NONLOCAL ELLIPTIC PROBLEMS WITH ASYMPTOTICALLY LINEAR NONLINEARITIES

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Abstract. In this work we consider the following class nonlocal elliptic problems

$$\begin{cases} -\Delta u + V(x)u = [I_\alpha * F(x,u)]f(x,u), & x \in \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N) & , \end{cases}$$

where $N \geq 3$ and $\alpha \in (0,N)$. The Riesz potential is denoted by I_α and $V : \mathbb{R}^N \to \mathbb{R}$ is a positive continuous potential. The nonlinearity f is asymptotically linear at infinity and at the origin in suitable sense. Our main results relies on the fact that nonlocal semilinear elliptic problems have nontrivial solutions whenever a kind o crossing of eigenvalues is allowed. Here we also consider an eigenvalue elliptic problem with a nonlocal term driven by the Choquard equation.

1. Introduction

In the present work we shall prove existence and multiplicity of solutions to the following class of nonlocal elliptic problems:

$$\begin{cases} -\Delta u + V(x)u = [I_\alpha * F(x,u)]f(x,u), & x \in \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N) & , \end{cases} \tag{P}$$

where $N \geq 3$ and $\alpha \in (0,N)$. The Riesz potential is denoted by I_α and $V : \mathbb{R}^N \to \mathbb{R}$ is a positive continuous function. The nonlinearity f is asymptotically linear at infinity and at the origin in suitable sense.

Now, for the potential $V : \mathbb{R}^N \to \mathbb{R}$ we consider the following assumptions:

(V_1) There exists $V_0 > 0$ such that $V(x) \geq V_0, x \in \mathbb{R}^N$;

(V_2) There holds $1/V \in L^1(\mathbb{R}^N)$, that is, we assume that

$$\int_{\mathbb{R}^N} \frac{1}{V(x)} dx < \infty.$$

In order to describe our main results we need to consider the following working space

$$X = \left\{ u : \mathbb{R}^N \to \mathbb{R} : \int_{\mathbb{R}^N} V(x)u^2 dx < \infty \right\}. \tag{1.1}$$

It is well know that X is a Hilbert space endowed with the norm and inner product given as follows:

$$\|u\| = \int_{\mathbb{R}^N} |\nabla u|^2 + V(x)u^2 dx, u \in X; \tag{1.2}$$

and

$$\langle u, v \rangle = \int_{\mathbb{R}^N} \nabla u \nabla v + V(x)uv dx, u, v \in X. \tag{1.3}$$

It follows follows we consider an standard variational point of view finding weak solutions to the elliptic Problem (P). In fact, we ensure that critical points for the functional $J : X \to \mathbb{R}$ given by

$$J(u) = \frac{1}{2}\|u\|^2 - \frac{1}{2}\int_{\mathbb{R}^N} [I_\alpha * F(x,u)]F(x,u) dx, u \in X, \tag{1.4}$$

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are precisely the weak solutions to the elliptic Problem (P). Recall also that $F(x,t)=\int_0^t f(x,s)ds, x\in\mathbb{R}^N, t\in\mathbb{R}$. It is important to stress that J is in C^1 class. Furthermore, we observe that

$$J'(u)v = \langle u,\, v\rangle + \int_{\mathbb{R}} [I_\alpha * F(x,u)] f(x,u) v dx, u, \phi \in X. \tag{1.5}$$

Now, we shall consider some eigenvalues problems in order to control the behavior of the functional J at infinity and at the origin. Namely, we consider the following minimization problems:

$$\lambda_1 = \inf_{u\in X}\left\{\|u\|^2, \int_{\mathbb{R}^N} [I_\alpha * u] u dx = 1\right\} \tag{1.6}$$

and

$$\lambda_k = \inf_{u\in X}\left\{\|u\|^2, \int_{\mathbb{R}^N} [I_\alpha * u] u dx = 1, u \in span\{\phi,\dots,\phi_{k-1}\}^{\perp}\right\}, k\geq 2. \tag{1.7}$$

More specifically, we denote by $(\lambda_k)_{k\in\mathbb{N}}$ the sequence of eigenvalues for the following nonlocal elliptic problem

$$\begin{cases} -\Delta u + V(x) u = \lambda [I_\alpha * u] u, & x\in\mathbb{R}^N,\\ u\in H^1(\mathbb{R}^N). \end{cases} \tag{LP}$$

The sequence of eigenfunctions for (LP) is denoted by $(\phi_k)_{k\in\mathbb{N}}$. Later on, we shall prove that eigenvalue problem (LP) admits a sequence of eigenvalues $(\lambda_k)_{k\in\mathbb{N}}$ in such way that $\lambda_k\to\infty$ as $k\to\infty$, see Proposition 2.5 ahead.

Throughout this work for the nonlinearity f we shall assume the following hypotheses:

(H_1) $f:\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}$ is continuous;

(H_2) there exist $p\in(2_\alpha, 2_\alpha^*)$ and $C>0$ such that

$$|f(x,t)|\leq C(1+|t|^{p-1}), \quad \forall\,(x,t)\in\mathbb{R}^N\times\mathbb{R},$$

where $2_\alpha := (N+\alpha)/N$ and $2_\alpha^* := (N+\alpha)/(N-2)$;

(H_3) there holds

$$\lim_{t\to 0}\frac{F(x,t)}{|t|} < \lambda_1^{1/2} < \lim_{|t|\to+\infty}\frac{F(x,t)}{|t|}, \quad \text{unifmorly in } x\in\mathbb{R}^N;$$

(H_4) there hold

$$\begin{cases} 0\leq F(x,t)\leq tf(x,t), & \forall\, x\in\mathbb{R}^N,\, t\in\mathbb{R},\\ \lim\limits_{|t|\to\infty}[tf(x,t)-F(x,t)] = +\infty, & \text{unifmorly in } x\in\mathbb{R}^N. \end{cases}$$

Theorem 1.1. *Suppose that V and f satisfy (V_1)–(V_2) and (H_1)–(H_4). Then problem (P) has nontrivial weak solution $u\in X$ provided that one of the following conditions holds:*

(AL) *there exists $C>0$ such that*

$$\frac{F(x,t)}{|t|}\leq C, \quad \forall\, x\in\mathbb{R}^N,\, t\neq 0;$$

(SL) *if $\mathcal{F}_\alpha(x,t) := [I_\alpha * F(x,t)]\,(tf(x,t)-F(x,t))$, then there exists $C>0$ such that*

$$\mathcal{F}_\alpha(x,st)\leq C\mathcal{F}_\alpha(x,t), \quad \forall\, x\in\mathbb{R}^N,\, t\in\mathbb{R},\, s\in[0,1].$$

Now, we shall prove that J is bounded from below under some assumptions on F. Firstly, we shall assume also that

(H_5) there holds

$$\lim_{|t|\to\infty}\frac{F(x,t)}{|t|} < \lambda_1^{1/2} < \lim_{|t|\to 0}\frac{F(x,t)}{|t|}, \quad \text{uniformly in } x\in\mathbb{R}^N.$$

Theorem 1.2. *Suppose that V and f satisfy (V_1)–(V_2), (H_1)–(H_2) and (H_5). Then problem (P) has a weak nontrivial solution $u\in X$ provided the following conditions holds:*

(SBL) there exists $q\in[2_\alpha/2, 1)$ such that

$$\lim_{|t|\to 0}\frac{F(x,t)}{|t|^q} < +\infty, \quad \textit{uniformly in } x\in\mathbb{R}^N.$$

For the next result we shall consider the function f interacting with higher eigenvalues. More precisely, we consider the following assumption:

(H_6) There exists $\lambda \in (\lambda_k, \lambda_{k+1})$ such that $F(x,t) = \lambda^{1/2} t + G(x,t), x \in \mathbb{R}^N, t \in \mathbb{R}$. Furthermore, we assume that

$$\lim_{t \to 0} \frac{G(x,t)}{t} = 0$$

holds uniformly in $x \in \mathbb{R}^N$.

