SIGN-CHANGING SOLUTION FOR AN ELLIPTIC EQUATION WITH CRITICAL GROWTH AT THE BOUNDARY

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Abstract. We prove the existence of sign-changing solution to the problem

$$-\Delta u - \frac{1}{2}(x \cdot \nabla u) = \lambda u, \text{ in } \mathbb{R}^N_+, \qquad \frac{\partial u}{\partial \nu} = |u|^{2_*-2}u, \text{ on } \partial\mathbb{R}^N_+,$$

where $\mathbb{R}^N_+ = \{(x', x_N) : x' \in \mathbb{R}^{N-1}, x_N > 0\}$ is the upper half-space, $2_* := 2(N-1)/(N-2)$, $N \geq 7$, $\frac{\partial u}{\partial \nu}$ is the partial outward normal derivative and the parameter $\lambda > 0$ interacts with the spectrum of the linearized problem. In the proof, we apply variational methods.

1. Introduction and main result

Let $\mathbb{R}^N_+ = \left\{(x', x_N) : x' \in \mathbb{R}^{N-1}, x_N > 0\right\}$ be the upper half-space and consider the nonlinear boundary value problem

$$-\Delta v = f(x, v), \text{ in } \mathbb{R}^N_+, \qquad \frac{\partial v}{\partial \nu} = g(x, v), \text{ on } \mathbb{R}^{N-1}, \tag{1.1}$$

where $\frac{\partial u}{\partial \nu}$ denotes the outer unit normal derivative and we have identified $\partial\mathbb{R}^N_+ \simeq \mathbb{R}^{N-1}$. Its mathematical importance arises, for instance, in the study of conformal deformation of Riemannian manifolds [10, 17, 22, 18], problems of sharp constant in Sobolev trace inequalities [16, 14] and blow-up properties of the solutions of related parabolic equations [26, 20]. This kind of equations also appears in several applied contexts like glaciology [30], population genetics [2], non-Newtonian fluid mechanics [15], nonlinear elasticity [13], among others.

There is a vast literature concerning nonnegative solutions for (1.1). Using the moving plane method, Hu [25] obtained nonexistence of positive solutions when $f \equiv 0$ and $g(v) = v^q$, with $1 < q < N/(N-2)$. Similar results were obtained by Chipot *et al.* in [12] in the case that $f(v) = av^p$ and $g(v) = v^q$ with $1 < p \leq (N+2)/(N-2)$, $1 < q \leq N/(N-2)$, with one of the inequalities being strict, and $a > 0$ (see also [36] for existence and multiplicity results in the double subcritical case). In dimension $N = 2$ and $f \equiv 0$, Cabré and Morales [7] presented necessary and sufficient conditions on $g(v)$ for the existence of *layer solutions*, that is, bounded solutions that satisfy some monotonicity properties. When $f \equiv 0$ and $g(v) = (N-2)v^{N/(N-2)}$, existence of positive solution decaying as $|x|^{2-N}$ at infinity was obtained by Escobar [16] using the conformal equivalence between the unit ball in $\mathbb{R}^N$ and the half-space (see also [34]). In the same paper, it was considered the case $f(v) = N(N-2)v^{(N+2)/(N-2)}$ and $g(v) = bv^{N/N(N-2)}$. Later, Chipot *et*

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al. [11] removed the decay assumption by using the shrinking sphere method to give a complete description of positive solutions when $f(v) = av^{(N+2)/(N-2)}$ and $g(v) = bv^{n/(N-2)}$. Similar results were obtained by Li and Zhu in [27], including a 2-dimensional version with exponential type nonlinearities.

In this paper, we deal with the boundary critical problem

$$(P_\lambda) \qquad \begin{cases} -\Delta u - \dfrac{1}{2}(x \cdot \nabla u) = \lambda u, & \text{in } \mathbb{R}^N_+, \\ \dfrac{\partial u}{\partial \nu} = |u|^{2_*-2}u, & \text{on } \mathbb{R}^{N-1}, \end{cases}$$

where $2_* := 2(N-1)/(N-2)$. Notice that, if u is a solution of (P_λ), then the function $v = \exp(|x|^2/8)u$ verifies (1.1) for

$$f(x,v) = \left(\lambda - \frac{N}{4} - \frac{|x|^2}{16}\right) v, \qquad g(x,v) = \exp\left(-\frac{|x|^2}{4(N-2)}\right)|v|^{2_*-2}v.$$

Differently from the former cases, this problem is not homogeneous and the nonlinearity f is unbounded in the spatial variable. Hence, the techniques used in the aforementioned works do not apply and we need to perform a different approach to deal with the drift term inside the domain.

Before presenting our result, it is essential to highlight the resemblance of our equation to the classical problem:

$$-\Delta u = \lambda u + |u|^{2^*-2}u, \quad u \in H_0^1(\Omega).$$

In this equation, $\Omega \subset \mathbb{R}^N$ is a bounded domain, with $N \geq 3$, and $\lambda > 0$ is a parameter. This equation has its origins in Yamabe's problem, which revolves around the existence of Riemannian metrics with constant scalar curvature. In a seminal paper by Brezis and Nirenberg [5], it was established that the existence of a positive solution is linked to the interplay between the parameter and the first eigenvalue $\lambda_{1,\Omega} > 0$ within the spectrum $\sigma(-\Delta, H_0^1(\Omega))$. Among various contributions, they demonstrated that the above equation possesses a positive solution when $N \geq 4$ and $0 < \lambda < \lambda_{1,\Omega}$. This marked the commencement of an extensive body of literature dedicated to this critical equation. Notably, we would like to mention the work of Capozzi, Fortunato, and Palmieri [8], who obtained solutions for $\lambda \geq \lambda_{1,\Omega}$, and Cerami, Solimini, and Struwe [9], who proved the existence of a sign-changing solution when $0 < \lambda < \lambda_{1,\Omega}$ and $N \geq 6$.

In addition to its natural connection with the Brezis and Nirenberg problem, our equation is closely related to the nonlinear heat equation:

$$w_t - \Delta w = 0, \text{ in } \mathbb{R}^N_+ \times (0,+\infty), \quad \frac{\partial w}{\partial \nu} = |w|^{p-2}w, \text{ on } \mathbb{R}^{N-1} \times (0,+\infty).$$

A solution with the special form $w(x,t) = t^{-\lambda}u(t^{-1/2}x)$ is referred to as a self-similar solution. It is well-known (see, for example, [23, 29, 24]) that such solutions provide qualitative insights into aspects like global existence, blow-up, and asymptotic behavior. Furthermore, they maintain the scaling of the partial differential equation, offering simultaneous information about small and large-scale behaviors. The connection with (P_λ) becomes evident when we substitute this form of w into the heat equation. We observe that the profile u must satisfy the same equation as in (P_λ) with $\lambda = 1/(2(p-2))$ and 2_* replaced by $p \in (2, 2_*]$.

Setting $K(x) = \exp(|x|^2/4)$ and noticing that $2\nabla K = xK$, the first equation in (P_λ) can be rewritten as

$$-\mathrm{div}(K(x)\nabla u) = \lambda K(x)u, \quad \text{in } \mathbb{R}^N_+.$$

Hence, it is natural to look for finite energy solutions belonging to the Sobolev space $\mathcal{D}^{1,2}_K(\mathbb{R}^N_+)$ defined as the closure of $C^\infty_c(\overline{\mathbb{R}^N_+})$ with respect to the norm

$$\|u\| = \left(\int_{\mathbb{R}^N_+} K(x)|\nabla u|^2 dx\right)^{1/2}.$$

This kind of space was first introduced by Escobedo and Kavian [19] who considered a problem in the whole space $\mathbb{R}^N$. The upper half-space case was presented in [20], where it is proved that $\mathcal{D}^{1,2}_K(\mathbb{R}^N_+)$ is compactly embedded into the weighted Lebesgue space

$$L^2_K(\mathbb{R}^N_+) = \left\{u \in L^2(\mathbb{R}^N_+) \, : \, \int_{\mathbb{R}^N_+} K(x)u^2 dx < +\infty\right\}.$$

So, we can solve the linear problem associated with (P_λ), namely

$$(LP) \qquad -\Delta u - \frac{1}{2}(x\cdot\nabla u) = \lambda u, \text{ in } \mathbb{R}^N_+, \qquad \frac{\partial u}{\partial \nu} = 0, \text{ on } \mathbb{R}^{N-1},$$

and use spectral theory to obtain an increasing sequence of eigenvalues $(\lambda_j)_{j\in\mathbb{N}}$ such that $\lambda_1 = N/2$ and $\lambda_j \to +\infty$, as $j \to +\infty$.

The authors in [20] considered the subcritical version of (P_λ), that is, the same problem with 2_* replaced by $p \in (2, 2_*)$. Among other results, they obtained the existence of a positive solution if $\lambda < \lambda_1$. As a consequence, self-similar solutions to the associated heat equation exist whenever $2 + (1/N) < p < 2_*$. The critical version was recently considered in [21] and the situation turns out to be more delicate. After proving a new trace embedding the authors showed that, in the critical case, there is no self-similar solution to the equation. Besides this, they obtained a positive solution whenever $N \geq 7$ and the parameter λ verifies

$$\lambda^*_N = \frac{N}{4} + \frac{N-4}{8} < \lambda < \lambda_1.$$

In the first part of this paper we complete the above study by considering the case $\lambda > \lambda_1$. Standard arguments show that positive solutions are not expected and therefore we look for sign-changing solutions. More specifically, we prove the following:

Theorem 1.1. *If $N \geq 7$ and $\lambda > \lambda_1$ is not an eigenvalue of (LP), then problem (P_λ) has a sign-changing solution.*

In the proof, we apply the Linking Theorem [31] to the energy functional associated to (P_λ). Since usual arguments do not imply that this functional is anticoercive in finite-dimensional subspaces, we need to perform a detailed study of the structure of solutions of the eigenvalue problem (LP) and prove a projection result (see Lemma 2.4 and Proposition 2.5). The assumption that λ is not an eigenvalue of (LP) is a non-resonant type condition of technical nature and assures that Palais-Smale sequences are bounded. Actually, the arguments used in [31, 8] do not work in unbounded domains and therefore we need to perform a different approach here (see Proposition 2.6).

In the second part of the paper, we come back to the range where positive solution exists and ask if it is possible to obtain another solution. In this new setting, we prove the following:

Theorem 1.2. *If $N \geq 7$ and $\lambda \in (\lambda_N^*, \lambda_1)$, then problem (P_λ) has a sign-changing solution.*

In order to explain the main steps for the proof, we first define $u^+(x) = \max\{u(x), 0\}$ and $u^- = u^+ - u$. After that, inspired by the paper of Cerami, Solimini and Struwe [9], we introduce the Nehari nodal set $\mathcal{M}_\lambda = \{u \in \mathcal{D}_K^{1,2}(\mathbb{R}_+^N) : u^\pm \neq 0,\ I'_\lambda(u^\pm)u^\pm = 0\}$ and prove that

$$d_\lambda = \inf_{u \in \mathcal{M}_\lambda} I_\lambda(u)$$

is attained by a solution $u \in \mathcal{M}_\lambda$. Since we are dealing with the critical case, the functional I_λ satisfies only a local Palais-Smale condition. So, we need to prove some fine estimates (see Lemmas 5.2 and 5.3) involving the positive solution obtained in [21], which justifies the high dimensions and the strong technical restrictions on the parameter λ, and a slight modification of the *instanton functions* founded independently by Escobar [16] and Beckner [3]. This is essential to guarantee that d_λ belongs to the range where we have compactness. Since $\mathcal{M}_\lambda$ is not a differentiable manifold, it is not easy to construct Palais-Smale sequences on the level d_λ. In order to do this, we adapt some ideas introduced by Tarantello in [33].

