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MULTIPLICITY OF SOLUTIONS FOR ELLIPTIC SYSTEMS VIA LOCAL MOUNTAIN PASS METHOD

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Abstract. We consider the system

$$\left\{\begin{array}{l} -\varepsilon^2\Delta u + W(x)u = Q_u(u,v) \text{ in } \mathbb{R}^N, \\ -\varepsilon^2\Delta v + V(x)v = Q_v(u,v) \text{ in } \mathbb{R}^N, \\ u,v \in H^1(\mathbb{R}^N), u(x), v(x) > 0 \text{ for each } x \in \mathbb{R}^N, \end{array}\right.$$

where $\varepsilon > 0$, W and V are positive potentials and Q is a homogeneous function with subcritical growth. We relate the number of solutions with the topology of the set where W and V attain their minimum values. In the proof we apply Ljusternik-Schnirelmann theory.

1. **Introduction.** In the last years, many papers have considered the scalar equation

$$(P_\varepsilon) \qquad -\varepsilon^2\Delta u + V(x)u = W(x)|u|^{p-2}u \text{ in } \mathbb{R}^N,$$

where $N \geq 3$, V and W are positive potentials and $2 < p < 2^* := 2N/(N-2)$. The main points considered by these papers were the existence and multiplicity of solutions; the concentration of maximum points of the solutions, which is strongly related with the shape of the potentials V and W; the relation between the number of solutions and the topology of the set of critical points of the potentials.

The first author to deal with (P_ε) via variational methods seems to be Rabinowitz [20]. Among other results, he studied the case $W \equiv 1$ and obtained a positive solution by assuming that

$$0 < V_0 := \inf_{x\in\mathbb{R}^N} V(x) < \liminf_{|x|\to\infty} V(x). \qquad (R)$$

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Later, Wang [21] showed that solutions found in [20] are concentrated around global minimum points of $V(x)$ as ε tends to zero.

In [12] del Pino and Felmer considered a local condition on V and established the existence of positive solutions to (P_ε) which concentrate around local minimum of $V(x)$. In order to do this, they introduced a penalization method and supposed that $W \equiv 1$ and

$$\inf_{\xi \in \Lambda} V(\xi) < \min_{\xi \in \partial\Lambda} V(\xi), \qquad (DF)$$

for some open and bounded set $\Lambda \subset \mathbb{R}^N$.

Cingolani and Lazzo [11] exploited the geometry of the function V to obtain multiplicity of solutions for (P_ε). By using Ljusternik-Schnirelmann theory and assuming condition (R), they related the number of positive solutions with the topology of set where V attained its minimum value. In [5], Alves and Figueiredo have been established similar results from that found in [11], for a class of problems involving the p-Laplacian operator.

In [22], Wang and Zeng studied the full version of (P_ε) by considering the ground energy function $d(\xi)$, defined as the least energy of the functional associated with

$$-\varepsilon^2 \Delta u + V(\xi)u = W(\xi)|u|^{p-2}u \text{ in } \mathbb{R}^N,$$

where $\xi \in \mathbb{R}^N$ acts as a parameter instead of an independent variable. Under suitable assumptions on the potentials V and W, the function $\xi \mapsto d(\xi)$ attains its global minimum at a point $y^* \in \mathbb{R}^N$. Moreover, for every $\varepsilon > 0$ sufficiently small, there exists a solution u_ε whose global maximum point moving toward y^* as ε tends to 0. Other results concerning the scalar equation can be found in [15, 6, 3, 4, 16, 17, 18, 19] and references therein.

Recently, Alves [1] extended some existence and concentration results of the scalar equation for the following class of elliptic systems

$$(S_\varepsilon) \qquad \begin{cases} -\varepsilon^2 \Delta u + W(x)u = Q_u(u,v) \text{ in } \mathbb{R}^N, \\ -\varepsilon^2 \Delta v + V(x)v = Q_v(u,v) \text{ in } \mathbb{R}^N, \\ u\ , v \in H^1(\mathbb{R}^N),\ u(x), v(x) > 0 \text{ for each } x \in \mathbb{R}^N, \end{cases}$$

where $\varepsilon > 0$, $N \geq 3$ and potentials W, V are Hölder continuous. In that paper, motivated by [12], it is developed a penalization method for the energy functional associated to (S_ε). So, it is natural to ask if we can obtain multiplicity results analogous to that of the scalar equation. In the present paper, motivated by the results and methods developed in [1, 2, 12, 11, 22], we give a first positive answer to this question.

Besides the regularity of W and V, we suppose a condition analogous to (DF). More specifically, we assume that there exist an open bounded set $\Lambda \subset \mathbb{R}^N$, $x_0 \in \Lambda$ and $\rho_0 > 0$ such that

(H_1) $W(x), V(x) \geq \rho_0$ for each $x \in \partial\Lambda$;
(H_2) $W(x_0), V(x_0) < \rho_0$;
(H_3) $W(x_0) \geq W(x_0) > 0,\ V(x) \geq V(x_0) > 0$ for each $x \in \mathbb{R}^N$.

