A PLANAR SCHRÖDINGER–POISSON SYSTEM WITH VANISHING POTENTIALS AND EXPONENTIAL CRITICAL GROWTH

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Abstract. In this paper we look for ground state solutions of the elliptic system

$$\begin{cases} -\Delta u + V(x)u + \gamma\phi K(x)u = Q(x)f(u), & x \in \mathbb{R}^2, \\ \Delta\phi = K(x)u^2, & x \in \mathbb{R}^2, \end{cases}$$

where $\gamma > 0$ and the continuous potentials V, K, Q satisfy some mild growth conditions and the nonlinearity f has exponential critical growth. The key point of our approach is a new version of the Trudinger-Moser inequality for weighted Sobolev space.

1. Introduction and main results

We are concerned with the existence of solution to the system

$$(\mathcal{S}) \qquad \begin{cases} -\Delta u + V(x)u + \gamma K(x)\phi u = Q(x)f(u), & \text{in } \mathbb{R}^2, \\ \Delta\phi = K(x)u^2, & \text{in } \mathbb{R}^2, \end{cases}$$

where $\gamma > 0$ and the continuous potentials V, K, Q satisfy some mild growth conditions. As quoted by Benci and Fortunato in [8, 9], this system works as a model describing solitary waves for the nonlinear stationary Schrödinger equation interacting with the electrostatic field and also in semiconductor theory, nonlinear optics and plasma physics.

In the past few years, many authors have considered the 3-dimensional case assuming different conditions on the potentials and the nonlinearity f. We could cite [26, 20, 24, 29, 6, 18, 27] and references therein. A common aspect in most of the works is the variational approach. It essentially consists in impose some regularity condition on K, use Lax-Milgram Theorem to solve the second equation and obtain ϕ as the convolution $\phi = \Gamma_3 * (Ku^2)$, where Γ_3 is the fundamental solution of the Laplacian in $\mathbb{R}^3$, namely $\Gamma_3(x) = (-1/4\pi)|x|^{-1}$.

For the planar case, we can use the same idea to conclude that

$$\phi_u(x) := \frac{1}{2\pi}\int_{\mathbb{R}^2} \log(|x-y|)K(y)u^2(y)\mathrm{d}y,$$

2010 *Mathematics Subject Classification.* 35J15, 35J20, 35J60.

Key words and phrases. Non-autonomous Schrödinger–Poisson system; unbounded potential; decaying potential; exponential critical growth; Trudinger–Moser inequality.

The second author was supported by Capes/Brasil.

The third author was partially supported by CNPq/Brazil and FAPDF/Brazil.

The fourth author was supported by CNPq/Brasil and by Grant 2019/2014 Paraíba State Research Foundation (FAPESQ).

where we have used that the fundamental solution in $\mathbb{R}^2$ is given by $\Gamma_2(x) := (1/2\pi)\log|x|$. Hence, we are leading to consider the nonlocal equation

$$(\mathcal{E}) \qquad -\Delta u + V(x)u + \frac{\gamma}{2\pi}[\log|\cdot| * (Ku^2)](x)K(x)u = Q(x)f(u), \quad x \in \mathbb{R}^2.$$

After obtaining a weak solution $u \in W$, where W is an appropriated space which depends on the potentials, we can prove that the pair (u, ϕ_u) weakly solves $(\mathcal{S})$.

When dealing with $(\mathcal{E})$ via variational methods, the first difficulty occurs due to the logarithmic kernel, which is unbounded and has no defined sign. It turns out that the formal energy functional associated to the equation is not well defined in $H^1(\mathbb{R}^2)$ even if all the potentials V, K and Q are positive and bounded. To overcome this trouble, Stubbe [30] and Cingolani & Weth [19] introduced a new space which is appropriated to deal with the nonlocal part of the energy functional, namely

$$u \mapsto \mathcal{V}(u) := \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log(|x-y|)K(y)u^2(y)K(x)u^2(x)\mathrm{d}y\mathrm{d}x.$$

Here, we adapt this former argument in order to consider potentials with no kind of prescribed symmetry. We just impose the following decay assumptions, which were inspired from [5] (see also [31, 32]):

(VKQ) V, K, $Q \in C(\mathbb{R}^2)$ and there exist $\gamma \leq 2$, $\eta > 2$, $\beta > 2$ and positive constants b_V, b_K, b_Q such that, for any $x \in \mathbb{R}^2$,

$$\frac{b_V}{(1+|x|)^\gamma} \leq V(x), \quad 0 < K(x) \leq \frac{b_K}{(1+|x|)^\eta}, \quad 0 < Q(x) \leq \frac{b_Q}{(1+|x|)^\beta}.$$

Because of the above condition on V and the linear part on the right-hand side of $(\mathcal{E})$, it is natural to consider the Hilbert space

$$E := \left\{ u \in L^2_{\mathrm{loc}}(\mathbb{R}^2) : |\nabla u| \in L^2(\mathbb{R}^2) \text{ and } \int_{\mathbb{R}^2} V(x)u^2\mathrm{d}x < \infty \right\}$$

endowed with the inner product $\langle u, v\rangle_E := \int_{\mathbb{R}^2}[\nabla u \nabla v + V(x)uv]\,\mathrm{d}x$. We are going to prove that, for any $2 \leq p < \infty$, the space E is compactly embedded into the weighted Lebesgue space (see Proposition 2.2)

$$L^p(\mathbb{R}^2; Q) := \left\{ u : \mathbb{R}^2 \to \mathbb{R} \text{ measurable} : \int_{\mathbb{R}^2} Q(x)|u|^p\mathrm{d}x < \infty \right\},$$

which is a Banach space with the norm $\|u\|_{L^p(\mathbb{R}^2;Q)} = \left(\int_{\mathbb{R}^2} Q(x)|u|^p\mathrm{d}x\right)^{1/p}$. The same holds for the space $L^2(\mathbb{R}^2; K)$. Although some related results have been appeared in [28], our proof is different and new. Moreover, our approach can be used to obtain similar embeddings in $\mathbb{R}^N$, with $N \geq 3$, as those presented in [5, 11, 12].

Motivated by the above embedding, we ask if it is possible to obtain, for the functions of the space E, exponential integrability conditions determined by a Trudinger-Moser type inequality involving the weight Q, see for instance [21, 13, 33] and references therein. The answer is positive, as we can see from our first main theorem:

Theorem 1.1. *For any $\alpha > 0$ and $u \in E$, the function $Q(\cdot)(e^{\alpha u^2} - 1)$ belongs to $L^1(\mathbb{R}^2)$. Moreover, there exists $\alpha_* \in (0, 4\pi)$ such that*

$$\sup_{u\in E,\, \|u\|_E\le 1} \int_{\mathbb{R}^2} Q(x)(e^{\alpha u^2} - 1)\mathrm{d}x < \infty,$$

for any $0 < \alpha \le \alpha_*$.

By using the above theorem we can deal here with nonlinearities which grow faster than any prescribed power $|s|^p$. More specifically, we suppose that f has exponential critical growth at infinity and is superlinear at the origin, i.e.,

(f_0) $f \in C(\mathbb{R})$ and there exists $\alpha_0 > 0$ such that

$$\lim_{|s|\to+\infty} \frac{|f(s)|}{e^{\alpha s^2}} = \begin{cases} 0 & \text{if} \quad \alpha > \alpha_0, \\ +\infty & \text{if} \quad \alpha < \alpha_0; \end{cases}$$

(f_1) $f(s) = o(|s|)$ as $s \to 0$;

(f_2) there exists $\theta > 4$ such that $0 < \theta F(s) \le f(s)s$ for all $s \neq 0$, where $F(s) := \int_0^s f(t)dt$;

(f_3) there exists $\nu > 0$ such that $F(s) \ge \nu|s|^4$, for all $s \in \mathbb{R}$;

(f_4) the function $s \mapsto f(s)/|s|^3$ is increasing in $|s| > 0$.