The paper is organized as follows: in the next section, we present the variational framework and some technical results for Theorem 1.1, which is proved after in Section 3. In Section 4, we establish the minimization scheme in order to deal with problem $(P\lambda)$ when $\lambda \in (\lambda_N^*, \lambda_1)$, and in the last section we present the proof of Theorem 1.2 .

2. Variational setting and preliminary results

We start this section setting $K(x) := \exp(|x|^2/4)$ and noticing that

$$\operatorname{div}(K(x)\nabla u) = K(x)\left(\Delta u + \frac{1}{2}(x \cdot \nabla u)\right),$$

for any regular function u. Hence, it is natural to define the Banach space $\mathcal{D}_K^{1,2}(\Omega)$ as being the closure of $C_0^\infty(\overline{\Omega})$ with respect to the norm

$$\|u\|_{\mathcal{D}_K^{1,2}(\Omega)} := \left(\int_\Omega K(x)|\nabla u|^2\, dx\right)^{\frac{1}{2}},$$

for any open set $\Omega \subset \mathbb{R}^N$. For simplicity, we denote $\mathcal{D}_K^{1,2}(\mathbb{R}_+^N)$ by X and $\|\cdot\|_{\mathcal{D}_K^{1,2}(\mathbb{R}_+^N)}$ by $\|\cdot\|$. We also define, for any $2 \leq r \leq 2^* := 2N/(N-2)$, the weighted Lebesgue space

$$L_K^r(\mathbb{R}_+^N) := \left\{ u \in L^r(\mathbb{R}_+^N) \,:\, \|u\|_r := \left(\int_{\mathbb{R}_+^N} K(x)|u|^r dx\right)^{1/r} < \infty \right\}.$$

According to [20, Lemma 2.2], the embedding $X \hookrightarrow L_K^r(\mathbb{R}_+^N)$ is continuous for $2 \leq r \leq 2^*$ and compact for $2 \leq r < 2^*$. Moreover, denoting by

$$L_K^r(\mathbb{R}^{N-1}) := \left\{ u \in L^r(\mathbb{R}^{N-1}) \,:\, |\!|\!| u |\!|\!|_r := \left(\int_{\mathbb{R}^{N-1}} K(x',0)|u|^r dx'\right)^{1/r} < \infty \right\},$$

it was proved in [20, Lemma 2.4] the compact trace embedding $X \hookrightarrow L^r_K(\mathbb{R}^{N-1})$, for $2 < r < 2_*$. Subsequently, the authors in [21, Theorem 1.1] extended this former result by proving that the embedding is really continuous for $2 \leq r \leq 2_*$ and compact for $2 \leq r < 2_*$. So, the natural range of the trace embedding is covered and we can define the best constant

$$S(K) := \inf_{\varphi \in X\setminus\{0\}} \frac{\|\varphi\|^2}{\|\varphi\|^2_{2_*}} > 0. \tag{2.1}$$

Actually, it is proved in [21] that the above infimum is achieved and it is equal to the best constant S of the Sobolev trace embedding $\mathcal{D}^{1,2}(\mathbb{R}^N_+) \hookrightarrow L^{2_*}(\mathbb{R}^{N-1})$.

The energy functional associated with our problem $I_\lambda : X \to \mathbb{R}$ is given by

$$I_\lambda(u) := \frac{1}{2}\|u\|^2 - \frac{\lambda}{2}\|u\|_2^2 - \frac{1}{2_*}\|u\|^{2_*}_{2_*}, \quad \forall\, u \in X.$$

Standard calculations show that $I_\lambda \in C^1(X, \mathbb{R})$ and the weak solutions of (P_λ) are precisely the critical points of I_λ.

For proving Theorem 1.1, we shall use the following variant of the Mountain Pass Theorem [31] (see also [35, Theorem 2.12]).

Theorem 2.1. *Let $E = V \oplus W$ be a real Banach space with $\dim V < \infty$. Suppose $I \in C^1(E, \mathbb{R})$ satisfies*

- (I_1) *there exist $\rho, \alpha > 0$ such that $I|_{W\cap\partial B_\rho(0)} \geq \alpha$;*
- (I_2) *there exists $e \in W \cap \partial B_1(0)$ and $R > \rho$ such that*

$$I|_{\partial Q} \leq 0,$$

with

$$Q := \left(\overline{B_R(0)} \cap V\right) \oplus \{te : 0 < t < R\}.$$

If

$$c := \inf_{\gamma\in\Gamma} \max_{u\in Q} I(\gamma(u)), \tag{2.2}$$

where $\Gamma := \left\{\gamma \in C(\overline{Q}, E) : \gamma \equiv Id \text{ on } \partial Q\right\}$, then there exists a sequence $(u_n) \subset E$ such that $I(u_n) \to c$ and $I'(u_n) \to 0$, as $n \to +\infty$.

We are intending to apply this abstract result with $E = X$ and $I = I_\lambda$. In order to present the decomposition of the space X we consider the linearized problem

$$(LP) \qquad \begin{cases} -\mathrm{div}(K(x)\nabla u) = \lambda K(x) u, & \text{in } \mathbb{R}^N_+, \\ \frac{\partial u}{\partial \eta} = 0, & \text{on } \mathbb{R}^{N-1}, \\ u \in \mathcal{D}^{1,2}_K(\mathbb{R}^N_+). \end{cases}$$

Thanks to the compact embedding $X \hookrightarrow L^2_K(\mathbb{R}^N_+)$, we can use standard spectral theory to obtain sequence of eigenvalues $(\lambda_j)_{j\in\mathbb{N}}$ such that

$$0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_j \leq \cdots$$

with $\lim\limits_{j\to\infty} \lambda_j = +\infty$. A straightforward computation shows that

$$\varphi_1(x) := \exp\left(-|x|^2/4\right)$$

satisfies (LP). Since this function is positive, its associated eigenvalue is the first one. Noticing that $\nabla\varphi_1 = -(x/2)K(x)^{-1}$, we can explicitly compute this first eigenvalue in the following way:

$$\lambda_1 = -\frac{\operatorname{div}(K(x)\nabla\varphi_1)}{K(x)\varphi_1} = \frac{1}{2}\operatorname{div}(x) = \frac{N}{2}.$$

Along this entire section we shall assume that $\lambda \in (\lambda_k, \lambda_{k+1})$, for some $k \in \mathbb{N}$. In order to apply Theorem 2.1, we pick φ_i the eigenfunction associated with the eigenvalue λ_i of problem (LP), for $1 \le i \le k$, and set

$$V := \operatorname{span}\{\varphi_1, \ldots, \varphi_k\}, \qquad W := V^{\perp}. \tag{2.3}$$

We have that $X = V \oplus W$. Besides this, it is well known from the variational characterization of the eigenvalue of (LP) that

$$\frac{1}{\lambda_k}\|v\|^2 \le \|v\|_2^2, \qquad \|w\|_2^2 \le \frac{1}{\lambda_{k+1}}\|w\|^2, \qquad \forall\, v \in V,\ w \in W. \tag{2.4}$$

The condition (I_1) easily follows from the above inequalities.

Lemma 2.2. *The functional I_λ satisfies assumption (I_1) of Theorem 2.1 .*

Proof. Using (2.4) and (2.1) we obtain, for any $w \in W$,

$$I_\lambda(w) \ge \frac{1}{2}\left(\frac{\lambda_{k+1}-\lambda}{\lambda_{k+1}}\right)\|w\|^2 - \frac{1}{2_*}\|w\|_{2_*}^{2_*} \ge \|w\|^2\left(\frac{C_1}{2} - \frac{1}{2_*}S^{-2_*/2}\|w\|^{2_*-2}\right),$$

where $C_1 := (\lambda_{k+1}-\lambda)/\lambda_{k+1} > 0$. Hence,

$$I_\lambda(w) \ge \frac{\rho^2 C_1}{4}, \qquad \forall\, w \in W \cap \partial B_\rho(0),$$

for $\rho := \left[(2_* C_1 S^{2_*/2})/4\right]^{1/(2_*-2)}$. The lemma is proved. □

The proof of (I_2) is more involved since the usual techniques are not sufficient to show that I_λ is anticoercive in general finite-dimensional subspaces. Thus, we need to construct a specific subspace where this property holds. In order to do this we need to perform a detailed study of the solutions of (LP). We start with an interesting result proved by Escobedo and Kavian [19, Proposition 2.3] via a Fourier Transform approach:

Proposition 2.3. *The eigenvalues of the problem*

$$\begin{cases} -\operatorname{div}(K(x)\nabla u) = \mu K(x)u, & \text{in } \mathbb{R}^N, \\ u \in \mathcal{D}_K^{1,2}(\mathbb{R}^N), \end{cases} \tag{2.5}$$

are $\mu_k = (N+k-1)/2$, with $k \in \mathbb{N}$. The associated eigenspaces are given by

$$\mathcal{V}_k := \mathit{span}\left\{D^\beta\varphi_1 \,:\, |\beta| = k-1\right\}$$

where $\varphi_1(x) = \exp(-|x|^2/4)$, $\beta \in (\mathbb{N}\cup\{0\})^N$, $|\beta| := \beta_1 + \cdots + \beta_N$ and $D^\beta := \partial^{\beta_1}\cdots\partial^{\beta_N}$. In particular, any eigenfunction can be written as $P(x)\varphi_1(x)$, for some polynomial function P.

As an application of the above result, we can describe the shape of the solutions of the problem (LP). More specifically, we have the following:

Lemma 2.4. *If $\varphi \in X$ is an eigenfunction of (LP), then there exists a polynomial $p(x)$ such that $\varphi(x) = p(x)\varphi_1(x)$, for any $x \in \mathbb{R}^N_+$.*

Proof. Suppose that $\varphi \in X$ is an eigenfunction of (LP) and define

$$v(x', x_N) := \begin{cases} \varphi(x', x_N), & \text{if } x_N \geq 0, \\ \varphi(x', -x_N), & \text{if } x_N < 0. \end{cases}$$

Since $\frac{\partial \varphi}{\partial x_N}(x', 0) = 0$ in $\mathbb{R}^{N-1}$, we can check that $v \in \mathcal{D}^{1,2}_K(\mathbb{R}^N)$. Moreover, $v_{|_{\mathbb{R}^N_-}} \in \mathcal{D}^{1,2}_K(\mathbb{R}^N_-)$ is a solution of a linear problem analogous to (LP) but with $\mathbb{R}^N_+$ replaced by $\mathbb{R}^N_- := \{(x', x_N) : x' \in \mathbb{R}^{N-1}, x_N < 0\}$.

