ON A FITZHUGH-NAGUMO NONLINEAR SYSTEM WITH EXPONENTIAL GROWTH

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Abstract. We consider the system

$$-\Delta u = \lambda Q(|x|)f(u) - V(|x|)v, \quad -\Delta v = V(|x|)u - V(|x|)v, \quad \text{in } \mathbb{R}^2,$$

where $\lambda > 0$, the potentials V and Q are continuous functions which can be singular at the origin, unbounded or decaying at infinity, and the nonlinearity f has exponential growth. Under appropriate hypotheses, we establish the existence, multiplicity and regularity of non-zero radial functions which solve the system for large values of λ.

1. Introduction and main results

In this work, we analyze the existence, multiplicity, and regularity of solutions to the following planar FitzHugh–Nagumo system:

$$\begin{cases} -\Delta u = \lambda Q(|x|)f(u) - V(|x|)v, & \text{in } \mathbb{R}^2, \\ -\Delta v = V(|x|)u - V(|x|)v, & \text{in } \mathbb{R}^2, \end{cases} \tag{$\mathcal{S}_\lambda$}$$

where $\lambda > 0$, the potentials $V, Q : (0, \infty) \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ are continuous functions meeting certain conditions specified later. This type of system, derived from activator-inhibitor dynamics, is significant in neurobiology for modeling nerve conduction and the transmission of electrical signals in neurons. Relevant background and studies can be found in [8, 9, 12, 19].

More broadly, our problem examines the steady-state of FitzHugh–Nagumo systems, which are described by the following ODE:

$$u_t = u^3 - v, \quad \tau v_t = u + a - bv, \tag{1.1}$$

initially proposed by Richard FitzHugh [8] and further developed by Jinichi Nagumo and collaborators [12]. This system models nerve impulse propagation through a simplified activator-inhibitor framework, capturing essential neurobiological processes. Further details on the physical background are available in [19].

Authors in [6, 10] point out that system (1.1) belongs to a more general class of reaction-diffusion systems, namely

$$\begin{cases} u_t = D_1\Delta u + g(u) - v, & \text{in } (0, \infty) \times \Omega, \\ v_t = D_2\Delta v + \varepsilon(u - \gamma v), & \text{in } (0, \infty) \times \Omega, \end{cases}$$

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where Ω is a bounded domain and D_1, D_2, ε and γ are positive constants. This type of problem has motivated the study of the system

$$\begin{cases} u_t = D_1 \Delta u + g(u) - kv, & \text{in } (0,\infty) \times \mathbb{R}^N, \\ v_t = D_2 \Delta v + u - \gamma v, & \text{in } (0,\infty) \times \mathbb{R}^N. \end{cases}$$

See, for example, [10, 13] for the one-dimensional case and more recently [7] for the n-dimensional case, which has strongly influenced our investigation.

From a mathematical perspective, researchers have focused on problems involving potentials and weights that may be either unbounded or vanish at infinity. We especially emphasize the paper by Su, Wang, and Willem [17] (see also [1, 2, 3]), which suppose, among other conditions, that V and Q satisfy the following:

(V) $V : (0,\infty) \to (0,\infty)$ is continuous and there exists $a > -2$ such that

$$\liminf_{r \to \infty} \frac{V(r)}{r^a} > 0;$$

(Q) $Q : (0,\infty) \to (0,\infty)$ is continuous and there exist b_0, $b > -2$ such that

$$\limsup_{r \to 0} \frac{Q(r)}{r^{b_0}} < \infty, \qquad \limsup_{r \to \infty} \frac{Q(r)}{r^b} < \infty.$$

In their paper, the authors consider the Schrödinger equation

$$-\Delta u + V(|x|)u = Q(|x|)|u|^{p-2}u, \quad \text{in } \mathbb{R}^N,$$

for $N \geq 2$, with an additional condition concerning the behavior of V near the origin. After establishing the appropriate functional framework involving radially symmetric functions, they proved some existence and non-existence results for solutions that approach zero at infinity.

Before presenting our main results, let us briefly outline our strategy for addressing the system. For a fixed radial function u in an appropriate subspace of $W^{1,2}(\mathbb{R}^N)$, we consider the linear problem

$$-\Delta v + V(|x|)v = V(|x|)u, \quad \text{in } \mathbb{R}^2.$$

After finding a solution to this problem, we can return to system ($\mathcal{S}_\lambda$) and replace v with $B[u]$ in the first equation. This substitution transforms the system into the following problem:

$$-\Delta u + V(|x|)B[u] = \lambda Q(|x|)f(u), \quad \text{in } \mathbb{R}^2,$$

in such a way that the solutions of this scalar equation provides solutions $(u, B[u])$ for system ($\mathcal{S}_\lambda$).

The aim of this paper is twofold: we show how to adapt the abstract ideas from [18] to address the system ($\mathcal{S}_\lambda$), and we also consider the problem in the two-dimensional case. In this setting, we expect to allow nonlinearities with exponential growth. Specifically, we shall assume the following conditions:

(f_1) $f \in C(\mathbb{R},\mathbb{R})$ and there exists $\alpha_0 > 0$ such that

$$\lim_{|s| \to \infty} \frac{|f(s)|}{e^{\alpha s^2}} = \begin{cases} 0, & \text{if} \quad \alpha > \alpha_0, \\ \infty, & \text{if} \quad \alpha < \alpha_0; \end{cases}$$

(f_2) $f(s) = o(|s|^{\gamma-1})$, as $s \to 0$, where

$$\gamma := \max\left\{2, \frac{4(b-a)}{(a+2)} + 2\right\};$$

(f_3) there exists $\mu > \gamma$ such that

$$0 < \mu F(s) := \mu \int_0^s f(t)\,\mathrm{d}t \le f(s)s, \qquad \forall\, s \neq 0;$$

(f_4) there exist $C > 0$ and $\nu > \gamma$ such that

$$F(s) \ge C|s|^\nu, \qquad \forall\, s \in \mathbb{R}.$$

The main results of this paper are:

Theorem 1.1. *Suppose that (V), (Q) and (f_1)-(f_4) hold. Then there exists $\lambda_0 > 0$ such that the system $(\mathcal{S}_\lambda)$ has a radial non-zero weak solution, provided $\lambda \ge \lambda_0$. Moreover, if we call (u, v) this solution, the following hold:*

(a) *if there exists $a_0 > -2$ such that*

$$\limsup_{r \to 0} \frac{V(r)}{r^{a_0}} < \infty,$$

then $u, v \in W^{2,p}_{\mathrm{loc}}(\mathbb{R}^2)$ for any $p > 1$ such that pa_0, $pb_0 > -2$. In particular, u, v are locally Hölder continuous;

(b) *if V is locally Hölder continuous, then $v \in C^{2,\sigma}_{\mathrm{loc}}(\mathbb{R}^2)$ for some $\sigma \in (0, 1)$.*

Theorem 1.2. *Suppose that (V), (Q) and (f_1)-(f_4) hold. If additionally f is odd then, for any given $m \in \mathbb{N}$, there exists $\lambda_m > 0$ such that the system $(\mathcal{S}_\lambda)$ has at least $2m$ radial non-zero weak solutions, provided $\lambda \ge \lambda_m$.*

For the proof of the first theorem, we apply the classical Mountain Pass Theorem. It is important to establish the variational framework to correctly define the energy functional. In particular, we prove a Trudinger-Moser type inequality (see Theorem 2.4), which is interesting in itself (see Remark 2.5). Our abstract results actually complement those of [15] and can be applied to other types of problems with exponential growth. For the second theorem, we exploit the symmetry of the functional to obtain multiple critical points. As the associated functional is even, the strategy is to obtain m distinct non-zero critical points as the parameter λ becomes large.