Remark 1.3. *It is worthwhile to mention that using hypothesis (H_6) we obtain that*

$$\lambda_k^{1/2} < \lim_{t \to 0} \frac{F(x,t)}{t} < \lambda_{k+1}^{1/2} \tag{1.8}$$

holds uniformly in $x \in \mathbb{R}^N$.

In order to control the behavior of G at the infinity we also assume that

(H_7) There holds

$$\lim_{|t| \to \infty} \frac{F(x,t)}{|t|} < \lambda_1^{1/2}$$

holds uniformly in $x \in \mathbb{R}^N$.

Remark 1.4. *Here we assume that*

$$\lim_{|t| \to \infty} \frac{G(x,t)}{|t|} < +\infty \tag{1.9}$$

holds uniformly in $x \in \mathbb{R}^N$. Suppose also that $G(x,t) \geq 0$ for each $x \in \mathbb{R}^N, t \in \mathbb{R}$.

Under these conditions, we shall state the following main result:

Theorem 1.5. *Suppose that the potential V satisfies hypotheses (V_1) and (V_2). Assume also that the function f satisfies assumptions (H_1)–(H_2), (H_6), (SBL) and (H_7). Then Problem (P) has at least two nontrivial weak solution $u \in X$.*

1.1. **Outline.** The remainder of this paper is organized as follows: In the forthcoming Section we consider some preliminary results and the variational setting for our main problem. Section 3 is devoted to ensure that J has a linking geometry. In Section 4 we prove some our main results taking into account the energy functional J together with some compactness used in variational methods.

Throughout the paper we will use the following notation: C, $\tilde{C}$, C_1, C_2,... denote positive constants (possibly different). The norm in $L^p(\mathbb{R}^N)$ and $L^\infty(\mathbb{R}^N)$, will be denoted respectively by $\|\cdot\|_p$ and $\|\cdot\|_\infty$ for each $p \in [1, \infty)$. The norm in $H^1(\mathbb{R}^N)$ is given by $\|u\| = (\|\nabla u\|_2^2 + \|u\|_2^2)^{1/2}, u \in X$. Furthermore, $|\cdot|$ denotes the Lebesgue measure on $\mathbb{R}^N$.

2. Preliminaries and variational framework

In this section, we shall give some definitions and properties which will be used along this work. We start recalling that, under conditions (V_1)-(V_2), we can define the space

$$X := \left\{ u : \mathbb{R}^N \to \mathbb{R} : \int_{\mathbb{R}^N} V(x) u^2 dx < \infty \right\},$$

which is a Hilber space if we consider the inner product

$$\langle u, v \rangle := \int_{\mathbb{R}^N} \left[\nabla u \cdot \nabla \phi + V(x) u \phi \right] dx, \quad u, v \in X,$$

and its induced norm

$$\|u\| := \left(\int_{\mathbb{R}^N} \left[|\nabla u|^2 + V(x) u^2 \right] dx \right)^{1/2}, \quad u \in X.$$

As proved in [5], we have the following Sobolev embeddings

Proposition 2.1. *If (V_1)–(V_2) hold, then the embedding $X \hookrightarrow L^s(\mathbb{R}^N)$ is continuous for each $s \in [1, 2^*]$. Furthermore, this embedding is compact for each $s \in [1, 2^*)$.*

In order to properly define the energy functional associated to our problem, we need to prove that its nonlocal part is well defined. Namely, we need to prove that

$$\int_{\mathbb{R}^N}[I_\alpha * F(x,u)]F(x,u)\,dx < +\infty,$$

for any $u \in X$. At this point, we recall the well known the Hardy-Littlewood-Sobolev inequality [13], which can be stated as follows:

Lemma 2.2. *Let r, $s > 1$ and $0 < \alpha < N$ be such that*

$$\frac{1}{r} + \frac{1}{s} + \frac{N-\alpha}{N} = 2.$$

Then there exists $C = C(N,\alpha,r,s) > 0$ such that

$$\int_{\mathbb{R}^N}[I_\alpha * \psi]\phi\,dx \le C\|\phi\|_r\|\psi\|_s, \quad \forall\, \phi \in L^r(\mathbb{R}^N),\, \psi \in L^s(\mathbb{R}^N).$$

In view of (H_1)-(H_2), there exists $C_1 > 0$ such that

$$|F(x,t)| \le C_1(|t| + |t|^p), \quad \forall\, x \in \mathbb{R}^N,\, t \in \mathbb{R}. \tag{2.1}$$

Hence, for any fixed $u \in X$, we have for a.e. $x \in \mathbb{R}^N$,

$$\Big|[I_\alpha * F(x,u)]\,F(x,u)\Big| \le C_1\Big([I_\alpha * |u|]\,|u|^p + [I_\alpha * |u|]\,|u| + [I_\alpha * |u|^p]\,|u|^p\Big)$$

and we shall verify that all terms of the righ-hand side above are integrable. Indeed, if we pick $r = s = 2N/(N+\alpha) \in (1, 2^*)$, then it follows from

$$\frac{N+\alpha}{N} < p < \frac{N+\alpha}{N-2}$$

that $2 \le rp \le 2^*$ and therefore we can use Lemma 2.2 and Proposition 2.1, to get

$$\int_{\mathbb{R}^N}[I_\alpha * |u|]\,|u|^p dx \le C\|u\|_r\|u\|_{rp}^p \le C_1\|u\|^{p+1}. \tag{2.2}$$

Analagously,

$$\int_{\mathbb{R}^N}[I_\alpha * |u|]\,|u| dx \le C\|u\|_r\|u\|_r \le C_2\|u\|^2 \tag{2.3}$$

and

$$\int_{\mathbb{R}^N}[I_\alpha * |u|^p]\,|u|^p dx \le C\|u\|_{rp}^p\|u\|_{rp}^p \le C_3\|u\|^{2p}. \tag{2.4}$$

Putting all these estimates together we can define the functional $J : X \to \mathbb{R}$ as follows:

$$J(u) := \frac{1}{2}\|u\|^2 - \frac{1}{2}\int_{\mathbb{R}^N}[I_\alpha * F(x,u)]\,F(x,u)\,dx, \quad u \in X.$$

Moreover, by using some extra calculations we can prove that $J \in C^1(X,\mathbb{R})$, with the derivative given by

$$J'(u)v = \langle u, v\rangle + \int_{\mathbb{R}^N}[I_\alpha * F(x,u)]\,f(x,u)v\,dx, \quad \forall\, u,\, v \in X.$$

Thus, critical points of J are precisely the weak solutions of Problem (P).