Let $\phi \in C_0^\infty(\mathbb{R}^N)$ and denote by $\phi_+ \in C_0^\infty(\overline{\mathbb{R}^N_+})$ the restriction of ϕ to $\mathbb{R}^N_+$. We define ϕ_- in an analogous way and compute

$$\begin{aligned} \int_{\mathbb{R}^N} K(x)(\nabla v \cdot \nabla \phi)\, dx &= \int_{\mathbb{R}^N_+} K(x)(\nabla \varphi \cdot \nabla \phi_+)\, dx \\ &\quad + \int_{\mathbb{R}^N_-} K(x)(\nabla \varphi(x', -x_N) \cdot \nabla \phi_-)\, dx \\ &= \lambda \int_{\mathbb{R}^N_+} K(x) v \phi_+\, dx + \lambda \int_{\mathbb{R}^N_-} K(x) v \phi_-\, dx \\ &= \lambda \int_{\mathbb{R}^N} K(x) v \phi\, dx, \end{aligned}$$

that is, v is an eigenfunction of (2.5). The result follows from Proposition 2.3. □

We are ready to prove a technical result which will be useful for verifying the geometric condition (I_2).

Proposition 2.5. *Suppose that $\phi \in C_0^\infty(\overline{\mathbb{R}^N_+}) \setminus \{0\}$ is such that $\phi_{|_{\mathbb{R}^{N-1}}} \neq 0$ and its orthogonal projection $\phi^\perp$ over W is nonzero. Then the functional I_λ satisfies assumption (I_2) of Theorem 2.1 for $e := \phi^\perp / \|\phi^\perp\|$.*

Proof. Since $\lambda > \lambda_k$, we can use the variational inequality (2.4) to check that $I_\lambda \leq 0$ in V. From the definition of Q given in Theorem 2.1, condition (I_2) holds if we can prove that

$$\lim_{\|z\| \to +\infty,\, z \in V \oplus \mathbb{R}e} I_\lambda(z) = -\infty. \tag{2.6}$$

In order to prove the above claim, we first notice that there exists a maximal set of indices $L = \{j_1, \ldots, j_l\} \subset \{1, \ldots, k\}$ such that $\mathcal{O} := \{\varphi_{j_1}(x', 0), \ldots, \varphi_{j_l}(x', 0)\}$ is linearly independent and

$$\operatorname{span} \mathcal{O} = \operatorname{span}\{\varphi_1(x', 0), \ldots, \varphi_k(x', 0)\} \tag{2.7}$$

After a rearrangement, we may assume that $L = \{1, 2, \ldots, m\}$, with $m \leq k$.

We first show that the function

$$|(b_1, \cdots, b_m, b_{m+1})|_1 := \| b_1 \varphi_1 + \cdots + b_m \varphi_m + b_{m+1} \phi^\perp \|_{2_*}$$

defines a norm in $\mathbb{R}^{m+1}$. Indeed, suppose that $|(b_1, \cdots, b_m, b_{m+1})|_1 = 0$, in such way that

$$b_1 \varphi_1(x', 0) + \cdots + b_m \varphi_m(x', 0) + b_{m+1} \phi^\perp(x', 0) = 0, \quad \forall\, x' \in \mathbb{R}^{N-1}.$$

If $b_{m+1} \neq 0$, then $\phi^\perp(\cdot, 0)$ is a linear combination of the elements of $\mathcal{O}$. By Lemma 2.4, there exists a polynomial q such that $\phi^\perp(x', 0) = q(x')\varphi_1(x', 0)$, for any $x' \in \mathbb{R}^{N-1}$. Since $X = V \oplus W$ and, consequently, $\phi - \phi^{\perp} \in \text{span}\{\varphi_1, \cdots, \varphi_k\}$, it follows again from Lemma 2.4 that there exists a polynomial r such that

$$\phi(x', 0) = [(\phi - \phi^{\perp}) + \phi^{\perp}](x', 0) = r(x')\varphi_1(x', 0), \quad \forall\, x' \in \mathbb{R}^{N-1}.$$

But $\phi_{|_{\mathbb{R}^{N-1}}} \neq 0$, $\varphi_1 > 0$ and ϕ has compact support, and therefore we could construct polynomials of type $t \mapsto p(x_1, \ldots, t, \ldots, x_{N-1})$ with infinitely many roots, which is absurd. Thus, we have that $b_{m+1} = 0$ and, since $\mathcal{O}$ is linearly independent, all the others coefficients are also null. The other properties of a norm can be easily verified.

Now we prove that there exist m polynomials $Q_i : \mathbb{R}^k \to \mathbb{R}$ of degree 1, $1 \leq i \leq m$, and $C_1 > 0$ such that

$$\text{(2.8)} \quad \| a_1\varphi_1 + \cdots + a_k\varphi_k + a_{k+1}\phi^{\perp} \|_{2_*} \geq C_1 \left[\left(\sum_{i=1}^{m} Q_i^2(a_1, \ldots, a_k) \right) + a_{k+1}^2 \right]^{1/2},$$

for any $a_1, \ldots, a_{k+1} \in \mathbb{R}$. Indeed, since $|\cdot|_1$ is a norm in $\mathbb{R}^{m+1}$, there exists $C_1 > 0$ such that

$$\text{(2.9)} \qquad |(b_1, \ldots, b_m, b_{m+1})|_1 \geq C_1 \left(\sum_{i=1}^{m+1} b_i^2 \right)^{1/2},$$

for any $(b_1, \ldots, b_{m+1}) \in \mathbb{R}^{m+1}$. For each $l = 1, \ldots, k$, we infer from (2.7) that $\varphi_l = \sum_{i=1}^{m} c_i^l \varphi_i$ in $\mathbb{R}^{N-1}$, and consequently

$$\left(\sum_{l=1}^{k} a_l\varphi_l \right) + a_{k+1}\phi^{\perp} = \left(\sum_{i=1}^{m} Q_i(a)\varphi_i \right) + a_{k+1}\phi^{\perp}, \quad \text{in } \mathbb{R}^{N-1},$$

where $Q_i(a) := \sum_{l=1}^{k} a_l c_i^l$ and $a = (a_1, \ldots, a_k) \in \mathbb{R}^k$. Setting $b_i := Q_i(a)$, $1 \leq i \leq m$, and $b_{m+1} = a_{k+1}$, (2.8) is a direct consequence of the above expression, (2.9) and the definition of $|\cdot|_1$.

We are ready to prove (2.6). Let $z = \left(\sum_{i=1}^{k} a_i\varphi_i \right) + a_{k+1}\phi^{\perp} \in V \oplus \mathbb{R}e$ and notice that, by (LP) and the orthogonality of the eigenfunctions, we have that

$$I_\lambda(z) = -\frac{1}{2} \sum_{i=1}^{k} a_i^2(\lambda - \lambda_i)\|\varphi_i\|_2^2 + \frac{a_{k+1}^2}{2}(\|\phi^{\perp}\|^2 - \lambda\|\phi^{\perp}\|_2^2) - \frac{1}{2_*}\| z \|_{2_*}^{2_*}.$$

Hence, if we set

$$C_2 := \min_{1 \leq i \leq k} (\lambda - \lambda_i)\|\varphi_i\|_2^2 > 0, \qquad C_3 := (\|\phi^{\perp}\|^2 - \lambda\|\phi^{\perp}\|_2^2) > 0,$$

it follows from (2.4) and (2.8) that

$$\text{(2.10)} \qquad I_\lambda(z) \leq -\frac{C_2}{2} \left(\sum_{i=1}^{k} a_i^2 \right) + \frac{C_3}{2} a_{k+1}^2 - \frac{C_1^{2_*}}{2_*}|a_{k+1}|^{2_*}.$$

Since $V \oplus \mathbb{R}e$ is finite-dimensional, there exists $C_4 > 0$ such that

$$C_4\|z\|^2 \leq |z|_1^2 = \left(\sum_{i=1}^{k} a_i^2 \right) + a_{k+1}^2.$$

So, if $\|z\| \to +\infty$, at least one of the terms on the right-hand side above goes to infinity and therefore (2.6) is a consequence of (2.10). The proposition is proved. □

In the final result of this section, we follow ideas of the celebrated paper of Brezis and Nirenberg [5] to get a local compactness result.

Proposition 2.6. *Suppose that* $(u_n) \in X$ *satisfies*

$$0 \neq \lim_{n\to\infty} I_\lambda(u_n) = d < \frac{1}{2(N-1)} S^{N-1}, \qquad \lim_{n\to\infty} I_\lambda'(u_n) = 0. \tag{2.11}$$

Then (u_n) *is bounded and, along a subsequence,* (u_n) *weakly converges to a nonzero weak solution to* (P_λ).

Proof. Since

$$I_\lambda(u) := \frac{1}{2}\|u\|^2 - \frac{\lambda}{2}\|u\|_2^2 - \frac{1}{2_*}|\!|\!| u |\!|\!|_{2_*}^{2_*}, \quad \forall\, u \in X,$$

we get that

$$I_\lambda'(u)v := \int_{\mathbb{R}^N_+} K(x)\nabla u \cdot \nabla v\, dx - \lambda \int_{\mathbb{R}^N_+} K(x) uv\, dx - \int_{\mathbb{R}^{N-1}} |u|^{2_*-2} uv\, dx',$$

for any $u, v \in X$. Hence, from (2.11), we obtain

$$\left(\frac{1}{2} - \frac{1}{2_*}\right) |\!|\!| u_n |\!|\!|_{2_*}^{2_*} = I_\lambda(u_n) - \frac{1}{2} I_\lambda'(u_n)u_n \leq C_1 + C_1\|u_n\|. \tag{2.12}$$

Using the decomposition $X = V \oplus W$, one can write $u_n = v_n + w_n$, with $v_n \in V$ and $w_n \in W$. Setting

$$J(u) := \frac{1}{2_*}\int_{\mathbb{R}^{N-1}} K(x',0)|u|^{2_*}\, dx', \quad \forall\, u \in X,$$

we can use (2.11) and (2.4) to get

$$\begin{aligned} C_2 + o_n(1)\|v_n\| &\geq I_\lambda(u_n) - \frac{1}{2} I_\lambda'(u_n)v_n \\ &\geq \frac{1}{2}\|w_n\|^2 - \frac{\lambda}{2}\|w_n\|_2^2 + \frac{1}{2}J'(u_n)v_n - \frac{1}{2_*}|\!|\!| u_n |\!|\!|_{2_*}^{2_*} \\ &\geq \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_{k+1}}\right)\|w_n\|^2 + \frac{1}{2}J'(u_n)v_n - \frac{1}{2_*}|\!|\!| u_n |\!|\!|_{2_*}^{2_*}, \end{aligned}$$

where $o_n(1)$ stands for a quantity approaching zero as $n \to +\infty$. If $A_1 := (\lambda_{k+1} - \lambda)/(2\lambda_{k+1}) > 0$, the above expression, (2.12), Holder's inequality imply that

$$\begin{aligned} A_1\|w_n\|^2 &\leq C_3 + o_n(1)\|v_n\| + C_3\|u_n\| - \frac{1}{2}\int_{\mathbb{R}^{N-1}} K(x',0)|u_n|^{2_*-2}u_n v_n\, dx' \\ &\leq C_3 + C_4\|u_n\| + C_5 |\!|\!| u_n |\!|\!|_{2_*}^{2_*-1} |\!|\!| v_n |\!|\!|_{2_*}. \end{aligned}$$

By using the trace embedding we obtain

$$A_1\|w_n\|^2 \leq C_3 + C_4\|u_n\| + C_6(C_1 + C_1\|u_n\|)^{(2_*-1)/2_*}\|u_n\|,$$

and therefore

$$A_1\|w_n\|^2 \leq C_3 + C_7\|u_n\| + C_8\|u_n\|^{2-(1/2_*)}. \tag{2.13}$$

On the other hand, from (2.4) we obtain

$$o_n(1)\|v_n\| = I_\lambda'(u_n)v_n \leq \left(1 - \frac{\lambda}{\lambda_k}\right)\|v_n\|^2 - \int_{\mathbb{R}^{N-1}} K(x',0)|u_n|^{2_*-2}u_n v_n\, dx'.$$

and we can argue as above to get

$$A_2\|v_n\|^2 \le C_9\|u_n\| + C_{10}\|u_n\|^{2-(1/2_*)},$$

where $A_2 := (\lambda - \lambda_k)/\lambda_k > 0$. Since $\|u_n\|^2 = \|v_n\|^2 + \|w_n\|^2$, the above expression and (2.13) imply that

$$\|u_n\|^2 \le C_{11} + C_{12}\|u_n\| + C_{13}\|u_n\|^{2-(1/2_*)}.$$

and therefore it follows from $2-(1/2_*) < 2$ that (u_n) is bounded in X.

