POSITIVE SOLUTIONS FOR A STATIONARY REACTION-DIFFUSION-ADVECTION MODEL WITH NONLINEAR ADVECTION TERM

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Abstract. In this work, we investigate a stationary reaction-diffusion-advection equation with a nonlinear term in the gradient, which entails several technical challenges in the analysis. By combining the method of sub- and supersolutions with bifurcation theory, we establish results on the existence and multiplicity of positive solutions.

Contents

1. Introduction

In this paper, we deal with the following stationary reaction-diffusion-advection equation

$$
\begin{cases} -\mathrm{div}(a(x)D_1(u)\nabla u) + D_2(u)[\vec{b}(x)\cdot\nabla u] = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}
$$

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where $\Omega \subset \mathbb{R}^N$ is a bounded regular domain, $a \in C^{1,\alpha}(\overline{\Omega}, [a_0, +\infty))$, for some $a_0 > 0$, and $\vec{b} \in C^{0,\alpha}(\overline{\Omega}; \mathbb{R}^N)$, $\alpha \in (0,1)$. The functions $D_1, D_2 \in C([0,+\infty), [0,+\infty))$ satisfy

(d_1) $D_i(s) > 0$, for any $s > 0$ and $i \in \{1,2\}$,

(d_2) $D_1(\infty) := \lim_{s\to+\infty} D_1(s) > 0$,

and suitable hypotheses at the origin and at infinity, which will be presented subsequently.

This type of equation is of interest both from a mathematical standpoint and from an applied perspective. For instance, it can be interpreted as a steady-state model of a reaction-diffusion-advection equation in Population Dynamics, where Ω represents the habitat of a species, and the population density at each point $x \in \Omega$ is given by $u(x)$. In this context, $-\mathrm{div}(a(x)D_1(u)\nabla u)$ is referred to as the diffusion term, which describes the spatial movement of the species. The function $d(x,u) := a(x)D_1(u)$ represents the diffusion rate, meaning that the movement speed depends on both the position x and the population density $u(x)$, making the model more realistic than in the semilinear case. On the other hand, $D_2(u)[\vec{b}(x)\cdot\nabla u]$ represents the advection term, which accounts for preferential movement of the species. This movement may result from individual behaviors or physical transport processes, such as wind or river currents. Finally, λu represents the reaction term, which can be interpreted as the local reproduction rate of individuals. In this case, we assume the reproduction rate is proportional to the population density. For further details, see [9, 13, 23] and the references therein.

From a mathematical standpoint, this is a quasilinear elliptic equation, where the nonlinear term in u appears both in the second derivatives and in the gradient term. This introduces significant technical challenges in the analysis, particularly due to the lack of sign information in the gradient term.

For instance, this problem does not possess a variational structure, which means that classical methods used to prove the nonexistence of positive solutions are not applicable here. Moreover, to the best of our knowledge, there are no existing a priori bounds for this class of problems. Additionally, since we allow the functions D_1 and D_2 to degenerate at the origin, this introduces further technical difficulties.

The case where $D_1 = D_2$ was studied in [19]. In that paper, the author applies a change of variables and transforms problem (1.1) into an equivalent semilinear equation. This transformation allows for the application of classical sub- and supersolution methods to this class of problems.

In [3], the authors develop global bifurcation theorems for the case where $D_2 \equiv 0$. Among other results, they study the following equation:

$$-\mathrm{div}(A(x,u)\nabla u) = \lambda u \ \text{ in } \Omega, \quad u = 0 \ \text{ on } \partial\Omega,$$

where $A(x,s)$ is a symmetric, positive-definite matrix whose coefficients are bounded Carathéodory functions. The study of this problem with unbounded coefficients was conducted in [10].

The logistic equation with a nonlinear diffusion term and linear advection was studied in [8]. We also highlight the works [6, 7], where the authors study the logistic equation with nonlinear diffusion in the absence of an advection term.

More recently, in [11], one of the authors examined the case where $D_1 \equiv 1$ and $D_2 = pu^{p-1}$, with $p > 1$, in the presence of the classical logistic reaction term. In that paper, the authors establish results on the existence and multiplicity of positive solutions, along with a novel uniqueness result for this class of problems. Subsequently, the case where $p < 1$ was analyzed in [22].

To state our main results, we need to introduce some notation. Throughout this work, we consider the function $h : [0, +\infty) \to (0, +\infty)$ defined by

$$h(s) = \frac{D_2(s)}{D_1(s)}, \text{ if } s > 0, \qquad h(0) := \lim_{s\to 0} \frac{D_2(s)}{D_1(s)}, \tag{1.2}$$

and we assume that

(H_1) The functions $D_1, D_2 : [0, +\infty) \to (0, +\infty)$ are of class C^1.

We also define the quantities

$$h(\infty) := \lim_{s\to+\infty} \frac{D_2(s)}{D_1(s)}, \qquad D_i(\infty) := \lim_{s\to+\infty} D_i(s), \text{ for } i \in \{1, 2\}.$$

and the operators

$$L_0 := -\mathrm{div}(a(x)\nabla) + h(0)\vec{b}(x) \cdot \nabla$$

and, when $h(\infty) \in \mathbb{R}$,

$$L_\infty := -\mathrm{div}(a(x)\nabla) + h(\infty)\vec{b}(x) \cdot \nabla.$$

Given a second-order elliptic operator L with Hölder continuous coefficients in a regular bounded domain $U \subset \mathbb{R}^N$, we denote by

$$\sigma_1^U[L]$$

the principal eigenvalue of L in U, under the homogeneous Dirichlet boundary condition. For simplicity, we also adopt the following notation:

$$\sigma_1^0 := \sigma_1^\Omega[L_0], \qquad \sigma_1^\infty := \sigma_1^\Omega[L_\infty].$$

In our first main theorems, we obtain the existence of a solution for the problem when $\sigma_1^0 D_1(0) < \sigma_1^\infty D_1(\infty)$. More specifically, the following results hold:

Theorem 1.1. *Suppose that (H_1) holds, $h(\infty) < +\infty$, and $\sigma_1^0 D_1(0) < \sigma_1^\infty D_1(\infty)$. Then, for any $\lambda \in (\sigma_1^0 D_1(0), \sigma_1^\infty D_1(\infty))$, problem (1.1) has a positive classical solution.*

Theorem 1.2. *Suppose that (H_1) holds, $h(\infty) = +\infty$, and*

(b_1) There exists $\psi \in C^2(\overline{\Omega})$ such that

$$[\vec{b}(x) \cdot \nabla\psi] > 0, \quad \textit{for all } x \in \overline{\Omega}.$$

Then, for any $\lambda > \sigma_1^0 D_1(0)$, problem (1.1) has a positive classical solution.

In the proof, we perform a known change of variables. However, since we do not assume $D_1 = D_2$, the equivalent problem remains quasilinear. Nevertheless, we prove that the sub- and supersolution methods can still be applied to obtain solutions. We emphasize that, unlike previous papers, we allow for the cases where $D_1(0) = 0$ or $D_1(\infty) = \infty$.

Condition (b_1) is used to construct a supersolution in the more delicate case where $h(\infty) = +\infty$. If $\vec{b}$ is a regular conservative field, meaning $\vec{b} = \nabla\psi$ for some $\psi \in C^2(\overline{\Omega})$, then $[\vec{b}\cdot\nabla\psi] = |\vec{b}|^2$, and therefore (b_1) holds as long as $\vec{b}$ does not vanish in Ω.

To complement the results of the above theorems, we need to consider the case where the inequality $\sigma_1^0 D_1(0) < \sigma_1^\infty D_1(\infty)$ does not hold. To address this, we apply bifurcation theory. In our first result, we present necessary conditions for bifurcation both from trivial solutions and from infinity. More specifically, we shall prove the following:

Theorem 1.3. *Suppose that* $D_1(0) > 0$*.*

(i) *Then there exists an unbounded component* $\mathfrak{C}_0 \subset \mathbb{R} \times C_0^1(\overline{\Omega})$ *of positive solutions to* (1.1) *emanating from the trivial solution at* $(\sigma_1^0 D_1(0), 0)$*;*

(ii) *If additionally* $0 < D_i(\infty) < +\infty$ *for any* $i \in \{1, 2\}$*, then there exists an unbounded component* $\mathfrak{C}_\infty \subset \mathbb{R} \times C_0^1(\overline{\Omega})$ *of solutions to* (1.1) *which meets* $(\sigma_1^\infty D_1(\infty), \infty)$*. Moreover, if* $\vec{b} \in C^1(\overline{\Omega}; \mathbb{R}^N)$ *and assumption* (H_1) *holds, then* $\mathfrak{C}_\infty$ *consists of positive solutions to* (1.1)*.*

As a matter of fact, we prove that bifurcation of positive solutions from the trivial solution (resp., infinity) cannot occur at any point other than $(\sigma_1^0 D_1(0), \infty)$ (resp., $(\sigma_1^\infty D_1(\infty), \infty)$).

Moreover, under one of the following assumptions

(b_2) There exists $\xi \in \big(H_0^1(\Omega) \cap L^4(\Omega)\big) \setminus \{0\}$ such that $\mathrm{div}(\xi^2 \vec{b})$ has a constant sign a.e. in Ω,

or

(d_3) There exists $C > 0$ such that $\int_0^\infty D_2(t) t^{-1} dt < C$,

we obtain a non-existence result of positive solutions of (1.1) for $\lambda > 0$ large (see Proposition 3.4). Actually, if we define

$$\underline{\lambda} := \min\{\sigma_1^0 D_1(0), \sigma_1^\infty D_1(\infty)\} \quad \text{and} \quad \overline{\lambda} := \max\{\sigma_1^0 D_1(0), \sigma_1^\infty D_1(\infty)\},$$

we can establish the following existence result:

Theorem 1.4. *Suppose that* (H_1) *holds,* $D_1(0) > 0$*,* $0 < D_i(\infty) < +\infty$*, for any* $i \in \{1, 2\}$*, and* $\vec{b} \in C^1(\overline{\Omega}; \mathbb{R}^N)$*. Then, for any* $\lambda \in (\underline{\lambda}, \overline{\lambda})$*, problem* (1.1) *has a positive classical solution. If additionally* (b_2) *or* (d_3) *is satisfied, then the continua* $\mathfrak{C}_0$ *and* $\mathfrak{C}_\infty$ *given by Theorem 1.3 coincide.*

We also point out that, by the Divergence Theorem, condition (b_2) implies that $\mathrm{div}(\xi^2 \vec{b}(x)) = 0$ a.e. in Ω

Finally, it is worth noting that we studied the bifurcation directions from both the trivial solution and infinity (see Theorems 4.1, 4.2 and 4.3). This analysis is particularly challenging due to the presence of the gradient term. Moreover, as a byproduct of the classification discussed above, we are able to establish the existence of at least two positive solutions to (1.1) under some suitable conditions on the function D_1 (see Theorem 4.4).