The paper is organized as follows: in the next section, we present the variational setting to address our problem, including the proof of the Trudinger-Moser inequality. In Section 3, we verify the required geometric and compactness properties for the energy functional. The main results are proved in the final Section 4.

2. Variational setting

Along all the paper, we assume that (V) and (Q) hold.

Consider the set

$$E := \left\{ u \in L^2_{\mathrm{loc}}(\mathbb{R}^2) : |\nabla u| \in L^2(\mathbb{R}^2),\ \int_{\mathbb{R}^2} V(|x|)u^2\,\mathrm{d}x < \infty \right\},$$

which is a Hilbert space when endowed with the scalar product

$$\langle u, w \rangle_E := \int_{\mathbb{R}^2} \Big(\nabla u \cdot \nabla w + V(|x|)uw \Big) \mathrm{d}x, \qquad \forall\, u,\, w \in E,$$

whose corresponding norm is $\|u\|_E := \langle u,u\rangle_E^{1/2}$. We also denote by E_{rad} the subspace of E consisted of the radial functions, that is,

$$E_{rad} := \{u \in E : u \circ g = u, \quad \forall\, g \in O(2)\}.$$

For any $u \in E$ fixed, we define the linear functional $T_u : E_{rad} \to \mathbb{R}$ given by

$$T_u(\varphi) := \int_{\mathbb{R}^2} V(|x|)u\varphi\,\mathrm{d}x.$$

Since $|T_u(\varphi)| \leq \|u\|_E\|\varphi\|_E$, we may invoke Riez's Theorem to obtain $B[u] \in E_{rad}$ such that $T_u(\varphi) = \langle B[u],\varphi\rangle_E$. Hence, $B[u]$ is a critical point of the C^1 functional $J_u : E \to \mathbb{R}$ defined by

$$J_u(w) := \frac{1}{2}\|w\|_E^2 - \int_{\mathbb{R}^2} V(|x|)uw\,\mathrm{d}x, \qquad \forall\, w \in E,$$

restricted to E_{rad}.

Given an orthogonal map $g \in O(2)$ and $w \in E$, we can define $(gw)(x) := w(g^{-1}x)$. Since V is radial, it is clear that $\|gw\|_E = \|w\|_E$. If additionaly $u \in E_{rad}$, then $J_u(gw) = J_u(w)$. So, by the Principle of Symmetric Criticality (see [14]), we conclude that $B[u]$ is a radial weak solution of the linear problem

$$-\Delta v + V(|x|)v = V(|x|)u, \quad \text{in } \mathbb{R}^2. \tag{2.1}$$

Hence, if we come back to system ($\mathcal{S}_\lambda$) and make the change of variable $v := B[u]$ in the first equation, we are lead consider the problem

$$-\Delta u + V(|x|)B[u] = \lambda Q(|x|)f(u), \quad \text{in } \mathbb{R}^2. \tag{2.2}$$

Actually, if $u \in E_{rad}$ is a solution of the above equation the couple $(u, B[u])$ of radial functions solve the system ($\mathcal{S}_\lambda$).

In order to address this last problem, we consider the bilinear form

$$\langle u,w\rangle_X := \int_{\mathbb{R}^2} \Big(\nabla u\cdot\nabla w + V(|x|)uB[w]\Big)\,\mathrm{d}x.$$

Using equation (2.1), we obtain

$$\langle u,u\rangle_X = \|B[u]\|_E^2 + \int_{\mathbb{R}^2} |\nabla u|^2\,\mathrm{d}x. \tag{2.3}$$

Hence, it is straightforward to prove that $\langle\cdot,\cdot\rangle_X$ defines a scalar product in E. From now on, we denote by X the vector space formed by the set E endowed with the norm induced by this inner product, that is

$$\|u\|_X := \left[\int_{\mathbb{R}^2} \Big(|\nabla u|^2 + V(|x|)uB[u]\Big)\,\mathrm{d}x\right]^{1/2}.$$

As before, we set

$$X_{rad} := \{u \in X,\ u \circ g = u, \quad \forall\, g \in O(2)\}$$

the subspace of X consisting of radial functions.

Proposition 2.1. *X is a Hilbert space.*

Proof. Let $u \in X$ and $v := B[u]$. By picking u as a test function in (2.1) and using Young's inequality, we get

$$\begin{aligned}\int_{\mathbb{R}^2} V(|x|)u^2 \,\mathrm{d}x &= \int_{\mathbb{R}^2} (\nabla v \cdot \nabla u)\,\mathrm{d}x + \int_{\mathbb{R}^2} V(|x|)uv\,\mathrm{d}x \\ &\le \frac{1}{2}\int_{\mathbb{R}^2} |\nabla v|^2\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2\,\mathrm{d}x + \int_{\mathbb{R}^2} V(|x|)uv\,\mathrm{d}x \\ &= -\frac{1}{2}\int_{\mathbb{R}^2} V(|x|)v^2\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2\,\mathrm{d}x + \frac{3}{2}\int_{\mathbb{R}^2} V(|x|)uv\,\mathrm{d}x,\end{aligned}$$

and therefore

$$\|u\|_E^2 \le \frac{3}{2}\|u\|_X^2, \qquad \forall\, u \in X. \tag{2.4}$$

Let (u_n) be a Cauchy sequence in X. From the above inequality we conclude that (u_n) is also a Cauchy sequence in the norm $\|\cdot\|_E$. Since E is a Hilbert space, there exists $u \in E$ such that $\|u_n - u\|_E = o_n(1)$, where $o_n(1) \to 0$ stands for a quantity approaching zero as $n \to \infty$. Denoting $v_n := B[u_n]$, we can use the linearity of B and (2.3) to get

$$\|v_n - v_m\|_E^2 = \|u_n - u_m\|_X^2 - \int_{\mathbb{R}^2} |\nabla(u_n - u_m)|^2\,\mathrm{d}x \le \|u_n - u_m\|_X^2,$$

which shows that (v_n) is a Cauchy sequence in the norm $\|\cdot\|_E$. Since E is a Hilbert space, there exists $v \in E$ such that $\|v_n - v\|_E = o_n(1)$. We claim that $v = B[u]$. If this is true, we can apply the above estimative to get

$$\begin{aligned}\|u_n - u\|_X^2 &= \int_{\mathbb{R}^2} |\nabla(u_n - u)|^2\,\mathrm{d}x + \int_{\mathbb{R}^2} V(|x|)\,(u_n - u)\,(v_n - v)\,\mathrm{d}x \\ &\le \|u_n - u\|_E^2 + \frac{1}{2}\|v_n - v\|_E^2 = o_n(1),\end{aligned}$$

and therefore $u_n \to u$ in X.