Up to a subsequence, we may assume that

$$\begin{cases} u_n \rightharpoonup u, & \text{weakly in } X,\\ u_n \to u, & \text{strongly in } L^2_K(\mathbb{R}^N_+),\\ u_n \to u, & \text{strongly in } L^s_K(\mathbb{R}^{N-1}),\end{cases}$$

for any $2 \le s < 2*$ and for some $u \in X$. Given $\phi \in C_0^\infty(\overline{\mathbb{R}^N_+})$, we can use the above convergences, Young's inequality and standard computations to show that

$$0 = \lim_{n\to+\infty} I'_\lambda(u_n)\phi = I'_\lambda(u)\phi,$$

and therefore u is a critical point of I_λ.

We prove now that $u \ne 0$. Suppose, by contradiction, that this is not the case. Then, $u_n \to 0$ in $L^2_K(\mathbb{R}^N_+)$ and we can use $I_\lambda(u_n) \to d$ and $I'_\lambda(u_n)u_n \to 0$ to obtain

$$\frac{1}{2}\|u_n\|^2 - \frac{1}{2_*}\|u_n\|_{2_*}^{2_*} = d + o_n(1) \tag{2.14}$$

and

$$\|u_n\|^2 - \|u_n\|_{2_*}^{2_*} = o_n(1).$$

Since we may assume that $\|u_n\|^2 \to l \ge 0$, the above expression shows that $\|u_n\|_{2_*}^{2_*} \to l$. Thus, it follows from (2.14) that

$$d = \left(\frac{1}{2} - \frac{1}{2_*}\right) l = \frac{1}{2(N-1)} l. \tag{2.15}$$

Recall that the constant $S(K)$ defined in (2.1) is equal to the best constant S of the trace embedding $\mathcal{D}^{1,2}(\mathbb{R}^N_+) \hookrightarrow L^{2_*}(\mathbb{R}^{N-1})$. So, passing the inequality $S\|u_n\|_{2_*}^2 \le \|u_n\|^2$ to the limit we obtain $Sl^{2/2_*} \le l$. If $l > 0$, we conclude that $l \ge S^{N-1}$. Combining this with (2.15), we obtain $d \ge S^{N-1}/[2(N-1)]$, which is a contradiction. Hence, $l = 0$ and therefore $u_n \to 0$ in X, which implies that $I_\lambda(u_n) \to d = 0$, contrary to the hypothesis. Thus, $u \ne 0$ and we have done. □

3. Nodal solution for $\lambda > \lambda_1$

We devote this section to the proof of Theorem 1.1. For any $\varepsilon > 0$, consider the function

$$U_\varepsilon(x', x_N) := \frac{\varepsilon^{(N-2)/2}}{[|x'|^2 + (x_N+\varepsilon)^2]^{(N-2)/2}}, \qquad (x', x_N) \in \mathbb{R}^N_+.$$

They are the so-called *instantons* which achieves the best constant of the Sobolev trace embedding $\mathcal{D}^{1,2}(\mathbb{R}^N_+) \hookrightarrow L^{2_*}(\mathbb{R}^{N-1})$ (see [16]).

We now fix $R > 0$, pick $\phi \in C^\infty(\overline{\mathbb{R}^N_+}, [0,1])$ such that $\phi \equiv 1$ in $\overline{\mathbb{R}^N_+} \cap B_R(0)$, $\phi \equiv 0$ in $\overline{\mathbb{R}^N_+} \backslash B_{2R}(0)$ and set, for each $\varepsilon > 0$,

$$\psi_\varepsilon(x) := K(x)^{-1/2}\phi(x)U_\varepsilon(x), \quad x \in \mathbb{R}^N_+.$$

This function ψ_ε was extensively exploited in [21], where it was proved that, if $N \geq 7$, then

$$\|\psi_\varepsilon\|^2 = A_N + O(\varepsilon^4) + \varepsilon^2 \gamma_N, \qquad \|\psi_\varepsilon\|_2^2 = O(\varepsilon^{N-2}) + \varepsilon^2 \alpha_N$$

and

$$|\psi_\varepsilon|_{2_*}^{2_*} = B_N^{2_*/2} - \varepsilon^2 D_N + o(\varepsilon^2), \tag{3.1}$$

where the constants A_N, B_N, D_N, α_N, $\gamma_N > 0$ depend only on the dimension N. Moreover, if we set

$$Q_\lambda(u) := \frac{\|u\|^2 - \lambda \|u\|_2^2}{|u|_{2_*}^2}, \quad \forall\, u \in X \setminus \{0\},$$

there exists $E_N > 0$, depending only on N, such that

$$Q_\lambda(\psi_\varepsilon) = S + \varepsilon^2 \left(-E_N + o(1)\right), \tag{3.2}$$

whenever $\lambda > \lambda_N^*$. It is worth mention that, along this section, the notations O and o refers to $\varepsilon \to 0^+$.

Remark 3.1. *We would like to emphasize that all the constants above can be explicitly computed in terms of the Beta function*

$$B(a,b) := \int_0^\infty \frac{s^{a-1}}{(s+1)^{a+b}} ds, \quad \forall\, a,\, b > 0,$$

the dimension N and the volume σ_{N-2} of the $(N-2)$-dimensional sphere. Actually,

$$A_N := \int_{\mathbb{R}_+^N} |\nabla U_\varepsilon|^2 dx, \qquad B_N := \left(\int_{\mathbb{R}^{N-1}} |U_\varepsilon|^{2_*} dx' \right)^{2/2_*},$$

$$D_N := \frac{\sigma_{N-2}}{8(N-2)} B\left(\frac{N+1}{2}, \frac{N-3}{2}\right), \qquad \alpha_N := \frac{\sigma_{N-2}}{2(N-4)} B\left(\frac{N-1}{2}, \frac{N-3}{2}\right)$$

$$\gamma_N := \frac{\sigma_{N-2}(N-2)}{4(N-4)} \left[B\left(\frac{N+1}{2}, \frac{N-3}{2}\right) + \frac{1}{(N-3)} B\left(\frac{N-1}{2}, \frac{N-1}{2}\right) \right]$$

and

$$E_N := \frac{\lambda \alpha_N - \gamma_N - (2/2_*) A_N B_N^{-2/2_*} D_N}{B_N}.$$

Before stating our next result, we need to introduce some useful notation. For any u_1, $u_2 \in X$, we denote

$$(u_1, u_2) := \int_{\mathbb{R}_+^N} K(x) \left(\nabla u_1 \cdot \nabla u_2\right) dx, \qquad (u_1, u_2)_2 := \int_{\mathbb{R}_+^N} K(x) u_1 u_2 \, dx. \tag{3.3}$$

Since ψ_ε has compact support, for any $\tau \geq 1$ it is well defined

$$|\psi_\varepsilon|_\tau := \left(\int_{\mathbb{R}^{N-1}} K(x', 0) |\psi_\varepsilon|^\tau dx' \right)^{1/\tau}.$$

Moreover, the following holds:

Lemma 3.2. *We have that*

$$|\psi_\varepsilon|_\tau^\tau = O(\varepsilon^{(N-1)-\tau(N-2)/2}), \qquad |\psi_\varepsilon|_1 = O(\varepsilon^{(N-2)/2}), \tag{3.4}$$

$$(v, \psi_\varepsilon) = \|v\|_2 O(\varepsilon^{(N-2)/2}), \qquad (v, \psi_\varepsilon)_2 = \|v\|_2 O(\varepsilon^{(N-2)/2}), \tag{3.5}$$

for any $v \in V$ and $\tau \in \mathbb{R}$ such that $(N-1)/(N-2) < \tau < 2_*$.

Proof. For saving notation, we write only K and ϕ to denote $K(x',0)$ and $\phi(x',0)$, respectively. Using the definition of ψ_ε and the change of variable $y' = (x'/\varepsilon)$, we get

$$\begin{aligned}
\int\limits_{\mathbb{R}^{N-1}} K|\psi_\varepsilon|^\tau dx' &= \varepsilon^{\tau(N-2)/2} \int\limits_{\mathbb{R}^{N-1}} \frac{K^{(2-\tau)/2}\phi^\tau}{[|x'|^2+\varepsilon^2]^{\tau(N-2)/2}} dx' \\
&\leq C_1 \varepsilon^{-\tau(N-2)/2} \int\limits_{B_{2R}(0)\cap\mathbb{R}^{N-1}} \frac{1}{[|x'/\varepsilon|^2+1]^{\tau(N-2)/2}} dx' \\
&\leq C_1 \varepsilon^{(N-1)-\tau(N-2)/2} \int\limits_{\mathbb{R}^{N-1}} \frac{1}{[|y'|^2+1]^{\tau(N-2)/2}} dy'.
\end{aligned}$$

Using $\tau > (N-1)/(N-2)$, we obtain

$$\begin{aligned}
\int\limits_{\mathbb{R}^{N-1}} \frac{1}{[|y'|^2+1]^{\tau(N-2)/2}} dy' &\leq C_2 + \int\limits_{\{|y'|\geq 1\}} \frac{1}{|y'|^{\tau(N-2)}} dy' \\
&\leq C_2 + C_3 \int_1^{+\infty} r^{-\tau(N-2)} r^{N-2} dr < +\infty,
\end{aligned}$$

and therefore the first equality in (3.4) holds. For the second one, notice that

$$\begin{aligned}
\int\limits_{\mathbb{R}^{N-1}} K|\psi_\varepsilon|\, dx' &= \varepsilon^{(N-2)/2} \int\limits_{\mathbb{R}^{N-1}} \frac{K^{1/2}\phi}{[|x'|^2+\varepsilon^2]^{(N-2)/2}} dx' \\
&\leq C_4 \varepsilon^{(N-2)/2} \int\limits_{B_{2R}(0)\cap\mathbb{R}^{N-1}} \frac{1}{[|x'|^2+\varepsilon^2]^{(N-2)/2}} dx' \\
&\leq C_4 \varepsilon^{(N-2)/2} \int\limits_{B_{2R}(0)\cap\mathbb{R}^{N-1}} \frac{1}{|x'|^{N-2}} dx'.
\end{aligned}$$

Again, the last integral above is finite.