Remark 1.5. *We emphasize that the condition* $D_2(s) > 0$ *for* $s > 0$ *is not strictly necessary. With minor adjustments, the same results can be obtained even if* D_2 *changes sign. In fact, what truly matters in the analysis is whether* $D_2(\infty)$ *is finite or not. For instance, the assumption* $D_2(\infty) < +\infty$ *can be replaced by* $|D_2(\infty)| < +\infty$*. Moreover, the case* $h(\infty) = -\infty$ *in Theorem 1.2 can also be treated by replacing the inequality in* (b_1) *with* $[\vec{b}(x) \cdot \nabla \psi] < 0$ *for all* $x \in \overline{\Omega}$*.*

The rest of the paper is organized as follows: In Section 2, we introduce a change of variables that will be used in the study of (1.1) and apply sub-supersolution methods to prove Theorems 1.1 and 1.2. Section 3 is dedicated to investigating the bifurcation from the curve of trivial solutions and from infinity. Theorems 1.3

and 1.4 are proven in this section. Finally, in Section 4, we analyze the bifurcation direction from the trivial solution and from infinity, and provide some results on the multiplicity of solutions.

2. The sub-supersolution approach

We start this section performing a change of variables in the following way: define the auxiliary function

$$g(s) := \int_0^s D_1(t)\mathrm{d}t, \qquad s \in \mathbb{R}_+ := \{s \in \mathbb{R} \,:\, s \geq 0\}.$$

Since $g'(s) = D_1(s) > 0$, for any $s > 0$, the function g is injective and belongs to the class C^1. Moreover, since $\int_0^\infty D_1(s)\mathrm{d}s = \infty$, thanks to hypothesis (d_2), we also have $g(\mathbb{R}_+) = \mathbb{R}_+$. So, the function g is invertible and, if we denote g^{-1} as its inverse, it is well defined the map

$$q(s) := \begin{cases} g^{-1}(s) & \text{if } s \geq 0, \\ 0 & \text{if } s < 0. \end{cases}$$

A direct computation shows that $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is a positive solution of (1.1) if, and only if, $w = g(u)$ is a positive solution of

$$\begin{cases} -\mathrm{div}(a(x)\nabla w) + h(q(w))[\vec{b}(x) \cdot \nabla w] = \lambda q(w) & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where h was defined in (1.2). The next result gathers some useful properties of q.

Lemma 2.1. *The map q is an increasing C^1 function, and it satisfies*

$$\lim_{s\to 0} \frac{q(s)}{s} = \begin{cases} \frac{1}{D_1(0)} & \text{if } D_1(0) > 0, \\ +\infty & \text{if } D_1(0) = 0, \end{cases} \tag{2.2}$$

and

$$\lim_{s\to +\infty} \frac{q(s)}{s} = \begin{cases} \frac{1}{D_1(\infty)} & \text{if } D_1(\infty) < +\infty, \\ 0 & \text{if } D_1(\infty) = +\infty. \end{cases} \tag{2.3}$$

Proof. Since q is the inverse function of g, it is of class C^1 and

$$q'(s) = \frac{1}{g'(q(s))} > 0, \quad \forall s > 0.$$

So, q is increasing. Moreover,

$$\lim_{s\to 0} \frac{q(s)}{s} = \lim_{s\to 0} q'(s) = \lim_{s\to 0} \frac{1}{g'(q(s))} = \lim_{s\to 0} \frac{1}{D_1(q(s))}$$

and (2.2) follows from L'Hospital's rule. The proof of (2.3) is analogous. □

Since we intend to apply the sub- and supersolution method, we present now the following definitions:

Definition 2.2. *We say that $\underline{w} \in C^2(\Omega) \cap C(\overline{\Omega})$ is a subsolution of (2.1) if*

$$\begin{cases} -\mathrm{div}(a(x)\nabla \underline{w}) + h(q(\underline{w}))[\vec{b}(x) \cdot \nabla \underline{w}] \leq \lambda q(\underline{w}) & \text{in } \Omega, \\ \underline{w} \leq 0 & \text{on } \partial\Omega. \end{cases}$$

A function $\overline{w} \in C^2(\Omega) \cap C(\overline{\Omega})$ *is called a supersolution of* (2.1) *if the above expression hold with reverse inequalities. Furthermore, a pair* $\underline{w}, \overline{w}$ *of sub-supersolution is called ordered if* $\underline{w} \leq \overline{w}$.

Now we will show that the sub- and supersolution methods can be applied to problem (2.1).

Theorem 2.3. *Suppose* (H_1) *and there exists an ordered pair* $\underline{w}, \overline{w}$ *of sub-supersolution of* (2.1)*. Then the problem has a minimal solution* w_* *and a maximal solution* w^* *in the order interval* $[\underline{w}, \overline{w}]$.

Proof. For any $\lambda > 0$, consider

$$f_\lambda(x, s, \eta) := \lambda q(s) - h(q(s))[\vec{b}(x) \cdot \eta], \quad (x, s, \eta) \in \Omega \times \mathbb{R}_+ \times \mathbb{R}^N.$$

Since $\vec{b} \in C^{0,\alpha}(\overline{\Omega}; \mathbb{R}^N)$, we have that $f_\lambda(\cdot, s, \eta) \in C^{0,\alpha}(\overline{\Omega})$ for every $(s, \eta) \in \mathbb{R}_+ \times \mathbb{R}^N$. By (H_1), the partial derivatives $\partial f_\lambda / \partial \eta$ and $\partial f_\lambda / \partial s$ are continuous. If we define

$$c(\rho) := \lambda q(\rho) + \|\vec{b}\|_{L^\infty(\Omega)} \max_{0 \leq s \leq \rho} h(q(s)), \quad \rho \geq 0,$$

it is clear that, for any $s \in [0, \rho]$, there holds

$$|f_\lambda(x, s, \eta)| \leq \lambda q(s) + |\vec{b}(x)| h(q(s)) |\eta| \leq c(\rho)(1 + |\eta|^2).$$

Thus, the result follows from [2, Theorem 1.1]. □

We aim now to obtain a ordered sub-supersolution pair of (2.1). We first establish the existence of a subsolution.

Lemma 2.4. *The problem* (2.1) *has a subsolution for any* $\lambda > \sigma_1^0 D_1(0)$.

Proof. Pick $m > 1$ in such a way that $\lambda > m\sigma_1^0 D_1(0)$. Let $\varphi_0 > 0$ be a principal eigenfunction of L_0 such that $\|\varphi_0\|_{L^\infty(\Omega)} = 1$. If we define, for $\epsilon > 0$, the function $\underline{w} := \epsilon \varphi_0^m$, we can use a direct computation of $\operatorname{div}(a(x)\nabla(\varphi_0^m))$ to conclude that $\underline{w}$ is a subsolution of (2.1) if, and only if,

$$\left(\frac{\lambda q(\epsilon \varphi_0^m)}{m \epsilon \varphi_0^m} - \sigma_1^0 \right) \varphi_0 \geq (1 - m) a(x) \varphi_0^{-1} |\nabla \varphi_0|^2 + (h(q(\epsilon \varphi_0^m)) - h(0))[\vec{b}(x) \cdot \nabla \varphi_0],$$

in Ω. Since $m > 1$, $a(x) \geq a_0 > 0$ in Ω and $\varphi_0^{-1} \geq \|\varphi_0\|_\infty^{-1} = 1$, the above inequality is true if

$$\left(\frac{\lambda q(\epsilon \varphi_0^m)}{m \epsilon \varphi_0^m} - \sigma_1^0 \right) \varphi_0 + (m-1) a_0 |\nabla \varphi_0|^2 - \|\vec{b} \cdot \nabla \varphi_0\|_{L^\infty(\Omega)} |h(q(\epsilon \varphi_0^m)) - h(0)| \geq 0, \tag{2.4}$$

for any $x \in \Omega$.

If $D_1(0) > 0$, we can use $q(0) = 0$ and (2.2) to conclude that, uniformly in Ω, the following holds

$$\lim_{\epsilon \to 0^+} |h(q(\epsilon \varphi_0^m)) - h(0)| = 0$$

and

$$\lim_{\epsilon \to 0^+} \left(\frac{\lambda q(\epsilon \varphi_0^m)}{m \epsilon \varphi_0^m} - \sigma_1^0 \right) = \mu_0 := \left(\frac{\lambda}{m D_1(0)} - \sigma_1^0 \right) > 0.$$

Denoting the left-hand side of (2.4) by Γ_ϵ, we have that

$$\Gamma_\epsilon(x) = (\mu_0 + o_\epsilon(1)) \varphi_0(x) + (m - 1) a_0 |\nabla \varphi_0(x)|^2 + o_\epsilon(1), \quad x \in \Omega, \tag{2.5}$$

where $o_\epsilon(1)$ stands for a quantity which uniformly approaches to 0 as $\epsilon \to 0^+$.