The following convergence results is a consequence of the compact embedding giveb in Proposition 2.1.

Proposition 2.3. *Suppose that V and f satisfy (V_1)–(V_2) and (H_1)–(H_2). If $(u_k) \subset X$ is such that $u_k \rightharpoonup u$ weakly in X, then*

$$\lim_{k\to+\infty}\int_{\mathbb{R}^N}[I_\alpha * F(x,u_k)]F(x,u_k)\,dx = \int_{\mathbb{R}^N}[I_\alpha * F(x,u)]F(x,u)\,dx, \tag{2.5}$$

$$\lim_{k\to+\infty}\int_{\mathbb{R}^N}[I_\alpha * F(x,u_k)]f(x,u_k)u_k\,dx = \int_{\mathbb{R}^N}[I_\alpha * F(x,u)]f(x,u)u\,dx, \tag{2.6}$$

and, for any $\phi \in X$,

$$\lim_{k\to+\infty}\int_{\mathbb{R}^N}[I_\alpha * F(x,u_k)]f(x,u_k)\phi\,dx = \int_{\mathbb{R}^N}[I_\alpha * F(x,u)]f(x,u)\phi\,dx. \tag{2.7}$$

Proof. According to Proposition 2.1, we may assume that

$$\begin{cases} u_k \to u, & \text{strongly in } L^s(\mathbb{R}^N), \\ u_k(x) \to u(x), & \text{for a.e. } x \in \mathbb{R}^N, \\ |u_k(x)| \le h_s(x), & \text{for a.e. } x \in \mathbb{R}^N, \end{cases} \tag{2.8}$$

for any $s \in [1, 2^*)$ and some $h_s \in L^s(\mathbb{R}^N)$. By using (2.1), we get

$$\Big| [I_\alpha * F(x,u_k)]\, F(x,u_k) \Big| \le C_1 \Big(D_{1,k} + D_{2,k} + D_{3,k} \Big),$$

with

$$D_{1,k} := [I_\alpha * |u_k|]\, |u_k|^p, \quad D_{2,k} := [I_\alpha * |u_k|]\, |u_k|, \quad D_{3,k} := [I_\alpha * |u_k|^p]\, |u_k|^p.$$

Setting $r := 2N/(N+\alpha) \in (1, 2^*)$ and using (2.8), we obtain

$$D_{1,k} \le [I_\alpha * h_r]\, h_{rp}^p.$$

Moreover, it follows from Lemma 2.2 that

$$\int_{\mathbb{R}^N} [I_\alpha * h_r]\, h_{rp}^p \, dx \le C \|h_r\|_r \|h_{rp}\|_{rp}^p < +\infty,$$

where we have used $2 \le rp \le 2^*$. The same argument show that $D_{2,k}$ and $D_{3,k}$ are bounded by integrable functions. So, the convergence in (2.5) follows from (2.8) and Lebesgue's theorem.

The proof of (2.6) can be done along the same lines but using the inequality

$$|f(x,t)t| \le C_1(|t| + |t|^p), \quad \forall\, x \in \mathbb{R}^N,\, t \in \mathbb{R}.$$

instead of (2.1).

In order to prove (2.7), we first use (2.1), (H_2) and the former argument to write

$$\Big| [I_\alpha * F(x,u_k)]\, f(x,u_k)\phi \Big| \le C_1 \Big(D_{4,k} + D_{5,k} + D_{6,k} \Big),$$

where

$$D_{4,k} := [I_\alpha * |u_k|]\, |u_k|^{p-1} |\phi|, \quad D_{5,k} := [I_\alpha * |u_k|^p]\, |u_k|^{p-1} |\phi|,$$

and $D_{6,k}$ is bounded by an integrable function. We can bound $D_{4,k}$ as

$$D_{4,k} \le [I_\alpha * |h_r|]\, |h_{rp}|^{p-1} |\phi|$$

and notice that

$$\begin{aligned} \int_{\mathbb{R}^N} [I_\alpha * |h_r|]\, |h_{rp}|^{p-1} |\phi| \, dx &\le C \|h_r\|_r \left(\int_{\mathbb{R}^N} |h_r|^{r(p-1)} |\phi|^r \, dx \right)^{1/r} \\ &\le C \|h_r\|_r \|h_{rp}\|^{p-1} \|\phi\|_{rp} < +\infty, \end{aligned}$$

where we have used Lemma 2.2 and Hölder's inequality with exponents $p/(p-1)$ and p. The same kind of calculation shows that $D_{5,k}$ is also bounded by an integrable function and therefore (2.7) is a consequence of (2.8) and Lebesgue's theorem. □

Lemma 2.4. *If (V_1) and (V_2) hold, then the first eigenvalue*

$$\lambda_1 := \inf_{u \in X} \left\{ \|u\|^2 \, : \, \int_{\mathbb{R}^N} [I_\alpha * u]\, u \, dx = 1 \right\} > 0$$

is attained by a nonnegative function $\phi_1 \in X$. In particular,

$$\lambda_1 \int_{\mathbb{R}^N} [I_\alpha * u]\, u \, dx \le \|u\|^2, \quad \forall\, u \in X. \tag{2.9}$$

Proof. Let $(u_k) \in X$ be such that

$$\lim_{k \to +\infty} \|u_k\|^2 = \lambda_1, \quad \int_{\mathbb{R}^N} [I_\alpha * u_k]\, u_k \, dx = 1.$$

We first claim that u_k can be assumed to be nonnegative. Indeed, if this is not the case, we may replace u_k by

$$\widetilde{u_k} := \frac{|u_k|}{\left(\int_{\mathbb{R}^N} [I_\alpha * |u_k|] |u_k| dx \right)^{1/2}},$$

which satisfies $\int_{\mathbb{R}^N}[I_\alpha * \widetilde{u_k}]\widetilde{u_k}\,dx = 1$, $\widetilde{u_k} \geq 0$ a.e. in $\mathbb{R}^N$ and

$$\lambda_1 \leq \|\widetilde{u_k}\|^2 = \frac{\||u_k|\|^2}{\int_{\mathbb{R}^N}[I_\alpha * |u_k|]|u_k|dx} \leq \frac{\|u_k\|^2}{\int_{\mathbb{R}^N}[I_\alpha * u_k]u_k dx} = \|u_k\|^2 \to \lambda_1.$$

that is, $(\widetilde{u_k}) \subset X$ is nonnegative minizazing sequence.