For proving (3.5) we pick $v = \sum_{i=1}^k a_i\varphi_i \in V$ and notice that, since each $\varphi_i \in X$ is a solution to the linear problem (LP) with $\lambda = \lambda_i$, then

$$\begin{aligned}
|(v,\psi_\varepsilon)| &= \left|\sum_{i=1}^k \lambda_i a_i (\varphi_i,\psi_\varepsilon)_2\right| \leq \lambda_k \sum_{i=1}^k |a_i||(\varphi_i,\psi_\varepsilon)_2| \\
&\leq \lambda_k \sum_{i=1}^k |a_i|\|\varphi_i\|_{L^\infty(\mathbb{R}^N_+)} \int_{\mathbb{R}^N_+} K(x)|\psi_\varepsilon|\, dx.
\end{aligned}$$

Since all the norms in V are equivalent, there exists $C_5 > 0$, independent of v, such that $\sum_{i=1}^k |a_i| \leq C_5\|v\|_2$. Hence, if we set $C_6 := \lambda_k \max\limits_{1\leq i\leq n} \|\varphi_i\|_{L^\infty(\mathbb{R}^N_+)}$, we obtain

$$\begin{aligned}
|(v,\psi_\varepsilon)| &\leq C_5 C_6 \|v\|_2 \varepsilon^{(N-2)/2} C_7 \int_{B_{2R}(0)\cap\mathbb{R}^N_+} \frac{1}{|x|^{(N-2)}}\, dx \\
&\leq C_8 \|v\|_2 \varepsilon^{(N-2)/2},
\end{aligned}$$

from which the first equality in (3.5) follows. The second one can be proved along the same lines. □

The following result is the keystone for proving Theorem 1.1.

Proposition 3.3. *For any $\varepsilon > 0$ small, there holds*

$$\max_{u\in V\oplus\mathbb{R}\psi_\varepsilon} I_\lambda(u) < \frac{1}{2(N-1)} S^{N-1}.$$

Proof. Given $u \neq 0$, a straightforward computation yields

$$\max_{t\geq 0} I_\lambda(tu) = \frac{1}{2(N-1)} \left(\frac{\|u\|^2 - \lambda\|u\|_2^2}{|\!|u|\!|_{2_*}^2} \right)^{N-1}.$$

Therefore, by homogeneity, we see that it is sufficient to prove that

$$\max_{u\in\Sigma_\varepsilon} \left(\|u\|^2 - \lambda\|u\|_2^2\right) < S, \tag{3.6}$$

where

$$\Sigma_\varepsilon := \{u = v + t\psi_\varepsilon \,:\, v \in V,\, t \in \mathbb{R},\, |\!|u|\!|_{2_*} = 1\}.$$

We first check that, for any $u = v + t\psi_\varepsilon \in \Sigma_\varepsilon$, there holds $t = O(1)$ as $\varepsilon \to 0^+$. Indeed, setting

$$A(u) := |\!|u|\!|_{2_*}^{2_*} - |\!|v|\!|_{2_*}^{2_*} - |\!|t\psi_\varepsilon|\!|_{2_*}^{2_*},$$

integrating the equality

$$\frac{d}{ds}\left(|sv + t\psi_\varepsilon|^{2_*} - |sv|^{2_*}\right) = 2_*\left[|sv + t\psi_\varepsilon|^{2_*-2}(sv + t\psi_\varepsilon) - |sv|^{2_*-2}(sv)\right] v$$

and using the Mean Value Theorem we obtain

$$\begin{aligned} A(u) &= \int_{\mathbb{R}^{N-1}} K(x',0)\left(|v + t\psi_\varepsilon|^{2_*} - |v|^{2_*} - |t\psi_\varepsilon|^{2_*}\right) dx' \\ &= 2_* \int_{\mathbb{R}^{N-1}} \int_0^1 K(x',0)\left(|sv + t\psi_\varepsilon|^{2_*-2}(sv + t\psi_\varepsilon) - |sv|^{2_*-2}(sv)\right) v\, ds\, dx' \\ &= 2_*(2_*-1) \int_{\mathbb{R}^{N-1}} \int_0^1 K(x',0)(|sv + t\psi_\varepsilon\theta|^{2_*-2} t\psi_\varepsilon v)\, ds\, dx', \end{aligned}$$

with $\theta(x) \in [0,1]$. Since $s \in [0,1]$, we get

$$\left||sv + t\psi_\varepsilon\theta|^{2_*-2} t\psi_\varepsilon v\right| \leq C_1(|t||v|^{2_*-1}|\psi_\varepsilon| + |t|^{2_*-1}|v||\psi_\varepsilon|^{2_*-1})$$

and therefore it follows from (3.4) with $\tau = 2_* - 1 = N/(N-2)$ that

$$\begin{aligned} |A(u)| &\leq C_1|t| \int_{\mathbb{R}^{N-1}} K(x',0)|v|^{2_*-1}|\psi_\varepsilon| dx' + C_1|t|^{2_*-1} \int_{\mathbb{R}^{N-1}} K(x',0)|v||\psi_\varepsilon|^{2_*-1} dx' \\ &\leq C_1|t|\|v\|_{L^\infty(\mathbb{R}^{N-1})}^{2_*-1} O(\varepsilon^{(N-2)/2}) + C_1|t|^{2_*-1}\|v\|_{L^\infty(\mathbb{R}^{N-1})} O(\varepsilon^{(N-2)/2}). \end{aligned}$$

Since V is finite-dimensional and the eigenfunctions φ_i of (LP) are regular up to the boundary (see Lemma 2.4), there exists $C_2 > 0$, independent of v, such that $\|v\|_{L^\infty(\mathbb{R}^{N-1})} \leq C_2 |\!|v|\!|_{2_*}$. So, we infer from the above expression that

$$|A(u)| \leq |t| |\!|v|\!|_{2_*}^{2_*-1} O(\varepsilon^{(N-2)/2}) + |t|^{2_*-1} |\!|v|\!|_{2_*} O(\varepsilon^{(N-2)/2}). \tag{3.7}$$

From Young's inequality with exponents $s = 2_*/(2_*-1)$ and $s' = 2_*$, we get

$$\begin{aligned} \|v\|_{2_*}^{2_*-1}|t|O(\varepsilon^{(N-2)/2}) &\leq \frac{1}{4}\|v\|_{2_*}^{2_*} + C_3|t|^{2_*}O(\varepsilon^{(N-2)/2})^{2_*} \\ &= \frac{1}{4}\|v\|_{2_*}^{2_*} + C_3|t|^{2_*}O(\varepsilon^{N-1}) \end{aligned}$$

and

$$\|v\|_{2_*}|t|^{2_*-1}O(\varepsilon^{(N-2)/2}) \leq \frac{1}{4}\|v\|_{2_*}^{2_*} + C_4|t|^{2_*}O(\varepsilon^{(N-1)(N-2)/N}).$$

Replacing the above expressions in (3.7) and using $(N-1)(N-2)/N < (N-1)$, we obtain

$$|A(u)| \leq \frac{1}{2}\|v\|_{2_*}^{2_*} + |t|^{2_*}O(\varepsilon^{(N-1)(N-2)/N}).$$

Hence, using (3.1) we get

$$\begin{aligned} 1 &= \|u\|_{2_*}^{2_*} = A(u) + \|v\|_{2_*}^{2_*} + \|t\psi_\varepsilon\|_{2_*}^{2_*} \\ &\geq -\frac{1}{2}\|v\|_{2_*}^{2_*} - |t|^{2_*}O(\varepsilon^{(N-1)(N-2)/N}) + \|v\|_{2_*}^{2_*} + |t|^{2_*}\|\psi_\varepsilon\|_{2_*}^{2_*} \\ &= \frac{1}{2}\|v\|_{2_*}^{2_*} + |t|^{2_*}\left(B_N^{2_*/2} + O(1)\right), \end{aligned}$$

and therefore $t = O(1)$ as $\varepsilon \to 0^+$.

For any given $u = v + t\psi_\varepsilon \in \Sigma_\varepsilon$, it follows from (2.4), (3.5) and $t = O(1)$ that

$$\begin{aligned} \|u\|^2 - \lambda\|u\|_2^2 &\leq (\lambda_k - \lambda)\|v\|_2^2 + \|v\|_2 O(\varepsilon^{(N-2)/2}) + \|t\psi_\varepsilon\|^2 - \lambda\|t\psi_\varepsilon\|_2^2 \\ &\leq \frac{1}{4(\lambda-\lambda_k)}O(\varepsilon^{N-2}) + Q_\lambda(t\psi_\varepsilon)\|t\psi_\varepsilon\|_{2_*}^2, \end{aligned} \tag{3.8}$$

where we have used, in the last inequality, that $as^2 + bs \leq -b^2/(4a)$ for $a < 0$ and $s \in \mathbb{R}$. Since $Q_\lambda(t\psi_\varepsilon) = Q_\lambda(\psi_\varepsilon)$, by (3.2) we obtain that

$$Q_\lambda(t\psi_\varepsilon) = S + \varepsilon^2\left(-E_N + o(1)\right). \tag{3.9}$$

In order to estimate $\|t\psi_\varepsilon\|_{2_*}^2$ we notice that, since the function $s \mapsto |s|^{2_*}$ is convex, we have that

$$\begin{aligned} 1 &= \int_{\mathbb{R}^{N-1}} K|v + t\psi_\varepsilon|^{2_*}dx' \\ &\geq \|t\psi_\varepsilon\|_{2_*}^{2_*} + 2_*\int_{\mathbb{R}^{N-1}} K|t\psi_\varepsilon|^{2^*-2}t\psi_\varepsilon v \\ &\geq \|t\psi_\varepsilon\|_{2_*}^{2_*} - 2_*\|v\|_{L^\infty(\mathbb{R}^{N-1})}|t|^{2_*-1}\|\psi_\varepsilon\|_{2_*-1}^{2_*-1} \end{aligned}$$

and therefore we infer from (3.4) that

$$\|t\psi_\varepsilon\|_{2_*}^2 \leq \left(1 + \|v\|_{2_*}O(\varepsilon^{(N-2)/2})\right)^{2/2_*} = 1 + O(\varepsilon^{(N-2)/2}).$$

Thus, it follows from (3.9) that

$$Q_\lambda(t\psi_\varepsilon)\|t\psi_\varepsilon\|_{2_*}^2 \leq S + \varepsilon^2\left[-E_N + O(\varepsilon^{(N-6)/2}) + o(1)\right].$$

Using this inequality, $N \geq 7$ and (3.8) we obtain

$$\|u\|^2 - \lambda\|u\|_2^2 \leq S + \varepsilon^2\left[-E_N + O(\varepsilon^{(N-6)/2}) + O(\varepsilon^{N-4}) + o(1)\right] < S,$$

for any $\varepsilon > 0$ sufficiently small. This establishes (3.6) and concludes the proof. □

We are ready to present the first part of the proof of our main result.