Under the above conditions, the energy functional associated to ($\mathcal{E}$) is, formally,

$$I(u) = \frac{1}{2}\|u\|_E^2 + \frac{\gamma}{8\pi}\mathcal{V}(u) - \int_{\mathbb{R}^2} Q(x)F(u)\mathrm{d}x.$$

Actually, it is necessary to guarantee that the nonlocal term $\mathcal{V}(u)$ is well defined. To do this, we consider a space smaller than E. As in [30], we can justify that the correct space to look for solutions is

$$W := \left\{ u \in E : \int_{\mathbb{R}^2} \log(1 + |x|)K(x)u^2\mathrm{d}x < \infty \right\},$$

which is a Hilbert space (see Lemma 2.1) with the norm

$$\|u\|_W^2 := \int_{\mathbb{R}^2} \left(|\nabla u|^2 + V(x)u^2\right)\mathrm{d}x + \int_{\mathbb{R}^2} \log(1 + |x|)K(x)u^2\mathrm{d}x.$$

Since clearly $W \hookrightarrow E$, we can define the numbers

$$S_4(Q) := \inf_{u\in W\setminus\{0\}} \frac{\|u\|_W^2}{\|u\|^2_{L^4(\mathbb{R}^2;Q)}}, \quad S_2(K) := \inf_{u\in W\setminus\{0\}} \frac{\|u\|_W^2}{\|u\|^2_{L^2(\mathbb{R}^2;K)}}.$$

Moreover, we can prove that $(f_0) - (f_1)$ implies that $I \in C^1(W)$. We call $u \in W$ a weak solution to equation ($\mathcal{E}$) if, for all $\varphi \in C_0^\infty(\mathbb{R}^2)$, it holds

$$\langle u, \varphi\rangle_E + \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log(|x - y|)K(y)u^2(y)K(x)u(x)\varphi(x)\mathrm{d}y\mathrm{d}x = \int_{\mathbb{R}^2} Q(x)f(u)\varphi\mathrm{d}x.$$

So, by using standard calculations we conclude that the weak solutions are exactly the critical points of I.

The main existence result for problem ($\mathcal{E}$) can be stated as follows:

Theorem 1.2. *Suppose that (VKQ) and $(f_0) - (f_4)$ hold. Then, there exists $\alpha_* \in (0, 4\pi)$ such that problem ($\mathcal{E}$) has a nonzero small energy solution provided*

$$\nu > S_4^2(Q) \max\left\{ \frac{1}{S_2(K)}, \frac{\alpha_0}{2\alpha_*} \right\}. \tag{1.1}$$

As a byproduct of Theorem 1.2, we can give a contribution concerning the existence of solutions to the system ($\mathcal{S}$), namely

Theorem 1.3. *Suppose the same hypotheses of Theorem 1.2 and let $u \in W$ be the solution obtained in that theorem. Then, the pair (u, ϕ_u) is a weak solution of system ($\mathcal{S}$), where $\phi_u = \Gamma_2 * (Ku^2)$.*

Unlike in the 3-dimensional case, the study of the planar version of Schrödinger-Poisson systems like ($\mathcal{S}$) is a very recent trend. Besides [30, 19], we could cite [22], where the authors consider an autonomous problem with $f(s) = |s|^{p-2}s$, $p > 2$, and obtained a ground state nonnegative solution and infinitely many sign-changing solutions. We refer to [14, 15, 16, 10] and its references for other results with f having polynomial growth. Concerning exponential type nonlinearities, we cite the works [3, 17]. In the first one, we have $V = K = Q \equiv 1$, while in the other one, the unique nonconstant potential V has axial symmetry. We also quote [2], where the authors obtained some results which are similar to ours but for radial potentials. Here, we consider potentials which depend on x and have no prescribed symmetry. In particular, we do not have any kind of radial decay property at infinity for the functions of our working space. Hence, our results do not follow as in these previous works.

It is worth noticing that, even if W provides a variational framework to our problems, some difficulties appear due to some unpleasant facts. The first one is that the norm in W does not appear explicitly in the expression of the functional. Moreover, it is not invariant under translations. Third, we can see that the quadratic part of I is not coercive on W. Besides all of these troubles, we can use condition (f_4) for obtaining a nonzero critical as a minimization argument in the Nehari manifold. As a final comment, we notice that we prove Theorem 1.1 to permit exponential growth for the function f. However, this weighted Trudinger-Moser inequality has an interest in its own and it can be used in many other contexts different from the Schrödinger-Poisson system.

The remainder of the paper is organized as follows: In Section 2, we present some basic results and prove the embedding of E into the weighted Lebesgue spaces. The Trudinger-Moser type inequality is proved in Section 3. The final section is devoted to the proof of our existence results.

2. Functional setting and embeddings results

In this section, we establish some preliminary results used in the proof of our main theorems. Throughout the paper, we shall assume that condition (VKQ) holds. For any $R > 0$ and $y \in \mathbb{R}^2$, we denote by $B_R(x)$ the open ball $\{x \in \mathbb{R}^2 : |y - x| < R\}$. If $x = 0$, we write only B_R. Finally, we denote by $C_1, C_2, \ldots$, positive constants (possibly different).

We are going to use the functional spaces E and W defined in the first section. Although E is well known, it is not so clear that W is a Hilbert space. For completeness, we present the proof of this fact (see also [1]).

Lemma 2.1. *$(W, \|\cdot\|_W)$ is a Hilbert space.*

Proof. Let $(u_n) \subset W$ be a Cauchy sequence and notice that

$$\left(\frac{\partial u_n}{\partial x_i}\right) \ (i = 1, 2), \quad \left(V^{1/2}(\cdot)u_n\right) \quad \text{and} \quad \left([\log(1 + |\cdot|)K(\cdot)]^{1/2}u_n\right),$$

are also Cauchy sequences in $L^2(\mathbb{R}^2)$. Hence, there exist u^1, u^2, v, $z \in L^2(\mathbb{R}^2)$ such that, as $n \to +\infty$,

$$\frac{\partial u_n}{\partial x_i} \to u^i \ (i=1,2), \quad V^{1/2}(\cdot)u_n \to v, \quad [\log(1+|\cdot|)K(\cdot)]^{1/2}u_n \to z, \tag{2.1}$$

strongly in $L^2(\mathbb{R}^2)$. Moreover, up to a subsequence,

$$\frac{\partial u_n}{\partial x_i} \to u^i \ (i=1,2), \quad u_n \to w := V^{-1/2}(x)v = [\log(1+|x|)K(x)]^{-1/2}\, z, \tag{2.2}$$

for a.e. $x \in \mathbb{R}^2$.

We shall prove that $w \in W$ and $u_n \to w$ in W. In order to do that, we pick $R > 0$ and consider $\varphi \in C_0^\infty(B_{R+1})$ such that $\varphi \equiv 1$ in B_R. Since $\varphi(u_n - u_m) \in H_0^1(B_{R+1})$, we can use Poincaré's inequality to get

$$\begin{aligned}\|u_n - u_m\|^2_{L^2(B_R)} &\le \int_{B_{R+1}} |\varphi(u_n-u_m)|^2 \mathrm{d}x \le C_1 \int_{B_{R+1}} |\nabla(\varphi(u_n-u_m))|^2 \mathrm{d}x \\ &\le C_2\left(\|u_n - u_n\|^2_{H_0^1(B_{R+1})} + \int_{B_{R+1}\setminus B_R} |u_n-u_m|^2\mathrm{d}x\right).\end{aligned}$$

Since $\inf_{B_{R+1}\setminus B_R} V > 0$, we obtain

$$\begin{aligned}\|u_n - u_m\|^2_{L^2(B_R)} &\le C_2\|u_n-u_m\|^2_{H_0^1(B_{R+1})} + C_3\int_{\mathbb{R}^2} V(x)|u_n-u_m|^2\mathrm{d}x \\ &\le C_4\|u_n-u_m\|^2_E \le C_4\|u_n-u_m\|^2_W,\end{aligned} \tag{2.3}$$

from which it follows that, for some $u_R \in L^2(B_R)$, there holds

$$u_n \to u_R \quad \text{in } L^2(B_R) \quad \text{and} \quad u_n \to u_R \quad \text{a.e. in } B_R, \tag{2.4}$$

as $n \to +\infty$. This and (2.2) imply that $w = u_R \in L^2(B_R)$ and so $w \in L^2_{\mathrm{loc}}(\mathbb{R}^2)$.

Next, we prove that w has weak derivate and $|\nabla w| \in L^2(\mathbb{R}^2)$. In fact, let $\psi \in C_0^\infty(B_R)$ and notice that, for any $n \in \mathbb{N}$,

$$\int_{\mathbb{R}^2} u_n \frac{\partial \psi}{\partial x_i}\mathrm{d}x = -\int_{\mathbb{R}^2} \frac{\partial u_n}{\partial x_i}\psi \mathrm{d}x, \quad i=1,2.$$

Passing to the limit, using (2.1), (2.4) and $u_R = w$ in B_R, we obtain

$$\int_{\mathbb{R}^2} w \frac{\partial \psi}{\partial x_i}\mathrm{d}x = -\int_{\mathbb{R}^2} u^i\psi \mathrm{d}x, \quad i=1,2,$$

and therefore w has weak derivative with $\nabla w = (u^1, u^2)$. The last equality, (2.1) and (2.2) guarantee that $|\nabla w| \in L^2(\mathbb{R}^2)$ and

$$\int_{\mathbb{R}^2} V(x)w^2\mathrm{d}x = \int_{\mathbb{R}^2} v^2\mathrm{d}x < \infty, \quad \int_{\mathbb{R}^2} \log(1+|x|)K(x)w^2\mathrm{d}x = \int_{\mathbb{R}^2} z^2\mathrm{d}x < \infty,$$

in such way that $w \in W$. For the same reasons

$$\int_{\mathbb{R}^2} |\nabla u_n - \nabla w|^2\mathrm{d}x \to 0, \quad \int_{\mathbb{R}^2} V(x)|u_n - w|^2\mathrm{d}x = \int_{\mathbb{R}^2}\left|V(x)^{1/2}u_n - v\right|^2\mathrm{d}x \to 0,$$

$$\int_{\mathbb{R}^2} \log(1+|x|)K(x)|u_n-w|^2\mathrm{d}x = \int_{\mathbb{R}^2}\left|[\log(1+|x|)K(x)]^{1/2}u_n - z\right|^2\mathrm{d}x \to 0,$$

as $n \to +\infty$. Hence, $u_n \to w$ in W and we have done. $\square$

Now, we quote an auxiliary result, which will be useful in the paper. The embedding can be view as a version of similar ones presented in [28]. However, the technique used here is a simple one.

