3/2, t) < 0,$$

for any $t \in [1/2, 3/2]$ and $R \geq R_0$ (see [4]). Analogously, for any $s \in [1/2, 3/2]$ and $R \geq R_0$, there hold

$$\Psi_-(s, 1/2) > 0, \quad \Psi_-(s, 3/2) < 0.$$

Since Ψ is continuous in D, we can apply Miranda's theorem to obtain $(s_R, t_R) \in D$ such that $\Psi(s_R, t_R) = (0,0)$ or, equivalently, $s_R u_0 - t_R v_R \in \mathcal{M}$.

We notice that

$$m_* = \inf_{u \in \mathcal{M}} I(u) \leq I(s_R u_0 - t_R v_R) \leq \max_{(s,t) \in D} I(su_0 - tv_R) \tag{4.12}$$

and we focus now on showing that the right-hand side above is smaller than $m_+ + m_\infty$. In order to achieve this objective, we first compute

$$\begin{aligned} I(su_0 - tv_R) = I(su_0) + I_\infty(tv_R) + \frac{t^2}{2}\Gamma_{R,2} + \frac{t^p}{p}\Gamma_{R,3} \\ + \frac{1}{4}\Gamma_{R,4} - st\Gamma_{R,5} - \frac{1}{p}\Gamma_{R,6} - \frac{1}{6}\Gamma_{R,7}, \end{aligned}$$

where $\Gamma_{R,2}$ and $\Gamma_{R,3}$ are defined in the proof of Theorem 1.1 and

$$\Gamma_{R,4} := \int K(x)\left[\phi_{su_0-tv_R}(su_0 - tv_R)^2 - \phi_{su_0}(su_0)^2\right],$$

$$\Gamma_{R,5} := \int (\nabla u_0 \cdot \nabla v_R + V(x)u_0 v_R),$$

$$\Gamma_{R,6} := \int a(x)\left(|su_0 - tv_R|^p - |su_0|^p - |tv_R|^p\right)$$

and

$$\Gamma_{R,7} := \int \left(|su_0 - tv_R|^6 - |su_0|^6 - |tv_R|^6\right).$$

Arguing as in the proof of Lemma 4.1, we can prove that the map $s \mapsto I(su_0)$ has a unique critical point in $(0, +\infty)$, which is a maximum point. So, recalling that $\langle I'(u_0), u_0\rangle = 0$, we conclude that $I(su_0) \le I(u_0) = m_+$, for any $s \ge 0$. Hence, it follows from $I_\infty(tv_R) \le m_\infty$, (4.12), (3.4), (3.5) and the above expressions that

$$m_* \le m_+ + m_\infty + C_1\left(e^{-\theta R} - e^{-\gamma R}\right) + \frac{1}{4}\Gamma_{R,4} - st\Gamma_{R,5} - \frac{1}{p}\Gamma_{R,6} - \frac{1}{6}\Gamma_{R,7}, \quad (4.13)$$

for some $C_1 = C_1(R) > 0$.

In what follows we estimate $\Gamma_{R,4}$ by first computing the decomposition

$$\Gamma_{R,4} := t^4\Gamma_{R,4,1} + 2s^2t^2\Gamma_{R,4,2} - 4s^3t\Gamma_{R,4,3} - 4st^3\Gamma_{R,4,4} + 4s^2t^2\Gamma_{R,4,5},$$

where

$$\Gamma_{R,4,1} := L_{\phi_{v_R}}(v_R), \quad \Gamma_{R,4,2} := \int K(x)\phi_{u_0}(v_R)^2,$$

$$\Gamma_{R,4,3} := \int K(x)\phi_{u_0}u_0 v_R, \quad \Gamma_{R,4,4} := \int K(x)\phi_{v_R}u_0 v_R,$$

and

$$\Gamma_{R,4,5} := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{K(x)K(y)u_0(x)u_0(y)v_R(x)v_R(y)}{|x-y|}\,dx\,dy.$$