We are going to obtain $\epsilon > 0$ and $m > 1$ such that Γ_ϵ is non-negative in Ω, which clearly implies (2.4). In order to do that, we first notice that L_0 has no term of order zero, and therefore we may apply the Strong Maximum Principle to conclude that $\varphi_0 > 0$ in Ω and $\frac{\partial \varphi_0}{\partial \nu} < 0$ on $\partial\Omega$. Then, given $r > 0$ small and setting

$$\Omega_r := \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) < r\},$$

we obtain $c_1 = c_1(r) > 0$ such that

$$\varphi_0 \geq c_1 > 0 \text{ in } \Omega \setminus \overline{\Omega}_r, \qquad |\nabla \varphi_0|^2 \geq c_1 > 0, \text{ in } \overline{\Omega}_r.$$

In the set $\Omega \setminus \overline{\Omega}_r$, we can use (2.5), the first inequality above, $m > 1$ and $\mu_0 > 0$, to obtain

$$\Gamma_\epsilon(x) \geq (\mu_0 + o_\epsilon(1))c_1 + o_\epsilon(1) \geq 0, \quad \forall\, 0 < \epsilon \leq \epsilon_1.$$

Analogously, since $\varphi_0 > 0$, in the set $\overline{\Omega}_r$ we have that

$$\Gamma_\epsilon(x) \geq (m-1)a_0 c_1^2 + o_\epsilon(1) \geq 0, \quad \forall\, 0 < \epsilon \leq \epsilon_2.$$

Consequently, if we fix $\epsilon := \min\{\epsilon_1, \epsilon_2\}$, we conclude that (2.4) is verified and therefore $\underline{w} = \epsilon \varphi_0^m$ is a subsolution of (2.1).

When $D_1(0) = 0$, it follows from (2.2) that $q(s)/s \to +\infty$, as $s \to 0$. Thus, for any $\mu_0 > 0$ we have that

$$\Gamma_\epsilon(x) \geq (\mu_0 + o_\epsilon(1))\,\varphi_0(x) + (m-1)a_0|\nabla\varphi_0(x)|^2 + o_\epsilon(1), \quad x \in \Omega,$$

and we can repeat the above argument. We omit the details. □

In the construction of the supersolution, we consider two distinct cases depending on $h(\infty)$. First, we will address the case where $h(\infty)$ is finite.

Lemma 2.5. *If $h(\infty) \in [0, +\infty)$, then problem (2.1) has a supersolution for any $\lambda < \sigma_1^\infty D_1(\infty)$.*

Proof. We first assume that $D_1(\infty) < \infty$. Let $U \subset \mathbb{R}^N$ be a regular domain such that $\overline{\Omega} \subset U$, and let $\widetilde{a}, \widetilde{b} : \overline{U} \to \mathbb{R}$ be smooth extensions of a and $\vec{b}$ in U, with $\widetilde{a} \geq a_0/2 > 0$. We may then consider the operator L_∞ acting on functions defined in U. Since $\lambda < \sigma_1^\infty D_1(\infty)$, we can use the continuity of the principal eigenvalue with respect to both the coefficients and the domain to choose U such that

$$\lambda < \sigma_1^U[L_\infty] D_1(\infty) \leq \sigma_1^\infty D_1(\infty).$$

Let $\widetilde{\varphi}_\infty > 0$ be an eigenfunction of L_∞ associated with $\sigma_1^U[L_\infty]$ and such that $\|\widetilde{\varphi}_\infty\|_{L^\infty(U)} = 1$. If we define, for $K > 0$, the function $\overline{w} := K\widetilde{\varphi}_\infty$, a direct computation shows that $\overline{w}$ is a supersolution of (2.1) if, and only if,

$$\left(\frac{\lambda q(K\widetilde{\varphi}_\infty)}{K\widetilde{\varphi}_\infty} - \sigma_1^U[L_\infty]\right)\widetilde{\varphi}_\infty \leq \left(h(q(K\widetilde{\varphi}_\infty)) - h(\infty)\right)[\vec{b}(x) \cdot \nabla\widetilde{\varphi}_\infty] \quad \text{in } \Omega. \tag{2.6}$$

We conclude from (2.3) that $q(s) \to +\infty$, as $s \to +\infty$. Since $\widetilde{\varphi}_\infty$ is positive, it follows from (2.3) again that

$$\mu_\infty := \lim_{K \to +\infty} \left(\frac{\lambda q(K\widetilde{\varphi}_\infty)}{K\widetilde{\varphi}_\infty} - \sigma_1^U[L_\infty]\right) = \frac{\lambda}{D_1(\infty)} - \sigma_1^U[L_\infty] < 0.$$

By construction, there exists $c_1 > 0$ such that $\widetilde{\varphi}_\infty \geq c_1 > 0$ in Ω. Hence, for $K > 0$ large, we have that

$$\left(\frac{\lambda q(K\widetilde{\varphi}_\infty)}{K\widetilde{\varphi}_\infty} - \sigma_1^U[L_\infty]\right)\widetilde{\varphi}_\infty = (\mu_\infty + o_K(1))\,\widetilde{\varphi}_\infty \leq \frac{\mu_\infty}{2} c_1 < 0,$$

in Ω. The inequality in (2.6) is a consequence of the above bound and the fact that

$$\lim_{K\to+\infty}(h(q(K\widetilde{\varphi}_\infty))-h(\infty))=0,$$

uniformly in Ω.

If $D_1(\infty)=+\infty$, we can repeat the argument noticing that $\mu_\infty=-\sigma_1^U[L_\infty]<0$. We omit the details. □

The case $h(\infty)=\infty$ is more delicate and requires the geometric condition (b_1). More specifically, we have the following:

Lemma 2.6. *If $h(\infty)=\infty$ and $\vec{b}$ satisfies (b_1), then problem (2.1) has a supersolution for any $\lambda>0$.*

Proof. Let $\psi\in C^2(\overline{\Omega})$ be given by (b_1) and $M>0$ such that $\psi+M>0$ in $\overline{\Omega}$. Define $\overline{w}=K(\psi+M)$, where $K>0$ is a constant to be chosen. By a direct calculation we see that $\overline{w}$ is a supersolution of (2.1) if, and only if,

$$\frac{\lambda q(K(\psi+M))}{K(\psi+M)}(\psi+M)+\operatorname{div}(a(x)\nabla\psi)\le h(q(K(\psi+M)))[\vec{b}(x)\cdot\nabla\psi]\quad\text{in }\Omega. \tag{2.7}$$

Since $h(\infty)=\infty$ and $[\vec{b}(x)\cdot\nabla\psi]>0$, it follows from (2.3) that

$$\lim_{K\to+\infty}h(q(K(\psi+M)))[\vec{b}(x)\cdot\nabla\psi]=+\infty$$

and

$$\lim_{K\to+\infty}\frac{\lambda q(K(\psi+M))}{K(\psi+M)}=\begin{cases}\frac{\lambda}{D_1(\infty)} & \text{if } D_1(\infty)<+\infty,\\ 0 & \text{if } D_1(\infty)=+\infty.\end{cases}$$

uniformly in Ω. Since $\operatorname{div}(a(x)\nabla\psi(x))$ is bounded, the above expressions imply that (2.7) holds for any $K>0$ large. □

We are now in position to prove our first existence results for (1.1).

Proof of Theorem 1.1. Let $\lambda\in(\sigma_0^1 D_1(0),\sigma_1^\infty D_1(\infty))$ be fixed. From Lemmas 2.4 and 2.5 we obtain a pair of sub-supersolutions $\underline{w}=\epsilon\varphi_0^m$, $\overline{w}=K\widetilde{\varphi}_\infty$ for (2.1). Since $\widetilde{\varphi}_\infty\ge c_1>0$ in Ω, we can take $K>0$ large in such a way that $\underline{w}\le\overline{w}$ in Ω. It follows from Theorem 2.3 that problem (2.1) has at least one solution w in $[\underline{w},\overline{w}]$. Since $\varphi_0>0$ in Ω, this solution is positive. Therefore, taking $u=q(w)$, we obtain a positive solution for (1.1). □

Proof of Theorem 1.2. The proof is analogous to that presented to Theorem 1.1, just using Lemma 2.6 instead of Lemma 2.5. We omit the details. □

3. The Bifurcation approach

This section is dedicated to the study of bifurcation points of positive solutions, as well as to prove Theorem 1.4. Before proving our results let us recall some basic facts and some abstract result of Bifurcation Theory. We refer to [12, 15, 17, 20, 21] for more details.

Leu $\mathcal{U}$ be a Banach space and $\mathcal{F}:\mathbb{R}\times\mathcal{U}\to\mathcal{U}$ be a continuous function such that $\mathcal{F}(\lambda,0)=0$, for any $\lambda\in\mathbb{R}$. We say that $(\lambda_0,0)$ is a bifurcation point of equation $\mathcal{F}(\lambda,u)=0$ from the curve of trivial solutions $(\lambda,0)$ if there exists a sequence $(\lambda_n,u_n)\subset\mathbb{R}\times\mathcal{U}\setminus\{0\}$ such that $\mathcal{F}(\lambda_n,u_n)=0$, $\lambda_n\to\lambda_0$ and $u_n\to 0$, as $n\to+\infty$.

In the proof of Theorem 1.3, we are going to use the following abstract result which is a compiled of the results in [17, Proposition 6.5.2, Lemma 6.5.3, Lemma 6.5.4 and Theorem 6.5.5]:

Theorem 3.1. *Let $\mathcal{U}$ be an ordered Banach space whose positive cone $\mathcal{P}_{\mathcal{U}}$ is normal and has nonempty interior. Let $\mathcal{K}:\mathcal{U}\to\mathcal{U}$ be linear, continuous, compact and such that $\mathcal{K}(\mathcal{P}_{\mathcal{U}}\setminus\{0\})\subset \operatorname{int}(\mathcal{P}_{\mathcal{U}})$. Let $\mathcal{G}:\mathbb{R}\times\mathcal{U}\to\mathcal{U}$ be a continuous operator, which is compact on bounded sets and such that, for any compact set $\Lambda\subset\mathbb{R}$,*

$$\lim_{\|u\|_{\mathcal{U}}\to 0}\frac{\mathcal{G}(\lambda,u)}{\|u\|_{\mathcal{U}}}=0,\qquad \text{uniformly for } \lambda\in\Lambda.$$

Finally, assume that the operator

$$\mathcal{F}(\lambda,u):=u-\lambda\mathcal{K}u-\mathcal{G}(\lambda,u),$$

satisfies the strong maximum principle, in the sense that

$$\left.\begin{array}{c}(\lambda,u)\in\mathbb{R}\times(\mathcal{P}_{\mathcal{U}}\setminus\{0\})\\ \mathcal{F}(\lambda,u)=0\end{array}\right\}\implies u\in\operatorname{int}(\mathcal{P}_{\mathcal{U}}).$$

Then there exists an unbounded component $\mathfrak{C}\subset\mathbb{R}\times\operatorname{int}(\mathcal{P}_{\mathcal{U}})$ of solutions of $\mathcal{F}(\lambda,u)=0$ emanating from $(\lambda_0,0)$, where λ_0 is the inverse of the spectral radius of $\mathcal{K}$. Moreover, this is the unique bifurcation point of positive solutions from the curve of trivial solutions.

