REMARKS ON A SOBOLEV EMBEDDING

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Abstract. In this paper, we prove a Hardy-Sobolev type inequality which is posteriorly used to give a characterization of an important class of Sobolev space. The abstract setting is applied to obtain the existence of solution for a class of elliptic equations with Neumann/Robin boundary conditions.

1. Introduction and main results

For any domain $\Omega \subset \mathbb{R}^N$, we denote by $C_\delta^\infty(\Omega)$ the set of all functions $v \in C_0^\infty(\mathbb{R}^N)$ restricted to Ω. For $N \geq 2$, let $\mathbb{R}_+^N := \big\{(x', x_N) : x' \in \mathbb{R}^{N-1},\, x_N > 0\big\}$ be the usual upper half-space. The main goal of this note is to give a characterization of the Sobolev space defined as the completion of $C_\delta^\infty(\mathbb{R}_+^N)$ with respect to the norm

$$v \mapsto \left(\int_{\mathbb{R}_+^N} |\nabla v|^2 dx + \int_{\mathbb{R}^{N-1}} v^2 dx' \right)^{1/2}.$$

Our starting motivation arises from the growing interest in partial differential with Neumann/Robin boundary conditions of the form

$$\begin{cases} \quad -\Delta v = & f(x,v), \quad \text{in } \mathbb{R}_+^N \\ \frac{\partial v}{\partial \nu} + \lambda v = & g(x,v), \quad \text{on } \partial\mathbb{R}_+^N, \end{cases}$$

see for instance, [1, 4, 8, 11] and references therein. One of the challenges in solving this kind of problem via variational methods is to obtain embedding into Lebesgue spaces. This step is crucial because it allows us to analyze the properties of the solutions and apply powerful tools from functional analysis. As it is well-known, Hardy-Sobolev inequality and its variants play an important role in this feature. For example, Opic-Kufner [10] (see also [2, 7, 8, 11, 14, 3]) and references therein provide different conditions on the weight functions w_1 and w_2 for the validity of the Hardy-Sobolev inequality:

$$\int_\Omega w_1(x)|u|^p dx \leq \int_\Omega w_2(x)|\nabla u|^p dx, \quad u \in C_0^\infty(\Omega).$$

In the aforementioned works, the authors deal with functions that vanish at the boundary. However, when dealing with PDEs with Neumann/Robin boundary conditions (see Section 2), it is important to consider functions that can take nonzero values on the boundary of Ω. Therefore, we first cite the papers by

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Janssen [8] and Pfluger [11], where they obtained a constant $C_0 > 0$, satisfying the following Hardy type inequality:

$$\int_\Omega \frac{|u|^p}{(1+|x|)^p}dx \leq C_0\left(\int_\Omega |\nabla u|^p dx + \int_{\partial\Omega} \frac{|x\cdot\nu|}{(1+|x|)^p}|u|^p d\sigma\right), \quad u \in C_\delta^\infty(\Omega).$$

Here, $1 < p < N$ and ν denotes the unit outward normal vector on $\partial\Omega$. For further results, we refer to the works [6, 7, 13, 14], and their references.

In order to state our results, we denote by $\mathcal{E}^{1,2}$ the completion of $C_\delta^\infty(\mathbb{R}^N_+)$ with respect to the norm

$$\|v\| := \left(\int_{\mathbb{R}^N_+} |\nabla v|^2 dx + \int_{\mathbb{R}^{N-1}} v^2 dx' + \int_{\mathbb{R}^N_+} \frac{v^2}{(1+x_N)^2}dx\right)^{1/2}.$$

In our first result, we prove the following Hardy-type inequality:

Theorem 1.1 (Hardy inequality)**.** *For any $v \in \mathcal{E}^{1,2}$, there holds*

$$\frac{1}{4}\int_{\mathbb{R}^N_+} \frac{v^2}{(1+x_N)^2}dx \leq \int_{\mathbb{R}^N_+} |\nabla v|^2 dx + \frac{1}{2}\int_{\mathbb{R}^{N-1}} v^2 dx'. \tag{1.1}$$

The above result is an improvement of [1, Theorem 1.1]. Moreover, it also complements some related results which can be found in the papers [12, 5, 9], where Hardy-type inequalities are stated in the context of $C_0^\infty(\mathbb{R}^N_+)$ functions.

