ELLIPTIC EQUATIONS WITH CRITICAL AND SUPERCRITICAL GROWTH AT THE BOUNDARY

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Abstract. In this work, we consider two classes of elliptic problems with nonlinear boundary conditions of concave-convex type. In the first problem, we obtain two nonzero and nonnegative solutions when the nonlinear term exhibits critical growth. In the second one, we obtain infinitely many solutions (with no prescribed sign) by assuming that the nonlinearity is even and subcritical near the origin but has no growth condition at infinity.

1. Introduction

Consider $N \geq 3$ and the problem

$$-\Delta v = g(x, v, \nabla v), \text{ in } \mathbb{R}^N_+, \qquad \frac{\partial v}{\partial \nu} = h(x', v), \text{ on } \partial\mathbb{R}^N_+,$$

where $\mathbb{R}^N_+ := \{(x', x_N) : x' \in \mathbb{R}^{N-1}, x_N > 0\}$ is the upper half-space and ν is the outward normal vector at the boundary $\partial\mathbb{R}^N_+$. The authors in [20] considered

$$g(x, v, \nabla v) = \mu v + \frac{1}{2}(x \cdot \nabla v), \qquad h(v) = |v|^{q-2}v,$$

with $2 < q < 2_* := 2(N-1)/(N-2)$. Besides obtaining the existence of solutions for certain values of $\mu > 0$, they presented the relationship between the problem and the existence of self-similar solutions of the nonlinear heat equation

$$w_t - \Delta w = 0, \text{ in } \mathbb{R}^N_+ \times (0, +\infty), \qquad \frac{\partial w}{\partial \nu} = |w|^{q-2}w, \text{ on } \mathbb{R}^{N-1} \times (0, +\infty).$$

For other appropriate choices of functions g and f, the problem models contexts such as glaciology [29], population genetics [4], non-Newtonian fluid mechanics [13], nonlinear elasticity [11], among others. From a mathematical perspective, it is also related to the study of sharp constants in Sobolev trace inequalities [15, 12] as well as the conformal deformation of Riemannian manifolds [16, 17].

In a recent paper, Furtado and Silva [22] considered

$$g(x, v, \nabla v) = \frac{1}{2}(x \cdot \nabla v), \quad h(x', v) = \mu|v|^{q-2}v + |v|^{2_*-2}v,$$

with $2 \leq q < 2_*$, and obtained the existence of a nonnegative nonzero solution in two cases: $2 < q < 2_*$, $\mu > 0$; and $q = 2$, $\mu \in (0, \mu_1)$, where $\mu_1 > 0$ is the first positive eigenvalue of a linear associated problem. In view of these results, it is natural to ask what happens in the sublinear case $1 < q < 2$. In the first part of

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this paper, we give a partial answer to this question. More specifically, we deal with the problem

$$(P_1) \qquad \begin{cases} -\Delta u - \dfrac{1}{2}(x\cdot\nabla u) = 0, & \text{in } \mathbb{R}^N_+,\\ \dfrac{\partial u}{\partial \nu} = \mu a(x')|u|^{q-2}u + b(x')|u|^{2_*-2}u, & \text{on } \mathbb{R}^{N-1}, \end{cases}$$

where $1<q<2$, $\mu>0$ is a real parameter and the potentials a, b are indefinite in sign and satisfy some mild conditions.

As it will be explained in the next section, the weak solutions of problem (P_1) belongs to the space X defined as the closure of $C_0^\infty(\overline{\mathbb{R}^N_+})$ with respect to the norm

$$\|u\| = \left(\int_{\mathbb{R}^N_+} K(x)|\nabla u|^2 dx\right)^{1/2},$$

where $K(x) := e^{|x|^2/4}$. This kind of space was first introduced by Escobedo and Kavian [18], who considered a problem in the whole space $\mathbb{R}^N$. The main point here is that this functional space is embedded into

$$L^r_K := \left\{ u \in L^1_{\text{loc}}(\mathbb{R}^{N-1}) : \|u\|_r := \left(\int_{\mathbb{R}^{N-1}} K(x',0)|u|^r dx'\right)^{1/r} < \infty \right\},$$

for any $r\in[2,2_*]$. With this space in hands, we are able to present the assumptions on the sign-changing potentials a and b.

In what follows, we denote by $s'>1$ the conjugated exponent of $s>1$, that is, $1/s+1/s'=1$. We also define

$$\Omega_a^+ := \{x'\in\mathbb{R}^{N-1} : a(x')>0\}, \quad \Omega_b^+ := \{x'\in\mathbb{R}^{N-1} : b(x')>0\}$$

and assume the following:

- (a_1) $a\in L_K^{\sigma_q}\cap L^{N-1}_{\text{loc}}(\mathbb{R}^{N-1})$, where $(2_*/q)'<\sigma_q\le(2/q)'$;
- (b_1) $b\in L^\infty(\mathbb{R}^{N-1})$;
- (ab) there exist $\delta>0$, $b_0>0$ and $\gamma>N-1$ such that
$$B'_\delta := \{x'\in\mathbb{R}^{N-1} : |x'|<\delta\} \subset \left(\Omega_a^+\cap\Omega_b^+\right)$$
and
$$\|b\|_{L^\infty(\mathbb{R}^{N-1})} \le b(x') + b_0|x'|^\gamma, \quad \text{a.e. in } B'_\delta.$$

We shall to prove the following:

Theorem 1.1. *Suppose that $N\ge4$ and the functions a and b verify (a_1), (b_1) and (ab). Then, there exists $\mu^*>0$ such that, for any $\mu\in(0,\mu^*)$, the problem (P_1) has at least two nontrivial and nonnegative weak solutions.*

The first solution will be obtained with a minimization argument, while the second one requires more challenging arguments, as the trace embedding we are going to use fails to be compact. The main point to overcome this difficulty comes from fine estimates of a modification of the *instanton functions* exploited by Escobar [15] and Beckner [9]. We notice that the one of the solutions above can be obtained even if $N=3$.

It is worth noticing that the energy functional of problem (P_1) is clearly even. Hence, as in Bartsch and Willem [8], it is expected that we could obtain infinitely many solutions (with no prescribed sign). In our secon result, inspired by the papers

[33, 26], we replace the critical term $b(x')|u|^{2_*-2}u$ by a general function f which is even near the origin. More specifically, we consider the problem

$$(P_2) \qquad \begin{cases} -\Delta u - \dfrac{1}{2}(x\cdot\nabla u) = 0, & \text{in } \mathbb{R}^N_+, \\ \dfrac{\partial u}{\partial \nu} = \mu a(x')|u|^{q-2}u + f(u), & \text{on } \mathbb{R}^{N-1}, \end{cases}$$

where $\mu > 0$ and $1 < q < 2$ are as before. Concerning the potential a and the nonlineairity $f : \mathbb{R} \to \mathbb{R}$, we suppose that

(f_1) $f \in C(\mathbb{R}, \mathbb{R})$;

(f_2) there exists $p \in (2, 2_*)$ such that

$$\lim_{s\to 0} \frac{f(s)}{|s|^{p-1}} = 0;$$

$(\widetilde{a_1})$ $a \in L^{\sigma_q}_K \cap L^\infty(\mathbb{R}^{N-1})$, where $(p/q)' < \sigma_q \le (2/q)'$;

(a_2) Ω^+_a has an interior point.

According to Proposition 2.1, we have the continuous trace embedding $X \hookrightarrow L^{2_*}_K$. So, it is well defined

$$S_{2_*} := \inf\left\{ \int_{\mathbb{R}^N_+} K(x)|\nabla u|^2 dx \, : \, u \in X, \ \|u\|_{2_*} = 1 \right\}$$

and we can state our second main result as follows:

Theorem 1.2. *Suppose that a and f verify $(\widetilde{a_1})$, (a_2), $(f_1) - (f_2)$ and f is odd in the interval $[-C_{N,p}, C_{N,p}]$, where*

$$(1.1) \qquad C_{N,p} := \max\left\{1, 2S^{-1}_{2_*}\right\}^{1/(2_*-p)} \left(\frac{2_* + 2 - p}{2}\right)^{2(2_*+2-p)/(2_*-p)^2}.$$

Then, there exists $\bar{\mu} > 0$ such that, for any $\mu \in (0, \bar{\mu})$, the problem (P_2) has infinitely many weak solutions.

The proof is also variational, but it presents a significant challenge that needs to be overcome. If $F(s) := \int_0^s f(t)dt$, the formal energy functional associated to the problem (P_2) contains the therm $\int F(u)dx'$, which could be infinite, since we have no control on the behaviour of f at infinity. Even in the definition of weak solution, we need to take this account in mind and consider solutions in the distributional sense. To overcome this difficult, we adopt ideas from the papers [5, 33, 26, 7]. This involves applying a truncation to the function f in such a way that the new (truncated) functional becomes well-defined and coercive in an appropriated Sobolev-type space. After demonstrating the existence of infinitely many critical points for the trucanted functional, we apply a Moser iteration technique [27] to prove that, if $\mu > 0$ is small, they have small L^∞-norm at the boudary and therefore they are solutions to the original problem.