We now recall that (λ_0,∞), with $\lambda_0\in\mathbb{R}$, is a bifurcation point from infinity for $\mathcal{F}(\lambda,u)=0$ if there exists a sequence $(\lambda_n,u_n)\subset\mathbb{R}\times\mathcal{U}$ such that $\mathcal{F}(\lambda_n,u_n)=0$, $\lambda_n\to\lambda_0$ and $\|u_n\|_{\mathcal{U}}\to+\infty$, as $n\to+\infty$.

In order to present the abstract result, we shall need to prove the second part of Theorem 1.3, we consider a uniformly elliptic operator $\mathfrak{L}$, a continuous function $\kappa:\overline{\Omega}\to\mathbb{R}$ such that $\kappa(x)\geq\kappa_0>0$ in Ω and denote by μ_1 the principal eigenvalue of

$$\mathfrak{L}u=\lambda\kappa(x)u\ \text{ in }\Omega,\qquad u=0\ \text{ on }\partial\Omega.$$

With these notation, we can state the following particular case of [21, Theorem 2.28 and Corollary 2.37]:

Theorem 3.2. *If $\mathcal{G}\in C(\Omega\times\mathbb{R}\times\mathbb{R}^N\times\mathbb{R})$ satisfies*

$$\lim_{(s,|\xi|)\to(+\infty,+\infty)}\frac{|\mathcal{G}(x,s,\xi,\lambda)|}{(s^2+|\xi|^2)^{1/2}}=0, \tag{3.1}$$

uniformly for $x\in\Omega$ and $\lambda\in\Lambda$ a compact set, then the equation

$$\mathfrak{L}u=\lambda\kappa(x)u+\mathcal{G}(x,u,Du,\lambda)\ \text{ in }\Omega,\qquad u=0 \text{ on }\partial\Omega,$$

has a continuum $\mathfrak{C}_\infty\subset\mathbb{R}\times C^1_0(\overline{\Omega})$ of solutions which meets (μ_1,∞). Moreover, there exists a neighborhood $\mathcal{M}$ of (μ_1,∞) such that either

(i) *$\mathfrak{C}_\infty\setminus\mathcal{M}$ is bounded in $\mathbb{R}\times C^1_0(\overline{\Omega})$ and meets $\mathbb{R}\times\{0\}$ or*
(ii) *$\mathfrak{C}_\infty\setminus\mathcal{M}$ is unbounded.*

If, additionally, $\mathcal{G}$ is continuously differentiable and

$$\mathcal{G}(x,s,\xi,\lambda)=\mathcal{G}_1(x,s,\xi,\lambda)s+\sum_{j=1}^N(\mathcal{G}_2)_j(x,s,\xi,\lambda)\xi_j, \tag{3.2}$$

with $\mathcal{G}_1$, $\mathcal{G}_2$ continuous at $(s,\xi)=(0,0)$, then the solutions can be assumed positive.

We devote the rest of this section for the study of the bifurcation from the curve of trivial solutions and from infinity.

3.1. **Bifurcation from the trivial solution.** Our goal now is to obtain the existence of an unbounded continuum $\mathfrak{C}_0 \subset \mathbb{R} \times C_0^1(\overline{\Omega})$ of positive solutions of (1.1) emanating from the point $(\sigma_1^0 D_1(0), 0)$. This is exactly the statement of Theorem 1.3, whose proof we present in what follows.

Proof of Theorem 1.3 item (i). The first step is to rewrite problem (2.1) is such a way we can apply Theorem 3.1. Since L_0 has no terms of order zero, we may consider the map $K = L_0^{-1} : C(\overline{\Omega}) \to C_0^1(\overline{\Omega})$, which is the resolvent operator associated with the linear problem

$$\begin{cases} L_0 u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

for each $f \in C(\overline{\Omega})$. It is easy to prove that K is linear and continuous. Elliptic regularity combined with the compact embedding $W^{2,p}(\Omega) \hookrightarrow C_0^1(\overline{\Omega})$, for $p > N$, show that K is compact. Since L_0 satisfies the Strong Maximum Principle, we have that K is strongly positive. Moreover, since $K = L_0^{-1}$, the inverse of the spectral radius of the operator K is exactly σ_1^0.

Notice that $w \in C^2(\Omega) \cap C(\overline{\Omega})$ is a classical solution of (2.1) if, and only if,

$$\begin{cases} L_0 w = \lambda q(w) + (h(0) - h(q(w)))\,[\vec{b}(x) \cdot \nabla w] & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

Since $D_1(0) > 0$, the above problem is equivalent to

$$F(\lambda, w) := w - \lambda \frac{1}{D_1(0)} K w - G(\lambda, w) = 0,$$

where $G : \mathbb{R} \times C_0^1(\overline{\Omega}) \to C_0^1(\overline{\Omega})$ is defined by

$$G(\lambda, w) = \lambda K \left(q(w) - \frac{w}{D_1(0)} \right) + K \Big((h(0) - h(q(w)))\,[\vec{b}(x) \cdot \nabla w] \Big).$$

We observe that G is continuous and compact. Moreover, for any $w \neq 0$, we may use that K is a linear continuous operator to get

$$\begin{aligned} \frac{\|G(\lambda, w)\|_{C_0^1(\overline{\Omega})}}{\|w\|_{C_0^1(\overline{\Omega})}} &\leq \frac{C_1 |\lambda|}{\|w\|_{C_0^1(\overline{\Omega})}} \left\| q(w) - \frac{w}{D_1(0)} \right\|_{C(\overline{\Omega})} + C_2 \|h(q(w)) - h(0)\|_{C(\overline{\Omega})} \\ &\leq C_1 |\lambda| \left\| \frac{q(w)}{w} - \frac{1}{D_1(0)} \right\|_{C(\overline{\Omega})} + C_2 \|h(q(w)) - h(0)\|_{C(\overline{\Omega})} \end{aligned}$$

with $C_1 = C_1(T) > 0$ and $C_2 = C_2(\vec{b}) > 0$. Thus, recalling that $q(0) = 0$ and using (2.2), we conclude that, for any compact set $\Lambda \subset \mathbb{R}$, there holds

$$\lim_{\|w\|_{C_0^1(\overline{\Omega})} \to 0} \frac{\|G(\lambda, w)\|_{C_0^1(\overline{\Omega})}}{\|w\|_{C_0^1(\overline{\Omega})}} = 0, \quad \text{uniformly in } \lambda \in \Lambda.$$

We finally recall that the positive cone $\mathcal{P}$ of the ordered Banach space $C_0^1(\overline{\Omega})$ verifies

$$\operatorname{int}(\mathcal{P}) = \left\{ u \in C_0^1(\overline{\Omega}) : u(x) > 0,\ x \in \Omega \text{ and } \frac{\partial u}{\partial \nu}(x) < 0\ x \in \partial\Omega \right\},$$

where $\nu = \nu(x)$ stands for the outward unit normal to Ω at $x \in \partial\Omega$. Suppose that $\lambda \in \mathbb{R}$ and $w \in \mathcal{P} \setminus \{0\}$ are such that $F(\lambda, w) = 0$. Then w is a non-negative and a non-zero solution of (2.1). Thus, for

$$\widetilde{L} := -\mathrm{div}(a(x)\nabla) + h(q(w(x)))\vec{b}(x) \cdot \nabla$$

we have that

$$\begin{cases} \widetilde{L}w = \lambda q(w) & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

If $\lambda \le 0$, it follows from the Strong Maximum Principle that $w \le 0$, which is a contradiction. Hence, $\lambda > 0$ and we can use the Strong Maximum Principle again to conclude that $w \in \mathrm{int}(\mathcal{P})$.

All together, the above considerations show that we can apply Theorem 3.1 to obtain the continuum of positive solutions $\mathfrak{C}_0$ stated in item (i) of Theorem 1.3. □

3.2. **Bifurcation from infinity.** In this subsection, we will obtain results on bifurcation from infinity for problem (1.1), complementing the study above. We begin by proving the second part of Theorem 1.3:

Proof of Theorem 1.3 item (ii). Since $D_i(\infty) \in (0, +\infty)$, for $i = 1, 2$, we have that $h(\infty) < +\infty$. Hence, we can rewrite (2.1) in the following way:

$$\begin{cases} L_\infty w = \lambda \dfrac{1}{D_1(\infty)} w + G(x, w, \nabla w, \lambda) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.3}$$

where

$$G(x, s, \xi, \lambda) := \lambda \left(q(s) - \frac{s}{D_1(\infty)} \right) + (h(\infty) - h(q(s)))\,[\vec{b}(x) \cdot \xi],$$

for any $(x, s, \xi, \lambda) \in \Omega \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}$. It is clear that the decomposition in (3.2) holds with

$$G_1(x, s, \xi, \lambda) := \begin{cases} \lambda \left(\frac{q(s)}{s} - \frac{1}{D_1(\infty)} \right) & \text{if } s \ne 0, \\ \lambda \left(\frac{1}{D_1(0)} - \frac{1}{D_1(\infty)} \right) & \text{if } s = 0, \end{cases}$$

and $(G_2)_j(x, s, \xi, \lambda) := (h(\infty) - h(q(s)))\, b_j(x)$, for any $j = 1, \dots, N$.