Proof of Theorem 1.1. Consider the decomposition $X = V \oplus W$, with V and W as in (2.3). Let $\varepsilon > 0$ and notice that the function $\phi = \psi_\varepsilon$ verifies all the conditions of Proposition 2.5. Hence, we can use Lemma 2.2 and Theorem 2.1 to obtain $(u_n) \subset X$ such that

$$I_\lambda(u_n) \to c, \qquad I'_\lambda(u_n) \to 0,$$

with the minimax level $c > 0$ defined in (2.2). We can pick $\varepsilon > 0$ so small in such way that Proposition 3.3 holds. Since $V \oplus \mathbb{R}e = V \oplus \mathbb{R}\psi_\varepsilon$, Proposition 3.3 and (2.2) imply that $c < S^{N-1}/(2(N-1))$. It follows from Proposition 2.6 that I_λ has a nonzero critical point $u \in X$. In order to prove that u changes its sign we consider $\varphi_1 > 0$ a first eigenfunction of (LP) and notice that, since $I'_\lambda(u)\varphi_1 = 0$, there holds

$$(\lambda_1 - \lambda) \int_{\mathbb{R}^N_+} K(x) u \varphi_1 dx = \int_{\mathbb{R}^{N-1}} K(x', 0)|u|^{2_*-2} u \varphi_1 dx'.$$

If $u \geq 0$ in $\mathbb{R}^N_+$, it follows from the above expression and $\int_{\mathbb{R}^N_+} K(x) u \varphi_1 dx > 0$ that $\lambda \leq \lambda_1$, which is not true. A similar argument discard $u \leq 0$ and therefore the proof is complete. □

4. A Nehari type approach for $\lambda^*_N < \lambda < \lambda_1$

We present in this section some preliminary results for the proof of Theorem 1.2. From now on, we suppose that $\lambda^*_N < \lambda < \lambda_1$. Hence, we can use [21, Theorem 1.5] to obtain a positive solution $u_0 \in X$ of the problem (P_λ). Since $\| u_0 \|_{2_*} \neq 0$, the number $R > 0$ appearing in the definition of the function ψ_ε in the Section 3 can be chosen in such way that

$$\int_{\mathbb{R}^{N-1} \setminus B_{2R}(0)} K(x', 0) u_0^{2_*} dx' > 0. \tag{4.1}$$

For any given $u \in X$, we define $u^+(x) := \max\{u(x), 0\}$, $u^- := u^+ - u$ and the sets

$$\mathcal{N}_\lambda := \{u \in X \setminus \{0\} \,:\, I'_\lambda(u)u = 0\}, \qquad \mathcal{M}_\lambda := \left\{u \in X \,:\, u^\pm \in \mathcal{N}_\lambda\right\}.$$

Notice that the Nehari manifold $\mathcal{N}_\lambda$ contains all the nonzero critical points of I_λ and $\mathcal{M}_\lambda \subset \mathcal{N}_\lambda$. The idea is to look for a critical point of I_λ which belongs to $\mathcal{M}_\lambda$ and therefore changes sign.

If $u \in \mathcal{N}_\lambda$, we have that

$$\|u\|^2 = \lambda \|u\|_2^2 + \| u \|_{2_*}^{2_*} \leq \frac{\lambda}{\lambda_1}\|u\|^2 + S^{-2_*/2}\|u\|^{2_*},$$

and therefore there exists $\gamma > 0$ such that

$$\|u\| \geq \gamma, \quad \forall\, u \in \mathcal{N}_\lambda. \tag{4.2}$$

Moreover, on $\mathcal{N}_\lambda$ we have that

$$I_\lambda(u) = \left(\frac{1}{2} - \frac{1}{2_*}\right)\|u\|^2 - \lambda\left(\frac{1}{2} - \frac{1}{2_*}\right)\|u\|_2^2 \geq \frac{1}{2(N-1)}\left(1 - \frac{\lambda}{\lambda_1}\right)\|u\|^2,$$

in such way that we can define the positive numbers

$$c_\lambda := \inf_{u \in \mathcal{N}_\lambda} I_\lambda(u), \qquad d_\lambda := \inf_{u \in \mathcal{M}_\lambda} I_\lambda(u).$$

Although $\mathcal{M}_\lambda$ is not a differentiable manifold, we can adapt an argument of [33] for proving the following:

Lemma 4.1. *There exists a sequence* $(u_n) \subset \mathcal{M}_\lambda$ *such that* $I_\lambda(u_n) \to d_\lambda$ *and* $I'_\lambda(u_n) \to 0$*, as* $n \to +\infty$*.*

Proof. Using Ekeland's Variational Principle, we obtain a sequence $(u_n) \subset \mathcal{M}_\lambda$ such that

$$I_\lambda(u_n) \le d_\lambda + \frac{1}{n}, \qquad I_\lambda(z) \ge I_\lambda(u_n) - \frac{1}{n}\|z - u_n\|, \quad \text{for all } z \in \mathcal{M}_\lambda. \tag{4.3}$$

Using (4.2) and recalling that I_λ is coercive over $\mathcal{M}_\lambda$ we obtain $\mu > \gamma > 0$ such that $\gamma \le \|u_n^\pm\| \le \mu$, for all $n \in \mathbb{N}$.

We claim that there exists $K = K(\lambda, \gamma, \mu) > 0$ such that $\|I'_\lambda(u_n)\| \le K/n$, for all $n \in \mathbb{N}$. If this is true we obtain $I'_\lambda(u_n) \to 0$ and the result follows from (4.3).

In order to prove the claim, we fix $n \in \mathbb{N}$ and $v \in X$ such that $\|v\| \le 1$ and notice that, since $(u_n - \delta v)^\pm \to u_n^\pm$ as $\delta \to 0$, then

$$\phi_{\delta,n}^\pm := (u_n - \delta v)^\pm \ne 0,$$

for any δ small. For simplicity, we drop the subscript n in what follows. The above expression and a direct computation shows that,

$$z_\delta := t_\delta^+ \phi_\delta^+ - t_\delta^- \phi_\delta^- \in \mathcal{M}_\lambda$$

where $t_\delta^\pm$ are given by

$$t_\delta^\pm = \left(\frac{\|(u - \delta v)^\pm\|^2 - \lambda \|(u - \delta v)^\pm\|_2^2}{\boldsymbol{|}(u - \delta v)^\pm \boldsymbol{|}_{2_*}^{2_*}} \right)^{1/(2_* - 2)}.$$

Setting $g_\pm(\delta) := t_\delta^\pm$, we obtain from the above expression that $g_\pm(0) = 1$ and

$$(2_* - 2) g'_\pm(0) = \frac{-2(u^\pm, v) + 2\lambda (u^\pm, v)_2 + 2_* \int_{\mathbb{R}^{N-1}} K(x', 0)(u^\pm)^{2_* - 1} v \, dx'}{\boldsymbol{|} u^\pm \boldsymbol{|}_{2_*}^{2_*}},$$

where the inner products $(\cdot, \cdot)$ and $(\cdot, \cdot)_2$ were defined in (3.3). Since $\|u^\pm\| \ge \gamma$, we have that

$$\boldsymbol{|} u^\pm \boldsymbol{|}_{2_*}^{2_*} = \|u^\pm\|^2 - \lambda \|u^\pm\|_2^2 \ge \gamma^2 \left(1 - \frac{\lambda}{\lambda_1}\right)$$

and therefore, using $\|u^\pm\| \le \mu$ and Hölder's inequality we obtain

$$|g'_\pm(0)| \le \frac{2\|u^\pm\| \|v\| + 2\lambda \|u^\pm\|_2 \|v\|_2 + 2_* \boldsymbol{|} u^\pm \boldsymbol{|}_{2_*}^{2_* - 1} \boldsymbol{|} v \boldsymbol{|}_{2_*}}{(2_* - 2)\gamma^2 (\lambda_1 - \lambda)/\lambda_1} \le C_1 \tag{4.4}$$

for

$$C_1 := \frac{2\mu + 2(\lambda/\lambda_1)\mu + 2_* S^{-2_*/2} \mu^{2_* - 1}}{(2_* - 2)\gamma^2 (\lambda_1 - \lambda)/\lambda_1}.$$

We now notice that

$$z_\delta - u = \left(t_\delta^+ - 1\right) \phi_\delta^+ - \left(t_\delta^- - 1\right) \phi_\delta^- - \delta v, \tag{4.5}$$

and therefore

$$\frac{\|z_\delta - u\|}{\delta} = \|g'_+(0) u^+ - g'_-(0) u^- - v\| + o_\delta(1), \tag{4.6}$$

as $\delta \to 0^+$. Thus, we can use (4.3) to get

$$I'_\lambda(u)(z_\delta - u) + o_\delta(\|z_\delta - u\|) = I_\lambda(z_\delta) - I_\lambda(u) \ge -\frac{1}{n}\|z_\delta - u\|.$$

It follows from (4.5) and $g_\pm(0)=1$ that

$$\begin{aligned} I'_\lambda(u)v \quad &\le \quad \left(\frac{g_+(\delta)-g_+(0)}{\delta}\right) I'_\lambda(u)\phi_\delta^+ - \left(\frac{g_-(\delta)-g_-(0)}{\delta}\right) I'_\lambda(u)\phi_\delta^- \\ &\quad + \frac{1}{n}\frac{\|z_\delta-u\|}{\delta} + \frac{o_\delta(\|z_\delta-u\|)}{\delta}. \end{aligned}$$

Passing to the limit, recalling that $I'_\lambda(u)\phi_\delta^\pm = I'_\lambda(u)(u^\pm + o_\delta(1)) = o_\delta(1)$, using (4.6), (4.4) and $\|u^\pm\|\le\mu$ we conclude that

$$I'_\lambda(u)v \le \frac{1}{n}\|g'_+(0)u^+ - g'_-(0)u^- - v\| \le \frac{1}{n}\left(2C_1\mu+1\right), \quad \forall\, v\in X,\ \|v\|\le 1.$$

and therefore $\|I'_\lambda(u_n)\|\le K/n$, for $K=2C_1\mu+1$. The lemma is proved. □

As in the first case, the energy functional satisfies a local compactness condition.

Proposition 4.2. *Suppose that $(u_n)\subset\mathcal{M}_\lambda$ satisfies*

$$\lim_{n\to+\infty} I_\lambda(u_n) = d < c_\lambda + \frac{1}{2(N-1)}S^{N-1}, \qquad \lim_{n\to\infty} I'_\lambda(u_n)=0.$$

Then (u_n) has a convergent subsequence.

Proof. Since I_λ restricted to $\mathcal{N}_\lambda$ is coercive the sequence (u_n) is bounded in X. So, up to a subsequence, we may assume that $u_n\rightharpoonup u$ weakly X, $u_n^\pm\rightharpoonup u^\pm$ weakly in X and $u_n^\pm\to u^\pm$ strongly in $L^2_K(\mathbb{R}^N_+)$, for some $u\in X$. Arguing as in the proof of Proposition 2.6 we obtain $I'_\lambda(u)=0$. Moreover, since $(u_n)\subset\mathcal{M}_\lambda$, we have that $o_n(1)=I'_\lambda(u_n^+)u_n^+ - I'_\lambda(u)u^+$ and therefore the above convergences imply that

$$\lim_{n\to+\infty}\|u_n^+-u^+\|^2 = l, \qquad \lim_{n\to+\infty} [\![u_n^+-u^+]\!]_{2_*}^{2_*} = l, \tag{4.7}$$

for some $l\ge 0$.