It is worth mentioning some valid examples for the nonlinearity f. Besides the classical example $f(s) = |s|^{r-2}s$, with $r > 2$, we may pick $f(s) = |s|^{r-2}se^{s^2}$ with $r > 2$, which has exponential growth. Actually, there is no growth restriction for large values of $|s|$.

Due to the asymmetric nature of the nonlinear boundary term, our problems fall into a broader class known as concave-convex type problems. With the aim of presenting a historical perspective, we may consider

$$-\Delta u = g(x,u), \text{ in } \Omega, \qquad \alpha_1 u + \alpha_2 \frac{\partial u}{\partial \nu} = h(x,u), \text{ on } \partial\Omega,$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 3$, is a bounded domain. In their seminal paper, Ambrosetti, Brezis, and Cerami [3] obtained two positive solutions when $\alpha_2 = 0$, $h \equiv 0$, and $g(x,s) = \mu s^{q-1} + s^{p-1}$, with $1 < q < 2 < p \leq 2^*$, and $\mu > 0$ is small. In [19], de Figueiredo, Gossez, and Ubilla generalized these results by considering $g(x,s) = \mu c(x)|s|^{q-2}s + d(x)|s|^{p-2}s$, with c and d having no constant sign. In this setting of Dirichlet boundary conditions, we can refer to [2, 28, 34], and references therein.

For the Neumann case, when $\alpha_1 = 0$, we can cite the paper of Azorezo, Peral, and Rossi [6], who consider $g(x,s) = |s|^{p-2}s - s$, $h(x,s) = \mu|s|^{q-2}s$ and have obtained results similar to those of [3]. In [23], the authors considered $g \equiv 0$ and $h(x,s) = \mu c(x)|s|^{q-2}s + d(x)|s|^{p-2}s$, with $1 < q < 2 < p < 2_*$. The bounded potentials c and d are supposed to verify the sign conditions $\int_{\partial\Omega} c(x) d\sigma_x < 0$, $\int_{\partial\Omega} d(x) d\sigma_x \neq 0$. Under some other technical conditions, they obtained two positive solutions if $\mu > 0$ is small. Some other results concerning the existence of infinitely many solutions can be found in [33, 32, 24, 26, 7], and their references.

The two results proved in this paper extend and/or complement the aforementioned works in several ways: we consider a different operator, sign-changing potentials, the upper half-space, and local symmetry conditions.

The paper contains four more sections: in the next one, we obtain the first solution of problem (P_1). The second solution is obtained in Section 3. In Section 4, we deal with a modified problem which is related with problem (P_2). In the final section, we prove that the solutions of the modified problem solve the original problem.

2. The first solution in Theorem 1.1

In this section, we start the proof of our first main theorem. In the next two sections, we assume that (a_1), (b_1) and (ab) hold. Moreover, along all the paper, we write $\|v\|_\infty$ for the $L^\infty(\mathbb{R}^{N-1})$ norm of an essentially bounded function v defined a.e. in $\mathbb{R}^{N-1}$.

As quoted in the introduction, we shall consider the weigth

$$K(x) := e^{|x|^2/4}, \quad x \in \mathbb{R}^N.$$

A formal computation shows that, if u is a smooth function, then

$$\operatorname{div}(K(x)\nabla u) = K(x)\left[\Delta u + \frac{1}{2}(x \cdot \nabla u)\right]$$

and therefore, for solving (P_1) with a variational approach, it is natural to consider the space X defined as the closure of $C_c^\infty(\overline{\mathbb{R}^N_+})$ with respect to the norm

$$\|u\| := \left(\int_{\mathbb{R}^N_+} K(x)|\nabla u|^2 dx\right)^{1/2}.$$

We also define, for $r \in [2, 2_*]$, the weighted Lebesgue space

$$L_K^r := \left\{ u \in L^1_{\rm loc}(\mathbb{R}^{N-1}) : \|u\|_r^r := \int_{\mathbb{R}^{N-1}} K(x',0)|u|^r dx' < +\infty \right\}.$$

The following abstract result was proved in [21, Theorem 1.1] (cf. also [20]).

Proposition 2.1. *For any $r \in [2, 2_*]$, we have that*

$$S_r := \inf_{u \in X \setminus \{0\}} \frac{\int_{\mathbb{R}^N_+} K(x)|\nabla u|^2 dx}{\left(\int_{\mathbb{R}^{N-1}} K(x',0)|u|^r dx'\right)^{2/r}} < +\infty \tag{2.1}$$

and therefore there holds the continuous Sobolev trace embedding $X \hookrightarrow L_K^r$. Moreover, the embedding is compact if $r \in [2, 2_*)$.

After multiplying the first equation in (P_1) by K, we reach the energy functional associated with the problem, namely

$$I(u) := \frac{1}{2}\|u\|^2 - \frac{\mu}{q} \int_{\mathbb{R}^{N-1}} K(x',0)a(x')(u^+)^q dx' - \frac{1}{2_*} \int_{\mathbb{R}^{N-1}} K(x',0)b(x')(u^+)^{2_*} dx'.$$

Standard calculations show that I is well defined, $I \in C^1(X, \mathbb{R})$ and its critical points are precisely the weak solutions of the problem.

We start with the following regularity result:

Lemma 2.2. *If $u \in X$ is a critical point of I, then $u \geq 0$ a.e. in $\mathbb{R}^N_+$. Moreover, if (a_1) and (b_1) hold, then $u \in L^\nu_{\rm loc}(\mathbb{R}^N_+) \cap L^\nu_{\rm loc}(\mathbb{R}^{N-1})$, for any $\nu \geq 1$.*

Proof. Let $u^+ := \max\{u, 0\}$ and $u^- := u^+ - u$ be the positive and negative part of u, respectively. Since $u^+u^- = 0$ a.e. in $\mathbb{R}^N_+$, a simple computation shows that $0 = I'(u)u^- = -\|u^-\|^2$. This proves that $u = u^+ \geq 0$, as stated.

For the regularity, we first notice that $v := K^{1/2}u \in W^{1,2}_{\rm loc}(\mathbb{R}^N_+)$ is a weak solution to the problem

$$\begin{cases} -\Delta v = g_1(x, v), & \text{in} \quad \mathbb{R}^N_+ \\ \dfrac{\partial v}{\partial \nu} = g_2(x', v), & \text{on} \quad \mathbb{R}^{N-1} \end{cases},$$

where $g_1 : \mathbb{R}^N_+ \times \mathbb{R} \to \mathbb{R}$ and $g_2 : \mathbb{R}^{N-1} \times \mathbb{R} \to \mathbb{R}$ are given by

$$g_1(x, s) := -\left(\frac{|x|^2 + 4N}{16}\right)s$$

and

$$g_2(x', s) := a(x')e^{(2-q)|x'|^2/8}|s|^{q-2}s + b(x')e^{(2-2_*)|x'|^2/8}|v(x',0)|^{2_*-2}s.$$

By setting

$$L_1(x) := \left(\frac{|x|^2 + 4N}{16}\right), \quad L_2(x') := |a(x')|e^{(2-q)|x'|^2/8} + \|b\|_\infty |v(x',0)|^{2_*-2},$$

we have, for any $x \in \mathbb{R}^N_+$, $x' \in \mathbb{R}^{N-1}$ and $s \in \mathbb{R}$,

$$|g_1(x, s)| \leq L_1(x)(1 + |s|), \qquad |g_2(x', s)| \leq L_2(x')(1 + |s|).$$

We know that $L_1 \in L^{N/2}_{\rm loc}(\mathbb{R}^N_+)$ and $L_2 \in L^{N-1}_{\rm loc}(\mathbb{R}^{N-1})$, this last one due to (a_1) and $v \in L^{2_*}_{\rm loc}(\mathbb{R}^{N-1})$. So, we may apply [1, Lemma 4.1] to conclude that $u \in L^\nu_{\rm loc}(\mathbb{R}^N_+) \cap L^\nu_{\rm loc}(\mathbb{R}^{N-1})$, for any $\nu \geq 1$. □

In the first part of the proof of Theorem 1.1, we are going to use a minimization argument to obtain a solution u_μ with negative energy. So, we need to prove the following:

Lemma 2.3. *There exist $\mu^* = \mu^*(q, \|a\|_{\sigma_q}, \|b\|_\infty) > 0$, $\rho = \rho(q, \|b\|_\infty) > 0$ and $\alpha = \alpha(\rho) > 0$ such that, for any $\mu \in (0, \mu^*)$, there holds*

$$I(u) \geq \alpha, \quad \forall\, u \in X \cap \partial B_\rho(0).$$

Proof. It follows from Hölder's inequality, (a_1) and (2.1) that

$$\int_{\mathbb{R}^{N-1}} K(x', 0) a(x')(u^+)^q dx' \leq \|a\|_{\sigma_q} \|u^+\|^q_{q\sigma_q'} \leq S^{-q/2}_{q\sigma_q'} \|a\|_{\sigma_q} \|u\|^q.$$

This and (2.1) again imply that

$$I(u) \geq \frac{1}{2}\|u\|^q \left[\|u\|^{2-q} - \mu \frac{2}{q} S^{-q/2}_{q\sigma_q'} \|a\|_{\sigma_q} - C_1 \|u\|^{p-q} \right],$$

where $C_1 := (2/2_*) S_{2_*}^{-2_*/2} \|b\|_\infty > 0$.