As a direct consequence of the above theorem, we see that the norm $\|\cdot\|$ is equivalent to

$$\|v\|_{\mathcal{E}^{1,2}} := \left(\int_{\mathbb{R}^N_+} |\nabla v|^2 dx + \int_{\mathbb{R}^{N-1}} v^2 dx'\right)^{1/2}.$$

In order to better understand this new norm, we denote by $X^{1,2}$ the completion of $C_\delta^\infty(\mathbb{R}^N_+)$ with respect to the norm

$$\|u\|_{X^{1,2}} := \left(\int_{\mathbb{R}^N_+} (1+x_N)^2|\nabla u|^2 dx\right)^{1/2}.$$

If $N \geq 3$, we clearly have that $X^{1,2} \hookrightarrow D^{1,2}(\mathbb{R}^N_+) \hookrightarrow L^{2^*}(\mathbb{R}^N_+)$. Hence, it is natural to ask if $X^{1,2}$ can be immersed into some Lebesgue spaces even for $N = 2$. Corollary 1.3 asserts that the answer is positive. It is a consequence of the next theorem, which is the main result of this note.

Theorem 1.2 (An isometric model for $\mathcal{E}^{1,2}$)**.** *The linear map*

$$\mathbf{T} : \left(\mathcal{E}^{1,2}, \|\cdot\|_{\mathcal{E}^{1,2}}\right) \to \left(X^{1,2}, \|\cdot\|_{X^{1,2}}\right)$$

defined by $\mathbf{T}(v) := v(1+x_N)^{-1}$ is an isometry, that is, it is bijective and

$$\|\mathbf{T}v\|_{X^{1,2}} = \|v\|_{\mathcal{E}^{1,2}}, \quad \forall\, v \in \mathcal{E}^{1,2}.$$

As a byproduct, we obtain the following:

Corollary 1.3. *For any $u \in X^{1,2}$, there holds*

$$\|u\|^2_{W^{1,2}(\mathbb{R}^N_+)} \leq 5\|u\|^2_{X^{1,2}}.$$

In particular, $X^{1,2} \hookrightarrow W^{1,2}(\mathbb{R}^N_+)$.

In the next section, we prove the results stated in this introduction. In Section 3, we present a simple application in PDE.

2. Proofs of Theorems 1.1, 1.2 and Corollary 1.3

Proof of Theorem 1.1. We first prove the inequality for $v \in C_\delta^\infty(\mathbb{R}_+^N)$. Integrating by parts, we obtain

$$\begin{aligned}\int_0^\infty \frac{v^2}{(1+x_N)^2}dx_N &= -\int_0^\infty v^2 \frac{d}{dx_N}\frac{1}{(1+x_N)}dx_N \\ &= -\frac{v^2}{(1+x_N)}\Big|_{x_N=0}^\infty + \int_0^\infty \frac{2vv_{x_N}}{(1+x_N)}dx_N.\end{aligned}$$

Since v has compact support, for any $\varepsilon > 0$ we get

$$\begin{aligned}\int_0^\infty \frac{v^2}{(1+x_N)^2}dx_N &= v(x',0)^2 + \int_0^\infty \frac{2vv_{x_N}}{(1+x_N)}dx_N \\ &\le v(x',0)^2 + \int_0^\infty \left[\varepsilon\frac{v^2}{(1+x_N)^2} + \frac{1}{\varepsilon}|\nabla v|^2\right]dx_N,\end{aligned}$$

where we have used $2ab \le \varepsilon a^2 + \varepsilon^{-1}b^2$. After integrating over $\mathbb{R}^{N-1}$, we obtain

$$\varepsilon(1-\varepsilon)\int_{\mathbb{R}_+^N}\frac{v^2}{(1+x_N)^2}dx \le \int_{\mathbb{R}_+^N}|\nabla v|^2dx + \varepsilon\int_{\mathbb{R}^{N-1}} v(x',0)^2dx'$$

and the result follows by picking $\varepsilon = 1/2$.