We shall prove that $l=0$ and therefore $u_n^+\to u^+$ in X. Suppose, by contradiction, that $l>0$. Passing the inequality $\|u_n^+-u^+\|^2\le S^{-1}[\![u_n^+-u^+]\!]_{2_*}^2$ to the limit we get $l\ge S^{N-1}$. On the other hand, the convergences of (u_n^+) just mentioned and Brezis-Lieb's lemma [4] implies that

$$I_\lambda(u_n^+) = I_\lambda(u_n^+-u^+) + I_\lambda(u^+) + o_n(1). \tag{4.8}$$

However, by (4.7) and the strong convergence we get

$$I_\lambda(u_n^+-u^+) = \frac{1}{2}\|u_n^+-u^+\|^2 - \frac{1}{2_*}[\![u_n^+-u^+]\!]_{2_*}^{2_*} + o_n(1) = \frac{1}{2(N-1)}l + o_n(1),$$

and therefore (4.8) implies that

$$I_\lambda(u_n^+) = \frac{1}{2(N-1)}l + I_\lambda(u^+) + o_n(1).$$

Recalling that $u_n^-\in\mathcal{N}_\lambda$, we conclude that $c_\lambda\le I_\lambda(u_n^-)$. Also, since $I'_\lambda(u)u^+=0$, we have that $I_\lambda(u^+)\ge 0$. So, we can use the above inequality and $l\ge S^{N-1}$ to get

$$d+o_n(1) = I_\lambda(u_n) = I_\lambda(u_n^-) + I_\lambda(u_n^+) \ge c_\lambda + \frac{1}{2(N-1)}S^{N-1} + o_n(1).$$

Passing to the limit we obtain a contradiction. Hence $l=0$ and $u_n^+\to u^+$ strongly in X. The same argument shows that $u_n^-\to u^-$ strongly in X and the proposition is proved. □

5. Nodal solution for $\lambda_N^* < \lambda < \lambda_1$

Since we already have a Palais-Smale sequence at level d_λ, we need only to show that d_λ belongs to the compactness range of the functional I_λ. We shall use the following intersection property.

Lemma 5.1. *There exists α_*, $\beta_* \in \mathbb{R}$ such that $(\alpha_* u_0 + \beta_* \psi_\varepsilon) \in \mathcal{M}_\lambda$.*

Proof. Define

$$J(u) := \frac{|\!|\!| u |\!|\!|_{2_*}^{2_*}}{\|u\|^2 - \lambda \|u\|_2^2}, \quad \forall\, u \in X \setminus \{0\},$$

and $J(0) = 0$. From $\lambda < \lambda_1$ and the continuous embedding $X \hookrightarrow L_K^{2_*}(\mathbb{R}^{N-1})$ we obtain $0 \le J(u) \le C_0 \|u\|^{2_*-2}$, and therefore J is continuous.

We now set

$$\sigma(r,s,t) := rt\,[(1-s)u_0 - s\psi_\varepsilon]\,, \quad \forall\, r \ge 0,\ s,\ t \in [0,1].$$

and

$$\Gamma(r) := \inf_{s\in[0,1]} J(\sigma(r,s,1)), \quad \forall\, r > 0.$$

If $\Gamma(1) = 0$, then there exists $s_0 \in [0,1]$ such that $J(\sigma(1,s_0,1)) = 0$, that is, $|\!|\!|(1-s_0)u_0 - s_0\psi_\varepsilon|\!|\!|_{2_*}^{2_*} = 0$. Since ψ_ε is positive in $B_R(0)\cap\mathbb{R}^{N-1}$ and (4.1) holds, we have that $s_0 \in (0,1)$. Thus, recalling that $\psi_\varepsilon \equiv 0$ outside $B_{2R}(0)\cap\mathbb{R}^{N-1}$, we obtain

$$0 = |\!|\!|(1-s_0)u_0 - s_0\psi_\varepsilon|\!|\!|_{2_*}^{2_*} \ge (1-s_0)^{2_*} \int_{\mathbb{R}^{N-1}\setminus B_{2R}(0)} K(x',0) u_0^{2_*}\, dx',$$

which contradicts (4.1). Hence, $\Gamma(1) > 0$ and we infer from $J(\sigma(r,s,1)) = r^{2_*-2} J(\sigma(1,s,1)) \ge r^{2_*-2}\Gamma(1)$ that

$$\lim_{r\to+\infty} \Gamma(r) = +\infty.$$

Let $r_0 > 0$ be such that

$$J(\sigma(r_0,s,1)) \ge \Gamma(r_0) > 2, \quad \forall\, s\in[0,1], \tag{5.1}$$

and define the functions f, $g : [0,1]\times[0,1] \to \mathbb{R}$ as

$$f(s,t) := J(\sigma^-(r_0,s,t)) - J(\sigma^+(r_0,s,t))$$

and

$$g(s,t) := J(\sigma^+(r_0,s,t)) + J(\sigma^-(r_0,s,t)) - 2.$$

Since $\sigma(r_0,0,t) = r_0 t u_0 \ge 0$ and $\sigma(r_0,1,t) = -r_0 t \psi_\varepsilon \le 0$, it follows that

$$f(0,t) = -J(\sigma^-(r_0,1,t)) \le 0, \qquad f(1,t) = J(\sigma^+(r_0,0,t)) \ge 0,$$

for any $t\in[0,1]$. Moreover, for any $s \in [0,1]$,

$$g(s,0) = -2 \le 0, \qquad g(s,1) = J(\sigma^+(r_0,s,1)) + J(\sigma^-(r_0,s,1)) - 2 \ge 0,$$

where we have used $J(u^+) + J(u^-) \ge J(u)$ and (5.1) in the last inequality.

Using the above inequalities and Miranda's Theorem [28] we obtain s_0, $t_0 \in [0,1]$ such that $f(s_0,t_0) = 0 = g(s_0,t_0)$ and so

$$J(\sigma^+(r_0,s_0,t_0)) = 1 = J(\sigma^-(r_0,s_0,t_0)).$$

Consequently, $I'_\lambda(\sigma^\pm(r_0,s_0,t_0))\sigma^\pm(r_0,s_0,t_0) = 0$. Since $J(0) = 0$, we also have that $\sigma^\pm(r_0,s_0,t_0) \ne 0$, and therefore the lemma holds for $\alpha_* := r_0 t_0 (1-s_0)$ and $\beta_* := r_0 t_0 s_0$. $\square$

The two next results are of technical nature and it will be useful to estimate d_λ.

Lemma 5.2. *If τ_1, $\tau_2 > 1$, then there exists $A_1 = A_1(u_0, R, \tau_1, \tau_2) > 0$ such that*

$$\left| \| \alpha u_0 + \beta\psi_\varepsilon \|_{2_*}^{2_*} - \|\alpha u_0\|_{2_*}^{2_*} - \|\beta\psi_\varepsilon\|_{2_*}^{2_*} \right| \leq A_1 \left(|\alpha|^{2_*-1} \|\beta\psi_\varepsilon\|_{\tau_1} + |\alpha| \|\beta\psi_\varepsilon\|_{(2_*-1)\tau_2}^{2_*-1} \right),$$

for any α, $\beta \in \mathbb{R}$.

Proof. For simplicity, we write only K to denote $K(x', 0)$. If we call $\Psi(\alpha, \beta)$ the term into modulus in the inequality above, we have that

$$\begin{aligned} \Psi(\alpha,\beta) &= \int_{\mathbb{R}^{N-1}} K \left(\int_0^1 \frac{d}{ds} \left[|s\alpha u_0 + \beta\psi_\varepsilon|^{2_*} - |s\alpha u_0|^{2_*} \right] ds \right) dx' \\ &= 2_* \int_{\mathbb{R}^{N-1}} K \left(\int_0^1 [g(1) - g(0)]\, \alpha u_0 \, ds \right) dx' \end{aligned}$$

for $g(t) := |s\alpha u_0 + t\beta\psi_\varepsilon|^{2_*-2}(s\alpha u_0 + t\beta\psi_\varepsilon)$. From the Mean Value Theorem we obtain $\theta(x, s) \in (0, 1)$ such that

$$\Psi(\alpha,\beta) = 2_*(2_* - 1) \int_{\mathbb{R}^{N-1}} K \left(\int_0^1 \left[|s\alpha u_0 + \theta\beta\psi_\varepsilon|^{2_*-2} \alpha u_0 \beta\psi_\varepsilon \right] ds \right) dx'.$$

Since s, $\theta \in [0, 1]$, we obtain

$$|\Psi(\alpha,\beta)| \leq C_1 \int_{\mathbb{R}^{N-1}} K|\alpha u_0|^{2_*-1} |\beta\psi_\varepsilon| \, dx' + C_1 \int_{\mathbb{R}^{N-1}} K|\alpha u_0| |\beta\psi_\varepsilon|^{2_*-1} \, dx'. \tag{5.2}$$

We now notice that the positive solution $u_0 \in X$ of problem (P_λ) given in [21, Theorem 1.5] belongs to $C^2(\mathbb{R}^N_+)$. Although regularity up to the boundary is a more complicated issue, we can adapt the proof of Brezis-Kato's theorem [6] presented by Struwe [32, Lemma B.3] (see also [1, Lemma 4.1] for the normal derivative version) to conclude that $u_0 \in L^\tau_{loc}(\mathbb{R}^{N-1})$, for any $\tau \geq 1$. So, if we set $\Omega := \{x' \in \mathbb{R}^{N-1} : |x'| < 2R\}$ and recall that ψ_ε vanishes outside $B_{2R}(0)$, we can use Hölder's inequality to get

$$\int_{\mathbb{R}^{N-1}} K|\alpha u_0|^{2_*-1} |\beta\psi_\varepsilon| \, dx' \leq |\alpha|^{2_*-1} |\beta| \|u_0\|^{2_*-1}_{L_K^{(2_*-1)\tau_1'}(\Omega)} \|\psi_\varepsilon\|_{\tau_1}$$

and

$$\int_{\mathbb{R}^{N-1}} K|\alpha u_0| |\beta\psi_\varepsilon|^{2_*-1} \, dx' \leq |\alpha| |\beta|^{2_*-1} \|u_0\|^{\tau_2'}_{L_K^{\tau_2'}(\Omega)} \|\psi_\varepsilon\|^{2_*-1}_{(2_*-1)\tau_2},$$

where $\|u_0\|_{L^r_K(\Omega)} := \left(\int_\Omega K(x', 0)|u_0|^r dx' \right)^{1/r}$, for $r > 1$. So, it is sufficient to define

$$A_1 := C_1 \left(\|u_0\|^{2_*-1}_{L_K^{(2_*-1)\tau_1'}(\Omega)} + \|u_0\|^{\tau_2'}_{L_K^{\tau_2'}(\Omega)} \right)$$

and use the two above inequalities together with (5.2). □

Lemma 5.3. *If τ_1, $\tau_2 > 1$, then there exists $A_i = A_i(u_0, R, \tau_1, \tau_2, N) > 0$, $i = 2, 3$, such that*

$$\|\alpha u_0 + \beta\psi_\varepsilon\|_{2_*}^{2_*} \geq \frac{1}{3} |\alpha|^{2_*} \|u_0\|_{2_*}^{2_*} + |\beta|^{2_*} \left(\|\psi_\varepsilon\|_{2_*}^{2_*} - A_2 \|\psi_\varepsilon\|_{\tau_1}^{2_*} - A_3 \|\psi_\varepsilon\|_{(2_*-1)\tau_2}^{2_*} \right),$$

for any α, $\beta \in \mathbb{R}$.