A simple computation shows that the function $h(t) = t^{2-q} - C_1 t^{2_*-q}$, for $t > 0$, achieves its global maximum at

$$\rho := \left[\frac{2-q}{C_1(2_* - q)} \right]^{1/(2_*-2)} > 0.$$

So, if we set $C_2 := h(\rho)$, we have that

$$I(u) \geq \frac{1}{2}\rho^q \left[C_2 - \mu \frac{2}{q} S^{-q/2}_{q\sigma_q'} \|a\|_{\sigma_q} \right] \geq \frac{C_2 \rho^q}{4} =: \alpha > 0,$$

whenever $\|u\| = \rho$ and

$$0 < \mu < \mu^* := \frac{q C_2}{4 \|a\|_{\sigma_q}} S^{q/2}_{q\sigma_q'}.$$

The lemma is proved. □

The next resul provides a first solution for the problem (P_1).

Proposition 2.4. *Let μ^* and $\rho > 0$ as in Lemma 2.3. Then, for any $\mu \in (0, \mu^*)$, the infimum*

$$c_\mu := \inf_{u \in \overline{B_\rho(0)}} I(u) < 0$$

is achivied at a nonnegative critical point $u_\mu \in B_\rho(0)$.

Proof. Using (a_1), (b_1) and Proposition 2.1, we can check $c_\mu > -\infty$. Let $\delta > 0$ be given by condition (ab) and $\varphi \in C_0^\infty(B_\delta(0))$ be a nonnegative function such that $\varphi \equiv 1$ in $B_{\delta/2}(0)$. Since $B_\delta(0) \cap \partial\mathbb{R}^N_+ \subset \Omega_a^+$, we obtain

$$\int_{\mathbb{R}^{N-1}} K(x') a(x') \varphi^q dx' \geq \int_{B_{\delta_1/2}(0) \cap \partial\mathbb{R}^N_+} K(x') a(x') dx' > 0,$$

from which it follows that

$$\limsup_{t \to 0^+} \frac{I(t\varphi)}{t^q} \leq -\frac{\mu}{q} \int_{\mathbb{R}^{N-1}} K(x') a(x') \varphi^q dx' < 0,$$

and therefore $I(t\varphi) < 0$, for any $t > 0$ small. This proves that $c_\mu < 0$.

Let $(u_n) \subset \overline{B_\rho(0)}$ be such that $I(u_n) \to c_\mu$. Since (u_n) is bounded, there exists $u_\mu \in X$ such that $u_n \rightharpoonup u_\mu$ weakly in X, stronlgy in L^r_K, for any $r \in [2, 2_*)$, and $u_n(x',0) \to u_\mu(x',0)$ a.e. in $\mathbb{R}^{N-1}$. Moreover, by Lemma 2.3, we have that $(u_n) \subset B_\rho(0)$, for any $n \geq n_0$. So, we may use Ekeland's Variational Principle [14] to assume that $I'(u_n) \to 0$, as $n \to +\infty$.

We are going to show that $I'(u_\mu) = 0$. Since $\sigma_q > (2_*/q)'$, we have that $1/\sigma_q + (q-1)/2_* < 1$. Hence, there exist $r \in (2, 2_*)$ and $\tau > 1$ such that

$$\frac{1}{\sigma_q} + \frac{q-1}{r} + \frac{1}{\tau} = 1.$$

From the strong convergence in L^r_K, we obtain $\eta_0 \in L^r_K$ such that $|u_n(x',0)| \leq \eta_0(x')$ for a.e. $x' \in \mathbb{R}^{N-1}$. Thus, for any $v \in C_0^\infty(\overline{\mathbb{R}^N_+})$, we can use Young's inequality to get,

$$|Ka(u_n^+)^{q-1}v| \leq K\left[\frac{|a|^{\sigma_q}}{\sigma_q} + \frac{q-1}{r}|\eta_0|^r + \frac{|v|^\tau}{\tau}\right] \quad \text{a.e. in } \mathbb{R}^{N-1}.$$

Since v has compact support, the right-hand side above belongs to $L^1(\mathbb{R}^{N-1})$. It follows from Lebesgue's Theorem that

$$\lim_{n\to+\infty} \int_{\mathbb{R}^{N-1}} K(x',0)a(x')(u_n^+)^{q-1}v\,dx' = \int_{\mathbb{R}^{N-1}} K(x',0)a(x')(u_\mu^+)^{q-1}v\,dx'.$$

By using $b \in L^\infty(\mathbb{R}^{N-1})$ and an easier argument, we may check that

$$\lim_{n\to+\infty} \int_{\mathbb{R}^{N-1}} K(x',0)b(x')(u_n^+)^{2_*-1}v\,dx' = \int_{\mathbb{R}^{N-1}} K(x',0)b(x')(u_\mu^+)^{2_*-1}v\,dx'.$$

Combining the last two convergences and the weak convergence, we conclude that

$$0 = \lim_{n\to+\infty} I'(u_n)v = I'(u_\mu)v, \quad \forall\, v \in C_0^\infty(\overline{\mathbb{R}^N_+})$$

and it follows by density that $I'(u_\mu) = 0$. By Lemma 2.2, we know that u_μ is nonnegative.

Since $q\sigma_q' \in [2, 2_*)$, there exists $\eta_1 \in L_K^{q\sigma_q'}$ such that $|u_n(x',0)| \leq \eta_1(x')$ a.e. in $\mathbb{R}^{N-1}$. So, we can use Young's inequality and Lebesgue's Theorem as before to conclude that

$$\lim_{n\to+\infty} \int_{\mathbb{R}^{N-1}} K(x')a(x')(u_n^+)^q dx' = \int_{\mathbb{R}^{N-1}} K(x')a(x')(u_\mu^+)^q dx'.$$

Thus,

$$\begin{aligned}
c_\mu &= \liminf_{n\to+\infty} \left[I(u_n) - \frac{1}{2_*}I'(u_n)u_n\right] \\
&= \liminf_{n\to+\infty} \left[\left(\frac{1}{2} - \frac{1}{2_*}\right)||u_n||^2 + \left(\frac{1}{2_*} - \frac{1}{q}\right)\mu \int_{\mathbb{R}^{N-1}} K(x',0)a(x')(u_n^+)^q dx'\right] \\
&\geq \left(\frac{1}{2} - \frac{1}{2_*}\right)||u_\mu||^2 + \left(\frac{1}{2_*} - \frac{1}{q}\right)\mu \int_{\mathbb{R}^{N-1}} K(x',0)a(x')(u_\mu^+)^q dx' \\
&= I(u_\mu) - \frac{1}{2_*}I'(u_\mu)u_\mu = I(u_\mu).
\end{aligned}$$

Since we already know that $I(u_\mu) \geq c_\mu$, we conclude that $I(u_\mu) = c_\mu$. □

3. The second solution in Theorem 1.1

In order to obtain a second solution, we adapt arguments from [24]. Given $c \in \mathbb{R}$, we recall that I satisfies the Palais-Smale condition at level c ($(PS)_c$ for short) if any sequence $(u_n) \subset X$ such that $I(u_n) \to c$ and $I'(u_n) \to 0$ has a convergent subsequence.

Lemma 3.1. *Suppose that u_μ given by Lemma 2.3 is the only nonzero critical point of I. Then I satisfies the $(PS)_c$ condition at any level*

$$c < \overline{c} := I(u_\mu) + \frac{1}{2(N-1)} \frac{1}{\|b\|_\infty^{N-2}} S_{2_*}^{N-1}.$$

Proof. Let (u_n) be such that $I(u_n) \to c < \overline{c}$ and $I'(u_n) \to 0$. By Hölder's inequality,

$$\begin{aligned} c + o_n(1)\|u_n\| &= I(u_n) - \frac{1}{2_*} I'(u_n)u_n \\ &\geq \left(\frac{1}{2} - \frac{1}{2_*}\right)\|u_n\|^2 - \left(\frac{1}{q} - \frac{1}{2_*}\right)\mu S_{q\sigma_q'}^{-q/2} \|a\|_{\sigma_q} \|u_n\|^2, \end{aligned}$$

and therefore (u_n) is bounded. So, there exists $u \in X$ such that $u_n \rightharpoonup u$ weakly in X. Arguing as in the proof of Proposition 2.4, we can verify that

$$\lim_{n\to+\infty} \int_{\mathbb{R}^{N-1}} K(x',0)a(x')(u_n^+)^q dx' = \int_{\mathbb{R}^{N-1}} K(x',0)a(x')(u^+)^q dx'.$$

Setting $v_n := u_n - u$ and applying Brezis-Lieb's Lemma, we get

$$0 = I'(u_n)u_n = I'(u)u + \|v_n\|^2 - \int_{\mathbb{R}^{N-1}} K(x',0)b(x')(v_n^+)^{2_*} dx' + o_n(1).$$

As in the proof of Proposition 2.4, we have that $I'(u) = 0$. Thus, there exists $l \geq 0$ such that

$$\lim_{n\to+\infty} \|v_n\|^2 = l = \lim_{n\to+\infty} \int_{\mathbb{R}^{N-1}} K(x',0)b(x')(v_n^+)^{2_*} dx'.$$

