EMBEDDING THEOREMS FOR WEIGHTED SOBOLEV SPACES IN A BORDERLINE CASE AND APPLICATIONS

J. L. CARVALHO, M. F. FURTADO, AND E. S. MEDEIROS

Abstract. We establish some embedding results for weighted Sobolev spaces. As an application, we obtain one nonzero solution for the equation

$$-\mathrm{div}(|\nabla u|^{N-2}\nabla u)+V(x)|u|^{N-2}u=\lambda Q(x)f(u),\quad \text{in } \mathbb{R}^N,$$

where V, Q are nonnegative potentials, $\lambda>0$ is a large parameter and f has critical growth in the Trudinger-Moser sense.

1. Introduction and main results

In this paper, we present a functional space to deal with the quasilinear equation

$$(\mathcal{P})\qquad -\Delta_N u+V(x)|u|^{N-2}u=\lambda Q(x)f(u),\quad \text{in } \mathbb{R}^N,$$

where $N\geq 2$, $\Delta_N u:=\mathrm{div}(|\nabla u|^{N-2}\nabla u)$ is the N-laplacian operator, $\lambda>0$ is a parameter and V, Q are nonnegative potentials. If $f(s)=|s|^{q-2}s$ and we are intending to apply variational methods, it is natural to look for an inequality like

$$(1.1)\qquad \int_{\mathbb{R}^N} Q(x)|u|^q dx\leq C_0\left(\int_{\mathbb{R}^N}\left[|\nabla u|^N+V(x)|u|^N\right]dx\right)^{q/N},\quad \forall\, u\in C_0^\infty(\mathbb{R}^N),$$

with $C_0=C_0(q)>0$ independent of u. Our purpose here is to present conditions on V and Q which guarantee the above inequality for any $q\geq N$.

In order to precisely present our variational setting, we shall suppose that the potentials V and $Q\not\equiv 0$ satisfy

(VQ) there exist c_V, $c_Q>0$ and $\gamma,\beta\in\mathbb{R}$ such that, for any $x\in\mathbb{R}^N$, there hold

$$\frac{c_V}{(1+|x|)^\gamma}\leq V(x),\qquad 0\leq Q(x)\leq\frac{c_Q}{(1+|x|)^\beta}.$$

Let $p>1$, $\gamma\in\mathbb{R}$ and consider the space $E^{1,p,\gamma}$ defined as the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_{E^{1,p,\gamma}}:=\left(\int_{\mathbb{R}^N}\left[|\nabla u|^p+\frac{|u|^p}{(1+|x|)^\gamma}\right]dx\right)^{1/p}.$$

2010 *Mathematics Subject Classification.* 35J66.

Key words and phrases. Weighted Sobolev embedding; Trace embedding; Trudinger-Moser inequality; Robin boundary condition; Neumann boundary condition.

The first author was supported by Capes/Brasil.

The second author was partially supported by CNPq/Brazil and FAPDF/Brazil.

The third author was supported by CNPq/Brasil and by Grant 2019/2014 Paraíba State Research Foundation (FAPESQ).

Given a positive function $\omega \in L^1_{loc}(\mathbb{R}^N)$ and $s \geq 1$, we define the weighted Lebesgue space

$$L^s(\mathbb{R}^N,\omega) := \left\{ u \in L^1_{loc}(\mathbb{R}^N) \, : \, \|u\|_{L^s(\mathbb{R}^N,\omega)} := \left(\int_{\mathbb{R}^N} \omega(x)|u|^s \, dx \right)^{1/s} < +\infty \right\}.$$

In the first part of this paper, we aim to obtain conditions on $\beta \geq 0$, $\gamma > 0$ and $q > 1$ in such a way that, for some constant $C_0 > 0$, the following inequality holds

$$\|u\|^q_{L^q(\mathbb{R}^N,(1+|\cdot|)^{-\beta})} \leq C_0 \|u\|^q_{E^{1,N,\gamma}}, \quad \forall\, u \in E^{1,N,\gamma}. \tag{1.2}$$

Before presenting our result we notice that, if $1 < p < N$ and $p^* := Np/(N-p)$, it follows from Gagliardo-Nirenberg inequality that

$$\|u\|^{p^*}_{L^{p^*}(\mathbb{R}^N,(1+|\cdot|)^{-\beta})} \leq \|u\|^{p^*}_{L^{p^*}(\mathbb{R}^N)} \leq C_1 \left(\int_{\mathbb{R}^N} |\nabla u|^p dx \right)^{p^*/p} \leq C_1 \|u\|^{p^*}_{E^{1,p,\gamma}},$$

for any $u \in C_0^\infty(\mathbb{R}^N)$. Moreover, if $1 < q < p^*$, we can use Hölder's inequality with exponents $s = p^*/q > 1$ and $s' = p^*/(p^*-q)$, to get

$$\|u\|^q_{L^q(\mathbb{R}^N,(1+|\cdot|)^{-\beta})} \leq \|u\|^q_{L^{p^*}(\mathbb{R}^N)} \left(\int_{\mathbb{R}^N} \frac{1}{(1+|x|)^\theta} dx \right)^{1/s'} \leq C_2 \|u\|^q_{L^{p^*}(\mathbb{R}^N)},$$

whenever $\theta := \beta p^*/(p^*-q) > N$. So, we can use Gagliardo-Nirenberg inequality again to show that (1.2) holds if we replace $E^{1,N,\gamma}$ by $E^{1,p,\gamma}$, with $p(N-\beta)/(N-p) < q \leq p^*$.

As we will see, the equation (1.2) represents a (more delicate) borderline case. There are some obstructions, since it can be proved that the inequality fails, for example, if $\max\{\beta,\gamma\} < N$ and $1 < q < N(N-\beta)/(N-\gamma)$ (see Remark 2.2). In our first result, we show that this borderline case can be considered for some appropriated range for the parameters. More specifically, we shall prove the following:

Theorem 1.1. *Suppose that $q \geq N \geq 2$. Then the inequality (1.2) is true if one of the following conditions holds:*

(i) $0 < \gamma \leq N \leq \beta$;

(ii) $N < \gamma \leq \beta$ *and* $q \leq N(\beta - N)/(\gamma - N)$.

Notice that, in the setting of the above theorem, the inequality (1.1) is a direct consequence of (VQ). If we think about that inequality with N replaced by $1 < p < N$ and potentials V and Q which are radials, many interesting papers have state the inequality in the context of radial functions, see for instance [21, 22, 23] and references therein. This seems to be the simplest case, since we may consider radial functions which has some prescribed decay rates at infinity. Even in the case $1 < p < N$, when have no symmetry, the situation is more involved and we need to control the behavior of the potentials at infinity. Some results were presented by Opic and Kufner [18], Ambrosseti, Felli and Machiodi [4], Ambrosetti and Wang [7], Ambrosetti, Malchiodi and Ruiz [5], Bonheure and Schaftingen [9], Alves and Souto [3], among others. We emphasize that we are considering the borderline case and we do not assume symmetry.