Setting $\mathbb{R}^2_+ := [0,\infty) \times [0,\infty)$, we can state our hypothesis on $Q \in C^2(\mathbb{R}^2_+, \mathbb{R})$ in the following way.

(Q_0) There exits $2 < p < 2^* := 2N/(N-2)$ such that

$$Q(tu, tv) = t^p Q(u,v) \text{ for each } t > 0,\ (u,v) \in \mathbb{R}^2_+.$$

(Q_1) There exists $c_1 > 0$ such that

$$|Q_u(u,v)| + |Q_v(u,v)| \le c_1 \left(u^{p-1} + v^{p-1}\right) \text{ for each } (u,v) \in \mathbb{R}^2_+.$$

(Q_2) $Q_u(0,1) = 0$, $Q_v(1,0) = 0$.
(Q_3) $Q_u(1,0) = 0$, $Q_v(0,1) = 0$.
(Q_4) $Q(u,v) > 0$ for each $u,v > 0$.
(Q_5) $Q_u(u,v), Q_v(u,v) \ge 0$ for each $(u,v) \in \mathbb{R}^2_+$.

We refer to [10, 13] for examples of functions verifying $(Q_0) - (Q_5)$ and for their main properties.

In order to get precise statements about our result we fix $\xi \in \mathbb{R}^N$ and consider the autonomous system associated to (S_ε), namely

$$\begin{cases} -\varepsilon^2 \Delta u + W(\xi) u = Q_u(u,v) \text{ in } \mathbb{R}^N, \\ -\varepsilon^2 \Delta v + V(\xi) v = Q_v(u,v) \text{ in } \mathbb{R}^N, \\ u\ , v \in H^1(\mathbb{R}^N),\ u(x), v(x) > 0 \text{ for each } x \in \mathbb{R}^N. \end{cases}$$

In view of conditions (H_3) and (Q_1), the above problem has a variational structure and the associated functional

$$I_\xi(u,v) := \frac{1}{2} \int_{\mathbb{R}^N} \left(\varepsilon^2 |\nabla u|^2 + \varepsilon^2 |\nabla v|^2 + W(\xi)|u|^2 + V(\xi)|v|^2\right) - \int_{\mathbb{R}^N} Q(u,v),$$

is well defined for $(u,v) \in E := H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$. Arguing as in [2, Section 2] we can show that I_ξ has the Mountain Pass geometry and therefore we can set the minimax level $C(\xi)$ in the following way

$$C(\xi) := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_\xi(\gamma(t)),$$

where $\Gamma := \{\gamma \in C([0,1], E) : \gamma(0) = 0,\ I_\xi(\gamma(1)) \le 0\}$. Moreover, the map $\xi \mapsto C(\xi)$ is continuous and $C(\xi)$ can be further characterized as

$$C(\xi) = \inf_{(u,v) \in \mathcal{M}_\xi} I_\xi(u,v),$$

with $\mathcal{M}_\xi$ being the Nehari manifold of I_ξ, that is

$$\mathcal{M}_\xi := \{(u,v) \in E \setminus \{(0,0)\} : I'_\xi(u,v)(u,v) = 0\}.$$

By using well known arguments, for each ξ fixed, the minimax level $C(\xi)$ is achieved and conditions $(H_1) - (H_3)$ yield

$$M := \left\{x \in \mathbb{R}^N : C(x) = \inf_{\xi \in \mathbb{R}^N} C(\xi)\right\} \neq \emptyset.$$

Furthermore, the same arguments found in [1] prove that

$$C^* = C(x_0) = \inf_{\xi \in \Lambda} C(\xi) < \min_{\xi \in \partial\Lambda} C(\xi). \tag{C_0}$$

If Y is a closed set of a topological space X, we denote by $\mathrm{cat}_X(Y)$ the Ljusternik-Schnirelmann category of Y in X, namely the least number of closed and contractible sets in X which cover Y. We are now ready to state the main result of this paper.

Theorem 1.1. *Suppose that potentials W and V satisfy $(H_1)-(H_3)$ and Q satisfies $(Q_0) - (Q_5)$. Then, for any $\delta > 0$ verifying*

$$M_\delta := \{x \in \mathbb{R}^N : \mathrm{dist}(x, M) < \delta\} \subset \Lambda,$$

there exists $\varepsilon_\delta > 0$ such that, for any $\varepsilon \in (0, \varepsilon_\delta)$, the system (S_ε) has at least $\mathrm{cat}_{M_\delta}(M)$ solutions.