Using the trace embedding $X \hookrightarrow L_K^{2_*}(\mathbb{R}^{N-1})$, we deduce

$$\int_{\mathbb{R}^{N-1}} K(x')b(x')(v_n^+)^{2_*} dx' \leq \|b\|_\infty S_{2_*}^{-2_*/2} \left(\int_{\mathbb{R}^N_+} K(x)|\nabla v_n|^2 dx \right)^{2_*/2}.$$

If $l > 0$, we can take the above expression to the limit to get

$$l \geq \frac{1}{\|b\|_\infty^{N-2}} S_{2_*}^{N-1}. \tag{3.1}$$

On the other hand,

$$c + o_n(1) = I(u_n) = I(u) + \frac{1}{2}\|v_n\|^2 - \frac{1}{2_*} \int_{\mathbb{R}^{N-1}} K(x',0)b(x')(v_n^+)^{2_*} dx'.$$

Since $I'(u) = 0$, it follows that $u \in \{0, u_\mu\}$, and therefore $I(u) \geq I(u_\mu)$. Thus, taking the limit in the above equality and using (3.1), we obtain

$$c \geq I(u) + \frac{1}{2(N-1)} \frac{1}{\|b\|_\infty^{N-2}} S_{2_*}^{N-1} \geq \overline{c},$$

which is a contraditcion. Hence, $l = 0$ and we have that

$$\lim_{n\to+\infty} \|u_n - u\|^2 = \lim_{n\to+\infty} \|v_n\|^2 = l = 0,$$

which proves that $u_n \to u$ strongly in X. □

Define, for each $\epsilon > 0$, the function

$$U_\epsilon(x', x_N) := \left(\frac{\epsilon}{|x'|^2 + (x_N + \epsilon)^2}\right)^{(N-2)/2}, \quad (x', x_N) \in \mathbb{R}^N_+.$$

The family $\{U_\epsilon\}_{\epsilon>0}$ consists of exactly the functions achieving the best constant of the Sobolev trace embedding $D^{1,2}(\mathbb{R}^N_+) \hookrightarrow L^{2_*}(\mathbb{R}^{N-1})$ (see [15] for more details). Now, consider

$$\psi_\epsilon(x) := K(x)^{-1/2}\varphi(x)U_\epsilon(x), \quad x \in \overline{\mathbb{R}^N_+}, \tag{3.2}$$

where $\varphi \in C_c^\infty(\overline{\mathbb{R}^N_+})$ is such that $0 \le \varphi \le 1$, $\varphi \equiv 1$ in $B_{\delta/2}(0) \cap \overline{\mathbb{R}^N_+}$, $\varphi \equiv 0$ outside $B_\delta(0) \cap \overline{\mathbb{R}^N_+}$, and $\delta > 0$ is given by the condition (ab).

If we now define

$$A_N := \int_{\mathbb{R}^N_+} |\nabla U_\epsilon|^2 dx, \quad B_N := \left(\int_{\mathbb{R}^{N-1}} |U_\epsilon|^{2_*} dx'\right)^{2/2_*},$$

it was proved in [22, Lemma 2.2] that $A_N/B_N = S_{2_*}$. Moreover, as $\epsilon \to 0^+$,

$$\|\psi_\epsilon\|^2 = A_N + \begin{cases} O(\epsilon^2|\ln\epsilon|), & \text{if } N = 4, \\ O(\epsilon^2), & \text{if } N \ge 5, \end{cases} \quad \text{and} \quad |\psi_\epsilon|_{2_*}^{2_*} = B_N^{2_*/2} + O(\epsilon^2).$$

The next result will be used to accurately determine the minimax level of the functional associated with problem (P_1).

Lemma 3.2. *Suppose that $N \ge 4$ and set*

$$v_\epsilon := \frac{\psi_\epsilon}{|\psi_\epsilon|_{2_*}},$$

where ψ_ϵ was defined in (3.2). Then, as $\epsilon \to 0^+$, we have that

$$\|v_\epsilon\|^{2(N-1)} = S_{2_*}^{N-1} + \begin{cases} O(\epsilon^2|\ln\epsilon|), & \text{if } N = 4, \\ O(\epsilon^2), & \text{if } N \ge 5, \end{cases}$$

and

$$|v_\epsilon|_s^s = O\left(\epsilon^{N-1-s(N-2)/2}\right),$$

for any $2_*/2 < s < 2_*$.

Proof. If $N \ge 5$, we can use the definition of v_ϵ, the Mean Value Theorem and the above expression to get

$$\begin{aligned} \|v_\epsilon\|^{2(N-1)} &= \frac{\|\psi_\epsilon\|^{2(N-1)}}{|\psi_\epsilon|_{2_*}^{2(N-1)}} = \frac{[A_N + O(\epsilon^2)]^{N-1}}{[B_N^{2_*/2} + O(\epsilon^2)]^{N-2}} = \frac{A_N^{N-1} + O(\epsilon^2)}{B_N^{2_*(N-2)/2} + O(\epsilon^2)} \\ &= \frac{A_N^{N-1} + O(\epsilon^2)}{B_N^{N-1} + O(\epsilon^2)} = \left(\frac{A_N}{B_N}\right)^{N-1} + O(\epsilon^2) = S_{2_*}^{N-1} + O(\epsilon^2). \end{aligned}$$

For the case $N = 4$, we can compute analogously

$$\|v_\epsilon\|^{2(N-1)} = \frac{A_N^{N-1} + O(\epsilon^2|\ln\epsilon|)}{B_N^{N-1} + O(\epsilon^2)} = \frac{A_N^{N-1} + O(\epsilon^2|\ln\epsilon|)}{B_N^{N-1} + O(\epsilon^2|\ln\epsilon|)} = S_{2_*}^{N-1} + O(\epsilon^2|\ln\epsilon|).$$

We now prove the estimative for $\|v_\epsilon\|_s^s$. Since $0 \le \varphi \le 1$ and φ vanishes outside a ball, we can use the change of variables $x' = \epsilon y'$ to obtain $C_1 > 0$ such that

$$\begin{aligned} \|\psi_\epsilon\|_s^s &\le C_1 \int_{\{|x'|\le\delta\}} \left[\frac{\epsilon}{|x'|^2+\epsilon^2}\right]^{s(N-2)/2} dx' \\ &\le C_1 \epsilon^{N-1-s(N-2)/2} \int_{\{|y'|\le\delta/\epsilon\}} \left[\frac{1}{|y'|^2+1}\right]^{s(N-2)/2} dy' \\ &\le C_1 \epsilon^{N-1-s(N-2)/2} \left[C_2 + \int_{\{|y'|\ge 1\}} |y'|^{-s(N-2)} dy'\right]. \end{aligned}$$

The term into the brackets above is finite if, and only if, $s > (N-1)/(N-2)$, which is according with our hypothesis. Thus, the last statement follows from the above expression and $\|\psi_\epsilon\|_{2_*}^s = B_N^{s/2} + o(1)$. □

Lemma 3.3. *Suppose that $N \ge 4$. For each $\epsilon > 0$, let $v_\epsilon \in X$ be given by Lemma 3.2 and define $t_\epsilon > 0$ as*

$$m_\epsilon := I(u_\mu + t_\epsilon v_\epsilon) = \max_{t\ge 0} I(u_\mu + t v_\epsilon),$$

where $u_\mu \in X$ is the local mimimun given by Proposition 2.4. Then (t_ϵ) remains bounded as $\epsilon \to 0^+$.

Proof. For each $\epsilon > 0$, set $h_\epsilon(t) := I(u_\mu + t v_\epsilon)$, for $t > 0$. It is easy to show that h_ϵ achieves its maximun at some $t_\epsilon > 0$. Suppose, by contradiction, that $t_{\epsilon_n} \to +\infty$, along some sequence $\epsilon_n \to 0^+$. By combining $0 = h'_{\epsilon_n}(t_{\epsilon_n}) = I'(u_\mu + t_{\epsilon_n} v_{\epsilon_n}) v_{\epsilon_n}$ with $I'(u_\mu) v_{\epsilon_n} = 0$, and recalling that a and b are positive in the support of v_{ϵ_n}, we obtain

$$\int_{\mathbb{R}^{N-1}} K(x',0) b(x') v_{\epsilon_n}^{2_*} dx' \le t_{\epsilon_n}^{2-2_*} \|v_{\epsilon_n}\|^2 + t_{\epsilon_n}^{1-2_*} \int_{\mathbb{R}^{N-1}} K(x',0) b(x') u_\mu^{2_*-1} v_{\epsilon_n} dx'.$$

It follows from Hölder's inequality, $\|v_{\epsilon_n}\|_{2_*} = 1$ and Lemma 3.2 that

$$\lim_{n\to+\infty} \int_{\mathbb{R}^{N-1}} K(x',0) b(x') v_{\epsilon_n}^{2_*} dx' = 0.$$

On the other hand, by condition (ab) we have that

$$b(x') \ge \|b\|_\infty - b_0 |x'|^\gamma, \quad \text{a.e. in } \{|x'| \le \delta\},$$

with $b_0 > 0$. Consequently,

$$o_n(1) = \int_{\mathbb{R}^{N-1}} K(x',0) b(x') v_{\epsilon_n}^{2_*} dx' \ge \|b\|_\infty - b_0 \int_{\mathbb{R}^{N-1}} K(x',0) |x'|^\gamma v_{\epsilon_n}^{2_*} dx'. \tag{3.3}$$

Since $\|\psi_{\epsilon_n}\|_{2_*}^{2_*} = B_N^{2_*/2} + o(1)$, we obtain

$$\begin{aligned} \int_{\mathbb{R}^{N-1}} K(x',0) |x'|^\gamma v_{\epsilon_n}^{2_*} dx' &\le \frac{C_1}{\|\psi_{\epsilon_n}\|_{2_*}^{2_*}} \int_{\{|x'|\le\delta\}} \frac{\epsilon_n^{N-1} |x'|^\gamma}{[|x'|^2 + \epsilon_n^2]^{N-1}} dx' \\ &\le O(\epsilon_n^{N-1}) \int_{\{|x'|\le\delta\}} |x'|^{\gamma-2(N-1)} dx' = O(\epsilon_n^{N-1}), \end{aligned}$$

where we have used $\gamma > N-1$ in the last equality above. It follows from (3.3) that $\|b\|_\infty = 0$, which contradicts $\Omega_a^+ \ne \varnothing$. □

The next lemma is crucial for the completion of the proof of Theorem 1.1.