Recall that the classical Sobolev Embedding Theorem assures that $W^{1,N}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$, for any $q \geq N$. Theorem 1.1 states an analogous result for our space

$E^{1,N,\gamma}$, namely, for any $0 < \gamma \le N \le \beta$ and $q \ge N$, we have the continuous embedding

$$E^{1,N,\gamma} \hookrightarrow L^q(\mathbb{R}^N, (1+|\cdot|)^{-\beta}). \tag{1.3}$$

Therefore, also in accordance with the $W^{1,N}(\mathbb{R}^N)$ case, it is natural to look for an embedding from $E^{1,N,\gamma}$ into Orlicz spaces. To be more precise, for $\alpha > 0$, we define the function

$$\Phi_\alpha(s) := e^{\alpha|s|^{N/(N-1)}} - \sum_{j=0}^{N-2} \frac{\alpha^j}{j!} |s|^{Nj/(N-1)}, \qquad s \in \mathbb{R}.$$

In our second result, we prove the following Trudinger-Moser type result:

Theorem 1.2. *Suppose that $N \ge 2$ and $0 < \gamma \le N \le \beta$. Then, for any $\alpha > 0$ and $u \in E^{1,N,\gamma}$, the function $(1+|\cdot|)^{-\beta}\Phi_\alpha(u)$ belongs to $L^1(\mathbb{R}^N)$. Moreover, there exists $\alpha^* = \alpha^*(N) > 0$ such that*

$$\sup_{\{u\in E^{1,N,\gamma}:\|u\|_{E^{1,N,\gamma}}\le 1\}} \int_{\mathbb{R}^N} \frac{1}{(1+|x|)^\beta} \Phi_\alpha(u)\, dx < +\infty,$$

for any $0 < \alpha < \alpha^*$.

The first results concerning Trudinger-Moser type inequalities have appeared in the papers of Yudovich, Moser, Trudinger [26, 17, 24], for the bounded domain case. Similar results for unbounded domains have been established by Cao [10] and Ruf [19] in $\mathbb{R}^2$, and by do Ó [11], Adachi and Tanaka [1], Li and Ruf [16], in higher dimensions. Concerning the case of weighted Sobolev spaces, we can refer the reader to [2, 14, 12, 15, 28, 8] and references therein. Some of these works considered radial weight functions, in such a way that rearrangement procedures work well. We finally mention that Theorem 1.2 is closely related to some results obtained in the 2-dimensional case by do Ó, Sani and Jianjun [13], where the authors used an approach borrowed from Opnic-Kurfner [18]. Here, we provide a different and simplified proof.

In the final part of the paper, we turn back to problem ($\mathcal{P}$) by assuming that the nonlinearity $f \in C(\mathbb{R},\mathbb{R})$ has exponential growth at infinity and is N-sublinear at the origin, that is,

(f_0) there exists $\alpha_0 > 0$ such that

$$\lim_{|s|\to+\infty} \frac{|f(s)|}{e^{\alpha|s|^{N/(N-1)}}} = \begin{cases} 0, & \text{if } \alpha > \alpha_0, \\ +\infty, & \text{if } \alpha < \alpha_0; \end{cases}$$

(f_1) $f(s) = o(|s|^{N-1})$ as $s \to 0$.

As we shall see, the above conditions imply that the map $u \mapsto \int_{\mathbb{R}^N} Q(x)F(u)dx$ is of class C^1, where $F(s) := \int_0^s f(t)\, dt$. Hence, we can apply standard variational techniques to obtain solutions for ($\mathcal{P}$).

With this basic assumption in hands, we can explore a huge variety of conditions on f to obtain existence of solutions for the problem. In order to illustrate this feature, we shall prove the following:

Theorem 1.3. *Suppose that (VQ) holds with $0 < \gamma \le N < \beta$. If f satisfies $(f_0)-(f_1)$ and*

(f_2) there exists $\theta > N$ such that

$$0 < \theta F(s) \le f(s)s, \quad s \in \mathbb{R};$$

(f_3) *there exist* $c_F > 0$ *and* $\nu > N$ *such that* $F(s) \geq c_F|s|^\nu$, *for any* $s \in \mathbb{R}$,

then there exists $\lambda_* > 0$ *such that, for any* $\lambda > \lambda_*$, *the equation* ($\mathcal{P}$) *has a nonzero weak solution.*

In the proof of this last theorem, we apply the classical Moutain Pass Theorem [6]. Condition (f_2) has been appeared in many papers and is related with the boundedness of Palais-Smale sequences. The technical condition (f_3) enable us to localize the minimax level in the range where we have compactness. This overcomes the lack of compactness which occurs due to the critical growth of f.

The rest of the paper is organized as follows: in the next two section we prove our two first theorems, one in each section. In the final Section 4, we obtain a solution for ($\mathcal{P}$).

2. Sobolev type embedding

In this section, we prove our first result for the space $E^{1,N,\gamma}$. The mains idea consists in a covering process where we write the entire space as a union of a ball with annular sets. For any $x_0 \in \mathbb{R}^N$ and $r > 0$, we denote along all the paper $B_R(x_0) := \{x \in \mathbb{R}^N : |x - x_0| < R\}$.

Recall that the Sobolev embedding $W^{1,N}(B_1(0)) \hookrightarrow L^q(B_1(0))$ holds for any $q \geq N$. Hence, since any function $u \in E^{1,N,\gamma}$ clearly belongs to $W^{1,N}_{loc}(\mathbb{R}^N)$, there exists $C_1 = C_1(N, q) > 0$ such that

$$\begin{aligned} \int_{B_1(0)} \frac{|u|^q}{(1+|x|)^\beta} dx &\leq C_1 \left(\int_{B_1(0)} \left[|\nabla u|^N + |u|^N \right] dx \right)^{q/N} \\ &\leq C_2 \left(\int_{B_1(0)} \left[|\nabla u|^N + \frac{|u|^N}{(1+|x|)^\gamma} \right] dx \right)^{q/N}, \end{aligned} \tag{2.1}$$

with $C_2 = C_2(N, q, \gamma) > 0$.

We now define, for each $j \in \mathbb{N} \cup \{0\}$, the annular set

$$A_j := \{z \in \mathbb{R}^N : 2^j < |z| < 2^{j+1}\},$$

and prove the following technical estimate.