Proposition 2.2. *The embedding $E \hookrightarrow L^p(\mathbb{R}^2; Q)$ is compact for any $2 \leq p < \infty$.*

Proof. Arguing as in the proof of (2.3) we can check that $E \hookrightarrow H^1_{loc}(\mathbb{R}^2)$. So, given $u \in E$ and $p \geq 2$, we can use (VKQ) to obtain C_1, $C_2 > 0$ such that

$$\begin{aligned}\int_{B_1} \frac{|u|^p}{(1+|x|)^\beta}\mathrm{d}x &< \int_{B_1} |u|^p \mathrm{d}x \leq C_1 \left(\int_{B_1} \left[|\nabla u|^2 + u^2 \right] \mathrm{d}x \right)^{p/2} \\ &\leq C_2 \left(\int_{B_1} \left[|\nabla u|^2 + V(x) u^2 \right] \mathrm{d}x \right)^{p/2}.\end{aligned} \tag{2.5}$$

If we define $A_j := \{z \in \mathbb{R}^2 : 2^j < |z| < 2^{j+1}\}$, for $j \in \mathbb{N} \cup \{0\}$, the change of variables $y := 2^{-j}x$ provides

$$\int_{A_j} \frac{|u|^p}{(1+|x|)^\beta}\mathrm{d}x \leq \frac{1}{2^{\beta j}} \int_{A_j} |u|^p \mathrm{d}x = 2^{(2-\beta)j} \int_{A_0} |u_j(y)|^p \mathrm{d}y,$$

where $u_j(y) := u(2^j y)$. Using the Sobolev embedding $H^1(A_0) \hookrightarrow L^p(A_0)$, we obtain $C_3 > 0$ such that

$$\begin{aligned}\int_{A_0} |u_j(y)|^p \mathrm{d}y &\leq C_3 \left(\int_{A_0} \left[|\nabla u_j(y)|^2 + u_j^2(y) \right] \mathrm{d}y \right)^{p/2} \\ &= C_3 \left(\int_{A_0} \left[|\nabla u(x)|^2 + 2^{-2j} u^2(x) \right] \mathrm{d}x \right)^{p/2}.\end{aligned}$$

Since $(1 + 2^{j+1}) \leq 2 \cdot 2^{j+1}$ and we may assume without loss of generality that $\gamma \geq 0$, one has

$$\int_{A_j} 2^{-2j} u^2(x) \mathrm{d}x \leq 2^{-2j}(1+2^{j+1})^\gamma \int_{A_j} \frac{u^2(x)}{(1+|x|)^\gamma}\mathrm{d}x \leq 2^{2\gamma+(\gamma-2)j} \int_{A_j} \frac{u^2(x)}{(1+|x|)^\gamma}\mathrm{d}x.$$

Using the above estimates and that $\gamma \leq 2 < \beta$, we get

$$\begin{aligned}\int_{A_j} \frac{|u|^p}{(1+|x|)^\beta}\mathrm{d}x &\leq 2^{(2-\beta)j} \left(\int_{A_j} \left[|\nabla u(x)|^2 + 2^{2\gamma+(\gamma-2)j} \frac{u^2(x)}{(1+|x|)^\gamma} \right] \mathrm{d}x \right)^{p/2} \\ &\leq C_4 \left(\int_{A_j} \left[|\nabla u(x)|^2 + \frac{u^2(x)}{(1+|x|)^\gamma} \right] \mathrm{d}x \right)^{p/2},\end{aligned} \tag{2.6}$$

where $C_4 := C_3 \cdot 2^{\gamma p}$ does not depend on j. Thus, recalling that the function $s \mapsto s^{p/2}$ is super additive for $p \geq 2$, we conclude that

$$\sum_{j=0}^{\infty} \int_{A_j} \frac{|u|^p}{(1+|x|)^\beta}\mathrm{d}x \leq C_4 \left(\int_{\mathbb{R}^2 \setminus B_1} \left[|\nabla u|^2 + \frac{u^2}{(1+|x|)^\gamma} \right] \mathrm{d}x \right)^{p/2}.$$

This, (2.5) and (VKQ) imply that

$$\int_{\mathbb{R}^2} Q(x)|u|^p \mathrm{d}x \leq b_Q \int_{\mathbb{R}^2} \frac{|u|^p}{(1+|x|)^\beta}\mathrm{d}x \leq C_5 \left(\int_{\mathbb{R}^2} \left[|\nabla u|^2 + V(x)u^2 \right] \mathrm{d}x \right)^{p/2},$$

which proves the continuous embedding.

For the compactness result, we take $(u_n) \subset E$ such that $u_n \rightharpoonup 0$ weakly in E. Since $\beta > 2$, for any given $\varepsilon > 0$ there exists $j_0 \in \mathbb{N}$ such that $2^{(2-\beta)j} < \varepsilon$, whenever $j \geq j_0$. Since $E \hookrightarrow H^1_{loc}(\mathbb{R}^2)$, it follows from Rellich-Kondrachov Theorem that

$$\int_{B_1} Q(x)|u_n|^p \mathrm{d}x + \sum_{j=0}^{j_0-1} \int_{A_j} Q(x)|u_n|^p \mathrm{d}x \leq \|Q\|_{L^\infty(\mathbb{R}^2)} \int_{B_{2^{j_0}}} |u_n|^p \mathrm{d}x = o_n(1),$$

where $o_n(1)$ stands for a quantity approaching zero as $n \to +\infty$. Using this, (VKQ), (2.6) and that $s \mapsto s^{p/2}$ is super additive, we obtain

$$\begin{aligned} \sum_{j=0}^{\infty} \int_{A_j} Q(x)|u_n|^p \mathrm{d}x &\leq o_n(1) + \varepsilon C_6 \sum_{j=j_0}^{\infty} \left(\int_{A_j} \left[|\nabla u_n(x)|^2 + V(x) u_n^2(x) \right] \mathrm{d}x \right)^{p/2} \\ &\leq o_n(1) + \varepsilon C_6 \|u_n\|_E^p = o_n(1), \end{aligned}$$

and the proposition is proved. □

Corollary 2.3. *The embeddings E, $W \hookrightarrow L^{8/3}(\mathbb{R}^2; K^{4/3})$ are compact.*

Proof. By assumption (VKQ), there exists $\eta > 2$ such that

$$K(x)^{4/3} \leq \frac{b_K^{4/3}}{(1+|x|)^{4\eta/3}}, \quad \forall\, x \in \mathbb{R}^2.$$

Since $4\eta/3 > 2$, we can apply Proposition 2.2 with $p = 8/3$ and $Q = K^{4/3}$ to get the result for E. Finally, take into account that $W \hookrightarrow E$, we conclude that the result also holds for the subspace W. □

3. Trudinger-Moser type inequality

We devote this section to the proof of Theorem 1.1. Before, we need two technical results.

Lemma 3.1. *Let $x_0 \in \mathbb{R}^2$ and $u \in H_0^1(B_R(x_0))$ be such that $\int_{B_R(x_0)} |\nabla u|^2 \mathrm{d}x \leq 1$. Then there exists $C > 0$ such that*

$$\int_{B_R(x_0)} \left(e^{4\pi u^2} - 1 \right) \mathrm{d}x \leq C \cdot R^2 \int_{B_R(x_0)} |\nabla u|^2 \mathrm{d}x.$$

Proof. See [34, Lemma 3.1]. □

The second auxiliary result is a version, for our functional space, of a previous result presented in [21]. In their proof, the authors used, among other things, Besicovitch covering lemma. Here we will use a similar approach used in [13] where the authors obtain a Trudinger Moser type inequality involving weight with logarithm growth.

Lemma 3.2. *There exist $C > 0$ and $\alpha_* \in (0, 4\pi)$ such that*

$$\int_{\mathbb{R}^2} Q(x)(e^{\alpha u^2} - 1)\mathrm{d}x \leq C\|u\|_E^2,$$

for any $0 < \alpha \leq \alpha_*$ and $u \in E$ verifying $\|u\|_E \leq 1$.