Since (u_k) is bounded, we may obtain $\phi_1 \in X$ such that $u_k \rightharpoonup \phi_1$ weakly in X, strongly in $L^r(\mathbb{R}^N)$, for $r = 2N/(N+\alpha)$ and $u_k(x) \to \phi_1(x) \geq 0$ for a.e. $x \in \mathbb{R}^N$. Arguing as in the proof of Proposition 2.3, we get

$$1 = \lim_{k\to+\infty}\int_{\mathbb{R}^N}[I_\alpha * u_k]\,u_k\,dx = \int_{\mathbb{R}^N}[I_\alpha * \phi_1]\,\phi_1\,dx.$$

from which it follows that $\phi_1 \neq 0$. Moreover, recalling that the norm is weakly lower semicontinuous, we also have

$$\lambda_1 \leq \|\phi_1\|^2 \leq \liminf_{k\to\infty}\|u_k\|^2 = \lambda_1.$$

Therefore, $\lambda_1 = \|\phi_1\|^2 > 0$ and $\int_{\mathbb{R}^N}[I_\alpha * \phi_1]\phi_1\,dx = 1$. The lemma is proved. □

At this stage we shall consider the sequence of eigenvalues $(\lambda_k)_k \in \mathbb{R}$ for the eigenvalue problem (LP). More specifically, we consider the minimization problem;

$$\lambda_k = \inf_{u\in X}\left\{\|u\|^2, \int_{\mathbb{R}^N}[I_\alpha * u]udx = 1, u \in span\{\phi,\ldots,\phi_{k-1}\}^\perp\right\}, k \geq 2. \tag{2.10}$$

Proposition 2.5. *Suppose* (V_1) *and* (V_2)*. Then we obtain the following assertions:*

i) λ_k *is attained for each* $k \geq 2$*, that is, there exists* $\phi_k \in X$ *such that*

$$\lambda_k = \|\phi_k\|^2, \int_{\mathbb{R}^N}[I_\alpha * \phi_k]\phi_k dx = 1. \tag{2.11}$$

ii) The function $\phi_k \in X$ *is a weak solution to the elliptic problem* (LP) *with* $\lambda = \lambda_k$*. Furthermore, the sequence* $(\lambda_k)_{k\in\mathbb{N}}$ *satisfies* $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots \leq \lambda_k \to \infty$ *as* $k \to \infty$*.*

iii) For each $u \in X$ *there exists a sequence* $(a_j) \in \mathbb{R}$ *such that*

$$u = \sum_{k=1}^{\infty} a_j\phi_j. \tag{2.12}$$

Furthermore, for each $u \in X$*, we obtain that* $a_j = \dfrac{\langle u, \phi_j\rangle}{\langle \phi_j, \phi_j\rangle}$ *with* $j \in \mathbb{R}$*.*

Proof. Let $(u_n) \in X$ be a minimizer sequence for the minimization problem given in (2.10). In other words, we have that

$$\|u_n\|^2 \to \lambda_k \text{ as } n \to \infty, \langle u_n, \phi_j\rangle = 0, j \in \{1,2,\ldots,k-1\} \tag{2.13}$$

and

$$\int_{\mathbb{R}^N}[I_\alpha * u_n]u_n dx = 1 \text{ holds for each } n \in \mathbb{N}. \tag{2.14}$$

Once again the sequence (u_n) is a bounded in X. Up to a subsequence there exists $u \in X$ such that $u_n \rightharpoonup u$ in X. Furthermore, by using Proposition 2.1, we infer that $u_n \to u$ in $L^s(\mathbb{R}^N)$ for each $s \in [1, 2^*)$. Hence, $u_n \to u$ a.e in $\mathbb{R}^N$ and there exists $h_s \in L^s(\mathbb{R}^N)$ in such way that $|u_n| \leq h_s$ in $\mathbb{R}^N$. In particular, arguing as was done in the proof of Proposition 2.3, it follows that

$$\int_{\mathbb{R}^N}[I_\alpha * u]udx = 1. \tag{2.15}$$

Moreover, by using the weak convergence $u_n \rightharpoonup u$ in X, we infer also that $< u, \phi_j >= 0$ holds for each $j \in \{1,2,\ldots,k-1\}$. Therefore, by using the weakly lower semicontinuity of the norm, we see that

$$\lambda_k \leq \|u\|^2 \leq \liminf_{n\to\infty}\|u_n\|^2 = \lambda_k. \tag{2.16}$$

As a consequence, denoting $\phi_k = u$, we obtain that $\|\phi_k\|^2 = \lambda_k, \int_{\mathbb{R}^N}[I_\alpha * \phi_k]\phi_k dx = 1$ and $\langle \phi_k, \phi_j\rangle = 0$ for each $j \in \{1,2,\ldots,k-1\}$. This finishes the proof of item i).

Now, we shall prove the item ii). Arguing as was done in the proof of Proposition 2.4 it follows that ϕ_k is a weak solution for the eigenvalue problem (LP). On the other hand, we mention that $u \in X$ is a weak solution for the eigenvalue problem (LP) if and only if u verifies

$$\langle u, \phi \rangle = \lambda \int_{\mathbb{R}^N} [I_\alpha * u]\phi dx, \phi \in X. \tag{2.17}$$

Furthermore, the map $\phi \mapsto \int_{\mathbb{R}^N} [I_\alpha * u]\phi dx$ define a linear and bounded operator from X into $\mathbb{R}$ for each $u \in X$ fixed. Hence, by using the Riesz representation Theorem, there exists a linear operator $T : X \to X$ such that

$$\langle u, \phi \rangle = \lambda \langle Tu, \phi \rangle, \phi \in X. \tag{2.18}$$

Under these conditions, u is a weak solution to the elliptic problem (LP) if and only if $1/\lambda$ is an eigenvalue for the linear operator T. Furthermore, T is a compact and self adjoint linear operator. The desired result follows from the theory of compact and self adjoint operators, see for instance [6]. This ends the proof. □

It is worthwhile to mention that Proposition 2.4 implies that

$$\|u\|^2 \geq \lambda_1 \int_{\mathbb{R}^N} [I_\alpha * u]udx, u \in X. \tag{2.19}$$

Similarly, by using Proposition 2.5, we obtain that

$$\|u\|^2 \geq \lambda_{k+1} \int_{\mathbb{R}^N} [I_\alpha * u]udx, u \in X_2 = X_1^\perp. \tag{2.20}$$

It is not hard to verify also that

$$\|u\|^2 \leq \lambda_k \int_{\mathbb{R}^N} [I_\alpha * u]udx, u \in X_1 = span\{\phi_1, \phi_2, \ldots, \phi_k\}. \tag{2.21}$$

Remark 2.6. *Notice that the sequence $(\phi_k) \in X$ of eigenfunctions for the Problem (LP) satisfies*

$$\langle \phi_i, \phi_j \rangle = 0 \text{ for each } i, j \in \mathbb{N}, i \neq j. \tag{2.22}$$

As a product, we obtain that

$$\int_{\mathbb{R}^N} [I_\alpha * \phi_i]\phi_j dx = 0, \text{ for each } i, j \in \mathbb{N}, i \neq j. \tag{2.23}$$

Furthermore, we know that

$$\langle \phi_i, \phi_i \rangle = \lambda_i, \int_{\mathbb{R}^N} [I_\alpha * \phi_i]\phi_i dx = 1, \text{ for each } i \in \mathbb{N}. \tag{2.24}$$

It is worthwhile to mention that Proposition 2.4 implies that

Proposition 2.7. *Suppose V and f satisfy (V_1)–(V_2) and (H_1)–(H_3). Then, there exist ρ, $\alpha > 0$ such that*

$$J(u) \geq \alpha, \quad \forall\, u \in X \cap \partial B_\rho(0).$$

Assume also that $F(x,t) \geq 0$ for each $x \in \mathbb{R}^N, t \in \mathbb{R}$. Then there exists $e \in X$ such that $\|e\| > \rho$ and $J(e) < 0$.