Lemma 3.4. *If $N \geq 4$ then, for any $\epsilon > 0$ small, the number m_ϵ defined in Lemma 3.3 verifies $m_\epsilon < \bar{c}$.*

Proof. By using $I'(u_\mu)v_\epsilon = 0$, we obtain

$$m_\epsilon = I(u_\mu) + \frac{t_\epsilon^2}{2}\|v_\epsilon\|^2 - \frac{\mu}{q}\Gamma_{1,\epsilon} - \frac{1}{2_*}\Gamma_{2,\epsilon}, \tag{3.4}$$

where

$$\Gamma_{1,\epsilon} := \int_{\mathbb{R}^{N-1}} K(x',0)a(x')\left[(u_\mu + t_\epsilon v_\epsilon)^q - u_\mu^q - t_\epsilon q u_\mu^{q-1} v_\epsilon\right] dx',$$

and

$$\Gamma_{2,\epsilon} := \int_{\mathbb{R}^{N-1}} K(x',0)b(x')\left[(u_\mu + t_\epsilon v_\epsilon)^{2_*} - u_\mu^{2_*} - t_\epsilon 2_* u_\mu^{2_*-1} v_\epsilon\right] dx'.$$

The Mean Value Theorem and the positivity of a in the support of v_ϵ imply that $\Gamma_{1,\epsilon} \geq 0$. Moreover, for any $r, s \geq 0$ and $\sigma \in (1, 2_* - 1)$, there exists $C_\sigma > 0$ such that

$$(r+s)^{2_*} \geq r^{2_*} + s^{2_*} + 2_* r^{2_*-1}s + 2_* r s^{2_*-1} - C_\sigma r^{2_*-\sigma}s^\sigma.$$

Picking $r = u_\mu$, $s = t_\epsilon v_\epsilon$ and $\sigma = 2_*/2$, we can use $\Gamma_{1,\epsilon} \geq 0$ and (3.4) to get

$$m_\epsilon \leq I(u_\mu) + \left(\frac{t_\epsilon^2}{2}\|v_\epsilon\|^2 - \frac{t_\epsilon^{2_*}}{2_*}\|b\|_\infty\right) + \Gamma_{2,\epsilon,1} - \Gamma_{2,\epsilon,2} + \Gamma_{2,\epsilon,3}, \tag{3.5}$$

where

$$\Gamma_{2,\epsilon,1} := \frac{t_\epsilon^{2_*}}{2_*}\int_{\mathbb{R}^{N-1}} K(x',0)\left[\|b\|_\infty - b(x')\right] v_\epsilon^{2_*}\, dx',$$

$$\Gamma_{2,\epsilon,2} := t_\epsilon^{2_*-1}\int_{\mathbb{R}^{N-1}} K(x',0)b(x')u_\mu v_\epsilon^{2_*-1} dx'$$

and

$$\Gamma_{2,\epsilon,3} := C_\sigma \frac{t_\epsilon^{2_*/2}}{2_*}\int_{\mathbb{R}^{N-1}} K(x',0)b(x')u_\mu^{2_*/2}v_\epsilon^{2_*/2} dx'.$$

We now notice that

$$\max_{t\geq 0}\left\{\frac{t^2}{2}\|v_\epsilon\|^2 - \frac{t^{2_*}}{2_*}\|b\|_\infty\right\} = \frac{1}{2(N-1)}\frac{\|v_\epsilon\|^{2(N-1)}}{\|b\|_\infty^{N-2}}$$

Assuming $N \geq 5$, from Lemma 3.2 we have that $\|v_\epsilon\|^{2(N-1)} = S_{2_*}^{N-1} + O(\epsilon^2)$. So, by definition of $\bar{c}$ and (3.5), we obtain

$$m_\epsilon \leq \bar{c} + O(\epsilon^2) + \Gamma_{2,\epsilon,1} - \Gamma_{2,\epsilon,2} + \Gamma_{2,\epsilon,3}. \tag{3.6}$$

We are going to estimate each of the therms $\Gamma_{2,\epsilon,i}$, for $i = 1, 2, 3$. For the first one, we can argue as in the proof of Lemma 3.3 to obtain $\Gamma_{2,\epsilon,1} = O(\epsilon^{N-1})$. The other two are more involved. So, pick $r_1 > 1$ in such a way that

$$\frac{1}{N+4} < \frac{r_1}{2(N-1)} < \frac{1}{N}.$$

Using Lemma 2.2 and Hölder's inequality, we get

$$\int_{\mathbb{R}^{N-1}} K(x',0)b(x')u_\mu v_\epsilon^{2_*-1} dx' \leq \|b\|_\infty \left(\int_{\{|x'|\leq\delta\}} K(x',0)u_\mu^{r_1'} dx'\right)^{1/r_1'} \|v_\epsilon\|_{(2_*-1)r_1}^{2_*-1}.$$

Since $1 < r_1 < 2(N-1)/N$, we have that $N/(N-2) < (2_* - 1)r_1 < 2_*$, and therefore we can use the above expression, Lemma 3.2 with $s = (2_* - 1)r_1$ and Lemma 3.3 to obtain

$$\Gamma_{2,\epsilon,2} = O\left(\epsilon^{(N-1)/r_1 - N/2}\right).$$

Now, we pick $r_2 \in (1,2)$ and argue as above to get

$$\Gamma_{2,\epsilon,3} \leq \|b\|_\infty \left(\int_{\{|x'| \leq \delta\}} K(x',0) u_\mu^{(2_*/2)r_2'} dx' \right)^{1/r_2'} \|v_\epsilon\|_{(2_*/2)r_2}^{2_*/2}.$$

Since $r_2 \in (1,2)$, we can use Lemma 3.3 with $s = (2_*/2)r_2 \in (2_*/2, 2_*)$ to obtain

$$\Gamma_{2,\epsilon,3} = O(\epsilon^{(N-1)/r_2 - (N-1)/2}).$$

Noticing that

$$\lim_{r \to 2(N-1)/N} \left(\frac{N-1}{r} - \frac{N}{2} \right) = 0 < \frac{N-1}{2} = \lim_{r \to 1} \left(\frac{N-1}{r} - \frac{N-1}{2} \right),$$

we can choose the numbers r_1, r_2 above in such a way that

$$\nu_1 := \frac{N-1}{r_1} - \frac{N}{2} < 2, \quad \nu_2 := \frac{N-1}{r_2} - \frac{N-1}{2} > \nu_1.$$

Since $\nu_1 < \min\{2, \nu_2\}$ and $N \geq 5$, we can use all the estimates performed above to rewrite (3.6) as

$$m_\epsilon \leq \bar{c} + O(\epsilon^2) + O(\epsilon^{N-1}) - O(\epsilon^{\nu_1}) + O(\epsilon^{\nu_2}) < \bar{c},$$

if $\epsilon > 0$ is small enough. This concludes the proof if $N \geq 5$.

In the case $N = 4$, the only change is that $\|v_\epsilon\|^{2(N-1)} = S_{2_*}^{N-1} + O(\epsilon^2 |\ln \epsilon|)$. By repeating all the previous steps, we obtain, for $\epsilon > 0$ small,

$$m_\epsilon \leq \bar{c} + O(\epsilon^2 |\ln \epsilon|) + O(\epsilon^{N-1}) - O(\epsilon^{\nu_1}) + O(\epsilon^{\nu_2}) < \bar{c},$$

because $\epsilon^{2-\nu_1} |\ln \epsilon| \to 0$, as $\epsilon \to 0^+$. □

We are ready to prove our first main theorem.