Proof. Let $u \in E$ be such that $\|u\|_E \leq 1$ and $\varphi \in C_0^\infty(B_2)$ satisfying $\varphi \equiv 1$ in B_1 and $|\nabla \varphi| \leq 2$ in B_2. As in the first part of the proof of Proposition 2.2, we can estimate

$$\int_{B_2} |\nabla (\varphi u)|^2 \, \mathrm{d}x \leq C_1 \int_{B_2} \left[|\nabla u|^2 + u^2\right] \mathrm{d}x \leq C_2 \int_{B_2} \left[|\nabla u|^2 + V(x) u^2\right] \mathrm{d}x.$$

Setting $v := (1/C_2)^{1/2} \varphi u$, we can apply Lemma 3.1 to obtain

$$\int_{B_2} \left(e^{4\pi v^2} - 1\right) \mathrm{d}x \leq C \cdot 2^2 \int_{B_2} |\nabla v|^2 \mathrm{d}x \leq C_3 \int_{B_2} \left[|\nabla u|^2 + V(x) u^2\right] \mathrm{d}x.$$

Thus, for any $0 < \alpha \leq 4\pi/C_2$, one has

$$\begin{aligned} \int_{B_1} \left(e^{\alpha u^2} - 1\right) \mathrm{d}x &= \int_{B_1} \left(e^{\alpha (\varphi u)^2} - 1\right) \mathrm{d}x \leq \int_{B_2} \left(e^{\alpha (\varphi u)^2} - 1\right) \mathrm{d}x \\ &= \int_{B_2} \left(e^{\alpha C_2 v^2} - 1\right) \mathrm{d}x \\ &\leq C_3 \int_{B_2} \left[|\nabla u|^2 + V(x) u^2\right] \mathrm{d}x. \end{aligned} \tag{3.1}$$

We claim that, for some $C_4 > 0$, there holds

$$\int_{A_j} Q(x) \left(e^{\alpha u^2} - 1\right) \mathrm{d}x \leq C_4 \int_{A_j} \left[|\nabla u|^2 + V(x) u^2\right] \mathrm{d}x, \tag{3.2}$$

for any $j \in \mathbb{N} \cup \{0\}$ and $0 < \alpha \leq \alpha_*$, with $\alpha_* > 0$ to be chosen later. If this is true, the statement of the lemma is a direct consequence the above inequality, (3.1) and

$$\int_{\mathbb{R}^2} Q(x) \left(e^{\alpha u^2} - 1\right) \mathrm{d}x = \int_{B_1} Q(x) \left(e^{\alpha u^2} - 1\right) \mathrm{d}x + \sum_{j=0}^{\infty} \int_{A_j} Q(x) \left(e^{\alpha u^2} - 1\right) \mathrm{d}x.$$

In order to prove (3.2), we first fix $j \in \mathbb{N} \cup \{0\}$ and use the change of variables $y := 2^{-j} x$, (VKQ) and $\beta > 2$ to obtain

$$\int_{A_j} Q(x) \left(e^{\alpha u^2} - 1\right) \mathrm{d}x \leq \frac{b_Q}{2^{\beta j}} \int_{A_j} \left(e^{\alpha u^2} - 1\right) \mathrm{d}x = b_Q \int_{A_0} \left(e^{\alpha u_j^2} - 1\right) \mathrm{d}y, \tag{3.3}$$

where $u_j(y) := u(2^j y)$. Consider $y \in A_0$, set $R_y := \operatorname{dist}(y, \partial A_0)$ and notice that $B_{R_y}(y) \subset A_0$. Moreover, from the compactness of $\overline{A_0}$, we obtain points $y_1, \ldots, y_k \in A_0$ such that $A_0 \subset \bigcup_{i=1}^k B_{R_i/2}(y_i)$, where $R_i := R_{y_i}$. For each $i = 1, \ldots, k$, we pick a function $\varphi_i \in C_0^\infty(B_{R_i}(y_i))$ such that $0 \leq \varphi_i \leq 1$ in $B_{R_i}(y_i)$, $\varphi_i \equiv 1$ in $B_{R_i/2}(y_i)$ and $|\nabla \varphi_i| \leq 4/R_i$ in $B_{R_i}(y_i)$. If we call $B^i := B_{R_i}(y_i)$, we have that

$$\begin{aligned} \int_{B^i} |\nabla (\varphi_i(y) u_j(y))|^2 \, \mathrm{d}y &\leq C_5 \int_{A_0} 2^{2j} |\nabla u(2^j y)|^2 \mathrm{d}y + C_6 R_i^{-2} \int_{A_0} u^2(2^j y) \mathrm{d}y \\ &\leq C_5 \int_{A_j} |\nabla u(x)|^2 \mathrm{d}x + \frac{C_6 R_i^{-2}}{2^{2j}} \int_{A_j} u^2(x) \mathrm{d}x. \end{aligned}$$

Since $(1 + 2^{j+1})^\gamma \leq 4^\gamma \cdot 2^{\gamma j}$ and we may assume without loss of generality that $\gamma \geq 0$, we get

$$\int_{A_j} u^2(x) \mathrm{d}x \leq 4^\gamma \cdot 2^{\gamma j} \int_{A_j} \frac{u^2(x)}{(1 + |x|)^\gamma} \mathrm{d}x.$$

All together, the above inequalities, $\gamma \le 2$ and (VKQ) imply that,

$$\int_{B^i} |\nabla(\varphi_i(y)u_j(y))|^2 \,\mathrm{d}y \le C_7 \int_{A_j} \left[|\nabla u(x)|^2 + V(x)u^2(x)\right] \mathrm{d}x.$$

At this point we define

$$\alpha_* := \min\left\{\frac{4\pi}{C_2}, \frac{4\pi}{C_7}\right\}.$$

If $v_{i,j} := (1/C_7)^{1/2}\,\varphi_i u_j$, we can use Lemma 3.1 to estimate

$$\int_{B^i} \left(e^{4\pi v_{i,j}^2} - 1\right) \mathrm{d}y \le C \cdot R_i^2 \int_{B^i} |\nabla v_{i,j}|^2 \mathrm{d}y \le C_8 \int_{A_j} \left[|\nabla u|^2 + V(x)u^2\right] \mathrm{d}x.$$

If $0 < \alpha \le \alpha_*$, we obtain

$$\int_{B^i} \left(e^{\alpha(\varphi_i u_j)^2} - 1\right) \mathrm{d}y \le C_8 \int_{A_j} \left[|\nabla u|^2 + V(x)u^2\right] \mathrm{d}x,$$

and thus

$$\begin{aligned}\int_{A_0} \left(e^{\alpha u_j^2} - 1\right) \mathrm{d}y &\le \sum_{i=1}^k \int_{B_{R_i/2}(y_i)} \left(e^{\alpha u_j^2} - 1\right) \mathrm{d}y = \sum_{i=1}^k \int_{B_{R_i/2}(y_i)} \left(e^{\alpha(\varphi_i u_j)^2} - 1\right) \mathrm{d}y \\ &\le C_9 \int_{A_j} \left[|\nabla u(x)|^2 + V(x)u^2(x)\right] \mathrm{d}x.\end{aligned}$$

The inequality (3.2) is a consequence of the last estimate and (3.3). □

We are ready to present the proof of our first main theorem.

Proof of Theorem 1.1. Let $\alpha > 0$ and $u \in E$. By density, there exists $u_0 \in C_0^\infty(\mathbb{R}^2)$ such that

$$\|u - u_0\|_E \le \delta,$$

with $\delta > 0$ to be chosen later. Since $u^2 \le 2(u - u_0)^2 + 2u_0^2$, we may estimate

$$\int_{\mathbb{R}^2} Q(x)\left(e^{\alpha u^2} - 1\right) \mathrm{d}x \le \int_{\mathbb{R}^2} Q(x)\left(e^{2\alpha(u-u_0)^2} e^{2\alpha u_0^2} - 1\right) \mathrm{d}x.$$

Recalling the elementary inequality

$$ab - 1 \le \frac{1}{2}(a^2 - 1) + \frac{1}{2}(b^2 - 1), \quad \forall\, a, b \ge 0,$$

setting $w := u - u_0$ and denoting by Ω_0 the support of u_0, we obtain

$$\begin{aligned}2\int_{\mathbb{R}^2} Q(x)\left(e^{\alpha u^2} - 1\right) \mathrm{d}x &\le \int_{\mathbb{R}^2} Q(x)\left(e^{4\alpha w^2} - 1\right) \mathrm{d}x + \int_{\Omega_0} Q(x)\left(e^{4\alpha u_0^2} - 1\right) \mathrm{d}x \\ &\le \int_{\mathbb{R}^2} Q(x)\left(e^{4\alpha\|w\|_E^2\left(\frac{w}{\|w\|_E}\right)^2} - 1\right) \mathrm{d}x + C_1 \int_{\Omega_0} 1 \mathrm{d}x,\end{aligned}$$

with $C_1 := \|Q\|_{L^\infty(\mathbb{R}^2)} e^{4\alpha\|u_0\|^2_{L^\infty(\mathbb{R}^2)}}$. We now pick $\delta > 0$ in such way that

$$4\alpha\|w\|_E^2 \le 4\alpha\delta^2 \le \alpha_*$$

and we use Lemma 3.2 to conclude that

$$\int_{\mathbb{R}^2} Q(x)\left(e^{\alpha u^2} - 1\right) \mathrm{d}x \le \frac{C}{2} + \frac{C_1}{2}|\Omega_0| < \infty.$$