Lemma 2.1. *Let* β, $\gamma > 0$ *and* $q \geq N$. *Then, for each* $j \in \mathbb{N} \cup \{0\}$, *there exists a constant* $C = C(N, \gamma, q) > 0$ *such that*

$$\int_{A_j} \frac{|u|^q}{(1+|x|)^\beta} dx \leq C \left(\int_{A_j} \left[2^{\zeta_1 j} |\nabla u|^N + 2^{\zeta_2 j} \frac{|u|^N}{(1+|x|)^\gamma} \right] dx \right)^{q/N}, \tag{2.2}$$

where

$$\zeta_1 := \frac{N}{q}(N - \beta), \quad \zeta_2 := (\gamma - N) + \frac{N}{q}(N - \beta).$$

Proof. By using the change of variables $y = 2^{-j}x$, we get

$$\int_{A_j} \frac{|u|^q}{(1+|x|)^\beta} dx \leq \frac{1}{2^{\beta j}} \int_{A_j} |u|^q dx = 2^{(N-\beta)j} \int_{A_0} |u_j(y)|^q dy,$$

where $u_j(y) := u(2^j y)$. From the Sobolev embedding $W^{1,N}(A_0) \hookrightarrow L^q(A_0)$, we obtain $C = C(N,q) > 0$ such that

$$\int_{A_0} |u_j(y)|^q dy \le C \left(\int_{A_0} \left[|\nabla u_j(y)|^N + |u_j(y)|^N \right] dy \right)^{q/N}$$
$$= C \left(\int_{A_j} \left[|\nabla u|^N + 2^{-Nj}|u|^N \right] dx \right)^{q/N}.$$

Since $(1 + 2^{j+1}) \le 2 \cdot 2^{j+1}$ and $\gamma > 0$, we get

$$\int_{A_j} 2^{-Nj}|u|^N dx \le 2^{-Nj}(1+2^{j+1})^{\gamma} \int_{A_j} \frac{|u|^N}{(1+|x|)^{\gamma}} dx$$
$$\le 2^{2\gamma + (\gamma - N)j} \int_{A_j} \frac{|u|^N}{(1+|x|)^{\gamma}} dx.$$

Combining the above estimates, we deduce that

$$\int_{A_j} \frac{|u|^q}{(1+|x|)^{\beta}} dx \le 2^{(N-\beta)j} 2^{2\gamma q/N} C \left(\int_{A_j} \left[|\nabla u|^N + \frac{|u|^N}{(1+|x|)^{\gamma}} \right] dx \right)^{q/N},$$

and the lemma is proved. □

We are ready to prove our first main result.

Proof of Theorem 1.1. If any of the conditions (i) or (ii) of the the theorem holds, the numbers ζ_1, ζ_2 obtained in Lemma 2.1 are nonpositive. Hence, for any $j \in \mathbb{N} \cup \{0\}$, we have that

$$\int_{A_j} \frac{|u|^q}{(1+|x|)^{\beta}} dx \le C \left(\int_{A_j} \left[|\nabla u|^N + \frac{|u|^N}{(1+|x|)^{\gamma}} \right] dx \right)^{q/N}.$$

Recalling that the function $s \mapsto s^{q/N}$ is super additive for $q \ge N$, we conclude that

$$\sum_{j=0}^{\infty} \int_{A_j} \frac{|u|^q}{(1+|x|)^{\beta}} dx \le C \sum_{j=0}^{\infty} \left(\int_{A_j} \left[|\nabla u|^N + \frac{|u|^N}{(1+|x|)^{\gamma}} \right] dx \right)^{q/N},$$
$$\le C \left(\int_{\mathbb{R}^N \setminus B_1(0)} \left[|\nabla u|^N + \frac{|u|^N}{(1+|x|)^{\gamma}} \right] dx \right)^{q/N}.$$

Combining the above estimate with (2.1), we obtain

$$\int_{\mathbb{R}^N} \frac{|u|^q}{(1+|x|)^{\beta}} dx \le (C_2 + C)\|u\|^q_{E^{1,N,\gamma}},$$

which concludes the proof. □

Remark 2.2. *It is worth mention that inequality* (1.2) *could fail for some choices of the parameters. Indeed, pick* $\omega \in C_0^{\infty}(\mathbb{R}^N)$ *a nonzero function vanishing near the origin, define* $u_\lambda(x) := \omega(\lambda x)$, *for* $\lambda > 0$, *and suppose that*

$$\max\{\gamma, \beta\} < N, \qquad 0 < q < \frac{N(N-\beta)}{(N-\gamma)}. \tag{2.3}$$

Using the change of variables $y = \lambda x$, we get

$$\int_{\mathbb{R}^N} |\nabla u_\lambda(x)|^N dx = C_1 > 0,$$

and, for $\gamma > 0$,

$$\int_{\mathbb{R}^N} \frac{|u_\lambda|^N}{(1+|x|)^\gamma} dx \le \int_{\mathbb{R}^N} \frac{|u_\lambda|^N}{|x|^\gamma} dx = \lambda^{\gamma-N} \int_{\mathbb{R}^N} \frac{|\omega(y)|^N}{|y|^\gamma} dy = C_2 \lambda^{\gamma-N}.$$

On the other hand, there exist C_3, $C_4 > 0$ such that,

$$\int_{\mathbb{R}^N} \frac{|u_\lambda|^q}{(1+|x|)^\beta} dx \ge C_3 \int_{\mathbb{R}^N \setminus B_1(0)} \frac{|u_\lambda|^q}{|x|^\beta} dx = C_4 \lambda^{\beta-N}.$$

If (1.2) holds, we obtain

$$\frac{1}{C_0} \le \frac{\left(\int_{\mathbb{R}^N} \left[|\nabla u_\lambda| + \frac{|u_\lambda|^N}{(1+|x|)^\gamma} \right] dx \right)^{q/N}}{\int_{\mathbb{R}^N} \frac{|u_\lambda|^q}{(1+|x|)^\beta} dx} \le \frac{\left(C_1 + C_2 \lambda^{\gamma-N}\right)^{q/N}}{C_4 \lambda^{\beta-N}} \le C_5 \left(\lambda^{\theta_1} + \lambda^{\theta_2} \right),$$

for some $C_5 > 0$ and

$$\theta_1 = N - \beta, \qquad \theta_2 = \frac{q}{N}(\gamma - N) + (N - \beta).$$

But (2.3) implies that θ_1, $\theta_2 > 0$, and we obtain a contradiction as $\lambda \to 0^+$.

3. A Trudinger-Moser type inequality

We present in this section the proof of Theorem 1.2. We start with two technical results.

Lemma 3.1. *Let $x_0 \in \mathbb{R}^N$ and $v \in W_0^{1,N}(B_R(x_0))$ be such that $\int_{B_R(x_0)} |\nabla v|^N dx \le 1$. Then there exists $C = C(N) > 0$ such that*

$$\int_{B_R(x_0)} \Phi_{\alpha_N}(v) dx \le C(N) \cdot R^N \int_{B_R(x_0)} |\nabla v|^N dx,$$

where $\alpha_N := N\omega_{N-1}^{1/(N-1)}$ and ω_{N-1} is the measure of the unit sphere in $\mathbb{R}^{N-1}$.

Proof. See [25, Lemma 3.1]. □

The second auxiliar result reads as

Lemma 3.2. *Suppose that $0 < \gamma \le N \le \beta$. Then, there exist $C_N > 0$ and $\alpha^* = \alpha^*(N) > 0$ such that*

$$\int_{\mathbb{R}^N} \frac{1}{(1+|x|)^\beta} \Phi_\alpha(u)\, dx \le C_N,$$

for any $0 < \alpha < \alpha^*$ and $u \in E^{1,N,\gamma}$ verifying $\|u\|_{E^{1,N,\gamma}} \le 1$.