Proof of Theorem 1.1. Suppose that $\mu \in (0, \mu^*)$, where μ^* is given by Lemma 2.3. According to Proposition 2.4, there exists a nonnegative solution u_μ such that $I(u_\mu) < 0$. Suppose, by contradiction, that this is the only nonzero critical point. Then, according to Lemma 3.1, I satisifies the $(PS)c$ at any level $c < \bar{c}$. Since v_ϵ vanishes outside a ball, a straightforward computation shows that

$$\lim_{t \to +\infty} I(u_\mu + t v_\epsilon) = -\infty,$$

and therefore there exists $t_* > 0$ such that that $I(u_\mu + t_* v_\epsilon) < 0$ and $\|u_\mu + t_* v_\epsilon\| > \rho$, where $\rho > 0$ comes from Lemma 2.3. This shows that it is well defined the Mountain Pass level

$$c_{MP} := \inf_{\theta \in \Gamma} \max_{0 \leq t \leq 1} I(\theta(t)) > 0,$$

where $\Gamma := \{\theta \in C([0,1], X) : \theta(0) = 0, \theta(1) = u_\mu + t_* v_\epsilon\}$. By Lemma 3.4, we have that $c_{MP} < \bar{c}$. Applying the Mountain Pass Theorem (cf. [30, Theorem 2.2]) we obtain a critical point $u_0 \neq 0$ such that $I(u_0) > 0$. Since u_μ has negative energy, we conclude that $u_0 \neq u_\mu$, which is absurd. This guarantees the existence of our second nonzero solution. As before, it is nonnegative in $\mathbb{R}^N_+$. □

4. Solving a modified problem

We start in this section the proof of our second main result. From now on, we assume that conditions $(\widetilde{a_1})$, (a_2) and $(f_1)-(f_2)$ hold.

As quoted in the introduction, if $F(s) := \int_0^s f(t)dt$, then the term $\int_{\mathbb{R}^{N-1}} K(x',0)F(u)dx'$ can be infite, since we have no control on the bahaviour of $F(s)$ for large values of s, this integral may not be finite. So, we draw upon the ideas of Azorezo and Alonso [5] and define $g : \mathbb{R} \to \mathbb{R}$ as

$$g(s) := \begin{cases} f(s), & \text{if } |s| \leq C_{N,p}, \\ \dfrac{f(C_{N,p})^{p-1}}{C_{N,p}}|s|^{p-2}s, & \text{if } |s| > C_{N,p}, \end{cases} \tag{4.1}$$

where $C_{N,p}$ is given in (1.1). Since f is odd in $[-C_{N,p}, C_{N,p}]$ we have that g is an odd function. Moreover, from (f_2), we obain a constant $C_g > 0$, depending on N and p, such that

$$|g(s)| \leq C_g|s|^{p-1}, \quad \forall\, s \in \mathbb{R}. \tag{4.2}$$

Define $G(s) := \int_0^s g(t)dt$ and $J : X \to \mathbb{R}$ by

$$J(u) := \frac{1}{2}\|u\|^2 - \frac{\mu}{q}\int_{\mathbb{R}^{N-1}} K(x',0)a(x')|u|^q dx' - \int_{\mathbb{R}^{N-1}} K(x',0)G(u)dx'$$

and notice that any critical point u of J such that $|u|_\infty \leq C_{N,p}$ is a weak solution of the problem (P_2). Moreover, since the nonquadratic part of J has subcritical growth, we can use a standard argument and the compact embedding of Proposition 2.1 to conclude that any bounded Palais-Smale sequence of J has a convergent subsequece.

Based on the previous remark, our objective is to establish the existence of an infinite number of critical points for J with small $L^\infty(\mathbb{R}^{N-1})$ norms.

Using Hölder's inequality as in Lemma 2.3, we obtain

$$J(u) \geq \frac{1}{2}\|u\|^2 - \mu M_1\|u\|^q - M_2\|u\|^p,$$

where

$$M_1 := (|a|_{\sigma_q}/q)S_{q\sigma_q'}^{-q/2}, \quad M_2 := (C_g/p)S_p^{-p/2},$$

and $C_g > 0$ comes from (4.2). Let

$$h(t) := \frac{1}{2}t^2 - \mu M_1 t^q - M_2 t^p, \quad t \geq 0, \tag{4.3}$$

and notice that $h(0) = 0$, $h < 0$ near the origin and $\lim_{t\to+\infty} h(t) = -\infty$. Moreover, since the map $t \to (1/2)t^{2-q} - M_2 t^{p-q}$ attains its positive maximum in $(0, +\infty)$, it is clear that there exists $\mu^{**} > 0$ such that, for any $\mu \in (0, \mu^{**})$, h has a positive maximum. For these values of μ, the function has at least two positive roots $R_1 < R_2$. By the generalized version of Descartes' rule of signs (see [25, Theorem 2.1]), it has no other positive roots. We assume from now on that $\mu \in (0, \mu^{**})$.

We observe that R_1 is dependent on μ, as the smaller the value of μ, the faster the function h assumes positive values. In fact, when $\mu = 0$, the function h begins to take positive values, resulting in $R_1 = 0$. Therefore, we can expect the following outcome:

Lemma 4.1. *The first root of h verifies $\lim_{\mu\to 0^+} R_1(\mu) = 0$.*

Proof. Recalling that $h(R_1) = 0$ and $h'(R_1) > 0$, we get

$$\frac{1}{2} = \mu M_1 R_1^{q-2} + M_2 R_1^{p-2} > \frac{q}{2}\mu M_1 R_1^{q-2} + \frac{p}{2} M_2 R_1^{p-2}.$$

From $p > 2$ and the above expression, we obtain $\alpha \geq 0$ such that $R_1 \to \alpha \geq 0$, as $\mu \to 0^+$. If $\alpha > 0$, passing the limit on both sides of the above expression and using $q < 2$, we get $M_2\alpha^{p-2} \geq (p/2)M_2\alpha^{p-2}$, which implies $p \leq 2$. This contradiction proves that $\alpha = 0$ and we have done. □

Whereas $0 < R_1 < R_2$, we can define a cutoff function $\phi \in C_c^\infty(\mathbb{R})$ such that $0 \leq \phi \leq 1$, $\phi \equiv 1$ on $[0, R_1]$ and $\phi \equiv 0$ on $[R_2, +\infty)$. Under these conditions, we consider the C^1-functional $\Phi : X \to \mathbb{R}$ given by

$$\Phi(u) := \frac{1}{2}\|u\|^2 - \frac{\mu}{q} \int\limits_{\mathbb{R}^{N-1}} K(x', 0)a(x')|u|^q dx' - \phi(\|u\|) \int\limits_{\mathbb{R}^{N-1}} K(x', 0)G(u)dx'.$$

Below, we present the key properties of Φ:

Lemma 4.2. *The following holds:*

(i) Φ *is coercive;*
(ii) *if* $\Phi(u) < 0$, *then* $\|u\| < R_1$ *and there exists a small neighborhood of* u *where* $\Phi \equiv J$*;*
(iii) Φ *satisfies* $(PS)_c$*, for any* $c < 0$.

Proof. Arguing as in the proof of Lemma 2.3 and using (4.2), we obtain

$$\Phi(u) \geq \frac{1}{2}\|u\|^2 - \mu M_1\|u\|^q - \phi(\|u\|)M_2\|u\|^p.$$

Since ϕ vanishes on $(R_2, +\infty)$ and $q < 2$, we conclude that Φ is coercive.

For proving item (ii), we define

$$h_\phi(t) := \frac{1}{2}t^2 - \mu M_1 t^q - \phi(t)M_2 t^p, \quad t \geq 0.$$

A simple computation shows that $h'_\phi(t) > 0$ for any $t > (\mu M_1)^{1/(2-q)}$. Since $h(R_2) = 0$, we have that

$$R_2 > (2\mu M_1)^{1/(2-q)} > (\mu M_1)^{1/(2-q)},$$

and therefore h_ϕ is increasing in $[R_2, +\infty)$.

Let $u \in X$ be such that $\Phi(u) < 0$ and suppose, by contradiction, that $\|u\| \geq R_1$. If $\|u\| > R_2$, then we can use $h(R_2)$ again to get

$$0 > \Phi(u) \geq h_\phi(\|u\|) \geq h_\phi(R_2) = h(R_2) + M_2(1 - \phi(R_2))R_2^p = M_2 R_2^p > 0,$$

which is absurd. Hence, we may have $R_1 \leq \|u\| \leq R_2$. But in this case $h(\|u\|) \geq 0$ and we obtain

$$0 > \Phi(u) \geq h_\phi(\|u\|) \geq h(\|u\|) \geq 0,$$

which also is a contradiction. This proves that $\|u\| < R_1$, and therefore $\Phi(u) = J(u)$. Using the continuity of Φ, we obtain $\nu > 0$ such that $\Phi < 0$ in $B_\nu(u)$. For any element in this ball the former argument shows that $\Phi = J$. This proves (ii).

Suppose $(u_n) \subset X$ is such that $\Phi(u_n) \to c < 0$ and $\Phi'(u) \to 0$. Since we may assume that $\Phi(u_n) < 0$, it follows from item (ii) that $\Phi(u_n) = J(u_n) \to c$ and $\Phi'(u_n) = J'(u_n) \to 0$, that is, (u_n) a Palais-Smale sequence of J. Since Φ is coercive, we have that (u_n) is bounded and therefore, as quoted before, (u_n) has a convergent subsequence. □

The next result is the keystone that enables us to apply critical point for symmetric functionals.