This proves the first statement of Theorem 1.1. The second one is a direct consequence of Lemma 3.2. ☐

4. Existence results

We prove in this section Theorems 1.2 and 1.3. The idea is looking for critical points of the energy functional defined in the introductory section. Since the value of $\gamma > 0$ is not important, we shall simplify the notation putting $\gamma = 2\pi$, in such a way that the functional becomes

$$I(u) := \frac{1}{2}\|u\|_E^2 + \frac{1}{4}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log(|x-y|)K(y)u^2(y)K(x)u^2(x)\mathrm{d}y\mathrm{d}x - \int_{\mathbb{R}^2} Q(x)F(u)\mathrm{d}x.$$

Using that $\log r = \log(1+r) - \log(1+r^{-1})$, for any $r > 0$, the nonlocal part of I can be decomposed as

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log(|x-y|)K(y)u^2(y)K(x)u^2(x)\mathrm{d}y\mathrm{d}x = \mathcal{V}_1(u) - \mathcal{V}_2(u),$$

with

$$\mathcal{V}_1(u) := \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log(1+|x-y|)K(y)u^2(y)K(x)u^2(x)\mathrm{d}y\mathrm{d}x,$$

and

$$\mathcal{V}_2(u) := \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log(1+|x-y|^{-1})K(y)u^2(y)K(x)u^2(x)\mathrm{d}y\mathrm{d}x.$$

So, we can rewrite I as

$$I(u) := \frac{1}{2}\|u\|_E^2 + \frac{1}{4}\mathcal{V}_1(u) - \frac{1}{4}\mathcal{V}_2(u) - \int_{\mathbb{R}^2} Q(x)F(u)\mathrm{d}x, \quad u \in W.$$

As proved in [19, Lemma 2.2], the nonlocal parts $\mathcal{V}_1$, $\mathcal{V}_2$ are well defined. Moreover, they belong to $C^1(W)$ and, for any u, $v \in W$, there hold

$$\mathcal{V}_1'(u)v = 4\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log(1+|x-y|)K(y)u^2(y)K(x)u(x)v(x)\mathrm{d}y\mathrm{d}x, \tag{4.1}$$

and

$$\mathcal{V}_2'(u)v = 4\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log(1+|x-y|^{-1})K(y)u^2(y)K(x)u(x)v(x)\mathrm{d}y\mathrm{d}x. \tag{4.2}$$

In particular,

$$\mathcal{V}_1'(u)u = 4\mathcal{V}_1(u), \quad \mathcal{V}_2'(u)u = 4\mathcal{V}_2(u), \quad \forall\, u \in W. \tag{4.3}$$

Since $1 + |x-y| \le (1+|x|)(1+|y|)$, for any $x, y \in \mathbb{R}^2$, we have that

$$\log(1+|x-y|) \le \log((1+|x|)(1+|y|)) = \log(1+|x|) + \log(1+|y|).$$

This inequality, Proposition 2.2 and a straightforward computation yield

$$\mathcal{V}_1(u) \le 2\|u\|^2_{L^2(\mathbb{R}^2;K)}\|u\|_W^2, \qquad \forall\, u \in W. \tag{4.4}$$

The argument for obtaining an estimate for $\mathcal{V}_2$ is more involved and use the following Hardy-Littlewood-Sobolev inequality:

Proposition 4.1. [25] *Let r, $s > 1$, $0 < \mu < 2$ with $1/r + 1/s + \mu/2 = 2$, $g \in L^r(\mathbb{R}^2)$ and $h \in L^s(\mathbb{R}^2)$. Then, there exists $C(r,s,\mu) > 0$ such that*

$$\left|\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{g(y)h(x)}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x\right| \le C(s,\mu,r)\|g\|_{L^r(\mathbb{R}^2)}\|h\|_{L^s(\mathbb{R}^2)}.$$

Using the elementary inequality $\log(1+r) \le r$, for any $r>0$, and the above result with $\mu = 1$ and $r = s = 4/3$, we can estimate

$$\begin{aligned} \mathcal{V}_2(u) &\le \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{K(y)u^2(y)K(x)u^2(x)}{|x-y|}\mathrm{d}y\mathrm{d}x \\ &\le C\left(\int_{\mathbb{R}^2}(K(x)u^2)^{4/3}\mathrm{d}x\right)^{3/4}\left(\int_{\mathbb{R}^2}(K(x)u^2)^{4/3}\mathrm{d}x\right)^{3/4}. \end{aligned} \tag{4.5}$$

From Corollary 2.3, we get $C>0$ such that

$$\mathcal{V}_2(u) \le C\|u\|_E^4, \qquad \forall\, u \in W. \tag{4.6}$$

In order to study the local part of I we notice that, for any given $\varepsilon>0$, $\alpha>\alpha_0$ and $p\ge 1$, we can use (f_0) and (f_1) to obtain $C>0$ such that

$$|f(s)| \le \varepsilon|s| + C|s|^{p-1}(e^{\alpha s^2}-1), \quad |F(s)| \le \varepsilon s^2 + C|s|^p(e^{\alpha s^2}-1), \tag{4.7}$$

for any $s\in\mathbb{R}$. This, the inequality

$$\left(e^{\alpha s^2}-1\right)^r \le \left(e^{r\alpha s^2}-1\right), \quad \forall\, r>0,\, s\in\mathbb{R}, \tag{4.8}$$

and Hölder's inequality imply that

$$\int_{\mathbb{R}^2} Q(x)F(u)\mathrm{d}x \le \varepsilon\|u\|^2_{L^2(\mathbb{R}^2;Q)} + C\|u\|^p_{L^{pr_1}(\mathbb{R}^2;Q)}\left(\int_{\mathbb{R}^2} Q(x)\left(e^{r_2\alpha u^2}-1\right)\mathrm{d}x\right)^{1/r_2}$$

whenever $r_1, r_2 > 1$ satisfy $1/r_1 + 1/r_2 = 1$ and $r_1 \ge 2$. Hence, we can use Theorem 1.1 and (4.7) to conclude that $u \mapsto \int_{\mathbb{R}^2} Q(x)F(u)\mathrm{d}x$ belongs to $C^1(W)$.

All together, the above considerations prove that the functional I is well defined in W. Moreover, it belongs to $C^1(W)$ with

$$I'(u)v = \langle u, v\rangle_E + \frac{1}{4}\mathcal{V}_1'(u)v - \frac{1}{4}\mathcal{V}_2'(u)v - \int_{\mathbb{R}^2} Q(x)f(u)v\,\mathrm{d}x,$$

for any $u,\, v \in W$.

In what follows, we denote by $\mathcal{N}$ the Nehari manifold associated to the functional I, namely

$$\mathcal{N} := \{u \in W\setminus\{0\} : I'(u)u = 0\}.$$

We first prove that $\mathcal{N}$ is a nonempty set.

Lemma 4.2. *Suppose that (f_1), (f_3) and (f_4) hold. If $u \in W\setminus\{0\}$, then there exists a unique $t_u>0$ such that $t_u u \in \mathcal{N}$.*

Proof. Let $u\in W\setminus\{0\}$ and define $\psi_u(t) := I(tu)$, for $t>0$. We have that $tu\in\mathcal{N}$ if, and only if, $\psi_u'(t) = 0$.