For an arbitrary $v \in \mathcal{E}^{1,2}$, we consider a sequence $(v_k) \subset C_\delta^\infty(\mathbb{R}_+^N)$ such that $v_k \to v$ in $\mathcal{E}^{1,2}$. Thus,

$$\begin{cases} |\nabla v_k| \to |\nabla v|, & \text{in } L^2(\mathbb{R}_+^N), \\ |v_k| \to |v|, & \text{in } L^2(\mathbb{R}^{N-1}), \\ \frac{|v_k|}{(1+x_N)} \to \frac{|v|}{(1+x_N)}, & \text{in } L^2(\mathbb{R}_+^N), \end{cases}$$

and we can take the inequality

$$\frac{1}{4}\int_{\mathbb{R}_+^N}\frac{v_k^2}{(1+x_N)^2}dx \le \int_{\mathbb{R}_+^N}|\nabla v_k|^2dx + \frac{1}{2}\int_{\mathbb{R}^{N-1}} v_k^2dx'$$

to the limit to conclude the proof.

□

Proof of Theorem 1.2. Let $v \in \mathcal{E}^{1,2}$ and denote $u := \mathbf{T}v = v(1+x_N)^{-1}$. A straightforward computation shows that

$$|\nabla v|^2 = (1+x_N)^2|\nabla u|^2 + 2(1+x_n)uu_{x_N} + u^2.$$

Using integration by parts, we get

$$\begin{aligned}\int_{\mathbb{R}_+^N}|\nabla v|^2dx &= \int_{\mathbb{R}_+^N}\left[(1+x_N)^2|\nabla u|^2 + (1+x_n)(u^2)_{x_N} + u^2\right]dx \\ &= \int_{\mathbb{R}_+^N}(1+x_N)^2|\nabla u|^2dx - \int_{\mathbb{R}^{N-1}} v^2dx',\end{aligned}$$

where we have used that the exterior normal to $\partial\mathbb{R}_+^N$ is $\nu = (0',-1)$. Thus,

$$\|v\|_{\mathcal{E}^{1,2}}^2 = \int_{\mathbb{R}_+^N}|\nabla v|^2dx + \int_{\mathbb{R}^{N-1}} v^2dx' = \int_{\mathbb{R}_+^N}(1+x_N)^2|\nabla u|^2dx = \|\mathbf{T}v\|_{X^{1,2}}^2 \tag{2.1}$$

and the proof is concluded. □

Proof of Corollary 1.3. Given $u \in X^{1,2}$, we set $v := \mathbf{T}^{-1}u = (1+x_N)u \in \mathcal{E}^{1,2}$ and compute

$$\begin{aligned}\int_{\mathbb{R}^N_+} (1+x_N)^2|\nabla u|^2 dx &= \int_{\mathbb{R}^N_+} |\nabla v|^2 dx + \int_{\mathbb{R}^{N-1}} v^2 dx' \\ &\geq \frac{1}{4}\int_{\mathbb{R}^N_+} \frac{v^2}{(1+x_N)^2} dx = \frac{1}{4}\int_{\mathbb{R}^N_+} u^2 dx,\end{aligned}$$

where we have used Theorem 1.2 and inequality (1.1). By using the above inequality and $1 \leq (1+x_N)^2$, we conclude that

$$\int_{\mathbb{R}^N_+} \left[|\nabla u|^2 + u^2\right] dx \leq \|u\|^2_{X^{1,2}} + \int_{\mathbb{R}^N_+} u^2 dx \leq 5\|u\|^2_{X^{1,2}},$$