Proof. Given $\varepsilon > 0$, it follows from (H_1)–(H_3) that

$$|F(x,t)| \leq (\lambda_1 - \varepsilon)^{1/2}|t| + C_1|t|^p, \quad \forall\, x \in \mathbb{R}^N,\, t \in \mathbb{R}. \tag{2.25}$$

for some $C_1 = C_1(\varepsilon) > 0$. Hence,

$$J(u) \geq \frac{1}{2}\|u\|^2 - \frac{1}{2}\int_{\mathbb{R}^N} [I_\alpha * ((\lambda_1 - \epsilon)^{1/2}|u| + C_1|u|^p)]((\lambda_1 - \epsilon)^{1/2}|u| + C_1|u|^p)dx$$

and therefore, we may use (2.19), (2.2) and (2.4), to get

$$\begin{aligned} J(u) \;&\geq\; \frac{1}{2}\|u\|^2 - \frac{\lambda_1 - \varepsilon}{2}\int_{\mathbb{R}^N} [I_\alpha * |u|]\,|u|\,dx \\ &\quad - C_2\int_{\mathbb{R}^N}\Big([I_\alpha * |u|]\,|u|^p + [I_\alpha * |u|^p]\,|u|^p\Big)dx \\ &\geq\; \frac{\varepsilon}{2\lambda_1}\|u\|^2 - C_3\Big[\|u\|^{p+1} + \|u\|^{2p}\Big] \end{aligned}$$

for constants C_2, $C_3 > 0$. Since $p > 1$, the term into brackets above is $o(\|u\|^2)$ as $\|u\| \to 0$. This proves the first statement of the proposition.

In order to finish the proof we pick $\phi_1 \in X$ given by Lemma 2.4. Define the set $\Omega^+ := \{x \in \mathbb{R}^N : \phi_1(x) > 0\}$. Once again we observe that $F(x,t) \geq 0$ for each $x \in \mathbb{R}^N, t \in \mathbb{R}$. It follows from (H_3), (H_4) and the Fatous' Lemma that, for $\varepsilon > 0$ small, there holds

$$\liminf_{t\to+\infty} \left[I_\alpha * \frac{F(x,t\phi_1)}{t\phi_1}\phi_1\right] \frac{F(x,t\phi_1)}{t\phi_1}\phi_1 \geq (\lambda_1+\varepsilon)\left[I_\alpha * \phi_1\right]\phi_1, \quad \forall x \in \Omega^+.$$

Since $F(x,0) = 0$, we have that

$$\frac{J(t\phi_1)}{t^2} = \frac{1}{2}\|\phi_1\|^2 - \frac{1}{2}\int_{\Omega^+}\left[I_\alpha * \frac{F(x,t\phi_1)}{t\phi_1}\phi_1\right]\frac{F(x,t\phi_1)}{t\phi_1}\phi_1 dx.$$

In view of Fatou's Lemma, (2.19) and the last identity we obtain that

$$\begin{aligned}\limsup_{t\to\infty}\frac{J(t\phi_1)}{t^2} &\leq \frac{1}{2}\|\phi_1\|^2 - \frac{\lambda_1+\varepsilon}{2}\int_{\mathbb{R}^N}\left[I_\alpha * \phi_1\right]\phi_1\,dx \\ &= \left(1 - \frac{\lambda_1+\varepsilon}{\lambda_1}\right)\|\phi_1\|^2 < 0.\end{aligned}$$

Hence, the second statement of the proposition hold for $e := t\phi_1$, with $t > 0$ sufficiently large. □

We now recall that $(u_k) \subset X$ is said to be a Cerami sequence at level $c \in \mathbb{R}$ if

$$\lim_{k\to+\infty} J(u_k) = c, \quad \lim_{k\to+\infty}(1+\|u_k\|)\|J'(u_k)\|_{X'} = 0. \tag{2.26}$$

We say that J satisfies the $(Ce)_c$ condition if any such sequence has a convergent subsequence. In the next result we shall prove that our energy functional J satifies the $(Ce)_c$ condition for any $c \in \mathbb{R}$. This can be done using some hypotheses on V and f which control the behavior of J at infinity.

Proposition 2.8. *Suppose V and f satisfy (V_1)–(V_2) and (H_1)–(H_4). If f also verifies (AL) or (SL), then J satisfies the $(Ce)_c$, for any level $c \in \mathbb{R}$.*

Proof. Let $(u_k) \in X$ be a Cerami sequence at level $c \subset X$. We first prove that (u_k) is bounded. In order to do that we suppose, by contradiction, that some subequence, still denoted (u_k), is such that $\|u_k\| \to \infty$. If we set the the normalized sequence $v_k := u_k/\|u_k\|$ we can use Proposition 2.1 to obtain $v \in X$ such that (up to a subsequence)

$$\begin{cases} v_k \to v, & \text{strongly in } L^s(\mathbb{R}^N), \\ v_k(x) \to v(x), & \text{for a.e. } x \in \mathbb{R}^N, \\ |v_k(x)| \leq h_s(x), & \text{for a.e. } x \in \mathbb{R}^N, \end{cases} \tag{2.27}$$

for any $s \in [1, 2^*)$ and some $h_s \in L^s(\mathbb{R}^N)$.

Suppose that $\Omega := \{x \in \mathbb{R}^N : v(x) \neq 0\}$ has positive measure. In this set, we have that $|u_k(x)| = |v_k(x)|\|u_k\| \to \infty$, and therefore it follows from (H_4) that

$$\begin{cases} \lim\limits_{k\to+\infty}\left[f(x,u_k(x)) - F(x,u_k(x))\right] = +\infty \\ \lim\limits_{k\to+\infty}\left[I_\alpha * F(\cdot,u_k)\right](x) = +\infty \end{cases}, \quad \forall\, x \in \Omega. \tag{2.28}$$

esse segundo limite me parece correto. mas sera que precisamos detalhar isso? Nao precisa falar disso. Ja esta bom assim.On the other hand, it follows from (2.26) that

$$C_1 \geq J(u_k) - J'(u_k)u_k = \frac{1}{2}\int_{\mathbb{R}^N}[I_\alpha * F(x,u_k][f(x,u_k)u_k - F(x,u_k)]dx,$$

for some $C_1 > 0$. Thus, by using (H_4) again, Fatou's lemma and (2.28), we obtain

$$C_1 \geq \frac{1}{2}\int_{\mathbb{R}^N}\liminf_{k\to\infty}[I_\alpha * F(x,u_k)][f(x,u_k)u_k - F(x,u_k)]\,dx = +\infty,$$

which is a contradiction and proves that $v = 0$.