Lemma 4.3. *For each $k \in \mathbb{N}$, there exist $r = r(k) > 0$, $\beta = \beta(k) > 0$ and a k-dimensional subspace $X_k \subset X$ such that*

$$\sup_{u \in X_k \cap \partial B_r(0)} \Phi(u) \leq -\beta < 0.$$

Proof. According to condition (a_2), there exists a ball $B' \subset \mathbb{R}^{N-1}$ such that $a > 0$ a.e. in B'. Let $\nu > 0$ and $y_1 = (y_1', 0), \cdots, y_k = (y_k', 0) \in \partial\mathbb{R}^N_+$ be such that $\left(B_\nu(y_i) \cap \partial\mathbb{R}^N_+\right) \subset B'$ and $\overline{B_\nu(y_i)} \cap \overline{B_\nu(y_j)} = \varnothing$, for any $i, j = 1, \ldots, k$, with $i \neq j$. For each $i = 1, \ldots, k$, we pick a smooth function ϕ_i such that $\phi_i \equiv 1$ in $B_{\nu/2}(y_i) \cap \overline{\mathbb{R}^N_+}$ and $\phi_i \equiv 0$ outside $B_\nu(y_i) \cap \overline{\mathbb{R}^N_+}$.

Considering that these functions have disjoint support, the set $\{\phi_1, ..., \phi_k\}$ is linearly independent, and the spanned space $X_k := \langle\{\phi_1, ..., \phi_k\}\rangle$ has dimension k. We assert that the mapping

$$Q(u) := \left(\int_{\mathbb{R}^{N-1}} K(x', 0)a(x')|u|^q dx'\right)^{1/q},$$

defines a norm in X_k. To verify this, we initially establish that $Q(u) > 0$ for any $u \neq 0$. Let $u = \sum_{i=1}^k \alpha_i \phi_i$ be a nonzero function. If we denote $B_i' := B_{\nu/2}(y_i) \cap \partial\mathbb{R}^N_+$, it follows from (a_2) that

$$\begin{aligned} Q(u)^q &= \int_{\mathbb{R}^{N-1}} K(x', 0)a(x')|\alpha_1\phi_1 + ... + \alpha_k\phi_k|^q dx' \\ &\geq \sum_{i=1}^k \int_{B_i'} K(x', 0)a(x')|\alpha_1\phi_1 + ... + \alpha_k\phi_k|^q dx' \\ &= \sum_{i=1}^k |\alpha_i|^q \int_{B_i'} K(x', 0)a(x')dx' > 0, \end{aligned}$$

since the sets B_i are disjoint, $\phi_i \equiv 1$ and $a > 0$ in B_i, and at least one of the α_i's is nonzero. The other properties that need to be verified by a norm can be easily deduced from the definition of Q.

Since $\dim X_k < \infty$, there exists $C_1 = C_1(k) > 0$ such that

$$C_1\|u\|^q \leq \int_{\mathbb{R}^{N-1}} K(x', 0)a(x')|u|^q dx', \quad \forall\, u \in X_k.$$

Hence, for some $C_2 > 0$, there holds

$$\Phi(u) \leq \frac{1}{2}\|u\|^q \left(\|u\|^{2-q} + C_2\|u\|^{p-q} - \frac{2\mu C_1}{q}\right) \leq -\beta < 0, \quad \forall\, u \in X_k.$$

for $r = r(k) > 0$ such that

$$r^{2-q} + C_2 r^{p-q} < \frac{\mu C_1}{q}$$

and $\beta = \beta(k) := r^q \mu C_1/(2q)$. The lemma is proved. □

Let Σ the class of all closed subsets of $X \setminus \{0\}$ that are symmetric with respect to the origin. If $A \subset \Sigma$, then the genus of $\gamma(A)$ is defined as

$$\gamma(A) := \inf\left\{k \in \mathbb{N} : \text{there exists } \varphi : A \to \mathbb{R}^k \text{ continuous and odd}\right\},$$

when this set is not empty. Otherwise, if it is empty, we just say that $\gamma(A) = +\infty$. We refer to [30, Chapter 7] for more details on this subject.

We are ready to prove the main result of this section.

Proposition 4.4. *The funcional Φ has infnitely many critical points with negative energy.*

Proof. For each $k \in \mathbb{N}$, let

$$\Gamma_k := \{A \in \Sigma \,:\, \gamma(A) \geq k\}$$

and

$$c_k := \inf_{A\in\Gamma_k} \sup_{u\in A} \Phi(u)$$

Since Φ remains bounded in balls and it is coercive, we conclude that $c_k \in \mathbb{R}$. Let X_k and $r > 0$ be given by Lemma 4.3. It is clear that we can define an odd homeomorphism between $X_k \cap \partial B_r(0)$ and the unit sphere $\mathbb{S}^{k-1} \subset \mathbb{R}^k$. Hence, we may use [30, Proposition 7.7] to conclude that $\gamma\left(X_k \cap \partial B_r(0)\right) = k$.

Since $X_k \cap \partial B_r(0)$ is closed and sysmmetric, it belongs to Γ_k. It follows from Lemma 4.3 that

$$c_k \leq \sup_{u\in X_k\cap\partial B_r(0)} \Phi(u) \leq -\beta < 0,$$

so that all the minimax levels c_k are negative. By Lemma 4.2(iii), Φ satisfies the Palais-Smale condition at each of this levels. By using $\Gamma_{k+1} \subset \Gamma_k$, we conclude that $c_k \leq c_{k+1}$. Moreover, since Φ is even and satisfies the Palais-Smale condition at any negative level, we can argue along the same lines of [30, Proposition 9.3] to prove that, if $c_k = \cdots = c_{k+j} = c$ and $K_c = \{u \in X \,:\, \Phi(u) = c,\, \Phi'(u) = 0\}$, then $\gamma(K_c) \geq j + 1$.

The aforementioned considerations prove that each $c_k < 0$ is a critical value of Φ. Furthermore, if one of these values repeats, namely $c_l = c_{l+1} < 0$, we have that $\gamma(K_{c_l}) \geq 2$, thereby implying that K_{c_l} has infinitely many elements (cf. [30, Remark 7.3]). Hence, we conclude that Φ has infinitely many critical points with negative energy. □

5. A priori estimates and the proof of Theorem 1.2

We dedicate this final section to proving our second main theorem.

Proof of Theorem 1.2. Let $u \in X$ be one of the critical points given by Proposition 4.4. Since $\Phi(u) < 0$, it follows from Lemma 4.2(ii) that $\|u\| < R_1$ and $J'(u) = 0$. We are going to show that, for any $\mu > 0$ sufficiently small, there holds

$$|u(x',0)| \leq C_{N,p}, \quad \text{a.e. in } \mathbb{R}^{N-1}, \tag{5.1}$$

ans therefore it follows from the definition of g (see (4.1)) that $f(u) = g(u)$. Thus, $J'(u) = 0$ implies that u is a solution of the original problem (P_2).

The idea for the proof of (5.1) is an adaptation of the classical Moser Iteration Method [27]. In what follows, we argue assuming that $u \geq 0$. If this is not the case, it is sufficient to perform all the calculations separately for the postive part u^+ and after for the negative part u^-.

Inspired by Stampacchia's truncation (see [31, 10]) we define, for any $0 < L < 1$ and for $x \in \overline{\mathbb{R}^N_+}$,

$$u_L(x) := \begin{cases} u(x) - L, & \text{if } u(x) > L, \\ 0, & \text{if } u(x) \le L. \end{cases}$$

For $\beta > 1$, we also define $\phi_L := u_L^{2(\beta-1)} u$. Since $\Phi(u) < 0$, it follows from Lemma 4.2 that $J'(u)\phi_L = 0$. Hence, we may use (4.2) to get

$$\begin{aligned} \int_{\mathbb{R}^N_+} K(x)(\nabla u \cdot \nabla \phi_L)dx &\le \mu \int_{\mathbb{R}^{N-1}} K(x',0)a(x')u^{q-1}\phi_L dx' \\ &+ C_g \int_{\mathbb{R}^{N-1}} K(x',0)u^{p-1}\phi_L dx'. \end{aligned} \tag{5.2}$$

Since $u\nabla u_L = u\nabla u$ in the set $\{u > L\}$ and

$$\nabla \phi_L = 2(\beta-1)u_L^{2\beta-3} u \nabla u_L + u_L^{2\beta-2}\nabla u,$$

we have that

$$\begin{aligned} \int_{\mathbb{R}^N_+} K(x)(\nabla u \cdot \nabla \phi_L)dx &= \int_{\{u>L\}} K(x)(\nabla u \cdot \nabla \phi_L)dx \\ &= \int_{\{u>L\}} K(x)\left[2(\beta-1)u_L^{2\beta-3}u + u_L^{2(\beta-1)}\right]|\nabla u|^2 dx \end{aligned}$$

Therefore, since $u_L = 0$ in $\{u \le L\}$, we get

$$\int_{\mathbb{R}^N_+} K(x)(\nabla u \cdot \nabla \phi_L)dx \ge \int_{\mathbb{R}^N_+} K(x)u_L^{2(\beta-1)}|\nabla u|^2 dx. \tag{5.3}$$