Proof. According to last result, we have that

$$\|\alpha u_0 + \beta\psi_\varepsilon\|_{2_*}^{2_*} \geq \frac{1}{3}|\alpha|^{2_*}\|u_0\|_{2_*}^{2_*} + |\beta|^{2_*}\|\psi_\varepsilon\|_{2_*}^{2_*} + f(|\alpha|) + g(|\alpha|), \tag{5.3}$$

for f, $g : [0, +\infty) \to \mathbb{R}$ given by

$$f(s) := \frac{1}{3}\|u_0\|_{2_*}^{2_*} s^{2_*} - A_1\|\beta\psi_\varepsilon\|_{\tau_1} s^{2_*-1},$$

and

$$g(s) := \frac{1}{3}\|u_0\|_{2_*}^{2_*} s^{2_*} - A_1\|\beta\psi_\varepsilon\|_{(2_*-1)\tau_2}^{2_*-1} s.$$

The function f attains its minimum at the point

$$s_0 := \frac{3(2_* - 1)}{2_*}\frac{A_1}{\|u_0\|_{2_*}^{2_*}}\|\beta\psi_\varepsilon\|_{\tau_1},$$

and therefore

$$f(|\alpha|) \geq f(s_0) = -A_2|\beta|^{2_*}\|\psi_\varepsilon\|_{\tau_1}^{2_*}, \quad \forall\, \alpha \in \mathbb{R},$$

with $A_2 := A_2(u_0, R, \tau_1, \tau_2, N) > 0$. Analogously, there exists $A_3 > 0$ such that

$$g(|\alpha|) \geq -A_3|\beta|^{2_*}\|\psi_\varepsilon\|_{(2_*-1)\tau_2}^{2_*}, \quad \forall\, \alpha \in \mathbb{R}.$$

The lemma follows from the two above inequalities and (5.3). □

We are ready to prove our second main theorem.

Proof of Theorem 1.2. Let $\lambda_N^* < \lambda < \lambda_1$. Invoking Lemma 4.1 we obtain $(u_n) \subset \mathcal{M}_\lambda$ such that $I_\lambda(u_n) \to d_\lambda$ and $I'_\lambda(u_n) \to 0$, as $n \to +\infty$. We claim that

$$d_\lambda < c_\lambda + \frac{1}{2(N-1)}S^{N-1}. \tag{5.4}$$

If this is true, it follows from Proposition 4.2 that, along a subsequence, $u_n \to u$ strongly in X. Since $\mathcal{M}_\lambda$ is closed, we have that $u \in \mathcal{M}_\lambda$, from which we conclude that $u^\pm \neq 0$. Moreover, recalling that $\mathcal{M}_\lambda \subset \mathcal{N}_\lambda$, we conclude that $I'_\lambda(u) = 0$ and therefore $u \in X$ is a sign-changing solution for (P_λ).

For proving (5.4) we first notice that, according to Lemma 5.1, there exists α_*, $\beta_* \in \mathbb{R}$ such that $(\alpha_* u_0 + \beta_*\psi_\varepsilon) \in \mathcal{M}_\lambda$. So, it is sufficient to show that, for some $\varepsilon > 0$,

$$\sup_{\alpha,\beta\in\mathbb{R}} I_\lambda(\alpha u_0 + \beta\psi_\varepsilon) < c_\lambda + \frac{1}{2(N-1)}S^{N-1}.$$

Arguing as in the proof of Lemma 5.1, we can check that $W := \operatorname{span}\{u_0, \psi_\varepsilon\}$ is a 2-dimensional subspace. Moreover, using (4.1) and the compact support of ψ_ε, we conclude that $\|\alpha u_0 + \beta\psi_\varepsilon\|_{2_*} = 0$ if, and only if, $(\alpha, \beta) = (0, 0)$. So, the function $(\alpha, \beta) \mapsto \|\alpha u_0 + \beta\psi_\varepsilon\|_{2_*}$ defines a norm in W. From the equivalence between norms in finite-dimensional subspaces, we get

$$\lim_{(|\alpha|+|\beta|)\to+\infty} I_\lambda(\alpha u_0 + \beta\psi_\varepsilon) = -\infty,$$

and therefore we can restrict our attention to points $(\alpha, \beta) \in \mathbb{R}^2$ such that

$$\|\alpha u_0 + \beta\psi_\varepsilon\|_{2_*}^{2_*} \leq C_1,$$

for some $C_1 > 0$ large enough.

Using Lemma 5.3, we get

$$C_1 \geq \frac{1}{3}|\alpha|^{2_*}\|u_0\|_{2_*}^{2_*} + |\beta|^{2_*}\left(\|\psi_\varepsilon\|_{2_*}^{2_*} - A_2\|\psi_\varepsilon\|_{\tau_1}^{2_*} - A_3\|\psi_\varepsilon\|_{(2_*-1)\tau_2}^{2_*}\right), \tag{5.5}$$

If we pick $(N-1)/(N-2) < \tau_1 < 2(N-1)/(N+2)$, it follows from (3.4) that

$$|\psi_\varepsilon|_{\tau_1} = O(\varepsilon^{2+\nu_1}), \qquad \nu_1 := \frac{2(N-1)-\tau_1(N+2)}{2\tau_1} > 0, \tag{5.6}$$

as $\varepsilon \to 0^+$. Moreover, for

$$\frac{N-1}{N} < 1 < \tau_2 < \frac{2(N-1)}{N+4} < \frac{2(N-1)}{N} < 2_*,$$

we can apply (3.4) with $\tau = (2_*-1)\tau_2$ to get

$$|\psi_\varepsilon|^{2_*-1}_{(2_*-1)\tau_2} = O(\varepsilon^{2+\nu_2}), \quad \nu_2 := \frac{2(N-1)-\tau_2(N+4)}{2\tau_2} > 0. \tag{5.7}$$

From the above inequalities we conclude that $|\psi_\varepsilon|^{2_*}_{\tau_1} = o(1)$ and $|\psi_\varepsilon|^{2_*}_{(2_*-1)\tau_2} = o(1)$, and therefore it follows from (5.5) and (3.1) that

$$C_1 \geq \frac{1}{3}|\alpha|^{2_*}|u_0|^{2_*}_{2_*} + |\beta|^{2_*}\left(B_N^{2_*/2} + o(1)\right).$$

Since $B_N > 0$, we conclude that $\alpha = O(1)$ and $\beta = O(1)$. It is worth mentioning that the above choices for τ_1 and τ_2 are possible because $N \geq 7$.

Notice that, since $I'_\lambda(u_0)\psi_\varepsilon = 0$, then

$$\int_{\mathbb{R}^N_+} K(x)\left(\nabla u_0 \cdot \nabla \psi_\varepsilon\right) dx - \lambda \int_{\mathbb{R}^N_+} K(x) u_0 \psi_\varepsilon dx = \int_{\mathbb{R}^{N-1}} K(x',0) u_0^{2_*-1}\psi_\varepsilon dx'.$$

Thus,

$$I_\lambda(\alpha u_0 + \beta\psi_\varepsilon) \leq I_\lambda(\alpha u_0) + \frac{\beta^2}{2}\left(\|\psi_\varepsilon\|^2 - \lambda\|\psi_\varepsilon\|_2^2\right) - \frac{|\beta|^{2_*}}{2_*}|\psi_\varepsilon|^{2_*}_{2_*} + \Phi(\varepsilon,\alpha,\beta) \tag{5.8}$$

with

$$\Phi(\varepsilon,\alpha,\beta) := A_1 O(1)\left(|\psi_\varepsilon|_{\tau_1} + |\psi_\varepsilon|^{2_*-1}_{(2_*-1)\tau_2}\right) + \alpha\beta\int_{\mathbb{R}^{N-1}} K(x',0)u_0^{2_*-1}\psi_\varepsilon\, dx',$$

with the number $A_1 > 0$ given by Lemma 5.2 and we have used that α and β remain bounded as $\varepsilon \to 0^+$. Arguing as in the proof of Lemma 5.2 and recalling that $\alpha = O(1)$ and $\beta = O(1)$ as $\varepsilon \to 0^+$, we get

$$\alpha\beta\int_{\mathbb{R}^{N-1}} K(x',0)u_0^{2_*-1}\psi_\varepsilon\, dx' \leq C_2 |u_0|^{2_*-1}_{L_K^{(2_*-1)\tau_1'}(\Omega)}|\psi_\varepsilon|_{\tau_1} = O(\varepsilon^{2+\nu_1}),$$

and therefore we can use (5.6), (5.7) and $\nu_1, \nu_2 > 0$ to conclude that

$$\Phi(\varepsilon,\alpha,\beta) = O(\varepsilon^{2+\nu_1}) + O(\varepsilon^{2+\nu_2}) = o(\varepsilon^2). \tag{5.9}$$

Since $\lambda > \lambda_1$, a straightforward computation shows that the function

$$f(\beta) := \frac{\beta^2}{2}\left(\|\psi_\varepsilon\|^2 - \lambda\|\psi_\varepsilon\|_2^2\right) - \frac{|\beta|^{2_*}}{2_*}|\psi_\varepsilon|^{2_*}_{2_*}, \quad \forall\, \beta \in \mathbb{R},$$

is such that

$$f(\beta) \leq \frac{1}{2(N-1)}\left[\frac{\|\psi_\varepsilon\|^2 - \lambda\|\psi_\varepsilon\|_2^2}{|\psi_\varepsilon|^2_{2_*}}\right]^{N-1} = \frac{1}{2(N-1)}Q_\lambda(\psi_\varepsilon)^{N-1},$$

for any $\beta \in \mathbb{R}$. Moreover, using (3.2) and the Mean Value Theorem, we obtain $\theta \in (0,1)$ such that

$$\begin{aligned} Q_\lambda(\psi_\varepsilon)^{N-1} &\leq \left[S + \varepsilon^2(-E_N + o(1))\right]^{N-1} \\ &= S^{N-1} + (N-1)\varepsilon^2 \left[-E_N + o(1)\right] \left[S + \theta\varepsilon^2(-E_N + o(1))\right]^{N-2} \end{aligned}$$

and therefore

$$f(\beta) \leq \frac{1}{2(N-1)} S^{N-1} + \varepsilon^2 \left[-\frac{E_N S}{2} + o(1) \right],$$

as $\varepsilon \to 0^+$. Since $I_\lambda(\alpha u_0) \leq I_\lambda(u_0) = c_\lambda$, for any $\alpha \in \mathbb{R}$, and $E_N > 0$, we can replace the above inequality and (5.9) in (5.8) to get

$$\begin{aligned} \sup_{\alpha,\, \beta \in \mathbb{R}} I_\lambda(\alpha u_0 + \beta \psi_\varepsilon) &\leq c_\lambda + \frac{1}{2(N-1)} S^{N-1} + \varepsilon^2 \left[-\frac{E_N S}{2} + o(1) \right] \\ &< c_\lambda + \frac{1}{2(N-1)} S^{N-1}, \end{aligned}$$

for any $\varepsilon > 0$ small. This finishes the proof of the second case of Theorem 1.2. □

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