and we have done. □

3. Applications to PDE's

In this section, we briefly discuss how we can apply our abstract results to look for weak solution of partial differential equations. As a simple example, we pick $a \in L^\infty(\mathbb{R}^N_+)$ and consider the zero-mass problem

$$(P) \qquad \begin{cases} -\Delta v = a(x)|v|^{q-2}v, & \text{in } \mathbb{R}^N_+ \\ \frac{\partial v}{\partial \nu} + v = 0, & \text{on } \partial\mathbb{R}^N_+. \end{cases}$$

Formally, the energy functional associated to the above problem is given by

$$\mathcal{I}(v) := \frac{1}{2}\int_{\mathbb{R}^N_+} |\nabla v|^2 dx + \int_{\mathbb{R}^{N-1}} v^2 dx' - \frac{1}{p}\int_{\mathbb{R}^N_+} a(x)|v|^q dx, \quad v \in \mathcal{E}^{1,2}.$$

Let $2^* := 2N/(N-2)$, if $N \geq 3$, and $2^* := \infty$, if $N = 2$. When $N \geq 3$, the above functional is well defined for $q = 2^*$ due to the embedding $\mathcal{D}^{1,2}(\mathbb{R}^N_+) \hookrightarrow L^{2^*}(\mathbb{R}^N_+)$. However, it is not clear how we can deal with the subcritical case $2 \leq q < 2^*$.

We are going to show that, using the isometry $\mathbf{T}$, we may consider the problem when the power q belongs to the classical subcritical range. More specifically, we assume that

(a_0) $2 \leq q < 2^*$ and there exists $C_1 > 0$, such that

$$0 \leq a(x) \leq \frac{C_1}{(1+x_N)^q}, \quad \forall\, x \in \mathbb{R}^N_+.$$

Under the above hypothesis, we can use Corollary 1.3 to guarantee that the functional

$$\mathcal{J}(u) := \frac{1}{2}\int_{\mathbb{R}^N_+} (1+x_N)^2|\nabla u|^2 dx - \frac{1}{2}\int_{\mathbb{R}^N_+} a(x)(1+x_N)^q|u|^q dx, \quad u \in X^{1,2},$$

is well defined and belongs to $C^1(X^{1,2}, \mathbb{R})$. It follows from Theorem 2.1 that $\mathcal{I}(\mathbf{T}^{-1}u) = \mathcal{J}(u)$, for any $u \in X^{1,2}$. Since $\mathbf{T}$ is linear, we conclude that $u \in X^{1,2}$ is a critical point of $\mathcal{J}$ if, and only if, $v = \mathbf{T}^{-1}u \in \mathcal{E}^{1,2}$ is a critical point of $\mathcal{I}$.

Remark 3.1. *If $v \in C^2(\mathbb{R}^N_+) \cap C^1(\overline{\mathbb{R}^{N-1}})$ is a classical solution of (P), we may use a straightforward computation to verify that $u = (1+x_N)^{-1}v$ is a solution of*

$$\begin{cases} -\mathrm{div}((1+x_N)^2\nabla u) = a(x)(1+x_N)^q|u|^{q-2}u, & \textit{in } \mathbb{R}^N_+ \\ \frac{\partial u}{\partial \nu} = 0, & \textit{on } \partial\mathbb{R}^N_+. \end{cases}$$