To prove that $v = B[u]$ we notice that, for any $\varphi \in E$, one has

$$\langle v_n, \varphi\rangle_E = \int_{\mathbb{R}^2} V(|x|)u_n\varphi\,\mathrm{d}x. \tag{2.5}$$

Since $v_n \to v$ in E, it follows that $\langle v_n, \varphi\rangle_E = \langle v, \varphi\rangle_E + o_n(1)$. Moreover, using Hölder's inequality, we obtain

$$\left|\int_{\mathbb{R}^2} V(|x|)(u_n - u)\varphi\,\mathrm{d}x\right| \le \|u_n - u\|_E\|\varphi\|_E = o_n(1).$$

These convergences combined with (2.5) imply that

$$\int_{\mathbb{R}^2} \nabla v \cdot \nabla\varphi\,\mathrm{d}x + \int_{\mathbb{R}^2} V(|x|)v\varphi\,\mathrm{d}x = \int_{\mathbb{R}^2} V(|x|)u\varphi\,\mathrm{d}x, \quad \forall\, \varphi \in E,$$

and therefore $v = B[u]$. □

Remark 2.2. *It follows from (2.4) that the embedding $X \hookrightarrow E$ is continuous. As quoted in [3, Remark 2.3], we also have E continuously immersed in $H^1_{loc}(\mathbb{R}^2)$. So, we have the continuous embedding $X \hookrightarrow L^q(B_R)$, for any $R > 0$ and $q \ge 1$.*

Next we define, for each $1 \le p < \infty$, the space

$$L^p_Q(\mathbb{R}^2) := \left\{u : \mathbb{R}^2 \to \mathbb{R} \text{ measurable} : \int_{\mathbb{R}^2} Q(|x|)|u|^p\,\mathrm{d}x < \infty\right\},$$

which is Banach space when endowed with the norm

$$\|u\|_{L^p_Q} := \left(\int_{\mathbb{R}^2} Q(|x|)|u|^p \,\mathrm{d}x \right)^{1/p}.$$

We notice that a version of the classical Radial Lemma of Strauss [16] holds in X_{rad}. In fact, it is proved in [17, Lemma 1] that there exist constants $C_r > 0$ and $R_0 > 0$ such that, for any $u \in X_{rad}$, the following holds:

$$|u(x)| \le C_r |x|^{-(2+a)/4} \|u\|_E, \qquad \text{for a.e. } |x| \ge R_0. \tag{2.6}$$

By taking advantage of this fact, we can obtain a range of compactness for the embedding of X_{rad} into the above weighted Lebesgue spaces. More specifically, the following holds:

Lemma 2.3. *If γ is given by (f_2), then the embedding $X_{rad} \hookrightarrow L^p_Q(\mathbb{R}^2)$ is continuous, for any $\gamma \le p < \infty$. Moreover, it is compact if $p > \gamma$.*

Proof. From (2.4), we have that $X_{rad} \hookrightarrow E_{rad}$. On other hand, it is proved in [17, Theorem 2] that the embedding $E_{rad} \hookrightarrow L^p_Q(\mathbb{R}^2)$ is continuous, for any $\gamma \le p < \infty$ and compact, whenever $\gamma < p < \infty$. The result is proved. □

We study now the embedding of the space X_{rad} into weighted Orlicz spaces. So, we pick $\alpha > 0$ and define the Young function

$$\Phi_\alpha(s) := e^{\alpha s^2} - \sum_{j=0}^{j_0-1} \frac{\alpha^j}{j!} s^{2j}, \qquad \forall\, s \in \mathbb{R},$$

where $j_0 := \inf\{j \in \mathbb{N} : j \ge \gamma/2\}$ and $\gamma > 0$ was defined in (f_2). We have that

$$\Phi_\alpha(s) = \Phi_{\alpha r^2}\left(\frac{s}{r}\right), \quad (\Phi_\alpha(s))^t \le \Phi_{t\alpha}(s), \qquad \forall\, s, r > 0,\ t \ge 1. \tag{2.7}$$

Indeed, the first expression above is a direct consequence of the definition of Φ_α as well as the second one was proved in [21, Lemma 2.1].

The following Trudinger-Moser type inequality complements the abstract results of [17]:

Theorem 2.4. *For each $u \in X_{rad}$ and $\alpha > 0$, we have that $Q(|\cdot|)\Phi_\alpha(u) \in L^1(\mathbb{R}^2)$. Furthermore,*

$$\sup_{\{u \in X_{rad}\,:\, \|u\|_X \le 1\}} \int_{\mathbb{R}^2} Q(|x|)\Phi_\alpha(u)\,\mathrm{d}x < \infty,$$

whenever $0 < \alpha < 4\pi(b_0/2 + 1)$.

Proof. Let $R_0 > 0$ be as in (2.6). We fix a number $R > R_0$ and divide the proof into three steps:

First step: for any $\alpha > 0$ and $u \in X_{rad}$, we have that $Q(|\cdot|)\Phi_\alpha(u) \in L^1(B_R)$.

Following [15] (see also [5]), we consider the function

$$v(|x|) := \beta^{-1/2} u(|x|^\beta), \qquad x \in \mathbb{R}^2,$$

with $\beta := 2/(b_0 + 2) > 0$. We claim that $v \in H^1(B_{R^{1/\beta}})$. In fact, a straightforward computation shows that

$$\int_{B_{R^{1/\beta}}} |\nabla v|^2 \,\mathrm{d}x = 2\pi \int_0^{R^{1/\beta}} |v'(s)|^2 s \,\mathrm{d}s = 2\pi\beta \int_0^{R^{1/\beta}} |u'(s^\beta)|^2 s^{2\beta - 1} \,\mathrm{d}s$$

and therefore the change of variables $t = s^\beta$ yelds

$$\int_{B_{R^{1/\beta}}} |\nabla v|^2 \, dx = 2\pi \int_0^R |u'(t)|^2 t \, dt = \int_{B_R} |\nabla u|^2 \, dx < \infty. \tag{2.8}$$

On the other hand,

$$\int_{B_{R^{1/\beta}}} v^2 \, dx = 2\pi\beta^{-1} \int_0^{R^{1/\beta}} u^2(s^\beta) s \, ds = 2\pi\beta^{-2} \int_0^R t^{2(1-\beta)/\beta} u^2(t) t \, dt,$$

where we have used the change of variables $t = s^\beta$ again. It follows from $2(1-\beta)/\beta = b_0$ that

$$\int_{B_{R^{1/\beta}}} v^2 \, dx = \beta^{-2} \int_{B_R} |x|^{b_0} u^2 \, dx. \tag{2.9}$$

We now recall that $\int_{B_R} |x|^t \, dx < \infty$, whenever $t > -2$. Since the parameter b_0 in (Q) verifies $b_0 > -2$, we can pick $t_1 > 1$ close to 1 in such a way that $|x|^{t_1 b_0} \in L^1(B_R)$. Thus, we may use Hölder's inequality and Remark 2.2 to obtain

$$\int_{B_{R^{1/\beta}}} v^2 \, dx \le \beta^{-2} \left(\int_{B_R} |x|^{t_1 b_0} \, dx \right)^{1/t_1} \left(\int_{B_R} |u|^{2t_2} \, dx \right)^{1/t_2} < \infty,$$

where $1/t_1 + 1/t_2 = 1$. This and (2.8) prove that $v \in H^1(B_{R^{1/\beta}})$, as claimed.