Suppose that $\Lambda \subset \mathbb{R}$ is a compact set and notice that, for any $\lambda \in \Lambda$,

$$\lim_{(s,|\xi|)\to(+\infty,+\infty)} \frac{\left|\lambda \left(q(s) - \dfrac{s}{D_1(\infty)} \right)\right|}{(s^2 + |\xi|^2)^{1/2}} \le c_1 \lim_{(s,|\xi|)\to(+\infty,+\infty)} \left| \frac{q(s)}{s} - \frac{1}{D_1(\infty)} \right| = 0.$$

Moreover,

$$\frac{\left|(h(\infty) - h(q(s)))\,[\vec{b}(x) \cdot \xi]\right|}{(s^2 + |\xi|^2)^{1/2}} \le |h(\infty) - h(q(s))| \|\vec{b}\|_{L^\infty(\Omega)},$$

and therefore

$$\lim_{(s,|\xi|)\to(+\infty,+\infty)} \frac{\left|(h(\infty) - h(q(s)))\,[\vec{b}(x) \cdot \xi]\right|}{(s^2 + |\xi|^2)^{1/2}} = 0.$$

Since the two above limits are uniform for $x \in \Omega$ and $\lambda \in \Lambda$, we conclude that

$$\lim_{(s,|\xi|)\to(+\infty,+\infty)} \frac{|G(x,s,\xi,\lambda)|}{(s^2+|\xi|^2)^{1/2}} = 0, \quad \text{uniformly for } x \in \Omega,\ \lambda \in \Lambda,$$

and therefore (3.1) holds. Thus, we can apply Theorem 3.2 for problem (3.3) and use the definition of σ_1^∞ to get the conclusions of Therem 1.3 item (ii). □

It is interesting to note that our problem has no other bifurcation points from infinity, as we can see from the next result:

Proposition 3.3. *Suppose that $D_1(0) > 0$ and $D_i(\infty) \in (0,+\infty)$, for $i = 1,2$. If $\lambda > 0$ is a bifurcation point from infinity of (2.1) for positive solutions in $\mathbb{R}\times C_0^1(\overline{\Omega})$, then $\lambda = \sigma_1^\infty D_1(\infty)$.*

Proof. Let $\lambda > 0$ be such that there exists $(\lambda_n, w_n) \subset \mathbb{R} \times C_0^1(\overline{\Omega})$ a sequence of solutions of (2.1) such that $w_n \geq 0$ in Ω and

$$(\lambda_n, \|w_n\|_{C_0^1(\overline{\Omega})}) \to (\lambda, +\infty).$$

By the Strong Maximum Principle one has $w_n(x) > 0$ for all $x \in \Omega$. Moreover, in view of the standard elliptic regularity, (λ_n, w_n) is a classical solution of (2.1) and

$$\|w_n\|_{L^2(\Omega)} \to +\infty.$$

Setting $v_n := w_n/\|w_n\|_{L^2(\Omega)}$ and using that (λ_n, w_n) is a solution of (2.1), we obtain

$$\int_\Omega a(x)[\nabla v_n \cdot \nabla\phi] + \int_\Omega h(q(w_n))[\vec{b}(x)\cdot\nabla v_n]\phi = \lambda_n \int_\Omega \frac{q(w_n)}{\|w_n\|_{L^2(\Omega)}}\phi, \tag{3.4}$$

for any $\phi \in H_0^1(\Omega)$.

If we choose $\phi = v_n$, we get

$$\int_\Omega a(x)|\nabla v_n|^2 + \int_\Omega h(q(w_n))[\vec{b}(x)\cdot\nabla v_n]v_n = \lambda_n \int_\Omega \frac{q(w_n)}{\|w_n\|_{L^2(\Omega)}} v_n. \tag{3.5}$$

In what follows, we consider $\vec{b} \neq 0$. The case $\vec{b} \equiv 0$ is similar. It follows from $D_1(\infty) \in (0,+\infty)$, $D_1(0) > 0$, (2.2) and (2.3) that

$$q(s) \leq c_1 s \quad \forall s \geq 0,$$

for some $c_1 > 0$. Using $D_1(0) > 0$ again together with $D_i(\infty) \in (0,+\infty)$, we obtain $c_2 > 0$ such that

$$|h(s)| \leq c_2 \quad \forall s \geq 0.$$

Moreover, there exists $c_3 > 0$ such that $|\lambda_n| \leq c_3$, for all $n \geq 1$. Given $\epsilon > 0$, we can apply these bounds in (3.5) to get

$$\begin{aligned} a_0\|v_n\|^2_{H_0^1(\Omega)} &\leq c_1c_3\int_\Omega v_n^2 + c_2\|\vec{b}\|_{L^\infty(\Omega)}\int_\Omega |\nabla v_n||v_n| \\ &\leq c_1c_3\int_\Omega v_n^2 + c_2\|\vec{b}\|_{L^\infty(\Omega)}\left(\epsilon\|v_n\|^2_{H_0^1(\Omega)} + \frac{1}{4\epsilon}\|v_n\|^2_{L^2(\Omega)}\right), \end{aligned}$$

where we also have used Young's inequality in the last line. If we choose $\epsilon = a_0/2c_2\|\vec{b}\|_{L^\infty(\Omega)} > 0$ and recall that $\|v_n\|_{L^2(\Omega)} = 1$, we obtain

$$\frac{a_0}{2}\|v_n\|^2_{H_0^1(\Omega)} \leq c_1c_3 + \frac{c_2^2\|\vec{b}\|^2_\infty}{2a_0}$$

and therefore (v_n) is bounded in $H_0^1(\Omega)$. Up to a subsequence, we have that

$$v_n \rightharpoonup v \text{ in } H_0^1(\Omega), \quad v_n \to v \text{ in } L^2(\Omega). \quad v_n(x) \to v(x) \text{ a.e. in } \Omega, \tag{3.6}$$

for some $v \in H_0^1(\Omega)$. Using $\phi = (v_n - v)$ as test function in (3.4) yields

$$\int_\Omega a(x)[\nabla v_n \cdot \nabla(v_n - v)] = \Gamma_{1,2} - \Gamma_{2,n} \tag{3.7}$$

where

$$\Gamma_{1,n} := \lambda_n \int_\Omega \frac{q(w_n)}{\|w_n\|_{L^2(\Omega)}}(v_n - v) \quad \text{and} \quad \Gamma_{2,n} := \int_\Omega h(q(w_n))(v_n - v)[\vec{b}(x) \cdot \nabla v_n].$$

From Hölder's inequality, we obtain

$$|\Gamma_{1,n}| \le c_1 c_3 \int_\Omega |v_n||v_n - v| \le c_1 c_3 \|v_n\|^2_{L^2(\Omega)} \|v_n - v\|^2_{L^2(\Omega)},$$

and

$$|\Gamma_{2,n}| \le c_2 \|\vec{b}\|_{L^\infty(\Omega)} \int_\Omega |\nabla v_n||v_n - v| \le c_2 \|\vec{b}\|_{L^\infty(\Omega)} \|v_n\|^2_{H_0^1(\Omega)} \|v_n - v\|^2_{L^2(\Omega)}.$$

It follows from (3.6) that both $\Gamma_{1,n}$ and $\Gamma_{2,n}$ goes to 0, as $n \to +\infty$. Thus, we may use (3.7), (3.6) and $a(x) \ge a_0$ to get

$$\lim_{n\to+\infty} \|v_n\|^2_{H_0^1(\Omega)} = \|v\|^2_{H_0^1(\Omega)}.$$

This and the weak convergence of (v_n) imply that $v_n \to v$ strongly in $H_0^1(\Omega)$.

Now, let us analyse the limit of each term in (3.4). Firstly, the weak convergence of (v_n) yields

$$\lim_{n\to+\infty} \int_\Omega a(x)[\nabla v_n \cdot \nabla\phi] = \int_\Omega a(x)[\nabla v \cdot \nabla \phi],$$

Since $v_n > 0$, we have that $v \ge 0$. Moreover, $v \ne 0$, because $\|v\|_{L^2(\Omega)} = 1$. Thus, the set

$$\Omega^+ = \{x \in \Omega : v(x) > 0\}$$

has positive measure. It is clear that $w_n(x) = v_n(x)\|w_n\|_{L^2(\Omega)} \to +\infty$ a.e. Ω^+ and therefore

$$\lim_{n\to+\infty} h(q(w_n(x))) = h(\infty),$$

$$\lim_{n\to+\infty} \frac{q(w_n)}{\|w_n\|_{L^2(\Omega)}} = \lim_{n\to+\infty} \frac{q(w_n)}{w_n} v_n = \frac{1}{D_1(\infty)} v,$$

for a.e. $x \in \Omega^+$. It follows from (3.6) and Lebesgue's Theorem that

$$\lim_{n\to+\infty} \int_{\Omega^+} h(q(w_n))[\vec{b}(x) \cdot \nabla v_n]\phi = \int_{\Omega^+} h(\infty)[\vec{b}(x) \cdot \nabla v]\phi,$$

and

$$\lim_{n\to+\infty} \int_{\Omega^+} \frac{q(w_n)}{\|w_n\|_{L^2(\Omega)}} \phi = \int_{\Omega^+} \frac{1}{D_1(\infty)} v\phi.$$

On the other hand,

$$\Big| \int_{\Omega\setminus\Omega_+} h(q(w_n))[\vec{b}(x) \cdot \nabla v_n]\phi \Big| \le c_2 \|\vec{b}\|_{L^\infty(\Omega)} \int_{\Omega\setminus\Omega^+} |\nabla v_n||\phi|$$

and

$$\Big| \int_{\Omega\setminus\Omega_+} \frac{q(w_n)}{\|w_n\|_{L^2(\Omega)}} \phi \Big| \le c_1 \int_{\Omega\setminus\Omega^+} |v_n||\phi|$$

Since $v \equiv 0$ in $\Omega \setminus \Omega_+$, we can argue as before to conclude that

$$\lim_{n\to+\infty} \int_{\Omega\setminus\Omega_+} h(q(w_n))[\vec{b}(x)\cdot\nabla v_n]\phi = \lim_{n\to+\infty} \int_{\Omega\setminus\Omega_+} \frac{q(w_n)}{\|w_n\|_{L^2(\Omega)}} = 0.$$

By combining the aforementioned convergences and letting $n \to \infty$ in equation (3.4), we conclude that

$$\int_\Omega a(x)[\nabla v\cdot\nabla\phi] + \int_\Omega h(\infty)[\vec{b}(x)\cdot\nabla v]\phi = \frac{\lambda}{D_1(\infty)}\int_\Omega v\phi, \qquad \forall\,\phi \in H_0^1(\Omega).$$

Thus, v weakly satisfies

$$\begin{cases} L_\infty v = \dfrac{\lambda}{D_1(\infty)} v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Since $v \neq 0$ and $v \geq 0$, it must be an eigenfunction associated with the principal eigenvalue of L_∞. Thus, $\lambda = \sigma_1^\infty D_1(\infty)$ and the proof is complete. □

3.3. **Proof of Theorem 1.4.** This subsection is dedicated to presenting the proof of Theorem 1.4. First, we present a result concerning the non-existence of solutions, which will be used to complement the study of the continua previously obtained. This result is also of independent interest.