In order to obtain a critical point for $\mathcal{J}$, we need some kind of compactness. So, suppose that $b \in L^\infty_{loc}(\mathbb{R}^N_+)$ is nonnegative and $b(x) \to 0$ as $|x| \to +\infty$. We shall prove that the embedding of $X^{1,2}$ into the weighted Lebesgue space

$$L^q_b(\mathbb{R}^N_+) := \left\{ u \in L^1_{loc}(\mathbb{R}^N_+) \, : \, \int_{\mathbb{R}^N_+} b(x)|u|^q dx < \infty \right\}$$

is compact. Indeed, let $(u_k) \subset X^{1,2}$ be such that $u_k \rightharpoonup 0$ weakly in $X^{1,2}$. For any given $\varepsilon > 0$, there exists $R = R_\varepsilon > 0$ such that $b(x) \le \varepsilon$ in $\mathbb{R}^N_+ \setminus B_R(0)$. Consequently, if we denote $B_R^+ := \mathbb{R}^N_+ \cap B_R(0)$, we have that

$$\int_{\mathbb{R}^N_+} b(x)|u_k|^q dx \le \|b\|_{L^\infty(B_R^+)} \int_{B_R^+} |u_k|^q dx + \varepsilon \int_{\mathbb{R}^N_+ \setminus B_R^+} |u_k|^q dx.$$

By using Corollary 1.3, the compact embedding $W^{1,2}(B_R^+) \hookrightarrow L^q(B_R^+)$, and the boundedness of (u_k), we can further derive:

$$\limsup_{k \to +\infty} \int_{\mathbb{R}^N_+} b(x)|u_k|^q dx \le C_2 \varepsilon,$$

where $C_2 := \sup_{k \in \mathbb{N}} \{ \|u_k\|_{L^q(\mathbb{R}^N_+)} \} > 0$. Since $\varepsilon > 0$ is arbitrary, we conclude that $u_k \to 0$ strongly in $L^q_b(\mathbb{R}^N_+)$.

We are now able to present the following application of our abstract results:

Theorem 3.2. *Suppose that $2 \le q < 2^*$. If $a : \mathbb{R}^N_+ \to \mathbb{R}$ satisfies (a_0) and*

(a_1) $\displaystyle\lim_{|x| \to +\infty} a(x)(1 + x_N)^q = 0,$

then the problem (P) has a nonzero weak solution.

Proof. As noticed before, we need only to obtain a nonzero critical point for $\mathcal{J}$. In order to do that, we define

$$\mathcal{M} := \left\{ u \in X^{1,2} \, : \, \int_{\mathbb{R}^N_+} a(x)(1 + x_N)^q |u|^q dx = 1 \right\}$$

and

$$c_0 := \inf_{u \in \mathcal{M}} \|u\|^2_{X^{1,2}}.$$

If $(u_k) \subset \mathcal{M}$ is a minimizing sequence for c_0, we may assume that $u_k \rightharpoonup u_0$ weakly in $X^{1,2}$. Since the norm is weakly lower semicontinuous, we get

$$\|u_0\|^2_{X^{1,2}} \le \liminf_{k \to +\infty} \|u_k\|^2_{X^{1,2}} = c_0.$$

Moreover, from (a_1) and the former considerations, along a subsequence we have that

$$1 = \lim_{k \to +\infty} \int_{\mathbb{R}^N_+} a(x)(1 + x_N)^q |u_k|^q dx = \int_{\mathbb{R}^N_+} a(x)(1 + x_N)^q |u|^q dx.$$

All together, the above expressions imply that c_0 is attained at $u_0 \in \mathcal{M}$. By using the Lagrange's Multiplier Theorem and a straightforward computation, we conclude that $u := c_0^{1/(q-2)} u_0 \in X^{1,2}$ is a nonzero critical point of $\mathcal{J}$. $\square$

We conclude the paper by noticing that we can deal with nonlinearities of type $a(x)f(u)$ provided we impose correct growth conditions on f. Moreover, we can use the isometry $\mathbf{T}$ and its consequences to deal with problems like

$$\begin{cases} -\mathrm{div}(b(x)\nabla u) = & a(x)f(u), \quad \text{in } \mathbb{R}^N_+ \\ \frac{\partial u}{\partial \nu} = & 0, \qquad\qquad \text{on } \partial\mathbb{R}^N_+. \end{cases}$$

We omit the details since the focus here is the abstract setting developed in the first section.

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