From the first statement in (Q), we obtain $C_1 > 0$ such that

$$Q(r) \le C_1 r^{b_0}, \qquad \forall\, r \in (0, R].$$

Hence, arguing as in the proof of (2.9), we get

$$\int_{B_R} Q(|x|)\Phi_\alpha(u) \, dx \le C_1 \int_{B_R} |x|^{b_0} e^{\alpha u^2} \, dx = C_1\beta \int_{B_R^{1/\beta}} e^{\alpha\beta v^2} \, dx. \tag{2.10}$$

We now define $\widetilde{v} \in H_0^1(B_{R^{1/\beta}})$ as

$$\widetilde{v}(|x|) := \begin{cases} v(|x|) - v(R^{1/\beta}), & \text{if } |x| \le R^{1/\beta}, \\ 0, & \text{if } |x| \ge R^{1/\beta}. \end{cases}$$

For any $\varepsilon > 0$, Young's inequality provides

$$v(|x|)^2 = \widetilde{v}(|x|)^2 + v(R^{1/\beta})^2 + 2\widetilde{v}(|x|)v(R^{1/\beta}) \le (1+\varepsilon)\widetilde{v}(|x|)^2 + C(\varepsilon)v(R^{1/\beta})^2,$$

with $C(\varepsilon) := (\varepsilon + 1)/\varepsilon$. This inequality, (2.10) and the classical Trudinger-Moser inequality (see [11, 20]) imply that

$$\int_{B_R} Q(|x|)\Phi_\alpha(u) \, dx \le C_2 e^{\alpha\beta C(\varepsilon) v(R^{1/\beta})^2} \int_{B_{R^{1/\beta}}} e^{\alpha\beta(1+\varepsilon)\widetilde{v}^2} \, dx < \infty, \tag{2.11}$$

where $C_2 := C_1\beta$. The first step is proved.

Second step: if $0 < \alpha < 4\pi(b_0/2 + 1)$, then

$$\sup_{\{u \in X_{rad} : \|u\|_X \le 1\}} \int_{B_R} Q(|x|)\Phi_\alpha(u) \, dx < \infty.$$

Let $0 < \alpha < 4\pi(b_0/2 + 1)$ and $u \in X_{rad}$, with $\|u\|_X \le 1$. Take $\varepsilon > 0$ such that $\alpha(1+\varepsilon) < 4\pi(b_0/2 + 1)$. Recalling that $\beta = 2/(b_0 + 2)$, we get

$$\alpha\beta(1+\varepsilon) < 4\pi. \tag{2.12}$$

Moreover, by (2.6) and (2.4), one deduces

$$\alpha\beta C(\varepsilon)v(R^{1/\beta})^2 = \alpha C(\varepsilon)u(R)^2 \le \frac{3}{2}\alpha C(\varepsilon)C_r^2 R^{-(2+a)/2}\|u\|_X^2 \le C_3 R^{-(2+a)/2}.$$

This and (2.11) imply that

$$\int_{B_R} Q(|x|)\Phi_\alpha(u)\,\mathrm{d}x \le C_2 e^{C_3 R^{-(2+a)/2}} \int_{B_{R^{1/\beta}}} e^{\alpha\beta(1+\varepsilon)\widetilde{v}^2}\,\mathrm{d}x. \tag{2.13}$$

From the definition of $\widetilde{v}$ and (2.8), we get

$$\int_{B_{R^{1/\beta}}} |\nabla\widetilde{v}|^2\,\mathrm{d}x = \int_{B_{R^{1/\beta}}} |\nabla v|^2\,\mathrm{d}x = \int_{B_R} |\nabla u|^2\,\mathrm{d}x \le 1.$$

Since $\widetilde{v} \in H_0^1(B_R^{1/\beta})$ and $\|\nabla\widetilde{v}\|_{L^2(B_{R^{1/\beta}})} \le 1$, we may use (2.12)-(2.13) and the classical Trudinger-Moser inequality to obtain

$$\sup_{\{u\in X_{rad}:\|u\|_X\le 1\}} \int_{B_R} Q(|x|)\Phi_\alpha(u)\,\mathrm{d}x < \infty.$$

The second step is finalized.

<u>Third step:</u> for any $\alpha > 0$, we have that

$$\sup_{\{u\in X_{rad}:\|u\|_X\le 1\}} \int_{\mathbb{R}^2\setminus B_R} Q(|x|)\Phi_\alpha(u)\,\mathrm{d}x < \infty.$$

Given $u \in X_{rad}$, we first prove that $Q(|\cdot|)\Phi_\alpha(u) \in L^1(\mathbb{R}^2 \setminus B_R)$. To do this, we notice that

$$\int_{\mathbb{R}^2\setminus B_R} Q(|x|)\Phi_\alpha(u)\,\mathrm{d}x = \sum_{j=j_0}^{\infty} \frac{\alpha^j}{j!} \int_{\mathbb{R}^2\setminus B_R} Q(|x|)|u|^{2j-\gamma}|u|^\gamma\,\mathrm{d}x.$$

Since $R \ge R_0$, it follows from (2.6) and (2.4) that

$$|u(x)|^{2j-\gamma} \le C_3^{2j-\gamma}|x|^{-(2j-\gamma)(2+a)/4}\|u\|_X^{2j-\gamma},$$

with $C_3 := \sqrt{3/2}C_r$. Thus

$$\int_{\mathbb{R}^2\setminus B_R} Q(|x|)\Phi_\alpha(u)\,\mathrm{d}x \le \left(\frac{C_3}{R^{(2+a)/4}}\right)^{-\gamma} \sum_{j=j_0}^{\infty} \frac{\left(\alpha C_3^2 R^{-(2+a)/2}\|u\|_E^2\right)^j}{j!} \left(\frac{\|u\|_{L_Q^\gamma}}{\|u\|_X}\right)^\gamma$$

where we have used $j \ge j_0 \ge \gamma/2$. This, together with Lemma 2.3 and (2.4), provides C_4, $C_5 > 0$ such that

$$\int_{\mathbb{R}^2\setminus B_R} Q(|x|)\Phi_\alpha(u)\,\mathrm{d}x \le C_4 e^{\alpha C_3^2 R^{-(2+a)/2}\|u\|_X^2} = C_4 e^{\alpha C_5\|u\|_X^2} < \infty.$$

Moreover,

$$\sup_{\{u\in X_{rad}\,:\,\|u\|_X\le 1\}} \int_{\mathbb{R}^2\setminus B_R} Q(|x|)\Phi_\alpha(u)\,\mathrm{d}x \le C_4 e^{\alpha C_5},$$

and the proof is finished. □

Remark 2.5. *Let* $\alpha_* := 4\pi(b_0/2+1)$. *As shown in [2, Proposition 2.5], we have*

$$\sup_{\{u\in E_{rad}\,:\,\|u\|_E\le 1\}} \int_{\mathbb{R}^2} Q(|x|)\Phi_\alpha(u)\,dx < \infty,$$

for $0<\alpha<\alpha_*$. *While this inequality, combined with* (2.4)*, could yield the conclusion of Theorem 2.4 for* $0<\alpha<2\alpha_*/3$*, we provide a different proof to encompass the entire range* $(0,\alpha_*)$.

We are currently unable to determine whether the exponent α_* *is optimal or whether it can actually be attained. Actually, we believe that the question is both interesting and challenging.*

Let $\mathcal{E}$ be a real Banach space and $\mathcal{I}\in C^1(\mathcal{E},\mathbb{R})$. Given $c\in\mathbb{R}$, we recall that a sequence $(u_n)\subset\mathcal{E}$ is called $(PS)_c$ sequence for $\mathcal{I}$ if

$$\lim_{n\to\infty}\mathcal{I}(u_n)=c, \qquad \lim_{n\to\infty}\mathcal{I}'(u_n)=0.$$

We say that $\mathcal{I}$ satisfies the $(PS)_c$ condition at level c if any such sequence has a convergent subsequence.

We finish this section presenting a version of the Symmetric Mountain Pass Theorem (see [4]). It will be used later in the proof of Theorem 1.2.