Proposition 3.4. *Suppose that $D_1(\infty) < +\infty$ and let $u \in H_0^1(\Omega)\setminus\{0\}$ be a weak non-negative solution of (1.1).*

(i) If D_2 satisfies (d_3), then

(3.8)
$$\lambda \leq \|D_1\|_{L^\infty(\mathbb{R})}\frac{\int_\Omega a(x)|\nabla\psi|^2}{\int_\Omega \psi^2} + C\int_\Omega |\mathrm{div}(\psi^2\vec{b}(x))| \quad \forall\psi \in [H_0^1(\Omega)\cap L^4(\Omega)]\setminus\{0\}.$$

(ii) If $\vec{b}$ satisfies (b_2), then

$$\lambda \leq \|D_1\|_{L^\infty(\mathbb{R})}\frac{\int_\Omega a(x)|\nabla\xi|^2}{\int_\Omega \xi^2}. \tag{3.9}$$

Proof. Let $u \in H_0^1(\Omega)\setminus\{0\}$ be a weak non-negative solution of (1.1) and $\psi \in H_0^1(\Omega)\cap L^4(\Omega)$. Given $\epsilon > 0$, we can use $\psi^2/(u+\epsilon) \in H_0^1(\Omega)$ as a test function to get

$$\begin{aligned} \lambda\int_\Omega \frac{u}{u+\epsilon}\psi^2 &= \int_\Omega \frac{D_2(u)}{u+\epsilon}\psi^2[\vec{b}(x)\cdot\nabla u] \\ &\quad - \int_\Omega a(x)D_1(u)\nabla u\cdot\left[\frac{\psi^2}{(u+\epsilon)^2}\nabla u - \frac{2\psi}{u+\epsilon}\nabla\psi\right]. \end{aligned} \tag{3.10}$$

Let $f_\epsilon : \mathbb{R}_+ \to \mathbb{R}_+$ be the C^1-function given by

$$f_\epsilon(s) := \int_0^s \frac{D_2(t)}{t+\epsilon}\,dt, \quad s \in \mathbb{R}_+.$$

Integrating by parts we find that

$$\begin{aligned}\int_\Omega \frac{D_2(u)}{u+\epsilon}\psi^2[\vec{b}(x)\cdot\nabla u] &= \int_\Omega \psi^2[\vec{b}(x)\cdot\nabla(f_\epsilon(u))]\\ &= \int_{\partial\Omega} f_\epsilon(u)\psi^2[\vec{b}(x)\cdot\vec{\nu}(x)]d\sigma - \int_\Omega f_\epsilon(u)\mathrm{div}(\psi^2\vec{b}(x))\\ &= -\int_\Omega f_\epsilon(u)\mathrm{div}(\psi^2\vec{b}(x)),\end{aligned}$$

Thus, returning to equation (3.10), it becomes aparent that

$$\begin{aligned}\lambda\int_\Omega \frac{u}{u+\epsilon}\psi^2 &= -\int_\Omega a(x)D_1(u)\nabla u\cdot\left[\frac{\psi^2}{(u+\epsilon)^2}\nabla u - \frac{2\psi}{u+\epsilon}\nabla\psi\right] - \int_\Omega f_\epsilon(u)\mathrm{div}(\psi^2\vec{b}(x))\\ &= -\int_\Omega a(x)D_1(u)\left|\nabla\psi - \frac{\psi}{u+\epsilon}\nabla u\right|^2 + \int_\Omega a(x)D_1(u)|\nabla\psi|^2 - \int_\Omega f_\epsilon(u)\mathrm{div}(\psi^2\vec{b}(x))\\ &\le \|D_1\|_{L^\infty(\mathbb{R})}\int_\Omega a(x)|\nabla\psi|^2 + \int_\Omega |f_\epsilon(u)\mathrm{div}(\psi^2\vec{b}(x))|.\end{aligned}$$

Letting $\epsilon\to 0^+$ and using Lebesgue Theorem we obtain

$$\lambda\int_\Omega \psi^2 \le \|D_1\|_{L^\infty(\mathbb{R})}\int_\Omega a(x)|\nabla\psi|^2 + \int_\Omega |f(u)\mathrm{div}(\psi^2\vec{b}(x))|.$$

The above inequality combined with the hypothesis implies the result. □

Remark 3.5. *If* $\mathrm{div}(\vec{b}(x)) = 0$ *for all* $x\in\Omega$ *and, for some nonzero function* $\psi\in H_0^1(\Omega)\cap L^4(\Omega)$, *we have that* $[\vec{b}\cdot\nabla\psi] = 0$ *a.e. in* Ω, *then* $\mathrm{div}(\psi^2\vec{b}(x)) = 0$ *a.e. in* Ω *and therefore* (b_2) *holds. This condition on the inner product means that* ψ *is the first integral of the vector field* $\vec{b}$. *It appears in several problems involving large advection. For instance, in* [4, Th. 0.3], *the existence of a first integral is a necessary and sufficient condition for determining the asymptotic behavior of an eigenvalue problem. See also* [1, 5].

We are able to prove our second main theorem concerning of existence of solutions.

Proof of Theorem 1.4. First, we notice that Theorem 1.3 provides the existence of the continua $\mathfrak{C}_0$ and $\mathfrak{C}_\infty$, bifurcating from the origin at $\sigma_1^0 D_1(0)$ and from infinity at $\sigma_1^\infty D_1(\infty)$, respectively. Moreover, by Theorem 3.1 and Proposition 3.3, these are the only bifurcation points for positive solutions of (1.1). Furthermore, by the Strong Maximum Principle, (1.1) does not possess positive solution for $\lambda = 0$. Thus,

$$\mathrm{Proj}_{\mathbb{R}}\mathfrak{C}_0, \mathrm{Proj}_{\mathbb{R}}\mathfrak{C}_\infty \subset (0,\infty).$$

If $\mathfrak{C}_0 = \mathfrak{C}_\infty$, then $(\underline{\lambda},\overline{\lambda})\subset \mathrm{Proj}_{\mathbb{R}}\mathfrak{C}_0$, which implies the result.

Now suppose that $\mathfrak{C}_0 \neq \mathfrak{C}_\infty$. Since $\mathfrak{C}_0$ is unbounded and $\sigma_1^\infty D_1(\infty)$ is the unique bifurcation point from infinity, it follows that $(\sigma_1^0 D_1(0),\infty)\subset \mathrm{Proj}_{\mathbb{R}}\mathfrak{C}_0$. In a similar way, we have that $(\sigma_1^\infty D(\infty),\infty)\subset \mathrm{Proj}_{\mathbb{R}}\mathfrak{C}_\infty$. Consequently, the result follows.

Finally, if we assume (b_2) or (d_3), we can apply Proposition 3.4 to conclude that problem (1.1) does not admit a positive solution for large $\lambda > 0$ and therefore the first coordinates of $\mathfrak{C}_0$ and $\mathfrak{C}_\infty$ are bounded. It follows from the global nature of these continua that $\mathfrak{C}_0 = \mathfrak{C}_\infty$.

□

Figure 1 illustrates the admissible bifurcation diagrams provided by Theorem 1.4. In part (a), we show a possible configuration when $\mathfrak{C}_0 \neq \mathfrak{C}_\infty$, and in part (b), a possibility when they are equal. For simplicity, we write $\lambda_0 = \sigma_1^0 D_1(0)$ and $\lambda_\infty = \sigma_1^\infty D_1(\infty)$

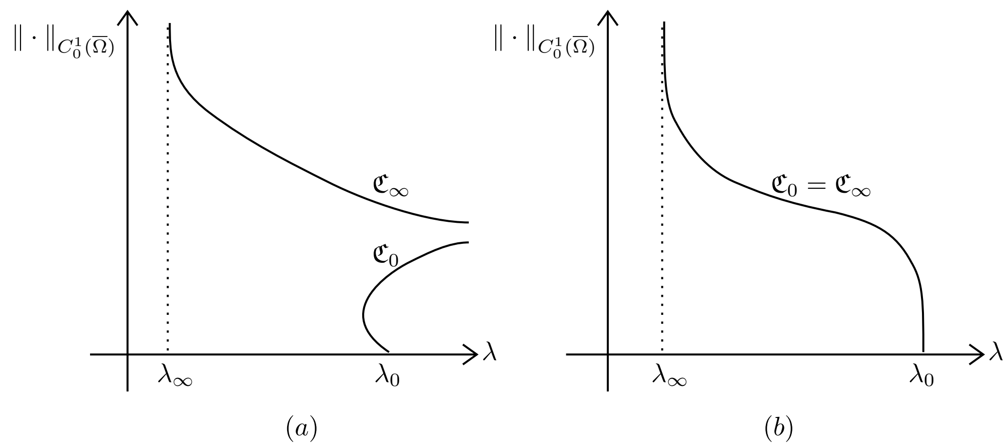

FIGURE 1. Possible bifurcation diagrams.

4. Bifurcation direction and multiplicity results

In this section, we study the bifurcation direction of positive solutions from both the origin and infinity. This analysis will allow us to obtain results on the multiplicity of positive solutions in certain cases.

In what follows we will analyze the bifurcation direction from the trivial solution.