Given $\varepsilon>0$, $\alpha>\alpha_0$ and $p>2$, we can argue as in the proof of the regularity of I and use Proposition 2.2 to obtain

$$\begin{aligned} \int_{\mathbb{R}^2} Q(x)F(tu)\mathrm{d}x &\le \varepsilon C_1 t^2\|u\|_E^2 \\ &\quad + C_2 t^p\|u\|_E^p\left(\int_{\mathbb{R}^2} Q(x)\left(e^{r_2\alpha\|tu\|_E^2\left(\frac{u}{\|u\|_E}\right)^2}-1\right)\mathrm{d}x\right)^{1/r_2}, \end{aligned}$$

where $r_1, r_2>1$ satisfy $1/r_1+1/r_2=1$. Choosing $t_*>0$ small in such way that $r_2\alpha\|t_*u\|_E^2 < \alpha_*$ and applying Theorem 1.1, we obtain

$$\int_{\mathbb{R}^2} Q(x)F(tu)\mathrm{d}x \le \varepsilon C_1 t^2\|u\|_E^2 + C_3 t^p\|u\|_E^p, \tag{4.9}$$

for all $t \in (0, t_*)$. Picking $0 < \varepsilon < 1/(2C_1)$, we can use the above expression, $\mathcal{V}_1 \geq 0$, (4.6) and $p > 2$, to get

$$\psi_u(t) \geq t^2 \left[\frac{1}{2}\|u\|_E^2 - \frac{C}{4}t^2\|u\|_E^4 - \varepsilon C_1 \|u\|_E^2 - C_3 t^{p-2}\|u\|_E^p \right] > 0,$$

for all $t > 0$ small. Moreover, it follows from (f_3) that, if $|s| \leq 1$, then $F(s) \geq \nu |s|^q$, for $q > 4$ fixed. This and (f_0) imply that, for some $C_4 > 0$, $F(s) \geq C_4 |s|^q$ for all $s \in \mathbb{R}$. Thus, from $\mathcal{V}_2 \geq 0$, (4.4) and $q > 4$, we infer that

$$\psi_u(t) \leq \frac{t^2}{2}\|u\|_E^2 + \frac{t^4}{2}\|u\|_{L^2(\mathbb{R}^2;Q)}^2 \|u\|_W^2 - \frac{C_4 t^q}{q}\int_{\mathbb{R}^2} Q(x)|u|^q \mathrm{d}x \to -\infty,$$

as $t \to +\infty$. So, the function ψ_u achieves its maximum value at $t_u > 0$ such that $\psi_u'(t_u) = 0$.

In order to prove that t_u is the unique critical point of ψ_u we notice that, from (4.1) and (4.2), we obtain

$$\mathcal{V}_1'(tu)u = 4t^3\mathcal{V}_1(u), \quad \mathcal{V}_2'(tu)u = 4t^3\mathcal{V}_2(u).$$

Hence,

$$\begin{aligned} \psi_u'(t) &= t\|u\|_E^2 + t^3\mathcal{V}_1(u) - t^3\mathcal{V}_2(u) - \int_{\mathbb{R}^2} Q(x) f(tu)u \mathrm{d}x \\ &= t^3 \left(\frac{1}{t^2}\|u\|_E^2 + \mathcal{V}_1(u) - \mathcal{V}_2(u) - \int_{\mathbb{R}^2} Q(x)\frac{f(tu)}{(tu)^3}u^4 \mathrm{d}x \right). \end{aligned}$$

It follows from (f_4) that $\psi_u'(t)/t^3$ is decreasing, and therefore it cannot vanish twice. This concludes the proof of the lemma. □

Remark 4.3. *As a byproduct of the above proof, we see that the point t_u which projects u in the Nehari manifold is exactly the maximum point of ψ_u. Since $\psi_u > 0$ near the origin and it has a unique critical point, we conclude that ψ_u' is positive in $(0, t_u)$ and negative in $(t_u, +\infty)$. In particular, we have that $t_u \in (0, 1]$ whenever $\psi_u'(1) = I'(u)u \leq 0$.*

The next result shows that $\mathcal{N}$ is the far way the origin.

Lemma 4.4. *Suppose that (f_1) holds. Then, there exists $\delta > 0$ such that*

$$\|u\|_E \geq \delta, \quad \forall\, u \in \mathcal{N}.$$

Proof. Otherwise, there exists $(u_n) \subset \mathcal{N}$ such that $u_n \to 0$ strongly in E. Since $I'(u_n)u_n = 0$ and $\mathcal{V}_1'(u_n)u_n = 4\mathcal{V}_1(u_n) \geq 0$, as $n \to +\infty$ we have that

$$\|u_n\|_E^2 \leq \frac{1}{4}\mathcal{V}_2'(u_n)u_n + \int_{\mathbb{R}^2} Q(x)f(u_n)u_n \mathrm{d}x \leq o_n(1) + \int_{\mathbb{R}^2} Q(x)f(u_n)u_n \mathrm{d}x, \tag{4.10}$$

where we have used $\mathcal{V}_2'(u_n)u_n = 4\mathcal{V}_2(u_n)$ and (4.5) in the last inequality.

Given $\varepsilon > 0$, we can use the first part of (4.7) and the same argument employed in the proof of Lemma 4.2, to get

$$\begin{aligned} \int_{\mathbb{R}^2} Q(x)f(u_n)u_n \mathrm{d}x \leq& \varepsilon C_1 \|u_n\|^2 \\ &+ C_2\|u_n\|_E^p \left(\int_{\mathbb{R}^2} Q(x) \left(e^{r_2 \alpha \|u_n\|_E^2 \left(\frac{u_n}{\|u_n\|_E}\right)^2} - 1 \right) \mathrm{d}x \right)^{1/r_2}, \end{aligned}$$

where $r_1, r_2 > 1$ satisfy $1/r_1 + 1/r_2 = 1$. From the convergence $u_n \to 0$ in E, we get $r_2\alpha\|u_n\|_E^2 < \alpha_*$ for large $n \in \mathbb{N}$. Hence, we can use Theorem 1.1, the above expression and (4.10) to conclude that

$$(1 - \varepsilon C_1)\|u_n\|_E^2 \leq C_3\|u_n\|_E^p + o_n(1).$$

Since $p > 2$, we obtain a contradiction choosing $0 < \varepsilon < 1/C_1$. □

If $u \in \mathcal{N}$ we can use the definition of $\mathcal{V}_i$ together with (4.3) to obtain

$$\begin{aligned} I(u) &= I(u) - \frac{1}{4}I'(u)u \\ &= \frac{1}{4}\|u\|_E^2 + \int_{\mathbb{R}^2} Q(x)\left(\frac{1}{4}f(u)u - F(u)\right)\mathrm{d}x \geq \frac{1}{4}\|u\|_E^2, \end{aligned} \tag{4.11}$$

where we also have used (f_2). So, it is well defined the number

$$c := \inf_{u \in \mathcal{N}} I(u).$$

The idea for proving Theorem 1.2 is to verify that c is attained. We shall need the following local compactness result:

Lemma 4.5. *Suppose that (f_1) and (f_2) hold. Let $(u_n) \subset \mathcal{N}$ be a minimizing sequence for $c < \alpha_*/(4\alpha_0)$. Then, up to a subsequence, $u_n \rightharpoonup u \neq 0$ weakly in E. Moreover*

$$\lim_{n\to+\infty} \int_{\mathbb{R}^2} Q(x)f(u_n)u_n\,\mathrm{d}x = \int_{\mathbb{R}^2} Q(x)f(u)u\,\mathrm{d}x$$

and

$$\lim_{n\to+\infty} \int_{\mathbb{R}^2} Q(x)F(u_n)\,\mathrm{d}x = \int_{\mathbb{R}^2} Q(x)F(u)\,\mathrm{d}x,$$

Proof. By (4.11) we conclude that (u_n) is bounded in E. So, up to a subsequence, we have that $u_n \rightharpoonup u$ weakly in E. For any $R > 0$, we can write

$$\int_{\mathbb{R}^2} Q(x)(f(u_n)u_n - f(u)u)\mathrm{d}x = J_1^R(n) + J_2^R(n), \tag{4.12}$$

where

$$J_1^R(n) := \int_{B_R} Q(x)(f(u_n)u_n - f(u)u)\,\mathrm{d}x,$$

$$J_2^R(n) := \int_{\mathbb{R}^2\setminus B_R} Q(x)(f(u_n)u_n - f(u)u)\,\mathrm{d}x.$$

Given $\varepsilon > 0$, we can apply Egorov's Theorem to otbain a measurable set $\Omega \subset B_R$ such that $|\Omega| < \varepsilon$ and $u_n \to u$ uniformly in $B_R \setminus \Omega$. Hence,

$$\left|J_1^R(n)\right| \leq \int_\Omega Q(x)f(u_n)u_n\mathrm{d}x + \int_\Omega Q(x)f(u)u\mathrm{d}x + o_n(1), \tag{4.13}$$

as $n \to +\infty$. Using (4.7) with $p \geq 2$, we see that

$$\int_\Omega Q(x)f(u_n)u_n\mathrm{d}x \leq \varepsilon\int_\Omega Q(x)u_n^2\mathrm{d}x + C_1\int_\Omega Q(x)|u_n|^p\left(e^{\alpha u_n^2} - 1\right)\mathrm{d}x. \tag{4.14}$$

From (4.11) and inequality $c < \alpha_*/(4\alpha_0)$, one has

$$\lim_{n\to\infty}\|u_n\|_E^2 \leq 4c < \frac{\alpha_*}{\alpha_0}.$$

Thus, we can pick $\alpha > \alpha_0$ and $r_1 > 1$ such that $r_1\alpha\|u_n\|_E^2 < \alpha_*$, for large $n \in \mathbb{N}$. Using Hölder's inequality with exponents $1/r_1 + 1/r_2 + 1/r_3 = 1$, (4.8), Proposition 2.2 and Theorem 1.1, we get

$$\begin{aligned}\int_\Omega Q(x)|u_n|^p \left(e^{\alpha u_n^2} - 1\right) \mathrm{d}x &\le \left(\int_\Omega Q(x)\left(e^{r_1\alpha\|u_n\|_E^2\left(\frac{u_n}{\|u_n\|_E}\right)^2} - 1\right)\mathrm{d}x\right)^{1/r_1} \\ &\quad \times \|u_n\|^p_{L^{pr_2}(\mathbb{R}^2;Q)}\left(\int_\Omega Q(x)\mathrm{d}x\right)^{1/r_3} \\ &\le C_2\left(\int_\Omega Q(x)\mathrm{d}x\right)^{1/r_3} \le C_3|\Omega|^{1/r_3} = C_3\varepsilon^{1/r_3}.\end{aligned}$$

Above estimate combined with (4.14) implies

$$\int_\Omega Q(x)f(u_n)u_n\mathrm{d}x \le C_4(\varepsilon + \varepsilon^{1/r_3}),$$

for n large. Since a similar estimate holds for $\int_\Omega Q(x)f(u)u\,\mathrm{d}x$, we infer from (4.13) that, for each fixed $R > 0$, there holds $J_1^R(n) \to 0$, as $n \to +\infty$.