Theorem 2.6. *Let* $\mathcal{E}$ *be a real Banach space and* $\mathcal{I}\in C^1(\mathcal{E},\mathbb{R})$ *be an even functional satisfying* $\mathcal{I}(0)=0$ *and*

($\mathcal{I}_1$) *there are constants* $\rho,\tau>0$ *such that* $\mathcal{I}(u)\ge\tau$*, whenever* $\|u\|_{\mathcal{E}}=\rho$*;*
($\mathcal{I}_2$) *there are* $\kappa>0$ *and a* m*-dimensional subspace* $\mathcal{V}$ *of* $\mathcal{E}$ *such that*

$$\max_{u\in\mathcal{V}}\mathcal{I}(u)\le\kappa.$$

If the functional $\mathcal{I}$ *satisfies the* $(PS)_d$ *condition for any* $0<d<\kappa$*, then it possesses at least* m *pairs of non-zero critical points.*

3. Some auxiliary results

Using the abstract results of the former section, we are able to define the Euler-Lagrange functional associated to equation (2.2). The first step is proving that $Q(|\cdot|)F(u)\in L^1(\mathbb{R}^2)$, for any $u\in X_{rad}$. By (f_1) and (f_2), given $\varepsilon>0$, $\alpha>\alpha_0$, and $q\ge 1$, there exists $C_f>0$ such that, for any $s>0$,

$$\begin{cases} |f(s)|\le\varepsilon|s|^{\gamma-1}+C_f|s|^{q-1}\Phi_\alpha(s),\\ |F(s)|\le\varepsilon|s|^{\gamma}+C_f|s|^{q}\Phi_\alpha(s). \end{cases} \tag{3.1}$$

Given $u\in X_{rad}$, it follows from the above estimate with $q\ge\gamma$, Hölder's inequality, Lemma 2.3, Theorem 2.4 and (2.7), that

$$\int_{\mathbb{R}^2} Q(|x|)F(u)\,\mathrm{d}x \le \varepsilon\|u\|^{\gamma}_{L^\gamma_Q} + C_f\|u\|^q_{L^{t_1q}_Q}\left(\int_{\mathbb{R}^2}Q(|x|)\Phi_{t_2\alpha}(u)\,\mathrm{d}x\right)^{1/t_2}<\infty, \tag{3.2}$$

where $1/t_1+1/t_2=1$. Hence, it is well defined the functional

$$I_\lambda(u) := \frac{1}{2}\|u\|_X^2 - \lambda\int_{\mathbb{R}^2}Q(|x|)F(u)\,\mathrm{d}x, \qquad u\in X_{rad}.$$

Moreover, by standard arguments one may conclude that $I_\lambda \in C^1(X_{rad}, \mathbb{R})$ with Gateaux derivative

$$I'_\lambda(u)\varphi = \langle u, \varphi\rangle_X - \lambda \int_{\mathbb{R}^2} Q(|x|) f(u)\varphi \, \mathrm{d}x, \qquad \forall\, u, \varphi \in X_{rad}. \tag{3.3}$$

Since the functional I_λ is not defined in the whole space X, we cannot directly apply Principle of Symmetric Criticality to conclude that critical points of I_λ weakly solves the first equation in ($\mathcal{S}_\lambda$). However, an indirect argument proves the following:

Proposition 3.1. *Suppose that* (f_1)-(f_2) *hold. If* $u \in X_{rad}$ *is a critical point of* I_λ*, then* u *is a weak solution of* (2.2).

Proof. Let $u \in X_{rad}$ be such that $I'_\lambda(u) = 0$ and consider the linear functional

$$T_u(w) := \langle u, w\rangle_X - \lambda \int_{\mathbb{R}^2} Q(|x|) f(u) w \, \mathrm{d}x, \qquad \forall\, w \in X.$$

We claim that T_u is continuous. If this is true, we may apply Riesz Representation Theorem to obtain a unique $\widetilde{u} \in X$ such that

$$T_u(w) = \langle \widetilde{u}, w\rangle_X, \qquad \forall\, w \in X. \tag{3.4}$$

It is clear that, for any orthogonal transformation $g \in O(2)$, there holds $gu = u$. Since $g^{-1}\mathbb{R}^2 = \mathbb{R}^2$, we can argue as in the beginning of Section 2 to conclude that $T_u(g\widetilde{u}) = T_u(\widetilde{u})$ and $\|g\widetilde{u}\|_X = \|\widetilde{u}\|_X$. This implies,

$$\|g\widetilde{u} - \widetilde{u}\|_X^2 = 2\|\widetilde{u}\|_X^2 - 2T_u(g\widetilde{u}) = 2\|\widetilde{u}\|_X^2 - 2T_u(\widetilde{u}) = 0$$

and therefore $g\widetilde{u} = \widetilde{u}$. Since $g \in O(2)$ is arbitrary, we conclude that $\widetilde{u} \in X_{rad}$. Hence, $0 = I'_\lambda(u)\widetilde{u} = T_u(\widetilde{u}) = \|\widetilde{u}\|_X^2$ and it follows from (3.4) that

$$I'_\lambda(u)w = T_u(w) = \langle 0, w\rangle_X = 0, \qquad \forall\, w \in X.$$

In order to prove the continuity of T_u, we first pick $\varepsilon = 1$ and $q = \gamma + 1$ in (3.1) to get

$$\left| \int_{\mathbb{R}^2} Q(|x|) f(u) w \, \mathrm{d}x \right| \le \int_{\mathbb{R}^2} Q(|x|)|u|^{\gamma-1}|w| \, \mathrm{d}x + C_f \int_{\mathbb{R}^2} Q(|x|)|u|^{\gamma}\Phi_\alpha(u)|w| \, \mathrm{d}x. \tag{3.5}$$

In view of (Q) and (V), there exists $C_1 > 0$ and $R \ge R_0 > 0$ such that

$$\begin{cases} Q(|x|) \le C_1|x|^{b_0}, & \text{if } |x| \le R, \\ Q(|x|) \le C_1|x|^{b},\ V(|x|) \ge C_2|x|^{a}, & \text{if } |x| \ge R. \end{cases} \tag{3.6}$$

By picking a $t_1 > 1$ such that $t_1 b_0 > -2$, it follows that $|x|^{t_1 b_0} \in L^1(B_R)$. So, we can use the above expression, Hölder's inequality and Remark 2.2, to get

$$\int_{B_R} Q(|x|)|u|^{\gamma-1}|w| \, \mathrm{d}x \le C_1 \left(\int_{B_R} |x|^{t_1 b_0} \, \mathrm{d}x \right)^{1/t_1} \|u\|_{L^{t_2(\gamma-1)}(B_R)}^{\gamma-1} \|w\|_{L^{t_3}(B_R)}$$

and therefore

$$\int_{B_R} Q(|x|)|u|^{\gamma-1}|w| \, \mathrm{d}x \le C_3\|w\|_X, \tag{3.7}$$

where $1/t_1 + 1/t_2 + 1/t_3 = 1$. Moreover, using (3.6) and (2.6) we obtain

$$\int_{\mathbb{R}^2\setminus B_R} Q(|x|)|u|^{\gamma-1}|w|\,\mathrm{d}x \le C_1 C_r^{\gamma-2}\|u\|_E^{\gamma-2}\int_{\mathbb{R}^2\setminus B_R}|x|^{\lambda_1}|x|^a|u||w|\,\mathrm{d}x,$$

where

$$\lambda_1 := (b-a) - (\gamma-2)\left(\frac{a+2}{4}\right).$$

From the definition of γ (see (f_2)), we deduce that $\lambda_1 \le 0$. Thus, we can use the last estimate, Hölder's inequality, (3.6) and (2.4) to obtain

$$\begin{aligned}\int_{\mathbb{R}^2\setminus B_R} Q(|x|)|u|^{\gamma-1}|w|\,\mathrm{d}x \le& C_4\left(\int_{\mathbb{R}^2\setminus B_R}|x|^a u^2\,\mathrm{d}x\right)^{1/2}\left(\int_{\mathbb{R}^2\setminus B_R}|x|^a w^2\,\mathrm{d}x\right)^{1/2}\\ \le& C_5\left(\int_{\mathbb{R}^2\setminus B_R}V(|x|)u^2\,\mathrm{d}x\right)^{1/2}\left(\int_{\mathbb{R}^2\setminus B_R}V(|x|)w^2\,\mathrm{d}x\right)^{1/2}\\ \le& C_6\|w\|_X.\end{aligned}$$