Proof. Since the above integral and $\|\cdot\|_{E^{1,N,\gamma}}$ are monotonic in β and γ, respectively, we may consider $\gamma = \beta = N$. Let $\varphi \in C_0^\infty(B_2(0))$ be such that $\varphi \equiv 1$ in $B_1(0)$ and

$|\nabla\varphi| \leq 2$ in $B_2(0)$. For any $u \in E^{1,N,\gamma}$ such that $\|u\|_{E^{1,N,\gamma}} \leq 1$, there hold

$$\begin{aligned}\int_{B_2(0)} |\nabla(\varphi u)|^N dx &\leq C_1 \int_{B_2(0)} |\nabla u|^N dx + C_1 \int_{B_2(0)} |u|^N dx \\ &\leq C_1 \int_{B_2(0)} |\nabla u|^N dx + C_1 \cdot 3^N \int_{B_2(0)} \frac{|u|^N}{(1+|x|)^N} dx \\ &\leq C_2 \int_{B_2(0)} \left[|\nabla u|^N + \frac{|u|^N}{(1+|x|)^N}\right] dx \leq C_2,\end{aligned}$$

with $C_2 = C_2(N) > 0$. Hence, if we set $v := (1/C_2)^{1/N}\varphi u$ and apply Lemma 3.1, we obtain

$$\int_{B_2(0)} \Phi_{\alpha_N}(v)\, dx \leq C(N) \cdot 2^N \int_{B_2(0)} |\nabla v|^N\, dx \leq C_3,$$

with $C_3 := C(N) \cdot 2^N$.

Form the definition of Φ_α, we easily conclude that

$$\Phi_\alpha(ts) = \Phi_{\alpha t^{N/(N-1)}}(s), \qquad s \in \mathbb{R},\ t > 0. \tag{3.1}$$

Thus, since $\varphi \equiv 1$ in $B_1(0)$ and $\Phi_\alpha \geq 0$ is monotonic in α, we have that

$$\begin{aligned}\int_{B_1(0)} \Phi_\alpha(u)\, dx &= \int_{B_1(0)} \Phi_\alpha(\varphi u)\, dx \leq \int_{B_2(0)} \Phi_\alpha(C_2^{1/N} v) dx \\ &= \int_{B_2(0)} \Phi_{\alpha C_2^{1/(N-1)}}(v)\, dx \leq \int_{B_2(0)} \Phi_{\alpha_N}(v)\, dx \leq C_3,\end{aligned}$$

whenever

$$0 < \alpha < \frac{\alpha_N}{C_2^{1/(N-1)}}.$$

Hence,

$$\int_{B_1(0)} \frac{1}{(1+|x|)^N} \Phi_\alpha(u)\, dx \leq C_3. \tag{3.2}$$

We now consider, for each $j \in \mathbb{N} \cup \{0\}$, the annulus $A_j = \{z \in \mathbb{R}^N : 2^j < |z| < 2^{j+1}\}$ and claim that, if $\alpha > 0$ is small enough, there exists $C_4 = C_4(N) > 0$, independent of j, such that

$$\int_{A_j} \frac{1}{(1+|x|)^N} \Phi_\alpha(u)\, dx \leq C_4 \int_{A_j} \left[|\nabla u|^N + \frac{|u|^N}{(1+|x|)^N}\right] dx. \tag{3.3}$$

By using this inequality and (3.2), we get

$$\begin{aligned}\int_{\mathbb{R}^N} \frac{1}{(1+|x|)^N} \Phi_\alpha(u)\, dx &\leq C_3 + \sum_{j=0}^{\infty} \int_{A_j} \frac{1}{(1+|x|)^N} \Phi_\alpha(u)\, dx. \\ &\leq C_3 + C_4 \int_{\mathbb{R}^N \setminus B_1(0)} \left[|\nabla u|^N + \frac{|u|^N}{(1+|x|)^N}\right] dx,\end{aligned}$$

and therefore

$$\int_{\mathbb{R}^N} \frac{1}{(1+|x|)^N} \Phi_\alpha(u)\, dx \leq C_N := C_3 + C_4,$$

whenever $u \in E^{1,N,\gamma}$ satisfies $\|u\|_{E^{1,N,\gamma}} \leq 1$.

It remains to be proved that (3.3) holds. First, we use the change of variables $y = 2^{-j}x$ to obtain, for $u_j(y) := u(2^j y)$, the following

$$\int_{A_j} \frac{1}{(1+|x|)^N} \Phi_\alpha(u)\, dx \le \frac{1}{2^{jN}} \int_{A_j} \Phi_\alpha(u)\, dx = \int_{A_0} \Phi_\alpha(u_j)\, dy. \tag{3.4}$$

For each $y \in A_0$, set $R_y := \mathrm{dist}(y, \partial A_0)$ and notice that $B_{R_y}(y) \subset A_0$. Moreover, from the compactness of $\overline{A_0}$, we obtain points $y_1, \ldots, y_k \in A_0$ such that $A_0 \subset \bigcup_{i=1}^k B_{R_i/2}(y_i)$, where $R_i := R_{y_i}$. For each $i = 1, \ldots, k$, we set $B^i := B_{R_i}(y_i)$ and pick a function $\varphi_i \in C_0^\infty(B^i)$ such that $0 \le \varphi_i \le 1$, $\varphi_i \equiv 1$ in $B_{R_i/2}(y_i)$ and $|\nabla \varphi_i| \le 4/R_i$ in B^i. We have that

$$\begin{aligned}
\int_{B^i} |\nabla(\varphi_i u_j)|^N dy &\le C_5 \int_{B^i} |\nabla u_j|^N dy + C_6 R_i^{-N} \int_{B^i} |u_j|^N dy \\
&\le C_5 \int_{A_0} |\nabla u(2^j y)|^N 2^{jN} dy + C_6 R_i^{-N} \int_{A_0} |u(2^j y)|^N dy \\
&= C_5 \int_{A_j} |\nabla u|^N dx + \frac{C_6}{R_i^N 2^{jN}} \int_{A_j} |u|^N dx.
\end{aligned}$$

Since $j \ge 0$, we have that $(1 + 2^{j+1}) \le 4 \cdot 2^j$, and therefore

$$\int_{A_j} |u|^N dx \le 4^N 2^{jN} \int_{A_j} \frac{|u|^N}{(1+|x|)^N} dx.$$

All together, the above inequalities imply that,

$$\int_{B^i} |\nabla(\varphi_i u_j)|^N dy \le C_7 \int_{A_j} \left[|\nabla u|^N + \frac{|u|^N}{(1+|x|)^N} \right] dx, \tag{3.5}$$

with $C_7 = C_7(N) > 0$.

Since $\|u\|_{E^{1,N,\gamma}} \le 1$, the above inequality shows that we can apply Lemma Lemma 3.1 with $v := (1/C_7)^{1/N} \varphi_i u_j$ to obtain $C_8 = C_8(N) > 0$ such that

$$\begin{aligned}
\int_{B^i} \Phi_{\alpha_N}(v)\, dy &\le C(N) \cdot R_i^N \int_{B^i} |\nabla v|^N dy \\
&\le C_8 \int_{A_j} \left[|\nabla u|^N + \frac{|u|^N}{(1+|x|)^N} \right] dx,
\end{aligned}$$

where we have used (3.5) and $R_i \le 1$.