In order to estimate $J_2^R(n)$, we take $s_1 > 1$ such that $s_1\alpha\|u_n\|_E^2 < \alpha_*$, for large $n \in \mathbb{N}$, and argue as before to obtain

$$\int_{\mathbb{R}^2\setminus B_R} Q(x)f(u_n)u_n\mathrm{d}x \le \varepsilon\int_{\mathbb{R}^2} Q(x)u_n^2\mathrm{d}x + C_5\left(\int_{\mathbb{R}^2\setminus B_R} Q(x)|u_n|^{ps_2}\mathrm{d}x\right)^{1/s_2},$$

where $1/s_1 + 1/s_2 = 1$. So, by Proposition 2.2,

$$\begin{aligned}\limsup_{n\to\infty}\int_{\mathbb{R}^2\setminus B_R} Q(x)f(u_n)u_n\mathrm{d}x &\le \varepsilon\|u\|^2_{L^2(\mathbb{R}^2;Q)} + C_5\left(\int_{\mathbb{R}^2\setminus B_R} Q(x)|u|^{p_2p}\mathrm{d}x\right)^{1/s_2} \\ &\le C_6\varepsilon,\end{aligned}$$

for $R > 0$ large enough. A similar argument provides $\int_{\mathbb{R}^2\setminus B_R} Q(x)f(u)u\mathrm{d}x < \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we conclude that $J_2^R(n) \to 0$, as $n \to +\infty$. Recalling that the same holds for $J_1^R(n)$, we infer from (4.12) that

$$\lim_{n\to+\infty}\int_{\mathbb{R}^2} Q(x)f(u_n)u_n\,\mathrm{d}x = \int_{\mathbb{R}^2} Q(x)f(u)u\,\mathrm{d}x.$$

The limit for $\int_{\mathbb{R}^2} Q(x)F(u_n)\,\mathrm{d}x$ is a consequence of the above expression, (f_2) and the Lebesgue Theorem.

It remains to check that $u \ne 0$. Suppose, by contradiction, that this is not the case. Then, (4.3), (4.5) and Corollary 2.3 imply that $\mathcal{V}_2'(u_n)u_n = o_n(1)$, as $n \to +\infty$. Hence,

$$\|u_n\|_E^2 + \frac{1}{4}\mathcal{V}_1'(u_n)u_n = o(1) + \int_{\mathbb{R}^2} Q(x)f(u_n)u_n\mathrm{d}x = o_n(1).$$

Recalling that $\mathcal{V}_1'(u_n)u_n = 4\mathcal{V}_1(u_n) \ge 0$, we conclude that $\|u_n\|_E \to 0$, which contradicts Lemma 4.4. Thus, $u \ne 0$ and the lemma is proved. □

We infer from (4.11) that minimizing sequences for c are bounded in E. In the next result, we prove that the same holds in the space W.

Lemma 4.6. *Suppose that f satisfies $(f_1) - (f_2)$. If $(u_n) \subset \mathcal{N}$ is a minimizing sequence for $c < \alpha_*/(4\alpha_0)$, then (u_n) is bounded in the norm $\|\cdot\|_W$.*

Proof. Using Lemma 4.5, we can assume that $u_n \rightharpoonup u \neq 0$ weakly in E. Since

$$\|u_n\|_W^2 = \|u_n\|_E^2 + \int_{\mathbb{R}^2} \log(1+|x|)K(x)u_n^2 \mathrm{d}x,$$

it is sufficient to bound the last integral above. In order to do that, we first notice that, since $I'(u_n)u_n = 0$, we get

$$\frac{1}{4}\mathcal{V}_1'(u_n)u_n \leq \|u_n\|_E^2 + \frac{1}{4}\mathcal{V}_1'(u_n)u_n = \frac{1}{4}\mathcal{V}_2'(u_n)u_n + \int_{\mathbb{R}^2} Q(x)f(u_n)u_n \mathrm{d}x. \tag{4.15}$$

By (4.3), $\mathcal{V}_2'(u_n)u_n = 4\mathcal{V}_2(u_n) \leq 4C\|u_n\|_E^2 \leq C_1$, for all $n \in \mathbb{N}$. Moreover, by Lemma 4.5, the last integral above is bounded, since it converges. So, we conclude that $\mathcal{V}_1'(u_n)u_n \leq C_2$, for some $C_2 > 0$.

Let $R > 0$ be such that $\int_{B_R} K(x)u^2\,\mathrm{d}x > 0$. For any $x \in \mathbb{R}^2 \setminus B_{2R}$ and $y \in B_R$, there holds

$$1 + |x-y| \geq 1 + |x| - |y| \geq 1 + |x| - R \geq 1 + \frac{|x|}{2} \geq \sqrt{1+|x|}.$$

Hence,

$$\begin{aligned}
\mathcal{V}_1'(u_n)u_n &\geq 4\int_{\mathbb{R}^2\setminus B_{2R}}\int_{B_R} \log(1+|x-y|)K(y)u_n^2(y)K(x)u_n^2(x)\mathrm{d}y\mathrm{d}x \\
&\geq 2\int_{\mathbb{R}^2\setminus B_{2R}}\int_{B_R} \log(1+|x|)K(y)u_n^2(y)K(x)u_n^2(x)\mathrm{d}y\mathrm{d}x \\
&= 2\left(\int_{B_R} K(y)u_n^2(y)\mathrm{d}y\right)\left(\int_{\mathbb{R}^2\setminus B_{2R}} \log(1+|x|)K(x)u_n^2(x)\mathrm{d}x\right),
\end{aligned}$$

and we can use Proposition 2.2 to get

$$\limsup_{n\to+\infty} \int_{\mathbb{R}^2\setminus B_{2R}} \log(1+|x|)K(x)u_n^2(x)\mathrm{d}x \leq \frac{C_2}{2}\left(\int_{B_R} K(x)u^2\,\mathrm{d}x\right)^{-1}.$$

On the other hand, since $\log(1+|x|) \leq 1+|x|$, we obtain

$$\int_{B_{2R}} \log(1+|x|)K(x)u_n^2(x)\mathrm{d}x \leq (1+2R)\|u_n\|_E^2 \leq C_3,$$

and we have done. ∎

We obtain in what follows the required estimate on the minimax level c.

Lemma 4.7. *Suppose that (f_3) holds and let $\alpha_* \in (0, 4\pi)$ be given by Theorem 1.1. If ν satisfies (1.1), then $c < \alpha_*/(4\alpha_0)$.*

Proof. Since $W \hookrightarrow E \hookrightarrow L^4(\mathbb{R}^2; Q)$ and this last embedding is compact (see Proposition 2.2), there exists $\omega \in W \setminus \{0\}$ such that

$$\|\omega\|_W^2 = S_4(Q), \quad \int_{\mathbb{R}^2} Q(x)\omega^4 \mathrm{d}x = 1.$$

We may assume $\omega \geq 0$, and therefore we obtain from Lemma 4.2 a number $t_\omega > 0$ such that $t_\omega\omega \in \mathcal{N}$. So, recalling that $\mathcal{V}_2 \geq 0$, using (4.4), (f_3) and the above equalities, we obtain

$$\begin{aligned} c \le I(t_\omega \omega) &\le \frac{t_\omega^2}{2} S_4(Q) + \frac{1}{4}\mathcal{V}_1(t_\omega \omega) - \int_{\mathbb{R}^2} Q(x) F(t_\omega \omega)\mathrm{d}x \\ &\le \frac{t_\omega^2}{2} S_4(Q) + \frac{t_\omega^4}{2}\|\omega\|^2_{L^2(\mathbb{R}^2;K)} S_4(Q) - t_\omega^4 \nu. \end{aligned}$$

But the definition of $S_2(K)$ and (1.1) provide

$$\|\omega\|_{L^2(\mathbb{R}^2;K)} \le \frac{1}{S_2(K)}\|\omega\|^2_W = \frac{S_4(Q)}{S_2(K)} \le \frac{\nu}{S_4(Q)},$$

and therefore

$$\begin{aligned} c &\le \frac{t_\omega^2}{2} S_4(Q) + \frac{t_\omega^4}{2}\nu - t_\omega^4 \nu \\ &\le \frac{1}{2}\max_{t>0}\left[t^2 S_4(Q) - t^4 \nu\right] = \frac{1}{2}\left(\frac{S_4^2(Q)}{4\nu}\right) = \frac{S_4^2(Q)}{8\nu} < \frac{\alpha_*}{4\alpha_0}, \end{aligned}$$

where we have used (1.1) again in the last part. The proof is complete. □

We are ready to present the proof of our existence theorems.