This inequality, combined with (3.5) and (3.7), imply that

$$\left|\int_{\mathbb{R}^2} Q(|x|)f(u)w\,\mathrm{d}x\right| \le (C_3+C_6)\|w\|_X + C_f\int_{\mathbb{R}^2}Q(|x|)|u|^\gamma\Phi_\alpha(u)|w|\,\mathrm{d}x. \tag{3.8}$$

We now proceed with the estimation of the last integral above. First, we apply Hölder's inequality, the second statement in (2.7), Lemma 2.3, and Theorem 2.4 to obtain

$$\begin{aligned}\int_{B_R} Q(|x|)|u|^\gamma\Phi_\alpha(u)|w|\,\mathrm{d}x \le& \|u\|^\gamma_{L_Q^{t_1\gamma}(B_R)}\left(\int_{B_R}Q(|x|)\Phi_{t_2\alpha}(u)\,\mathrm{d}x\right)^{1/t_2}\|w\|_{L_Q^{t_3}(B_R)}\\ \le& C_7\|w\|_{L_Q^{t_3}(B_R)}.\end{aligned}$$

By choosing $t_4 > 1$ such that $|x|^{t_4 b_0} \in L^1(B_R)$, we can combine Hölder's inequality and (3.6) to obtain

$$\int_{B_R} Q(|x|)|w|^{t_3}\,\mathrm{d}x \le C_1\left(\int_{B_R}|x|^{t_4 b_0}\,\mathrm{d}x\right)^{1/t_4}\|w\|^{t_3}_{L^{t_5 t_3}(B_R)}.$$

These last two estimates and Remark 2.2 again imply that

$$\int_{B_R} Q(|x|)|u|^\gamma\Phi_\alpha(u)|w|\,\mathrm{d}x \le C_8\|w\|_X. \tag{3.9}$$

From Hölder's inequality, (2.7) and Theorem 2.4, we get

$$\int_{\mathbb{R}^2\setminus B_R} Q(|x|)|u|^\gamma\Phi_\alpha(u)|w|\,\mathrm{d}x \le C_9\left(\int_{\mathbb{R}^2\setminus B_R}Q(|x|)|u|^{2\gamma}w^2\,\mathrm{d}x\right)^{1/2},$$

where

$$C_9 := \left(\int_{\mathbb{R}^2\setminus B_R}Q(|x|)\Phi_{2\alpha}(u)\,\mathrm{d}x\right)^{1/2}.$$

Once again, using (2.6) and (3.6), we can conclude that

$$\int_{\mathbb{R}^2\setminus B_R} Q(|x|)|u|^\gamma\Phi_\alpha(u)|w|\,\mathrm{d}x \le C_{10}\left(\int_{\mathbb{R}^2\setminus B_R}|x|^{\lambda_2}|x|^a w^2\,\mathrm{d}x\right)^{1/2},$$

where

$$\lambda_2 := (b-a) - \gamma\left(\frac{a+2}{2}\right).$$

The definition of γ (see (f_2)) and $a > -2$, yields $\lambda_2 \le 0$. So, we may argue as before to conclude that

$$\int_{\mathbb{R}^2\setminus B_R} Q(|x|)|u|^\gamma \Phi_\alpha(u)|w|\,\mathrm{d}x \le C_{11}\|w\|_X.$$

This, (3.8), (3.9) and the fact that $\lambda > 0$ imply that T_u is continuous on X. □

We prove in the sequel a local compactness result for our energy functional.

Lemma 3.2. *Suppose that (f_1)-(f_3) hold. Then, I_λ satisfies $(PS)_c$ condition at any level*

$$0 < c < \frac{(\mu-2)}{2\mu}\frac{4\pi(b_0/2+1)}{\alpha_0}.$$

Proof. Let $(u_n) \subset X_{rad}$ be a $(PS)_c$ sequence. From condition (f_3), we get

$$c + o_n(1)(1+\|u_n\|_X) = I_\lambda(u_n) - \frac{1}{\mu}I'_\lambda(u_n)u_n \ge \left(\frac{1}{2}-\frac{1}{\mu}\right)\|u_n\|_X^2 \tag{3.10}$$

and therefore we may use $\mu > 2$ to conclude that (u_n) is bounded in X_{rad}. Thus, up to a subsequence, $u_n \rightharpoonup u$ weakly in X_{rad}.

We claim that

$$\int_{\mathbb{R}^2} Q(|x|)f(u_n)(u_n-u)\,\mathrm{d}x = o_n(1). \tag{3.11}$$

If this is true, it follows that

$$o_n(1) = I'_\lambda(u_n)(u_n-u) = \|u_n\|_X^2 - \|u\|_X^2 + o_n(1)$$

and therefore $\|u_n\|_X \to \|u\|_X$. This, together with the weak convergence, implies that $u_n \to u$ strongly in X.

For proving (3.11), we first use (3.1) with $q = 1$ to get

$$\left|\int_{\mathbb{R}^2} Q(|x|)f(u_n)(u_n-u)\,\mathrm{d}x\right| \le \varepsilon A_n + C_f D_n,$$

where

$$A_n := \int_{\mathbb{R}^2} Q(|x|)|u_n|^{\gamma-1}|u_n-u|\,\mathrm{d}x, \qquad D_n := \int_{\mathbb{R}^2} Q(|x|)\Phi_\alpha(u_n)|u_n-u|\,\mathrm{d}x.$$

Taking the limit in (3.10), we conclude that

$$\limsup_{n\to\infty}\|u_n\|_X^2 \le \frac{2\mu}{(\mu-2)}c < \frac{4\pi(b_0/2+1)}{\alpha_0}.$$

Let $1 < t_1 < \gamma/(\gamma-1)$ and $\alpha > \alpha_0$ be such that $t_1\alpha\|u_n\|_X^2 < 4\pi(b_0/2+1)$, for any $n \ge n_0$. Using Hölder's inequality, (2.7) and Theorem 2.4, we obtain

$$\begin{aligned} D_n &\le \left(\int_{\mathbb{R}^2} Q(|x|)\Phi_{t_1\alpha}(u_n)\,\mathrm{d}x\right)^{1/t_1}\|u_n-u\|_{L_Q^{t_2}} \\ &= \left(\int_{\mathbb{R}^2} Q(|x|)\Phi_{t_1\alpha\|u_n\|_X^2}\left(\frac{u_n}{\|u_n\|_X}\right)\mathrm{d}x\right)^{1/t_1}\|u_n-u\|_{L_Q^{t_2}} \le C_1\|u_n-u\|_{L_Q^{t_2}}, \end{aligned}$$

where $1/t_1 + 1/t_2 = 1$, with $t_2 > \gamma$. This expression and the compactness of the embedding $X_{rad} \hookrightarrow L_Q^{t_2}(\mathbb{R}^2)$ (see Lemma 2.3) proves that $D_n = o_n(1)$.