Theorem 4.1. *Suppose that $D_1 \in C^2(\mathbb{R}_+)$, $D_2 \in C^1(\mathbb{R}_+)$ and $D_1(0) > 0$. Then $(\sigma_1^0 D_1(0), 0)$ is a bifurcation point of (1.1) from the curve of trivial solutions $(\lambda, 0)$. Moreover, if we denote by φ_0^* a principal positive eigenfunction of the adjoint operator L_0^*, this bifurcation point is subcritical if*

$$\mathcal{I} := D_1'(0) \int_\Omega a(x)\varphi_0 \left[\nabla\varphi_0 \cdot \nabla\varphi_0^*\right] + D_2'(0) \int_\Omega \varphi_0\varphi_0^* \left[\vec{b}(x) \cdot \nabla\varphi_0\right] < 0, \tag{4.1}$$

and supercritical if $\mathcal{I} > 0$.

Proof. Let $F : \mathbb{R} \times C_0^2(\overline{\Omega}) \to C(\overline{\Omega})$ be given by

$$F(\lambda, u) := -\text{div}(a(x)D_1(u)\nabla u) - D_2(u)[\vec{b}(x) \cdot \nabla u] - \lambda u.$$

It is clear that F is of class C^1 and the solutions of $F(\lambda, u) = 0$ are solutions of (1.1). Moreover, by a direct calculation,

$$\mathcal{L}(\lambda) := D_u F(\lambda, 0) = D_1(0)L_0 - \lambda I.$$

Since σ_1^0 is a simple eigenvalue of L_0, we have that

$$\ker\left[\mathcal{L}(\sigma_1^0 D_1(0))\right] = \text{span}\{\varphi_0\}.$$

On the other hand, by classical elliptic results (see, for instance, [**?**, Ch. 6]) it follows that $D_1(0)L_0 + M : C_0^2(\overline{\Omega}) \to C(\overline{\Omega})$ is an isomorphism, for $M > 0$ large enough.

Moreover, by compact embeddings, we have that $(\sigma_1^0 D_1(0)+M)I : C_0^2(\overline{\Omega}) \to C(\overline{\Omega})$ is a compact operator. Consequently, by [**?**, Th. 5.26], we conclude that $\mathcal{L}(\sigma_1^0 D_1(0)) = D_1(0)L_0 + MI - (\sigma_1^0 D_1(0)+M)I$ is a Fredholm operator of index zero. In particular, we have $\operatorname{codim}\operatorname{Rg}\left[\mathcal{L}(\sigma_1^0 D_1(0))\right] = \dim\ker\left[\mathcal{L}(\sigma_1^0 D_1(0))\right] = 1$.

We claim that

$$\mathcal{L}'(\sigma_1^0 D_1(0))\varphi_0 \notin R[\mathcal{L}(\sigma_1^0 D_1(0))], \tag{4.2}$$

where $\mathcal{L}' := D_\lambda \mathcal{L}$. Indeed, if this is not the case, we may use $\mathcal{L}'(\sigma_1^0 D_1(0)) = -I$ to obtain $\xi \in C_0^2(\overline{\Omega})$ such that

$$-\varphi_0 = \mathcal{L}(\sigma_1^0 D_1(0))\xi = D_1(0)\left[L_0\xi - \sigma_1^0\xi\right].$$

Multiplying this equality by φ_0^* and integrating over Ω, we obtain

$$\begin{aligned} 0 > \int_\Omega -\varphi_0\varphi_0^* &= D_1(0)\int_\Omega \left[L_0\xi - \sigma_1^0\xi\right]\varphi_0^* \\ &= D_1(0)\int_\Omega \left[L_0^*\varphi_0^* - \sigma_1^0\varphi_0^*\right]\xi = 0, \end{aligned}$$

which is a contradiction. This proves (4.2).

By using the Crandall-Rabinowitz Theorem [12, Theorem 1.7] we conclude that $(\sigma_1^0 D_1(0), 0)$ is a bifurcation point of $F(\lambda, u) = 0$ from the curve of trivial solutions. Moreover, if we denote by Z the topological complement of $\ker\left[\mathcal{L}(\sigma_1^0 D_1(0))\right]$ in $C_0^2(\overline{\Omega})$, there exist $\epsilon > 0$ and continuous functions

$$\lambda : (-\epsilon, \epsilon) \to \mathbb{R}, \qquad \psi : (-\epsilon, \epsilon) \to Z$$

such that $\lambda(0) = 0$, $\psi(0) = 0$ and the non-trivial solutions of $F(\lambda, u) = 0$ in a neighborhood of $(\sigma_1^0 D_1(0), 0)$ are given by

$$(\mu(s), u(s)) := \left(\sigma_1^0 D_1(0) + \lambda(s),\, s(\varphi_0 + \psi(s))\right) \quad s \in (-\epsilon, \epsilon),\ s \neq 0.$$

Since $\varphi_0 \in \operatorname{int}(\mathcal{P})$, then $u(s) \in \operatorname{int}(\mathcal{P})$ for $s > 0$ small enough, where $\mathcal{P}$ stands for the positive cone of $C_0^1(\overline{\Omega})$. This implies that the unique positive solutions in a neighborhood $(\sigma_1^0 D_1(0), 0)$ are given by $(\mu(s), u(s))$, for $s > 0$ small. Once that $F(\mu(s), u(s)) = 0$, we can take φ_0^* as test function to get

$$\begin{aligned} \int_\Omega \left[\sigma_1^0 D_1(0) + \lambda(s)\right] u(s)\varphi_0^* &= \int_\Omega a(x) D_1(u(s))[\nabla u(s)\cdot\nabla\varphi_0^*] \\ &\quad + \int_\Omega D_2(u(s))[\vec{b}(x)\cdot\nabla u(s)]\varphi_0^*. \end{aligned} \tag{4.3}$$

Moreover,

$$D_1(0)\int_\Omega \varphi_0^* L_0 u(s) = D_1(0)\int_\Omega u(s) L_0^*\varphi_0^* = \sigma_1^0 D_1(0)\int_\Omega u(s)\varphi_0^*,$$

that is,

$$D_1(0)\int_\Omega a(x)[\nabla u(s)\cdot\nabla\varphi_0^*] + D_2(0)\int_\Omega [\vec{b}(x)\cdot\nabla u(s)]\varphi_0^* = \sigma_1^0 D_1(0)\int_\Omega u(s)\varphi_0^*. \tag{4.4}$$

On the other hand, Taylor's expansion yields, for each $i = 1, 2$,

$$D_i(u(s)) = D_i(0) + sD_i'(0)u'(0) + o(s) = D_i(0) + sD_i'(0)\varphi_0 + o(s),$$

as $s \to 0$. Replacing the above equation in (4.3), using (4.4) and recalling that $u(s)/s = (\varphi_0 + \psi(s))$, we get

$$\frac{\lambda(s)}{s}\int_\Omega (\varphi_0+\psi(s))\,\varphi_0^* = \int_\Omega a(x)\left(D_1'(0)\varphi_0+\frac{o(s)}{s}\right)[\nabla\varphi_0^*\cdot\nabla(\varphi_0+\psi(s))]$$
$$+\int_\Omega\left(D_2'(0)\varphi_0+\frac{o(s)}{s}\right)[\vec{b}(x)\cdot\nabla(\varphi_0+\psi(s))]\varphi_0^*$$

Letting $s\to 0^+$, we obtain $\lambda_+'(0)\int_\Omega\varphi_0\varphi_0^*=\mathcal{I}$, that is, the sign of $\lambda_+'(0)$ is given by the sign of $\mathcal{I}$. This finishes the proof. □

As a consequence of this theorem, it is possible to show that the bifurcation direction of positive solutions from $(\lambda,0)$ is determined by the sign of $D_1'(0)$, assuming an appropriate hypothesis on $\vec{b}$. Specifically, we have the following result:

Theorem 4.2. *Suppose that $D_1\in C^2(\mathbb{R}_+)$, $D_2\in C^1(\mathbb{R}_+)$, $D_1(0)>0$ and define*

$$L_0' := -\mathrm{div}(a(x)D_1(0)\nabla\cdot).$$

Suppose that $\vec{b}\in C^1(\overline{\Omega};\mathbb{R}^N)$ satisfies $\mathrm{div}(\vec{b}(x))=0$ and $\vec{b}(x)\cdot\nabla z_0=0$ a.e. in Ω, where $z_0>0$ is the principal eigenfunction associated with $\sigma_1^\Omega[L_0']$. Then the bifurcation of positive solutions from $(\sigma_1^0 D_1(0),0)$ is subcritical if $D_1'(0)<0$, and supercritical if $D_1'(0)>0$.

Proof. Denote by simplicity $\lambda'=\sigma_1^\Omega[L_0']$. Since $\vec{b}(x)\cdot\nabla z_0=0$ for a.e. $x\in\Omega$, then

$$L_0 z_0=\frac{\lambda'}{D_1(0)}z_0 \ \text{ in } \Omega, \qquad z_0=0 \text{ on } \partial\Omega.$$

From $z_0>0$, we conclude that $\lambda'=\sigma_1^0 D_1(0)$ and $z_0=\varphi_0$, where φ_0 is a positive eigenfunction associated with σ_1^0. Furthermore, since $\vec{b}$ is divergence-free vector field, L_0 is a self-adjoint elliptic operator. In particular, $\varphi_0^*=\varphi_0$ and we can use Divergence Theorem to get

$$3\int_\Omega\varphi_0\varphi_0^*\left[\vec{b}(x)\cdot\nabla\varphi_0\right]=\int_\Omega\vec{b}(x)\cdot\nabla(\varphi_0^3)$$
$$=-\int_\Omega\mathrm{div}(\vec{b}(x))\varphi_0^3+\int_{\partial\Omega}\varphi_0^3\left[\vec{b}(x)\cdot\eta\right]d\sigma_x=0.$$

Thus,

$$\mathcal{I}=D_1'(0)\int_\Omega a(x)\varphi_0\left[\nabla\varphi_0\cdot\nabla\varphi_0^*\right]=D_1'(0)\int_\Omega a(x)\varphi_0|\nabla\varphi_0|^2$$

has the same sign of $D_1'(0)$. □

The same kind of result can be obtained when we are concerned with bifurcation from infinity, as we can see from:

Theorem 4.3. *Suppose that $D_1(0)>0$, $D_i(\infty)\in(0,+\infty)$, for $i=1,2$, and define*

$$L_\infty' := -\mathrm{div}(a(x)D_1(\infty)\nabla\cdot).$$

Suppose that $\vec{b}\in C^1(\overline{\Omega};\mathbb{R}^N)$ satisfies $\mathrm{div}(\vec{b}(x))=0$ and $\vec{b}(x)\cdot\nabla z_\infty(x)=0$ a.e. in Ω, where $z_\infty>0$ is a principal eigenfunction associated with $\sigma_1^\Omega[L_\infty']$. Then the bifurcation of positive solutions from infinity in $\lambda=\sigma_1^\infty D_1(\infty)$ is

(i) *subcritical, if $D_1(s)\le D_1(\infty)$ for every $s>0$.*

(ii) *supercritical, if $D_1(s)>D_1(\infty)$ for every $s>0$.*

Proof. Arguing as in the proof of Theorem 4.2 we obtain

$$\begin{cases} L_\infty z_\infty = \dfrac{\sigma_1^\Omega[L'_\infty]}{D_1(\infty)} z_\infty & \text{in } \Omega, \\ z_\infty = 0 & \text{on } \partial\Omega, \end{cases}$$

and therefore $\sigma_1^\Omega[L'_\infty] = \sigma_1^\infty D_1(\infty)$ and $z_\infty = \varphi_\infty$. So, it follows from Proposition 3.4 and Remark 3.5 that, if $(\lambda, u) \in \mathbb{R} \times H_0^1(\Omega)$ is a positive weak solution of (1.1), then

$$0 < \lambda \leq \|D_1\|_{L^\infty(\mathbb{R})} \frac{\int_\Omega a(x)|\nabla\varphi_\infty|^2}{\int_\Omega \varphi_\infty^2} = \|D_1\|_{L^\infty(\mathbb{R})}\sigma_1^\infty.$$

If $D_1(s) \leq D_1(\infty) < +\infty$, then $\|D_1\|_{L^\infty(\mathbb{R})} \leq D_1(\infty)$, and hence

$$\lambda \leq \sigma_1^\infty D_1(\infty),$$

which implies that the bifurcation at infinity of positive solutions is subcritical. This proves item (i).

To prove (ii), we proceed by contradiction. Suppose that $D_1(s) > D_1(\infty)$ for every $s > 0$ and assume that exists a sequence (λ_n, u_n) of classical solutions of (1.1) such that

$$(\lambda_n, \|u_n\|_{C_0^1(\overline{\Omega})}) \to (\sigma_1^\infty D_1(\infty), +\infty)$$

and $\lambda_n \leq \sigma_1^\infty D_1(\infty)$, for any $n \in \mathbb{N}$. Since $\varphi_\infty \in \text{int}(\mathcal{P})$, we can take u_n^2/φ_∞ as the test function in the equation satisfied by φ_∞ to get

$$\int_\Omega a(x)D_1(\infty)\nabla\varphi_\infty \cdot \left[\frac{2u_n}{\varphi_\infty}\nabla u_n - \frac{u_n^2}{\varphi_\infty^2}\nabla\varphi_\infty\right] = \sigma_1^\infty D_1(\infty)\int_\Omega u_n^2, \tag{4.5}$$

where we have used $\vec{b}(x)\cdot\varphi_\infty = 0$. Since (λ_n, u_n) verifies (1.1), we may pick u_n as test function in that equation and use $\lambda_n \leq \sigma_1^\infty D_1(\infty)$ to obtain

$$\int_\Omega a(x)D_1(u_n)|\nabla u_n|^2 + \int_\Omega D_2(u_n)[\vec{b}(x)\cdot\nabla u_n]u_n \leq \sigma_1^\infty D_1(\infty)\int_\Omega u_n^2. \tag{4.6}$$

Setting $f(s) := \int_0^s D_2(t)t\,\mathrm{d}t$, using $\text{div}(\vec{b}(x)) = 0$ in Ω and integrating by parts, we obtain

$$\int_\Omega D_2(u_n)[\vec{b}(x)\cdot\nabla u_n]u_n = \int_{\partial\Omega} f(u_n)[\vec{b}(x)\cdot\vec{\nu}(x)]d\sigma_x - \int_\Omega f(u_n)\text{div}(\vec{b}(x)) = 0.$$

Thus, it follows from (4.5) and (4.6) that

$$\int_\Omega a(x)D_1(\infty)\nabla\varphi_\infty \cdot \left[\frac{2u_n}{\varphi_\infty}\nabla u_n - \frac{u_n^2}{\varphi_\infty^2}\nabla\varphi_\infty\right] \geq \int_\Omega a(x)D_1(u_n)|\nabla u_n|^2.$$

Hence, we may use $D_1(s) > D_1(\infty)$ to get

$$\begin{aligned} 0 &< \int_\Omega a(x)|\nabla u_n|^2[D_1(u_n) - D_1(\infty)] \\ &\leq -\int_\Omega a(x)D_1(\infty)\left|\nabla u_n - \frac{u_n}{\varphi_\infty}\nabla\varphi_\infty\right|^2 \leq 0 \end{aligned}$$

which is a contradiction. ∎

Finally, we can combine Theorem 4.1 and 4.3 to establish a multiplicity result for some cases.

Theorem 4.4. *Suppose that all conditions of Theorems 4.1 and 4.3 hold and let $\mathcal{I}$ be the real number given by (4.1).*

(i) *If $\sigma_1^0 D_1(0) > \sigma_1^\infty D_1(\infty)$, $\mathcal{I} > 0$, and $D_1(s) > D_1(\infty)$ for every $s > 0$, then there exists $\lambda^* > \sigma_1^0 D_1(0)$ such that problem (1.1) has at least two positive classical solutions for any $\lambda \in (\sigma_1^0 D_1(0), \lambda^*)$.*

(ii) *If $\sigma_1^0 D_1(0) < \sigma_1^\infty D_1(\infty)$, $\mathcal{I} < 0$, and $D_1(s) < D_1(\infty)$ for every $s > 0$, then there exists $0 < \lambda^* < \sigma_1^0 D_1(0)$ such that problem (1.1) has at least two positive classical solutions for any $\lambda \in (\lambda^*, \sigma_1^0 D_1(0))$.*

Proof. We consider first item (i). Since $\mathcal{I} > 0$ and $D_1(s) > D_1(\infty)$ for every $s > 0$, by Theorems 4.1 and 4.3, we have that both bifurcations (at the origin and at infinity) are supercritical. Thus, using that $\sigma_1^0 D_1(0) > \sigma_1^\infty D_1(\infty)$, we obtain $\lambda^* > \sigma_1^0 D_1(0)$ such that (1.1) has two positive classical solutions for each $\lambda \in (\sigma_1^0 D_1(0), \lambda^*)$. The proof of part (ii) is analogous. □

In Figure 2, we illustrate the possible behaviors of the continuum $\mathfrak{C}_0 = \mathfrak{C}_\infty$ under the hypotheses of Theorem 4.4 (i) and (ii). For simplicity, we denote $\lambda_0 = \sigma_1^0 D_1(0)$ and $\lambda_\infty = \sigma_1^\infty D_1(\infty)$.

Remark 4.5. *Note that all the hypotheses of Theorem 4.4 are satisfied if the following conditions occur: assume $D_1(s) = D_2(s) = D(s)$. In this case, $h \equiv 1$ and $\sigma_1^0 = \sigma_1^\infty$. Moreover, suppose that $\operatorname{div}(\vec{b}(x)) = 0$ and that $\vec{b} \cdot \nabla\varphi_a = 0$ a.e. in Ω, where $\varphi_a > 0$ stands for the principal eigenfunction of the operator $-\operatorname{div}(a(x)\nabla\cdot)$. Thus, if $D(s) > D(\infty) > 0$ for every $s > 0$ and $D'(0) > 0$, we can apply item (i) to obtain two solutions for λ in a specific range. The function $D(s) := e^{-(s-1)^2}e^{-(s-1)} + 1$, $s \geq 0$, satisfies all the above conditions. Similarly, if $0 < D(s) < D(\infty)$ for every $s \geq 0$ and $D'(0) < 0$, all the conditions stated in item (ii) holds.*

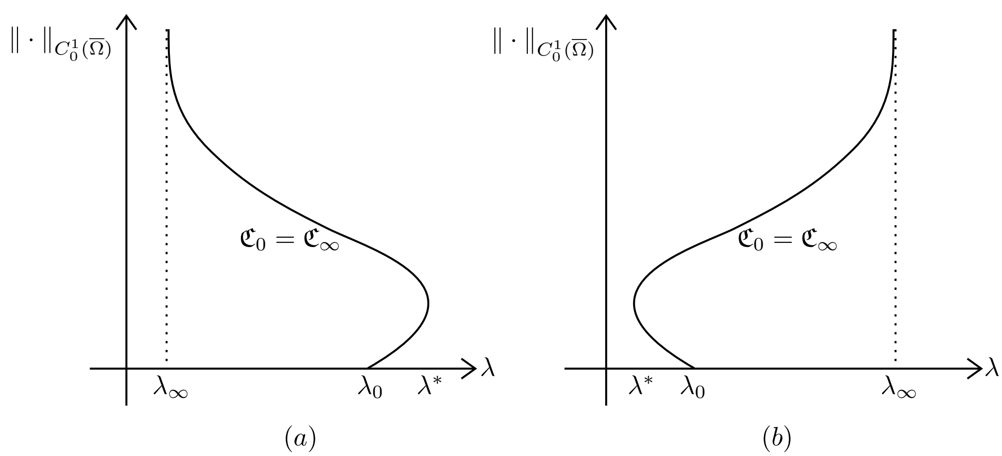

FIGURE 2. Possible bifurcation diagrams.

Remark 4.6. *Two interesting questions remain open:*

(i) *To complement the description of positive solutions of (1.1) when $h(\infty) = \infty$, without hypothesis (b_1);*

(*ii*) *Obtaining results regarding the existence or non-existence of positive solutions to* (1.1) *for large* λ *without assuming hypotheses* (b_2) *or* (d_3).

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