Proof of Theorem 1.2. Let $(u_n) \subset \mathcal{N}$ be a minimizing sequence for c. According to Lemma 4.7, we have that $c < \alpha_*/(4\alpha_0)$. Hence, by Lemma 4.5, we may assume that $u_n \rightharpoonup u \ne 0$ wealy in E. Since (u_n) is also bounded in W (Lemma 4.6) we also have that $u_n \rightharpoonup v$ weakly in W. From the compactness of the embeddings given by Proposition 2.2, we conclude that $u_n(x) \to u(x)$ and $u_n(x) \to v(x)$ for a.e. $x \in \mathbb{R}^2$ and consequently $v = u \in W$.

Let $t_u > 0$ be such that $t_u u \in \mathcal{N}$. Arguing as in (4.11), we conclude that

$$c \le I(t_u u) = \frac{1}{4}\|t_u u\|^2_E + \int_{\mathbb{R}^2} Q(x)\left(\frac{1}{4} f(t_u u) t_u u - F(t_u u)\right)\mathrm{d}x. \tag{4.16}$$

By hypothesis (f_4), the function $h(s) := (1/4) f(s)s - F(s)$ is increasing in $(0,\infty)$ and decreasing in $(-\infty, 0)$ (see [3, Lemma 2.4]). Hence, after we proving that $t_u \in (0,1]$, we can use $I(u_n) = c + o_n(1)$, (4.11), the weak semicontinuity of the norm, Lemma 4.5, $t_u \le 1$, the monotonicity of h and (4.16), to get

$$c = \liminf_{n\to+\infty} I(u_n) \ge \frac{1}{4}\|u\|^2_E + \int_{\mathbb{R}^2} Q(x)\left(\frac{1}{4} f(u)u - F(u)\right)\mathrm{d}x \ge I(t_u u) \ge c.$$

Thus, $I(t_u u) = c$ and we can use a (by now standard) deformation argument as in [7, Proposition 3.1] (see also [4, pp. 1163]) to conclude that $I'(t_u u) = 0$, that is, $t_u u \ne 0$ is the desired solution.

It remains to prove that $t_u \le 1$ or, equivalently, that $I'(u)u \le 0$ (see Remark 4.3). We first check that

$$\liminf_{n\to+\infty} \mathcal{V}_1'(u_n)u_n \ge \mathcal{V}_1'(u)u, \qquad \lim_{n\to+\infty} \mathcal{V}_2'(u_n)u_n = \mathcal{V}_2'(u)u. \tag{4.17}$$

Indeed, for any $R > 0$, a simple computation shows that

$$\mathcal{V}_1'(u_n)u_n \ge 4\Gamma_n(R) + 4\int_{B_R}\int_{B_R} \log(1+|x-y|)K(y)u^2(y)K(x)u^2(x)\mathrm{d}y\mathrm{d}x,$$

where $\Gamma_n := \Gamma_{n,1} + \Gamma_{n,2}$ and

$$\Gamma_{n,1} := \int_{B_R}\int_{B_R} K(y)K(x)u_n^2(y)\Big(u_n^2(x) - u^2(x)\Big)\mathrm{d}y\mathrm{d}x,$$

and

$$\Gamma_{n,2} := \int_{B_R}\int_{B_R} K(y)K(x)\Big(u_n^2(y) - u^2(y)\Big)u^2(x)\mathrm{d}y\mathrm{d}x.$$

Using Hölder's inequality and Proposition 2.2, we may write

$$|\Gamma_{n,1}| \le \|u_n\|^2_{L^2(\mathbb{R}^2;K)}\|u_n - u\|_{L^2(\mathbb{R}^2;K)}\|u_n + u\|_{L^2(\mathbb{R}^2;K)} = o_n(1),$$

as $n \to +\infty$. Since a similar estimate holds for $\Gamma_{n,2}$, we conclude that

$$\liminf_{n\to\infty} \mathcal{V}_1'(u_n)u_n \ge 4\int_{B_R}\int_{B_R} \log(1+|x-y|)K(y)u^2(y)K(x)u^2(x)\mathrm{d}y\mathrm{d}x.$$

For proving the first statement in (4.17), it is sufficient to let $R \to +\infty$ in the above expression and use Monotone Convergence Theorem together with the first equality in (4.3).

In order to prove the second part of (4.17), we first recall that $\mathcal{V}_2'(u_n)u_n = 4\mathcal{V}_2(u_n)$ to see that $\Sigma_n := |\mathcal{V}_2'(u_n)u_n - \mathcal{V}_2'(u)u|$ can be written as

$$\Sigma_n = 4\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log(1+|x-y|^{-1})K(y)K(x)\Big(u_n^2(y)u_n^2(x) - u^2(y)u^2(x)\Big)\mathrm{d}y\mathrm{d}x.$$

Arguing as in the estimate of Γ_n and using $\log(1+r) \le r$, for $r > 0$, we obtain

$$|\mathcal{V}_2'(u_n)u_n - \mathcal{V}_2'(u)u| \le 4\Sigma_{1,n} + 4\Sigma_{2,n},$$

where

$$\Sigma_{1,n} := \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{K(y)K(x)u_n^2(y)\,|u_n(x)-u(x)|\,|u_n(x)+u(x)|}{|x-y|}\mathrm{d}y\mathrm{d}x$$

and

$$\Sigma_{2,n} := \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{K(y)K(x)\,|u_n(y)-u(y)|\,|u_n(y)+u(y)|\,u^2(x)}{|x-y|}\mathrm{d}y\mathrm{d}x.$$

Using Proposition 4.1 with $\mu = 1$, $r = s = 4/3$, Hölder's inequality and Corollary 2.3, we obtain

$$|\Sigma_{1,n}| \le C\|u_n\|^2_{L^{8/3}(\mathbb{R}^2;K^{4/3})}\|u_n - u\|_{L^{8/3}(\mathbb{R}^2;K^{4/3})}\|u_n + u\|_{L^{8/3}(\mathbb{R}^2;K^{4/3})} = o_n(1),$$

as $n \to +\infty$. The same can be done with $\Sigma_{2,n}$ and therefore $\mathcal{V}_2'(u_n)u_n = \mathcal{V}_2'(u)u + o_n(1)$, as claimed.

Recalling that the norm is weakly lower semicontinuous, we can pass the lim inf in the equality $0 = I'(u_n)u_n$, use (4.17) and Lemma 4.5, to obtain

$$I'(u)u = \|u\|_E^2 + \frac{1}{4}\mathcal{V}_1'(u)u - \frac{1}{4}\mathcal{V}_2'(u)u - \int_{\mathbb{R}^2} Q(x)f(u)u\,\mathrm{d}x \le 0.$$

This concludes the proof of Theorem 1.2. □

Using the solution obtained in Theorem 1.2 and elliptic regularity, we can easily obtain a weak solution for the system ($\mathcal{S}$).

Proof of Theorem 1.3. Let $u \in W$ be the solution given by Theorem 1.2, $\varphi \in C_0^\infty(\mathbb{R}^2)$ and $R > 0$ be such that the support of φ is contained in B_R. For any $1 < p < \infty$, we have that

$$\int_{B_R} |K(x)u^2|^p\mathrm{d}x \le \|K\|^p_{L^\infty(\mathbb{R}^2)}\int_{B_R}|u|^{2p}\mathrm{d}x < \infty,$$

since $W \hookrightarrow L^{2p}(B_R)$. It follows from the classical potential theory (see [23, Theorem 9.9]) that $\phi_u := \Gamma_2 * (Ku^2) \in W^{2,p}(B_R)$ and $\Delta\phi_u = K(x)u^2$ for a.e. $x \in B_R$. This and Divergence Theorem ensure that

$$-\int_{B_R} \nabla\phi_u \cdot \nabla\varphi \,\mathrm{d}x = \int_{B_R} (\Delta\phi_u)\varphi \,\mathrm{d}x = \int_{B_R} (K(x)u^2)\varphi \,\mathrm{d}x.$$

Therefore, the pair $(u, \phi_u) \in W \times W^{2,p}_{\mathrm{loc}}(\mathbb{R}^2)$ is a weak solution of system ($\mathcal{S}$) and the theorem is proved. □

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