From Hölder's inequality and Lemma 2.3, it follows that

$$A_n \leq \|u_n\|_{L^\gamma_Q}^{\gamma-1}\|u_n-u\|_{L^\gamma_Q} \leq C_2\|u_n\|_X^{\gamma-1}\|u_n-u\|_X.$$

Thus, there exists $C_3>0$ such that $|A_n|\leq C_3$, for any $n\in\mathbb{N}$. Hence, we can use $D_n\to 0$ to conclude that

$$\limsup_{n\to\infty}\left|\int_{\mathbb{R}^2} Q(|x|)f(u_n)(u_n-u)\,\mathrm{d}x\right| \leq \varepsilon C_3.$$

Since $\varepsilon>0$ is arbitrary, it follows that (3.11) holds. □

We now verify that I_λ satisfies the geometry of the Mountain Pass Theorem.

Lemma 3.3. *Suppose that (f_1)-(f_3) hold. Then,*

(i) there exist $\tau,\rho>0$ such that $I_\lambda(u)\geq\tau$, whenever $\|u\|_X=\rho$;

(ii) there exists $e\in X_{rad}$ such that $\|e\|_X>\rho$ and $I_\lambda(e)<0$.

Proof. Let $\varepsilon>0$, $q>\gamma$ and t_1, $t_2>1$ be such that $1/t_1+1/t_2=1$. Using (3.2) and Lemma 2.3, we obtain

$$\int_{\mathbb{R}^2} Q(|x|)F(u)\,\mathrm{d}x \leq \varepsilon C_1\|u\|_X^\gamma + C_1\|u\|_X^q\left(\int_{\mathbb{R}^2} Q(|x|)\Phi_{t_2\alpha}(u)\,\mathrm{d}x\right)^{1/t_2}.$$

If $\rho_1>0$ is small in such a way that $t_2\alpha\rho_1^2<4\pi(b_0/2+1)$, we can use (2.7) and Theorem 2.4 to get

$$\int_{\mathbb{R}^2} Q(|x|)\Phi_{t_2\alpha}(u)\,\mathrm{d}x = \int_{\mathbb{R}^2} Q(|x|)\Phi_{t_2\alpha\|u\|_X^2}\left(\frac{u}{\|u\|_X}\right)\mathrm{d}x \leq C_2, \qquad \forall\,\|u\|_X\leq\rho_1,$$

where we also have used that $\Phi_s(t)$ is increasing in $s>0$. If $\|u\|_X\leq\rho_1$ and $\varepsilon=1/(4\lambda C_1)$, we obtain

$$I_\lambda(u)\geq\|u\|_X^2\left(\frac{1}{2}-\frac{1}{4}\|u\|_X^{\gamma-2}-C_3\|u\|_X^{q-2}\right).$$

Since $q>\gamma\geq 2$, the term into the parentheses above goes $1/2$, as $\|u\|_X\to 0$. This shows that *(i)* holds.

Now, let $K\subset\mathbb{R}^2$ the support of $\varphi\in C^\infty_{0,rad}(\mathbb{R}^2)$. By (f_3), there exist $C_4, C_5>0$ such that $F(s)\geq C_4|s|^\mu - C_5$, for any $s\in\mathbb{R}$. Consequently, for $t>0$,

$$I_\lambda(t\varphi)\leq\frac{t^2}{2}\|\varphi\|_X^2 - C_4t^\mu\int_K Q(|x|)|\varphi|^\mu\,\mathrm{d}x + C_5\int_K Q(|x|)\,\mathrm{d}x.$$

Since $\mu>\gamma\geq 2$, item *(ii)* holds for $e:=t_0\varphi$, with $t_0>0$ large enough. □

4. Proof of the main theorems

We start this section by presenting the proof of our existence (and regularity) result.

Proof of Theorem 1.1. In view of Lemma 3.3, we can define the minimax level

$$c^\lambda_{MP} := \inf_{g\in\mathcal{G}}\max_{t\in[0,1]} I_\lambda(g(t))\geq\tau>0,$$

where $\mathcal{G}:=\{g\in C([0,1],X_{rad}) : g(0)=0,\ I_\lambda(g(1))<0\}$. By using the Mountain Pass Theorem [4] we obtain a sequence $(u_n)\subset X_{rad}$ such that

$$\lim_{n\to\infty} I_\lambda(u_n)=c^\lambda_{MP}, \qquad \lim_{n\to\infty} I'_\lambda(u_n)=0.$$

We claim that, for $\lambda > 0$ large,

$$c_{MP}^{\lambda} < \frac{(\mu-2)}{2\mu}\frac{4\pi(b_0/2+1)}{\alpha_0}.$$

If this is true, it follows from Lemma 3.2 that, along a subsequence, $u_n \to u$ strongly in X. From the regularity of I_λ we obtain $I'_\lambda(u) = 0$ and $I_\lambda(u) \geq \tau > 0$, and therefore it follows from Proposition 3.1 that $u \neq 0$ is a weak solution of problem (2.2).

For proving the existence of solution, it remains to prove the upper bound on c_{MP}^{λ}. In order to do that, we consider $\nu > \gamma$ given by (f_4). A standard minimization argument together with the compactness of the embedding $X_{rad} \hookrightarrow L_Q^{\nu}(\mathbb{R}^2)$ provides $w_0 \in X_{rad}$ such that

$$\|w_0\|_X^2 = S_\nu := \inf\left\{\|u\|_X^2 : u \in X_{rad},\, \int_{\mathbb{R}^2} Q(|x|)|u|^{\nu}\,\mathrm{d}x = 1\right\}.$$

It follows from (f_4) that

$$I_\lambda(w_0) \leq \frac{1}{2}\|w_0\|_X^2 - \lambda C\int_{\mathbb{R}^2} Q(|x|)|w_0|^{\nu}\,\mathrm{d}x = \frac{1}{2}S_\nu - \lambda C < 0,$$

whenever $\lambda > S_\nu/2C$. This shows that the curve $g_0(t) := tw_0$ belongs to $\mathcal{G}$. Therefore

$$c_{MP}^{\lambda} \leq \max_{t\in[0,1]} I_\lambda(g_0(t)) \leq \max_{t\geq 0}\left\{\frac{t^2}{2}S_\nu - \lambda\int_{\mathbb{R}^2} Q(|x|)F(tw_0)\,\mathrm{d}x\right\}.$$

By (f_4), we have that $F(tw_0) \geq Ct^{\nu}|w_0|^{\nu}$, for any $t \geq 0$. Thus,

$$c_{MP}^{\lambda} \leq \max_{t\geq 0}\left\{\frac{t^2}{2}S_\nu - \lambda C t^{\nu}\right\} = h(\lambda) := \frac{\nu}{(\lambda C)^{2/(\nu-2)}}\left(\frac{S_\nu}{\nu}\right)^{\nu/(\nu-2)}\left(\frac{\nu-2}{2\nu}\right).$$

Since $\nu > \gamma \geq 2$, we have that $h(\lambda) \to 0$, as $\lambda \to \infty$, and the claim is proved.