The same argument used to prove (3.3) yields

$$\Gamma_{R,4,1} \le C_2 e^{-\alpha R}, \quad \Gamma_{R,4,2} \le C_2 e^{-\alpha R},$$

for some $C_2 = C_2(R) > 0$. As in the proof of Theorem 1.1, we may assume that $\alpha < \sqrt{V_\infty}$. Thus, we may pick $0 < \bar\eta < \sqrt{V_\infty} - \alpha$ and use $u_0 \in L^\infty(\mathbb{R}^3)$ to obtain

$$\begin{aligned} \Gamma_{R,4,3} &\le \|\phi_{u_0}\|_6 \|u_0\|_\infty \left(\int K(x)^{6/5} v_R^{6/5}\right)^{5/6} \\ &\le C_3 e^{-\alpha R}\left(\int e^{(6/5)\left(\alpha - (\sqrt{V_\infty} - \bar\eta)\right)|x|}\right)^{5/6} \le C_4 e^{-\alpha R}. \end{aligned}$$

Analogously,

$$\Gamma_{R,4,4} \le C_5 e^{-\alpha R}, \quad \Gamma_{R,4,5} = \int K(x)\phi_{\sqrt{u_0 v_R}} u_0 v_R \le C_6 e^{-\alpha R},$$

with C_4, C_5 and C_6 depending on $R > 0$. All together, the above inequalities imply that

$$\Gamma_{R,4} \le C_7 e^{-\alpha R}.$$

We now turn our attention to the term $\Gamma_{R,5}$. By recalling that $I'(u_0)v_R = 0$, we get

$$\begin{aligned} -\Gamma_{R,5} &= \int \left(K(x)\phi_{u_0} u_0 v_R - a(x)|u_0|^{p-2} u_0 v_R - |u_0|^4 u_0 v_R\right) \\ &\le C_8 e^{-\alpha R} + \|a\|_\infty \int |u_0|^{p-2} u_0 v_R, \end{aligned}$$

where we also have used u_0, $v_R > 0$ and the former calculations. We now pick $0 < \bar{\eta} < \eta < \sqrt{V_\infty} - \alpha$ and use the decay property of the positive solution u_0 (see Remark 3.1) to obtain

$$\begin{aligned} \int |u_0|^{p-2} u_0 v_R &\le \|u_0\|_\infty^{p-2} \int u_0 v_R \\ &\le C_9 \int e^{-(\sqrt{V_\infty}-\eta)|x+R\nu|} e^{-(\sqrt{V_\infty}-\bar{\eta})|x|} \\ &= C_9 e^{-(\sqrt{V_\infty}-\eta)R} \int e^{(\bar{\eta}-\eta)|x|}, \end{aligned}$$

and therefore

$$-\Gamma_{R,5} \le C_{10} e^{-\alpha R}.$$

Finally, using the inequality (see [1, Lemma 2.4])

$$\Big| |s-t|^q - s^q - t^q \Big| \le 2q(s^{q-1}t + st^{q-1}),$$

for any s, $t \ge 0$ and $4 \le q \le 6$, we infer that

$$\Gamma_{R,6} \le 2p\|a\|_\infty \int \left(u_0^{p-1} v_R + u_0 v_R^{p-1}\right) \le C_{11} e^{-\alpha R}$$

and

$$\Gamma_{R,7} \le 12\|a\|_\infty \int \left(u_0^5 v_R + u_0 v_R^5\right) \le C_{12} e^{-\alpha R}.$$

By replacing all the above inequalities in (4.13), we get

$$m_* \le m_+ + m_\infty + C_{13}\left(e^{-\alpha R} + e^{-\theta R} - e^{-\gamma R}\right)$$

and therefore, choosing $R > 0$ large enough, we can use $\gamma < \min\{\alpha, \theta\}$ to obtain $m_* < m_+ + m_\infty$, as desired. □

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