We now split the proof in two cases. Suppose first that f satisfies (AL), in such a way that the ratio $|F(x,t)/t| \le C_2$, for any $x \in \mathbb{R}^N$ and $t \neq 0$. Setting $\Omega_k := \{x \in \mathbb{R}^N \, : \, u_k(x) \neq 0\}$, we have that

$$o_k(1) = \frac{J(u_k)}{\|u_k\|^2} = \frac{1}{2} - \int_{\Omega_k} \left[I_\alpha * \frac{F(x,u_k)}{u_k} v_k \right] \frac{F(x,u_k)}{u_k} v_k \, dx, \tag{2.29}$$

where $o_k(1)$ stands for a quatity approaching zero as $k \to +\infty$. From (2.27), we obtain

$$\left| \left[I_\alpha * \frac{F(x,u_k)}{u_k} v_k \right] \frac{F(x,u_k)}{u_k} v_k \right| \le C_2 \left[I_\alpha * |v_k| \right] |v_k| \le C_2 \left[I_\alpha * h_r \right] h_r,$$

for $r = 2N/(N+\alpha)$. As in the proof of Proposition 2.3, we can check that the last function above is integrable. Thus, recalling that $v_k(x) \to 0$ for a.e. $x \in \mathbb{R}^N$, we conclude that

$$\lim_{k \to +\infty} \int_{\Omega_k} \left[I_\alpha * \frac{F(x,u_k)}{u_k} v_k \right] \frac{F(x,u_k)}{u_k} v_k \, dx = 0,$$

which contradicts (2.29). So, we conclude that (u_k) is bounded whenever condition (AL) holds.

We now assume that (SL) is verified. Consider, for each $k \in \mathbb{N}$, the number $t_k \in [0,1]$ such that

$$J(t_k u_k) = \max_{t \in [0,1]} J(t u_k).$$

If $t_k \in (0,1)$, then $\frac{d}{dt} J(tu_k)|_{t=t_k} = 0$. Therefore it follows from (SL) and (2.26) that

$$\begin{aligned} J(t_k u_k) &= J(t_k u_k) - J'(t_k u_k)(t_k u_k) \\ &= \frac{1}{2} \int_{\mathbb{R}^N} \left[I_\alpha * F(x, t_k u_k) \right] \left(t_k u_k f(x, t_k u_k) - F(x, t_k u_k) \right) dx \\ &\le C_3 \int_{\mathbb{R}^N} \left[I_\alpha * F(x,u_k) \right] \left(u_k f(x,u_k) - F(x,u_k) \right) dx \\ &= C_3 \left[J(u_k) + J'(u_k) u_k \right] \le C_4 \end{aligned}$$

for some $C_4 > 0$ independent of k. It is clear that this same kind of bound holds if $t_k = 0$ or $t_k = 1$.

By using the above expression and the definition of v_k and t_k, we obtain

$$C_4 \ge J(t_k u_k) \ge J(s v_k) = \frac{s^2}{2} - \int_{\mathbb{R}^N} \left[I_\alpha * F(x, s v_k) \right] F(x, s v_k) \, dx, \quad \forall \, s \le \|u_k\|.$$

So, for any $s > 0$ fixed, we can use $v = 0$, Proposition 2.1 and Proposition 2.3 to obtain

$$C_4 \ge \frac{s^2}{2} - \lim_{k \to \infty} \int_{\mathbb{R}^N} \left[I_\alpha * F(x, s v_k) \right] F(x, s v_k) \, dx = \frac{s^2}{2},$$

and we get a contradition by picking $s > \sqrt{C_4}$. This proves that (u_k) is bouded if condition (SL) holds.

Since (u_k) is bounded, there exists $u \in X$ such that $u_k \rightharpoonup u$ weakly in X and $u_k \to u$ stronly in $L^s(\mathbb{R}^N)$ for any $s \in [1, 2^*)$. We have that

$$J'(u_k)(u_k - u) = \langle u_k, u_k - u \rangle - \int_{\mathbb{R}^N} \left[I_\alpha * F(x,u_k) \right] f(x,u_k)(u_k - u) \, dx \tag{2.30}$$

Moreover, using (H_2) and (2.1)

$$\left| \int_{\mathbb{R}^N} \left[I_\alpha * F(x,u_k) \right] f(x,u_k)(u_k - u) \, dx \right| \le C_5 \left(D_{1,k} + D_{2,k} + D_{3,k} \right),$$

where

$$D_{1,k} := \int_{\mathbb{R}^N} \left[I_\alpha * |u_k| \right] |u_k - u| \, dx, \quad D_{2,k} := \int_{\mathbb{R}^N} \left[I_\alpha * |u|^p \right] |u_k - u| \, dx$$

and

$$D_{3,k} := \int_{\mathbb{R}^N} \left[I_\alpha * |u_k|^p \right] |u_k|^{p-1} |u_k - u| \, dx.$$

By using Lemma 2.2 with $r = s = 2N/(N+\alpha)$, Proposition 2.1 and the strong convergence in the Lebesgue spaces, we obtain

$$D_{1,k} \le C_6 \|u_k\|_r^r \|u_k - u\|_r^r \le C_7 \|u_k - u\|_r = o_k(1)$$

since (u_k) is bounded and $r \in (1, 2^*)$. The same argument shows that $D_{2,k} = o_k(1)$, since $rp \in (2, 2^*)$. The third term can be estimated in the following way:

$$D_{3,k} \leq C_6 \|u_k\|_{rp}^p \left(\int_{\mathbb{R}^N} |u_k|^{r(p-1)} |u_k - u| \, dx \right)^{1/r} \leq C_6 \|u_k\|_{rp}^{2p-1} \|u_k\|_{rp} = o_k(1).$$

By replacing all the above expressions into (2.30) and recalling that $J'(u_k) \to 0$, we conclude that $\langle u_k, u_k - u \rangle = o_k(1)$. Therefore $\|u_k\|^2 \to \|u\|^2$, as $k \to +\infty$. This and the weak convergence imply that $u_k \to u$ strongly in X and the proposition is proved. □

We are ready to prove our first theorems.