If we define

$$\alpha^* := \min \left\{ \frac{\alpha_N}{C_2^{1/(N-1)}}, \frac{\alpha_N}{C_7^{1/(N-1)}} \right\},$$

we can use the definition of v and (3.1) as before to get

$$\int_{B^i} \Phi_{\alpha^*}(\varphi_i u_j)\, dy \le \int_{B^i} \Phi_{\alpha_N}(v)\, dy \le C_8 \int_{A_j} \left[|\nabla u|^N + \frac{|u|^N}{(1+|x|)^N} \right] dx.$$

For any $0 < \alpha < \alpha^*$, we can proceed as in the first part of the proof to get

$$\begin{aligned}\int_{A_0} \Phi_\alpha(u_j)\,dy &\le \sum_{i=1}^{k} \int_{B_{R_i/2}(y_i)} \Phi_\alpha(u_j)\,dy \\ &= \sum_{i=1}^{k} \int_{B_{R_i/2}(y_i)} \Phi_\alpha(\varphi_i u_j)\,dy \\ &\le \sum_{i=1}^{k} \int_{B_{R_i/2}(y_i)} \Phi_{\alpha^*}(\varphi_i u_j)\,dy \\ &\le C_8 \int_{A_j} \left[|\nabla u|^N + \frac{|u|^N}{(1+|x|)^N} \right] dx.\end{aligned}$$

The inequality (3.3) is now a consequence of the above expression and (3.4). □

We are ready to present the proof of our second theorem.

Proof of Theorem 1.2. If we consider $\alpha^* > 0$ as in Lemma 3.2, we have that

$$\sup_{\{u\in E^{1,N,\gamma}:\|u\|_{E^{1,N,\gamma}}\le 1\}} \int_{\mathbb{R}^N} \frac{1}{(1+|x|)^\beta} \Phi_\alpha(u)\,dx < C_N, \tag{3.6}$$

for any $0 < \alpha < \alpha^*$. So, we need only to verify that the function $(1+|\cdot|)^{-\beta}\Phi_\alpha(u)$ is integrable for each $u \in E^{1,N,\gamma}$. In order to do that, we pick $u_0 \in C_0^\infty(\mathbb{R}^N)$ such that

$$\|u - u_0\|_{E^{1,N,\gamma}} \le \delta,$$

with $\delta > 0$ to be chosen later.

A simple computation shows that, for any $s \ge 0$,

$$|\Phi'_\alpha(s)| \le \frac{\alpha N}{N-1} |s|^{1/(N-1)} e^{\alpha |s|^{N/(N-1)}}.$$

Thus, for any $s, t \ge 0$, we can use the Mean Value Theorem to obtain $\theta \in [\min\{s,t\}, \max\{s,t\}]$ such that

$$\Phi_\alpha(s) \le \Phi_\alpha(t) + \frac{\alpha N}{N-1} |\theta|^{1/(N-1)} e^{\alpha|\theta|^{N/(N-1)}} |t - s|.$$

Using this inequality with $s = |u|$ and $t = |u - u_0|$, we obtain a function $x \mapsto \theta(x)$ such that, for a.e. $x \in \mathbb{R}^N$,

$$\Phi_\alpha(|u|) \le \Phi_\alpha(|u-u_0|) + \frac{\alpha N}{N-1} |\theta(x)|^{1/(N-1)} \psi(x) e^{\alpha|\theta(x)|^{N/(N-1)}}, \tag{3.7}$$

where $\psi := \Big| |u - u_0| - |u| \Big| \in E^{1,N,\gamma}$ has compact support Ω.

We now notice that,

$$\int_{\mathbb{R}^N} \frac{1}{(1+|x|)^\beta} \Phi_\alpha(|u-u_0|)dx \le \int_{\mathbb{R}^N} \frac{1}{(1+|x|)^\beta} \Phi_{\alpha \|u-u_0\|_{E^{1,N,\gamma}}^{N/(N-1)}} \left(\frac{|u-u_0|}{\|u-u_0\|_{E^{1,N,\gamma}}} \right) dx,$$

and therefore we can choose $\delta > 0$ small in such way that we can apply (3.6) to conclude that

$$\int_{\mathbb{R}^N} \frac{1}{(1+|x|)^\beta} \Phi_\alpha(|u-u_0|)dx < C_N. \tag{3.8}$$

Since u_0 is bounded and θ lives between $|u-u_0|$ and $|u|$, it is clear that

$$|\theta(x)| \leq |u-u_0| + |u| \leq C_1(|u|+1),$$

for a.e. $x \in \Omega$ and some $C_1 > 0$. Thus, we can use Hölder's inequality to obtain

$$\begin{aligned}\int_\Omega \frac{1}{(1+|x|)^\beta}|\theta|^{1/(N-1)}\psi e^{\alpha|\theta|^{N/(N-1)}}dx &\leq C_2\int_\Omega (|u|+1)^{1/(N-1)}\psi e^{C_3|u|^{N/(N-1)}}dx \\ &\leq C_4\left(\int_\Omega e^{r_3C_3|u|^{N/(N-1)}}dx\right)^{1/r_3},\end{aligned}$$

where $C_4 := \|(|u|+1)\|^{1/(N-1)}_{L^{r_1/(N-1)}(\Omega)}\|\psi\|^{r_2}_{L^{r_2}(\Omega)}$ and r_1, r_2, r_3 are such that $1/r_1 + 1/r_2+1/r_3 = 1$, $r_1 \geq N(N-1)$ and $r_2 \geq N$. It follows from the classical Trudinger-Moser inequality in $W^{1,N}(\Omega)$ that

$$\int_\Omega \frac{1}{(1+|x|)^\beta}|\theta|^{1/N}\psi e^{\alpha|\theta|^{N/(N-1)}}dx < +\infty.$$

Since $\Phi_\alpha(|u|) = \Phi_\alpha(u)$, we can use (3.7), (3.8) and the above expression to conclude that $(1+|\cdot|)^{-\beta}\Phi_\alpha(u) \in L^1(\mathbb{R}^N)$. The theorem is proved. □

4. Existence of solution for $(\mathcal{P})$

In this section, we prove Theorem 1.3. From now on, we shall assume that $(f_0)-(f_3)$ and (VQ) hold with $0 < \gamma \leq N < \beta$, and we look for solutions in the subspace of $E^{1,N,\gamma}$ defined as

$$E_V := \Big\{u \in E^{1,N,\gamma} : \|u\|_{E_V} < +\infty\Big\},$$

where $\|u\|_{E_V} := \left(\int_{\mathbb{R}^N}[|\nabla u|^N + V(x)|u|^N]dx\right)^{1/N}$.

As a byproduct of Theorem 1.1, we have the following embedding result:

Lemma 4.1. *Suppose that $0 < \gamma \leq N < \beta$. Then, the embedding $E_V \hookrightarrow L^q(\mathbb{R}^N, Q)$ is continuous and compact for any $q \geq N$.*

Proof. For any $u \in E_V$, we can use (VQ) and Theorem 1.1 to get

$$\begin{aligned}\int_{\mathbb{R}^N} Q(x)|u|^q dx \leq c_Q\int_{\mathbb{R}^N}\frac{|u|^q}{(1+|x|)^\beta}dx &\leq C_1\left(\int_{\mathbb{R}^N}\Big[|\nabla u|^N + \frac{|u|^N}{(1+|x|)^\gamma}\Big]dx\right)^{q/N} \\ &\leq C_2\left(\int_{\mathbb{R}^N}\Big[|\nabla u|^N + V(x)|u|^N\Big]dx\right)^{q/N} = C_2\|u\|^q_{E_V}.\end{aligned}$$

This shows that we have continuous embedding.