If we call Γ_1 the term multiplying μ in (5.2), we have that

$$\Gamma_1 = \int_{\{L<u<1\}} K(x',0)a(x')u^{q-1}\phi_L dx' + \int_{\{u\ge 1\}} K(x',0)a(x')u^{q-1}\phi_L dx'.$$

Since $u_L \le u < 1$, in the set $\{L < u < 1\}$ we have that

$$u^{q-1}\phi_L = u^{q-1}u_L^{2(\beta-1)}u \le u^q.$$

Moreover, in $\{u \ge 1\}$ there holds $u^{q-1} \le u^{p-1}$. Thus

$$\Gamma_1 \le \int_{\mathbb{R}^{N-1}} K(x',0)a(x')u^q dx' + \|a\|_\infty \int_{\mathbb{R}^{N-1}} K(x',0)u^{p-1}\phi_l dx'.$$

Using Hölder's inequality in the first integral above, (5.3) and (5.2), we obtain

$$\begin{aligned} \int_{\mathbb{R}^N_+} K(x)u_L^{2(\beta-1)}|\nabla u|^2 dx &\le \mu S_{q\sigma_q'}^{-q/2}\|a\|_{\sigma_q}\|u\|^q \\ &+ (\mu\|a\|_\infty + C_g)\int_{\mathbb{R}^{N-1}} K(x',0)u^{p-1}\phi_L dx' \end{aligned} \tag{5.4}$$

From $u_L \le u$ and Hölder's inequality with exponents $s = 2_*/(p-2)$ and $s' = 2_*/(2_* + 2 - p)$, we obtain

$$\begin{aligned} \int_{\mathbb{R}^{N-1}} K(x',0)u^{p-1}\phi_L dx' &\le \int_{\mathbb{R}^{N-1}} K(x',0)u^{p-2}u^{2\beta}dx' \\ &\le \|u\|_{2_*}^{p-2}\|u\|_{m\beta}^{2\beta} \le S_{2_*}^{(2-p)/2}\|u\|^{p-2}\|u\|_{m\beta}^{2\beta}, \end{aligned}$$

where

$$m := 2s' = \frac{2 \cdot 2_*}{2_* + 2 - p} < 2_*.$$

By definition, $u_L(x) \to u(x)$, as $L \to 0^+$. Moreover, we know that $\|u\| \leq R_1$. So, we can replace the above inequality in (5.4) and use Fatou's lemma to obtain

$$\begin{aligned} \int_{\mathbb{R}^N_+} K(x) u^{2(\beta-1)} |\nabla u|^2 dx &\leq \mu S_{q\sigma_q'}^{-q/2} |a|_{\sigma_q} R_1^q \\ &+ (\mu |a|_\infty + C_g) S_{2_*}^{(2-p)/2} R_1^{p-2} |u|_{m\beta}^{2\beta} \end{aligned} \tag{5.5}$$

At this point, we define $z_L := u_L^{\beta-1} u$ and notice that

$$\left(\int_{\mathbb{R}^{N-1}} K(x',0) z_L^{2_*} dx' \right)^{2/2_*} \leq S_{2_*}^{-1} \int_{\mathbb{R}^N_+} K(x) |\nabla z_L|^2 dx.$$

Since $u_L = 0$ in $\{u \leq L\}$,

$$\nabla z_L = \left[(\beta - 1) u_L^{\beta-2} u + u_L^{\beta-1} \right] \nabla u.$$

Hence, using $u_L \leq u$ and $(\beta-1)^2 + 1 + 2(\beta-1) = \beta^2$, we obtain

$$\left(\int_{\mathbb{R}^{N-1}} K(x',0) z_L^{2_*} dx' \right)^{2/2_*} \leq S_{2_*}^{-1} \beta^2 \int_{\mathbb{R}^N_+} K(x) u^{2(\beta-1)} |\nabla u|^2 dx.$$

Since $z_L(x) \to u^\beta(x)$, as $L \to 0^+$, it follows from the above estimate, (5.5) and Fatou's lemma that

$$|u|_{2_*\beta}^{2\beta} \leq S_{2_*}^{-1} \beta^2 \left[\mu S_{q\sigma_q'}^{-q/2} |a|_{\sigma_q} R_1^q + (\mu |a|_\infty + C_g) S_{2_*}^{(2-p)/2} R_1^{p-2} |u|_{m\beta}^{2\beta} \right]$$

We now recall that we are assuming that $\mu < \mu^{**}$, where this last number was introduced in the beginning of the last section and it assures that the function h defined in (4.3) has exactly 2 positive roots $R_1 < R_2$. We need to reduce the value of μ once more. Actually, since $R_1(\mu) \to 0$, as $\mu \to 0^+$ (see Lemma 4.1) we know that there exists $0 < \bar{\mu} < \mu^{**}$ such that, for any $\mu \in (0, \bar{\mu})$, all the inequalities below are satisfied:

$$\mu S_{q\sigma_q'}^{-q/2} |a|_{\sigma_q} R_1^q < 1, \quad (\mu |a|_\infty + C_g) S_{2_*}^{(2-p)/2} R_1^{p-2} < 1, \quad S_{2_*}^{-1} R_1 < 1. \tag{5.6}$$

With the above restriction on μ, we have that

$$|u|_{2_*\beta}^{2\beta} \leq C_1 \beta^2 \max\left\{1, |u|_{m\beta}^{2\beta}\right\}, \tag{5.7}$$

where $C_1 := 2S_{2_*}^{-1}$. This shows that, once we know that $u \in L_K^{m\beta}$, then $u \in L_K^{2_*\beta}$. So, if we fix $\beta := 2_*/m > 1$, we have that $2_*\beta > 2_* = m\beta$, and we can improve the regularity of u. Moreover, $2_*\beta = m\beta^2$ and we can repeat the previous calculations, replacing β by β^2, and use (5.7) to get

$$\begin{aligned} |u|_{2_*\beta^2}^{2\beta^2} &\leq C_1 \beta^4 \max\left\{1, |u|_{m\beta^2}^{2\beta^2}\right\} = C_1 \beta^4 \max\left\{1, |u|_{2_*\beta}^{2\beta^2}\right\} \\ &\leq C_1 \beta^4 \max\left\{1, \left(C_1 \beta^2 \max\{1, |u|_{2_*}^{2\beta}\}\right)^\beta\right\}. \end{aligned}$$

Setting $C_2 := \max\{1, C_1\}$, we can rewrite the above estimate as

$$|u|_{2_*\beta^2}^{2\beta^2} \leq C_2^{1+\beta} \beta^{2(2+\beta)} \max\left\{1, |u|_{2_*}^{2\beta^2}\right\}.$$

Repeating this process $k \in \mathbb{N}$ times, we get

$$\left| u \right|_{2_*\beta^k} \le C_2^{\frac{1}{2\beta^k}\sum_{i=0}^{k-1}\beta^i} (\beta^2)^{\frac{1}{2\beta^k}\sum_{i=1}^{k} i\beta^{k-i}} \max\{1, \left| u \right|_{2_*}\}. \tag{5.8}$$

Since $\beta > 1$, we have that

$$\frac{1}{\beta^k}\sum_{i=0}^{k-1}\beta^i \le \sum_{i=1}^{\infty}\left(\frac{1}{\beta}\right)^i = \frac{1}{(\beta-1)}, \qquad \frac{1}{\beta^k}\sum_{i=1}^{k} i\beta^{k-i} \le \sum_{i=1}^{\infty} i\left(\frac{1}{\beta}\right)^i = \frac{\beta}{(\beta-1)^2}.$$

Moreover, from the last inequality in (5.6) and $\|u\| \le R_1$, we get

$$\left| u \right|_{2^*} \le S_{2_*}^{-1}\|u\| \le S_{2_*}^{-1}R_1 < 1.$$

These remarks, $\beta = (2_* + 2 - p)/2$, $C_2 > 1$, (5.8) and (1.1) imply that

$$\left| u \right|_{2_*\beta^k} \le C_2^{1/[2(\beta-1)]}\beta^{\beta/(\beta-1)^2} = C_{N,p}, \quad \forall\, k \in \mathbb{N}, \tag{5.9}$$

from which we conclude that (5.1) holds. Indeed, suppose by contradiction that there exist $C_3 > C_{N,p}$ and $\Omega \subset \mathbb{R}^{N-1}$ with positive and finite measure in $\mathbb{R}^{N-1}$ such that $|u(x',0)| > C_3$ for a.e. $x' \in \Omega$. Thus,

$$\left| u \right|_{2_*\beta^k} \ge \left(\int_\Omega |u|^{2_*\beta^k}dx'\right)^{1/(2_*\beta^k)} \ge C_3|\Omega|^{1/(2_*\beta^k)},$$

which implies $\liminf_{n\to+\infty}\left| u \right|_{2_*\beta^k} \ge C_3 > C_{N,p}$, contrary to (5.9). This contradiction concludes the proof of Theorem 1.2. □

Remark 5.1. *In [7], the authors state a multiplicity result which can be viewed as a version of our Theorem 1.2 for a different problem. Unfortunately, it seems that the proof has a gap, as the equation (11) in that paper is false. We believe that the truncation defined in the proof of our Theorem 1.2 can be used to fix the proof of the main result in [7].*

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