Proof of Theorem 1.1. According to Proposition 2.7, it is well defined

$$c_{MP} := \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} J(\gamma(t)) \geq \alpha > 0, \tag{2.31}$$

where $\Gamma := \{\gamma \in C^0([0,1], X) : \gamma(0) = 0, \gamma(1) = e\}$. It follows from the Mountain Pass Theorem that there exists $(u_k) \subset X$ such that

$$\lim_{k \to +\infty} J(u_k) = c_{MP}, \quad \lim_{k \to +\infty} (1 + \|u_k\|) \|J'(u_k)\|_{X'} = 0.$$

In particular, $J'(u_k) \to 0$. Hence, we can use Proposition 2.8 to obtain $u_{MP} \in X$ such that $u_k \to u_{MP}$ stongly in X. Thus, $J'(u_{MP}) = 0$, $J(u_{MP}) = c_{PM} > 0$ and therefore we have obtained a nonzero weak solution of (P). □

Proof of Theorem 1.2. Given $\varepsilon > 0$, we can use (H_1), (H_2), (SBL) and argue as in the proof of Proposition 2.7 to get

$$\begin{aligned} J(u) \quad \geq \quad & \frac{1}{2}\|u\|^2 - \frac{\lambda_1 - \varepsilon}{2} \int_{\mathbb{R}^N} [I_\alpha * |u|] \, |u| \, dx \\ & - C_1 \int_{\mathbb{R}^N} \Big([I_\alpha * |u|] \, |u|^q + [I_\alpha * |u|^q] \, |u|^q \Big) dx \end{aligned}$$

Since $q \geq 2_\alpha/2$, we can use Lemma 2.2 and Proposition 2.1, to obtain

$$\int_{\mathbb{R}^N} \Big([I_\alpha * |u|] \, |u|^q + [I_\alpha * |u|^q] \, |u|^q \Big) dx \leq C_2 \left(\|u\|^{q+1} + \|u\|^{2q} \right).$$

It follows from (2.19) that

$$J(u) \geq \frac{\varepsilon}{2\lambda_1} \|u\|^2 - C_2 \left(\|u\|^{q+1} + \|u\|^{2q} \right).$$

Recalling that $q < 1$, we conclude that $J(u) \to +\infty$, as $\|u\| \to +\infty$. confiram essa conta ai, porque na versao inicial tinha umas coisa bem complicadas mas acho que esse argumento simples funciona. contas conferidas prof. Ficou show. Aqui destaco a necessidade grande das imersoes comecarem em s = 1. Por isso tivemos que colocar a hipotese de $1/V$ ser integravel. Em varios momentos usamos isso.

At this point, we consider the following minimization problem

$$c_0 := \inf_{u \in X} J(u).$$

By applying Ekland's Variacional Principle [7], we obtain $(u_k) \in X$ such that

$$\lim_{k \to +\infty} J(u_k) = c_0, \quad \lim_{k \to +\infty} J'(u_n) = 0.$$

Since J is coercive, the sequence (u_k) is a bounded. Now we may assume that there exists $u_0 \in X$ such that $u_n \rightharpoonup u_0$ in X. By computing $J'(u_k)(u_k - u)$ and arguing as in the proof of Proposition 2.8, we conclude that $u_k \to u$ strongly in X. Therefore $J'(u_0) = 0$.

Up to now, we have that $u_0 \in X$ is a critical point such that $J(u_0) = c_0$. In order to conclude that $u_0 \neq 0$, we shall prove that $c_0 < 0$. Indeed, first notice that

$$J(t\phi_1) = t^2 \left\{ \frac{1}{2}\|\phi_1\|^2 - \frac{1}{2} \int_{\mathbb{R}^N} \left[I_\alpha * \frac{F(x, t\phi_1)}{t\phi_1} \phi_1 \right] \frac{F(x, t\phi_1)}{t\phi_1} \phi_1 \, dx. \right\}.$$

Given $\varepsilon > 0$ small, we can use (H_5) and the Fatou's Lemma to get

$$\begin{aligned}\limsup_{t\to 0}\frac{J(t\phi_1)}{t^2} &\leq \frac{1}{2}\|\phi_1\|^2 - \frac{1}{2}\int_{\mathbb{R}^N}\liminf_{t\to\infty}\left[I_\alpha * \frac{F(x,t\phi_1)}{t\phi_1}\phi_1\right]\frac{F(x,t\phi_1)}{t\phi_1}\phi_1\,dx\\ &\leq \frac{1}{2}\|\phi_1\|^2 - \frac{\lambda_1+\varepsilon}{2}\int_{\mathbb{R}^N}[I_\alpha * \phi_1]\,\phi_1\,dx\\ &= \frac{1}{2}\left(1-\frac{\lambda_1+\varepsilon}{\lambda_1}\right)\|\phi_1\|^2 < 0.\end{aligned}$$

Under these conditions, for $t > 0$ small, there holds $c_0 \leq J(t\phi_1) < 0$. This concludes the proof. □

3. The high energy

In the present section we shall consider the Problem (P) using the Local Linking Theorem [8]. On this subject we refer the reader also to [20]. For the geometry conditions we write $X = X_1 \bigoplus X_2$ where $X_1 = span\{\phi_1,\dots,\phi_j\}$, $X_2 = X_1^{\perp}$. The function ϕ_k denote the eigenfunction corresponding to the eigenvalue λ_k for each $k \in \mathbb{N}$. It is important to recall that the functional J admits a Local Linking at the origin with respect to X_1 and X_2 if, some $\rho > 0$, there holds

$$\begin{aligned}J(u) &\leq 0, \quad u \in X_1 \text{ and } \|u\| \leq \rho,\\ J(u) &\geq 0, \quad u \in X_2 \text{ and } \|u\| \leq \rho.\end{aligned}$$

Hence, we shall use the Local Linking Theorem provided by [8] as follows

Theorem 3.1 (Willem, Li [8])**.** *Let X be a Banach where $X = X_1 \bigoplus X_2$. Suppose that $J \in C^1(X,\mathbb{R})$ satisfies the following assumptions:*

- (A_1) *J has a local linking at the origin.*
- (A_2) *J satisfies the Cerami condition.*
- (A_3) *J maps bounded sets into bounded sets.*
- (A_4) *J is bounded from below and $\inf_X J < 0$.*

Then the functional J admits at least two nontrivial critical points.

From now on, we shall consider the local linking geometry for the functional J. In order to do that we consider X $= X_1 \oplus X_2$ where $X_1 = span\{\phi_1,\phi_2,\dots,\phi_{k-1}\}$ and $X_2 = X_1^{\perp}$, $k \geq 2$. More specifically, we consider the following result:

Proposition 3.2. *Assume that V satisfies (V_1) and (V_2). Suppose also that f verifies (H_1)–(H_2), (H_6) and (H_7) are satisfied. Then we obtain the following assertions:*

i) There exist $\rho_1 > 0$ and $\alpha > 0$ in such way that

$$J(u) \geq \alpha, u \in X_2, \|u\| = \rho_1.$$

ii) There exists ρ_2 such that

$$J(u) \leq 0, u \in X_1, \|u\| = \rho_2.$$

Proof. It follows from (H_1), (H_2) and (H_6) that for each $\epsilon > 0$ there exists $C_\epsilon > 0$ such that

$$|G(x,t)| \leq \epsilon|t| + C_\epsilon|t|^p, x \in \mathbb{R}^N, t \in \mathbb{R}. \tag{3.1}$$