In order to prove the compactness, consider $(u_n) \subset E_V$ such that $u_n \rightharpoonup 0$. Given $\varepsilon > 0$, we can choose $N \leq \beta_0 < \beta$ and $R > 0$ such that $(1+|x|)^{\beta_0-\beta} < \varepsilon$ for $|x| \geq R$. This, together with (VQ) and the continuous embedding $E^{1,N,\gamma} \hookrightarrow L^q(\mathbb{R}^N, (1+|\cdot|)^{-\beta_0})$ imply that

$$\begin{aligned}\int_{\mathbb{R}^N\setminus B_R(0)} Q(x)|u_n|^q dx &\leq \varepsilon c_Q\int_{\mathbb{R}^N\setminus B_R(0)}\frac{|u_n|^q}{(1+|x|)^{\beta_0}}dx \\ &\leq \varepsilon C_3\left(\int_{\mathbb{R}^N}\Big[|\nabla u_n|^N + \frac{|u_n|^N}{(1+|x|)^\gamma}\Big]dx\right)^{q/N} \\ &\leq \varepsilon C_4\|u_n\|^q_{E_V} \leq \varepsilon C_5.\end{aligned}$$

On the other hand, from the embedding $E_V \subset E^{1,N,\gamma} \subset W^{1,N}_{loc}(\mathbb{R}^N)$ and Rellich-Kondrachov Theorem, we get

$$\int_{B_R(0)} Q(x)|u_n|^q dx \leq \|Q\|_{L^\infty(B_R(0))} \int_{B_R(0)} |u_n|^q dx = o_n(1),$$

where $o_n(1)$ stands for a quantity approaching zero as $n \to +\infty$. It follows from the above inequalities that $u_n \to 0$ strongly in $L^q(\mathbb{R}^N, Q)$. □

In order to introduce the energy functional which is associated with our problem we notice that, for any given $\varepsilon > 0$, $\alpha > \alpha_0$ and $r \geq 1$, we can use $(f_0) - (f_1)$ to obtain $C > 0$ such that

$$|f(s)| \leq \varepsilon |s|^{N-1} + C|s|^{r-1}\Phi_\alpha(s), \quad |F(s)| \leq \varepsilon |s|^N + C|s|^r \Phi_\alpha(s), \tag{4.1}$$

for any $s \in \mathbb{R}$. Given $u \in E_V$ and $r_1, r_2 > 1$ such that $1/r_1 + 1/r_2 = 1$, $r_1 \geq N$, we can use Hölder's inequality, Lemma 4.1 and Theorem 1.2 to get

$$\int_{\mathbb{R}^N} Q(x)F(u)dx \leq \varepsilon \|u\|^N_{L^N(\mathbb{R}^N,Q)} + C\|u\|^r_{L^{r_1 r}(\mathbb{R}^N,Q)} \left(\int_{\mathbb{R}^N} Q(x)\Phi_{r_2\alpha}(u)dx \right)^{1/r_2} < +\infty,$$

where we also have used the inequality (see [27, Lemma 2.1])

$$[\Phi_\alpha(s)]^r \leq \Phi_{r\alpha}(s). \tag{4.2}$$

By the above considerations, the functional $I_\lambda : E_V \to \mathbb{R}$ given by

$$I_\lambda(u) := \frac{1}{N}\|u\|^N_{E_V} - \lambda \int_{\mathbb{R}^N} Q(x)F(u)\, dx$$

is well defined. Moreover, using some standard calculations, we can prove that $I_\lambda \in C^1(E_V, \mathbb{R})$ and the weak solutions of problem $(\mathcal{P})$ are precisely the critical points of I_λ.

In our first auxiliary result, we prove a local compactness result.

Lemma 4.2. *Let $(u_n) \subset E_V$ be such that*

$$\lim_{n\to+\infty} I_\lambda(u_n) = d < \left(\frac{\alpha^*}{\alpha_0}\right)^{N-1} \left(\frac{\nu - N}{N\nu}\right), \qquad \lim_{n\to+\infty} I'_\lambda(u_n) = 0.$$

Then (u_n) has a convergent subsequence.

Proof. By computing $I_\lambda(u_n) - (1/\theta)I'_\lambda(u_n)u_n$ and using (f_2), we obtain $C_1, C_2 > 0$ such that

$$\begin{aligned} C_1 + C_2\|u_n\|_{E_V} &\geq \left(\frac{1}{N} - \frac{1}{\theta}\right)\|u_n\|^N_{E_V} + \lambda \int_{\mathbb{R}^N} Q(x)\left(\frac{1}{\theta}f(u_n)u_n - F(u_n)\right)dx \\ &\geq \left(\frac{1}{N} - \frac{1}{\theta}\right)\|u_n\|^N_{E_V}, \end{aligned}$$

which implies that (u_n) is bounded in E_V, since $\theta > N$. So, up to a subsequence, $u_n \rightharpoonup u$ weakly in E_V.

We claim that

$$\int_{\mathbb{R}^N} Q(x)f(u_n)(u_n - u)dx = o_n(1). \tag{4.3}$$

Indeed, by using (4.1), we get

$$\left| \int_{\mathbb{R}^N} Q(x)f(u_n)(u_n - u)dx \right| \leq \varepsilon A_n + CB_n,$$

where

$$A_n := \int_{\mathbb{R}^N} Q(x)|u_n|^{N-1}|u_n - u|\, dx,$$

$$D_n := \int_{\mathbb{R}^N} Q(x)|u_n|^{r-1}\Phi_\alpha(u_n)|u_n - u|\, dx.$$

Hölder's inequality and Lemma 4.1 provide

$$A_n \le \|u_n\|^{N-1}_{L^N(\mathbb{R}^N,Q)}\|u_n - u\|_{L^N(\mathbb{R}^N,Q)} \le C_3\|u_n\|^{N-1}_{E_V}\|u_n - u\|_{E_V} \le C_4.$$

Since $\varepsilon > 0$ is arbitrary, it is sufficient to prove that $D_n = o_n(1)$. Since we may assume that $\theta \ge \nu$, it follows from (f_2) that

$$d = \lim_{n\to+\infty}\left(I_\lambda(u_n) - \frac{1}{\theta}I'_\lambda(u_n)u_n\right) \ge \left(\frac{1}{N} - \frac{1}{\nu}\right)\lim_{n\to+\infty}\|u_n\|^N_{E_V},$$

and therefore the hypothesis on d implies that

$$\lim_{n\to+\infty}\|u_n\|^{N/(N-1)}_{E_V} \le \left(\frac{N\nu}{\nu - N}\right)^{1/(N-1)} d^{1/(N-1)} < \frac{\alpha^*}{\alpha_0}.$$