In order to obtain the regularity result, we call $(u, v) \in X_{rad} \times X_{rad}$ the solution given by the former argument and fix $p > 1$. For a fixed $R > 0$, define the function $\widetilde{v}(|x|) := v(|x|) - v(R)$. From Remark 2.2, we can infer that $\widetilde{v} \in H_0^1(B_R)$ weakly solves

$$-\Delta\widetilde{v} = h, \text{ in } B_R, \qquad \widetilde{v} = 0, \text{ on } \partial B_R, \tag{4.1}$$

where $h(x) := V(|x|)u(|x|) - V(|x|)v(|x|)$. We shall prove that $h \in L^p(B_R)$. Indeed, using that $\limsup_{r\to 0} V(r)/r^{a_0} < \infty$, we obtain $C_1 > 0$ such that

$$\int_{B_R} |h(x)|^p\,\mathrm{d}x \leq C_1\int_{B_R} |x|^{pa_0}|u|^p\,\mathrm{d}x + C_1\int_{B_R} |x|^{pa_0}|v|^p\,\mathrm{d}x.$$

Since $pa_0 > -2$, we can pick $t_1 > 1$ such that $|x|^{t_1pa_0} \in L^1(B_R)$. This, together with Hölder's inequality and Remark 2.2, yield

$$\int_{B_R} |h(x)|^p\,\mathrm{d}x \leq C_1\left(\int_{B_R} |x|^{t_1pa_0}\,\mathrm{d}x\right)^{1/t_1}\left(\|u\|_{L^{t_2p}(B_R)}^{t_2} + \|v\|_{L^{t_2p}(B_R)}^{t_2}\right) < \infty,$$

where $1/t_1 + 1/t_2 = 1$, proving the claim. Therefore, by classical elliptic regularity theory we conclude that $v = \widetilde{v} + v(R) \in W^{2,p}(B_R)$.

Now, considering $\widetilde{u}(|x|) := u(|x|) - u(R)$, then $\widetilde{u} \in H_0^1(B_R)$ is a solution of problem

$$-\Delta\widetilde{u} = g, \text{ in } B_R, \qquad \widetilde{u} = 0, \text{ on } \partial B_R,$$

where $g(x) := \lambda Q(|x|)f(u(|x|)) - V(|x|)v(|x|)$. Arguing as above, we can prove that $V(|\cdot|)v \in L^p(B_R)$. Moreover, from (3.1) with $q = 1$ and (2.7), we obtain

$$\begin{aligned}\int_{B_R} |Q(|x|)f(u)|^p \,\mathrm{d}x \le C_2 \int_{B_R} |Q(|x|)|^p |u|^{p(\gamma-1)} \,\mathrm{d}x \\ + C_2 \int_{B_R} |Q(|x|)|^p \Phi_{p\alpha}(u) \,\mathrm{d}x.\end{aligned} \tag{4.2}$$

Using (3.6), Hölder's inequality and Remark 2.2, we get

$$\begin{aligned}\int_{B_R} |Q(|x|)|^p |u|^{p(\gamma-1)} \,\mathrm{d}x &\le C_3 \int_{B_R} |x|^{pb_0} |u|^{p(\gamma-1)} \,\mathrm{d}x \\ &\le C_3 \left(\int_{B_R} |x|^{t_3 p b_0} \,\mathrm{d}x\right)^{1/t_3} \|u\|^{p(\gamma-1)}_{L^{t_4 p(\gamma-1)}(B_R)} < \infty,\end{aligned} \tag{4.3}$$

where $1/t_3 + 1/t_4 = 1$ and $t_3 p b_0 > -2$. On other hand, Young's inequality yields

$$u(|x|)^2 \le 2\widetilde{u}(|x|)^2 + 2u(R)^2.$$

So, we can use (3.6), Hölder's inequality and the classical Trudinger-Moser inequality to obtain

$$\begin{aligned}\int_{B_R} |Q(|x|)|^p \Phi_{p\alpha}(u) \,\mathrm{d}x &\le C_4 e^{2p\alpha u(R)^2} \int_{B_R} |x|^{pb_0} e^{2p\alpha \widetilde{u}^2} \,\mathrm{d}x \\ &\le C_5 \left(\int_{B_R} e^{2t_4 p\alpha \widetilde{u}^2} \,\mathrm{d}x\right)^{1/t_4} < \infty.\end{aligned}$$

The above estimate, (4.2) and (4.3), show that $Q(|\cdot|)f(u) \in L^p(B_R)$. Hence, we conclude as before that $u \in W^{2,p}(B_R)$. Since the embedding $W^{2,p}(B_R) \hookrightarrow C^\sigma(\overline{B_R})$ is continuous, for some $\sigma \in (0,1)$, then u, v are locally Hölder continuous.

Suppose now that V is locally Hölder continuous. By the former proof the functions u, v are locally Hölder continuous, and hence $h(x) := V(|x|)u(|x|) - V(|x|)v(|x|)$ belongs to $C^\sigma(\overline{B_R})$, for some $\sigma \in (0,1)$. Since $\widetilde{v}$ solves (4.1), by classical elliptic regularity theory $v = \widetilde{v} + v(R) \in C^{2,\sigma}(\overline{B_R})$. □

We prove in the sequel our multiplicity result.

Proof of Theorem 1.2. We are intending to apply Theorem 2.6 for the functional I_λ. It is clear that $I_\lambda(0) = 0$ and I_λ is even, since we are supposing f odd. Moreover, condition $(\mathcal{I}_1)$ is a consequence of the first statement in Lemma 3.3.

Given $m \in \mathbb{N}$, consider

$$V_m := \operatorname{span}\{\varphi_1, ..., \varphi_m\},$$

where $\{\varphi_i\}_{i=1}^m \subset C_0^\infty(\mathbb{R}^2)$ have disjoint supports. Since all norms are equivalent in V_m, we obtain a positive constant $C_1 = C_1(m) > 0$ such that

$$\|u\|_X^\nu \le C_1 \|u\|^\nu_{L^\nu_Q}, \qquad \forall\, u \in V_m.$$

Hence, it follows from (f_4) that

$$I_\lambda(u) \le \frac{1}{2}\|u\|_X^2 - \lambda C \|u\|^\nu_{L^\nu_Q} \le \frac{1}{2}\|u\|_X^2 - \lambda \frac{C_2}{\nu}\|u\|_X^\nu, \qquad \forall\, u \in V_m,$$

where $C_2 = C_1 C$.

We now consider the function

$$g(t) := \frac{t^2}{2} - \lambda \frac{C_2}{\nu} t^{\nu}, \qquad t \geq 0.$$

Since $\nu > 2$, it attains its maximum value at the point $t_* = (\lambda C_2)^{-1/(\nu-2)}$, which implies

$$I_\lambda(u) \leq A_{m,\lambda} := g(t_*) = \left(\frac{1}{2} - \frac{1}{\nu}\right)\left(\frac{1}{\lambda C_2}\right)^{2/(\nu-2)}, \qquad \forall\, u \in V_m.$$

Since $A_{m,\lambda} \to 0$, as $\lambda \to \infty$, we can find $\lambda_m > 0$ such that

$$0 < A_{m,\lambda} < \frac{(\mu-2)}{2\mu} \frac{4\pi(b_0/2+1)}{\alpha_0},$$

for any $\lambda > \lambda_m$. It follows from Lemma 3.2 and Theorem 2.6 that I_λ has at lest m pairs of non-zero critical points. □

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