Recall also that $F(x,t) = \lambda^{1/2}t + G(x,t), x \in \mathbb{R}, t \in \mathbb{R}$. Therefore, we obtain that

$$\begin{aligned}J(u) &\geq \frac{1}{2}\|u\|^2 - \frac{\lambda}{2}\int_{\mathbb{R}^N}[I_\alpha * u]udx - \epsilon\lambda^{1/2}\int_{\mathbb{R}^N}[I_\alpha * u]G(x,u)dx - c_\epsilon\int_{\mathbb{R}^N}[I_\alpha * G(x,u)]G(x,u)dx\\ &\geq \frac{1}{2}\|u\|^2 - \frac{\lambda}{2}\int_{\mathbb{R}^N}[I_\alpha * u]udx - \frac{\epsilon\lambda}{2}\int_{\mathbb{R}^N}[I_\alpha * |u|]|u|dx\\ &- c_\epsilon\int_{\mathbb{R}^N}[I_\alpha * |u|]|u|^p dx - c_\epsilon\int_{\mathbb{R}^N}[I_\alpha * |u|^p]|u|^p dx\end{aligned} \tag{3.2}$$

holds for some $c_\epsilon > 0$ and for any $u \in X$. Now, by using the last estimate and (2.20), we obtain that

$$J(u) \geq \frac{1}{2}\left(1-\frac{\lambda}{\lambda_{k+1}}\right)\|u\|^2 - \frac{\epsilon\lambda}{2}\int_{\mathbb{R}^N}[I_\alpha * |u|]|u|dx - c_\epsilon\int_{\mathbb{R}^N}[I_\alpha * |u|]|u|^p dx - c_\epsilon\int_{\mathbb{R}^N}[I_\alpha * |u|^p]|u|^p dx, u \in X_2. \tag{3.3}$$

Now, using the estimates given in (2.2), (2.3) and (2.4), we obtain that there exists $C > 0$ such that

$$J(u) \geq \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_{k+1}} - \epsilon C\right)\|u\|^2 - C_\epsilon \|u\|^{p+1} - C_\epsilon \|u\|^{2p}, u \in X_2. \tag{3.4}$$

As a product, we see that

$$J(u) \geq \frac{(\lambda_{k+1} - \lambda)}{4\lambda_{k+1}}\|u\|^2 - C_\epsilon \|u\|^{p+1} - C_\epsilon \|u\|^{2p}, u \in X_2. \tag{3.5}$$

Hence,

$$J(u) \geq \frac{(\lambda_{k+1} - \lambda)}{8\lambda_{k+1}}\|u\|^2, \|u\| = \rho_\epsilon, u \in X_2. \tag{3.6}$$

Here was used the fact that $\rho_e > 0$ is small enough. This finishes the proof of item $i)$ with $\rho = \rho_\epsilon$ and $\alpha = \epsilon\rho_\epsilon^2/(8\lambda_{k+1})$ where $\epsilon > 0$ is small enough. This ends the proof of item $i)$.

Now we shall prove the item $ii)$. Since $F(x,t) = \lambda^{1/2}t + G(x,t), x \in \mathbb{R}^N, t \in \mathbb{R}$ and taking into account (3.1) we deduce that

$$\begin{aligned} J(u) &= \frac{1}{2}\|u\|^2 - \frac{\lambda}{2}\int_{\mathbb{R}^N}[I_\alpha * u]u dx - \lambda^{1/2}\int_{\mathbb{R}^N}[I_\alpha * u]G(x,u)dx - \frac{1}{2}\int_{\mathbb{R}^N}[I_\alpha * G(x,u)]G(x,u)dx \\ &\leq \frac{1}{2}\|u\|^2 - \frac{\lambda}{2}\int_{\mathbb{R}^N}[I_\alpha * u]u dx + \frac{\epsilon\lambda}{2}\int_{\mathbb{R}^N}[I_\alpha * |u|]|u| dx \\ &+ c_\epsilon \int_{\mathbb{R}^N}[I_\alpha * |u|]|u|^p dx + c_\epsilon \int_{\mathbb{R}^N}[I_\alpha * |u|^p]|u|^p dx, u \in X. \end{aligned} \tag{3.7}$$

As a consequence, by using (2.21) and $\lambda \in (\lambda_k, \lambda_{k+1})$ together with the estimates (2.2), (2.3) and (2.4), we deduce that

$$J(u) \leq \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_k} + \epsilon C\right)\|u\|^2 + C_\epsilon \|u\|^{p+1} + C_\epsilon \|u\|^{2p}, u \in X_1. \tag{3.8}$$

In view of the last estimate we infer that

$$J(u) \leq \frac{1}{4}\left(1 - \frac{\lambda}{\lambda_k}\right)\|u\|^2 + C_\epsilon \|u\|^{p+1} + C_\epsilon \|u\|^{2p}, u \in X_1 \tag{3.9}$$

holds for each $\epsilon > 0$ small enough. Therefore, taking $\|u\| = \rho_2 > 0$ where $\rho_2 > 0$ is also small enough, we see that

$$J(u) \leq \frac{1}{8}\frac{(\lambda_k - \lambda)}{\lambda_k}\|u\|^2 = \frac{1}{8}\frac{(\lambda_k - \lambda)}{\lambda_k}\rho_2^2 < 0, u \in X_1. \tag{3.10}$$

This finishes the proof. □

Hence, we can state the following result:

Proposition 3.3. *Assume that V satisfies (V_1) and (V_2). Suppose also that f verifies (H_1)–(H_2), (H_6), (SBL) and (H_7) are satisfied. Let $(u_n) \in X$ be a $(Ce)_c$ sequence for the functional J. Then (u_n) is a bounded sequence in X.*

Proof. Firstly, using the same ideas discussed in the proof of Theorem 1.2, we infer that the functional J is coecive. In fact, given $\epsilon > 0$, we obtain that

$$J(u) \geq \frac{\varepsilon}{2\lambda_1}\|u\|^2 - C_2\left(\|u\|^{q+1} + \|u\|^{2q}\right)$$

where $2_\alpha/2 < q < 1$. Here was used the hypothesis (SBL) together with the same ideas employed in the proof of Theorem 1.2. Hence, any $(Ce)_c$ sequence is bounded in X. We omit the details. □

Proposition 3.4. *Assume that V satisfies (V_1) and (V_2). Suppose also that f verifies (H_1)–(H_2) and (H_6) are satisfied. Then the functional J satisfies the Cerami condition.*

Proof. The proof for this result follows arguing as was done in the proof of Proposition 2.8. We omit the details. □

Proof of Theorem 1.5. According to Proposition 3.4 the functional J satisfies the $(Ce)_c$ condition for each $c \in \mathbb{R}$. Furthermore, the functional J maps bounded sets into bounded sets which is proved using the continuous embedding of X into the Lebesgue spaces $L^r(\mathbb{R}^N), r \in [2, 2^*_s]$. Notice also that the the functional J has the local linking geometry, see for instance Proposition 3.2. Furthermore, by using the same ideas discussed in the proof of Theorem 1.2, we know that $c = \inf_{w \in X} J(w) = J(u) < 0$ where $u \in X$. In particular, by using the Ekeland variational Principle, we obtain that $J'(u)\phi = 0$ holds true for each $\phi \in X$. Hence, $u \in X$ is a nontrivial weak solution to the elliptic problem (P). Now, by using the Local Linking Theoreom, see for instance Theorem 3.1, we obtain a second nontrivial critical point $v \in X$ for the energy functional J. Hence, v is a nontrivial weak solution for the Problem (P). This finishes the proof.

□

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