We now pick $r_1 > 1$ and $\alpha > \alpha_0$ in such way that $r_1\alpha\|u_n\|^{N/(N-1)}_{E_V} < \alpha^*$, for all $n \in \mathbb{N}$ large enough. By using Hölder's inequality, Lemma 4.1, (VQ), Theorem 1.2 and (4.2), we deduce

$$\begin{aligned} D_n &\le \left(\int_{\mathbb{R}^N} Q(x)\Phi_{r_1\alpha\|u_n\|^{N/(N-1)}_{E_V}}\left(\frac{u_n}{\|u_n\|_{E_V}}\right)dx\right)^{1/r_1} \\ &\quad \times \|u_n\|^{r-1}_{L^{r_2(r-1)}(\mathbb{R}^N,Q)}\|u_n - u\|_{L^{r_3}(\mathbb{R}^N,Q)} \\ &\le C_5\|u_n\|^{r-1}_{E_V}\|u_n - u\|_{L^{r_3}(\mathbb{R}^N,Q)} = o_n(1), \end{aligned}$$

where $r > 1$ and r_2, r_3 are such that $1/r_1 + 1/r_2 + 1/r_3 = 1$, $r_3 \ge N$, $r_2 > 1$ and $r_2(r-1) \ge N$. This concludes the proof of (4.3).

Since $I'(u_n)(u_n - u) = o_n(1)$, we can use (4.3) to get

$$\int_{\mathbb{R}^N} |\nabla u_n|^{N-2}\nabla u_n \cdot \nabla(u_n - u)\, dx + \int_{\mathbb{R}^N} V(x)|u_n|^{N-2}u_n(u_n - u)\, dx = o_n(1).$$

Moreover, from the weak convergence, we have that

$$\int_{\mathbb{R}^N} |\nabla u|^{N-2}\nabla u \cdot \nabla(u_n - u)dx + \int_{\mathbb{R}^N} V(x)|u|^{N-2}u(u_n - u)dx = o_n(1).$$

Hence,

$$\int_{\mathbb{R}^N} [T_N(\nabla u_n, \nabla u)\cdot\nabla(u_n - u) + V(x)T_1(u_n, u)(u_n - u)]\, dx = o_n(1), \tag{4.4}$$

where

$$T_k(y_1, y_2) := \left(|y_1|^{N-2}y_1 - |y_2|^{N-2}y_2\right), \quad y_1, y_2 \in \mathbb{R}^k,$$

for $k \in \{1, N\}$. But we know that (see [20, inequality (2.2)])

$$T_k(y_1, y_2)\cdot(y_1 - y_2) \ge C(k, N)|y_1 - y_2|^N, \quad \forall\, y_1,\, y_2 \in \mathbb{R}^k.$$

From this inequality and (4.4) we obtain $C_6 > 0$ such that

$$C_6\|u_n - u\|^N_{E_V} \le o_n(1),$$

and therefore $u_n \to u$ strongly in E_V and the lemma is proved. □

In our next step, we prove that I_λ verifies the geometric conditions of the classical Mountain Pass Theorem.

Lemma 4.3. *There are constants $\rho, \tau > 0$ such that $I_\lambda(u) \geq \tau$, for any $\|u\|_{E_V} = \rho$. Moreover, there exists $e \in E_V$ such that $\|e\|_{E_V} > \rho$ and $I_\lambda(e) < 0$.*

Proof. From (4.1), Hölder's inequality, (3.1) and (4.2), we deduce

$$\int_{\mathbb{R}^N} Q(x)F(u)dx \leq \varepsilon \int_{\mathbb{R}^N} Q(x)|u|^N dx + C\left(\int_{\mathbb{R}^N} Q(x)|u|^{r_1 r} dx\right)^{1/r_1}$$
$$\times \left(\int_{\mathbb{R}^N} Q(x)\Phi_{r_2\alpha\|u\|_{E_V}^{N/(N-1)}}\left(\frac{u}{\|u\|_{E_V}}\right)dx\right)^{1/r_2},$$

for any $u \in E_V$. By picking $\rho_1 > 0$ such that $r_2\alpha\rho_1^{N/(N-1)} < \alpha^*$, we can use Lemma 4.1, (VQ) and Theorem 1.2 to get

$$\int_{\mathbb{R}^N} Q(x)F(u)dx \leq \varepsilon C_1\|u\|_{E_V}^N + C_2\|u\|_{E_V}^r,$$

whenever $\|u\|_{E_V} \leq \rho_1$. Thus,

$$I_\lambda(u) \geq \|u\|_{E_V}^N\left(\left(\frac{1}{N} - \varepsilon C_1\lambda\right) - C_2\lambda\|u\|_{E_V}^{r-N}\right).$$

By picking $r > N$ and $0 < \varepsilon < 1/(C_1\lambda N)$, we can easily use the above expression to obtain the first statement of the lemma.

In order to prove the second one, we fix $\varphi \in C_0^\infty(\mathbb{R}^N)$ such that $\int_{\mathbb{R}^N} Q(x)|\varphi|^\theta dx > 0$. From (f_2), there exist constants $C_3, C_4 > 0$ such that $F(s) \geq C_3|s|^\theta - C_4$, for any $s \in \mathbb{R}$. So, if we call Ω the support of φ, we obtain

$$I_\lambda(t\varphi) \leq \frac{t^N}{N}\|\varphi\|_{E_V}^N - C_3\lambda t^\theta \int_{\mathbb{R}^N} Q(x)|\varphi|^\theta dx + C_4\lambda\int_\Omega Q(x)dx,$$

for any $t > 0$. Since $\theta > N$, it is suficient to take $e = t\varphi$, with $t > 0$ sufficiently large. The lemma is proved. □

We are ready to prove our existence result.

Proof of Theorem 1.3. According to Lemma 4.3, it is well defined the Mountain Pass level

$$c_\lambda := \inf_{g\in\Gamma}\max_{t\in[0,1]} I_\lambda(g(t)) \geq \tau,$$

where $\Gamma := \{g \in C([0,1], E_V) : g(0) = 0 \text{ and } I_\lambda(g(1)) < 0\}$. We claim that there exists $\lambda_* > 0$ such that, for any $\lambda > \lambda_*$, there holds

$$c_\lambda < \left(\frac{\alpha^*}{\alpha_0}\right)^{N-1}\left(\frac{\nu - N}{N\nu}\right). \tag{4.5}$$

If this is true, we can use Lemma 4.2 and the Mountain Pass Theorem [6] to obtain a nonzero critical point of I_λ.

In order to prove the claim, we notice that there exists $\omega \in E_V$ such that

$$\|\omega\|_{E_V}^N = S_\nu := \inf\left\{\|u\|_{E_V}^N : \int_{\mathbb{R}^N} Q(x)|u|^\nu dx = 1\right\},$$

since the embedding $E_V \hookrightarrow L^\nu(\mathbb{R}^N, Q)$ is compact. From (f_3), we know that $F(s) \geq c_F|s|^\nu$, for all $s \in \mathbb{R}$. Thus, for any $\lambda > S_\nu/(Nc_F)$, we obtain

$$I_\lambda(\omega) \leq \frac{S_\nu}{N} - \lambda c_F \int_{\mathbb{R}^N} Q(x)|\omega|^\nu dx = \frac{S_\nu}{N} - \lambda c_F < 0,$$

and therefore the path $g(t) := t\omega$ belongs to Γ, which implies that

$$c_\lambda \leq \max_{t\geq 0} I_\lambda(t\omega) \leq \max_{t\geq 0}\left(\frac{S_\nu}{N}t^N - \lambda c_F t^\nu\right) = \frac{S_\nu^{\nu/(\nu-N)}}{(\lambda c_F \nu)^{N/(\nu-N)}}\left(\frac{\nu - N}{N\nu}\right).$$

Since the right-hand side above goes to zero as $\lambda \to +\infty$, there exists $\lambda_* > S_\nu/(Nc_F)$ such that (4.5) is verified, for any $\lambda > \lambda_*$. This concludes the proof. □

References

[1] S. Adachi, K. Tanaka, *Trudinger type inequalities in $\mathbb{R}^N$ and their best exponents.* Proc. Amer. Math. Soc. **128** (2000), 2051–2057. 3

[2] F. S. B. Albuquerque, C. O. Alves, E. S. Medeiros, *Nonlinear Schrödinger equation with unbounded or decaying radial potentials involving exponential critical growth in $\mathbb{R}^2$.* J. Math. Anal. Appl. **409** (2014), 1021–1031. 3

[3] C. O. Alves, M. A. S. Souto, *Existence of solutions for a class of nonlinear Schrödinger equations with potential vanishing at infinity.* J. Differential Equations **254** (2013), 1977–1991. 2

[4] A. Ambrosetti, V. Felli, A. Malchiodi, *Ground states of nonlinear Schrödinger equations with potentials vanishing at infinity.* J. Eur. Math. Soc. **7** (2005), 117–144. 2

[5] A. Ambrosetti, A. Malchiodi, D. Ruiz, *Bound states of nonlinear Schrödinger equations with potentials vanishing at infinity.* J. Anal. Math. **98** (2006), 317–348. 2

[6] A. Ambrosetti, P. H. Rabinowitz, *Dual variational methods in critical point theory and applications.* J. Funct. Anal. **14** (1973), 349–381. 4, 13

[7] A. Ambrosetti, Z.-Q. Wang, *Nonlinear Schrödinger equations with vanishing and decaying potentials.* Differential Integral Equations **18** (2005), 1321–1332. 2

[8] S. Aouaoui, R. Jlel, *New weighted sharp Trudinger-Moser inequalities defined on the whole euclidean space $\mathbb{R}^N$ and applications.* Calc. Var. Partial Differential Equations **60** (2021), no. 1, 50. 3

[9] D. Bonheure, J. V. Schaftingen, *Bound state solutions for a class of nonlinear Schrödinger equations.* Rev. Mat. Iberoam. **24** (2008), 297–351. 2

[10] D. M. Cao, *Nontrivial solution of semilinear elliptic equation with critical exponent in $\mathbb{R}^2$.* Comm. Partial Differential Equations **17** (1992), 407–435. 3

[11] J. M. do Ó, *N-Laplacian equations in $\mathbb{R}^N$ with critical growth.* Abstr. Appl. Anal. **2** (1997), 301–315. 3

[12] J. M. do Ó, E. Medeiros, U. Severo, *On a quasilinear nonhomogeneous elliptic equation with critical growth in $\mathbb{R}^N$.* J. Differential Equations **246** (2009), 1363–1386. 3

[13] J. M. do Ó, F. Sani, J. Zhang, *Stationary nonlinear Schrödinger equations in $\mathbb{R}^2$ with potentials vanishing at infinity.* Ann. Mat. Pura Appl. **196** (2017), 363–393. 3

[14] M. F. Furtado, E. S. Medeiros, U. B. Severo, *A Trudinger-Moser inequality in a weighted Sobolev space and applications.* Math. Nachr. **287** (2014), 1255–1273. 3

[15] Q. Han, *Compact embedding results of Sobolev spaces and existence of positive solutions to quasilinear equations.* Bull. Sci. Math. **141** (2017), 46–71. 3

[16] Y. Li, B. Ruf, *A sharp Trudinger-Moser type inequality for unbounded domains in $\mathbb{R}^n$.* Indiana Univ. Math. J. **57** (2008), 451–480. 3

[17] J. Moser, *A sharp form of an inequality by N. Trudinger.* Indiana Univ. Math. J. **20** (1985), 185-201. 3

[18] B. Opic, A. Kufner, *Hardy-Type Inequalities*, Pitman Research Notes in Mathematics Series. **219**. Longman Scientific and Technical, Harlow (1990). 2, 3

[19] B. Ruf, *A sharp Trudinger-Moser type inequality for unbounded domains in $\mathbb{R}^2$.* J. Funct. Anal. **219** (2005), 340–367. 3

[20] J. Simon, *Regularité de la Solution D'Une Equation Non Lineaire Dans $\mathbb{R}^N$.* Lecture Notes in Math, vol. 665, Springer, Heidelberg, (1978). 12

[21] W. Strauss, *Existence of solitary waves in higher dimensions*. Comm. Math. Phys. **55** (1977), 149–162. 2

[22] J. Su, Z. Q. Wang, M. Willem, *Nonlinear Schrödinger equations with unbounded and decaying radial potentials*. Commun. Contemp. Math. **9** (2007), 571–583. 2

[23] J. Su, Z. Q. Wang, M. Willem, *Weighted Sobolev embedding with unbounded and decaying radial potentials*. J. Differential Equations **238** (2007), 201–219 . 2

[24] N. S. Trudinger, *On the imbedding into Orlicz spaces and some applications*. J. Math. Mech. **17** (1967), 473–484. 3

[25] Y. Yang, X. Zhu, *A new proof of subcritical Trudinger-Moser inequalities on the whole Euclidean space*. J. Partial Differ. Equ. **26** (2013), 300–304. 6

[26] V. I. Yudovich, *Some estimates connected with integral operators and with solutions of elliptic equations* Dok. Akad. Nauk SSSR **138** (1961), 804–808, [English translation in Soviet Math. Doklady **2** (1961), 746–749.] 3

[27] Y. Yunyan, *Existence of positive solutions to quasi-linear elliptic equations with exponential growth in the whole Euclidean space*. J. Funct. Anal. **262** (2012), 1679–1704. 11

[28] M. C. Zhu, J. Wang, X. Y. Qian, *Existence of solutions to nonlinear Schrödinger equations involving N-laplacian and potentials vanishing at infinity*. Acta Mathematica Sinica **36** (2020), 1151–1170. 3

Universidade Federal da Paraíba, Departamento de Matemática, 58051-900 João Pessoa-PB, Brazil

Email address: `jonison.mat@gmail.com`

Universidade de Brasília, Departamento de Matemática, 70910-900 Brasília-DF, Brazil

Email address: `mfurtado@unb.br`

Universidade Federal da Paraíba, Departamento de Matemática, 58051-900 João Pessoa-PB, Brazil

Email address: `everaldo@mat.ufpb.br`