The proof of Theorem 1.1 will be done in three main steps. First, we apply the penalization method found in [1], modifying the function $Q(u,v)$ outside the set Λ in such way that the energy functional of the modified problem satisfies the Palais-Smale condition. It is worthwhile to emphasize that, since we deal with the functional restricted to an appropriated manifold, the calculations performed to get compactness are much more involved to those of [1, 12] (see Section 2.1 for details). In the second step, by using a technique due to Benci and Cerami [9], we relate the category of the set M with the number of positive solutions for the modified problem. This objective is achieved by a detailed study of the energy functional restricted to its Nehari manifold. Finally we prove that, for $\varepsilon > 0$ small, the solutions for the modified problem are in fact solutions for (S_ε).

Our theorem extends the first result in [14]. Moreover, since we obtain multiple solutions, we complement the papers [1, 2, 10]. As far we know, it is the first time that penalization methods together with Ljusternik-Schnirelmann theory are used to get multiple solutions for gradient systems.

We finish the introduction by mentioning the recent paper of Avila and Wan [7] (see also [8]) where the authors obtain multiple solutions for a Hamiltonian version of (S_ε) but with $W \equiv V$.

The paper is organized as follows. In Section 2 we present the abstract framework and proves the Palais-Smale condition for the modified functional. Section 3 is devoted to the proof of a multiplicity result for a modified problem. Finally, we prove Theorem 1.1 in Section 4.

2. **Variational framework and a compactness result.** Since we are interested in positive solutions we extend the function Q to the whole $\mathbb{R}^2$ by setting $Q(u,v) = 0$ if $u \le 0$ or $v \le 0$. For simplicity, we write only $\int u$ instead of $\int_{\mathbb{R}^N} u(x)\mathrm{d}x$. We also note that, since Q is p-homogeneous, for each $(s,t) \in \mathbb{R}^2$ we have

$$pQ(s,t) = sQ_s(s,t) + tQ_t(s,t) \tag{1}$$

and

$$p(p-1)Q(s,t) = s^2 Q_{ss}(s,t) + t^2 Q_{tt}(s,t) + 2st Q_{st}(s,t). \tag{2}$$

Hereafter, we will work with the following system equivalent to (S_ε).

$$(\widehat{S}_\varepsilon) \qquad \begin{cases} -\Delta u + W(\varepsilon x)u = Q_u(u,v) \text{ in } \mathbb{R}^N, \\ -\Delta v + V(\varepsilon x)v = Q_v(u,v) \text{ in } \mathbb{R}^N, \\ u, v \in H^1(\mathbb{R}^N),\ u(x), v(x) > 0 \text{ for each } x \in \mathbb{R}^N. \end{cases}$$

In order to overcome the lack of compactness originated by the unboundedness of $\mathbb{R}^N$ we use a penalization method. Such kind of idea has first appeared in the paper of delPino and Felmer [12]. Here, we use an adaptation for systems introduced by the first author in [1].

We start by choosing $a > 0$ and considering $\eta : \mathbb{R} \to \mathbb{R}$ a non-increasing function of class C^2 such that

$$\eta \equiv 1 \text{ on } (-\infty, a],\ \eta \equiv 0 \text{ on } [5a, +\infty),\ |\eta'(s)| \le \frac{C}{a} \text{ and } |\eta''(s)| \le \frac{C}{a^2}, \tag{3}$$

for each $s \in \mathbb{R}$ and for some positive constant $C > 0$. Using the function η, we define $\widehat{Q} : \mathbb{R}^2 \to \mathbb{R}$ by

$$\widehat{Q}(s,t) := \eta(|(s,t)|)Q(s,t) + (1 - \eta(|(s,t)|)A(s^2 + t^2)$$

where

$$A := \max\left\{\frac{Q(s,t)}{s^2+t^2} : (s,t) \in \mathbb{R}^2,\ a \le |(s,t)| \le 5a\right\}.$$

Notice that, since $A > 0$ tends to zero as $a \to 0^+$, we may suppose that $A < \min\{W(x_0), V(x_0)\}$.

Finally, denoting by χ_Λ the characteristic function of the set Λ, we define $H : \mathbb{R}^N \times \mathbb{R}^2 \to \mathbb{R}$ by setting

$$H(x,s,t) := \chi_\Lambda(x)Q(s,t) + (1-\chi_\Lambda(x))\widehat{Q}(s,t).$$

For future reference we note that, as proved in [1, Lemma 2.2], for any $a > 0$ small and $(s,t) \in \mathbb{R}^2$ we have

$$sH_s(x,s,t) + tH_t(x,s,t) \le \frac{1}{2}\Big(W(x)s^2 + V(x)t^2\Big) \text{ for each } x \in \mathbb{R}^N \setminus \Lambda. \quad (4)$$

From now on we assume that a is chosen in such way that the above inequality holds.

As an immediate consequence of the above notations, we have the following lemma

Lemma 2.1. *Let $u_\varepsilon, v_\varepsilon \in H^1(\mathbb{R}^N)$ be positive functions such that*

$$-\varepsilon^2\Delta u + W(x)u = H_u(x,u,v),\ -\varepsilon^2\Delta v + V(x)v = H_v(x,u,v),$$

for each $x \in \mathbb{R}^N$. Moreover, suppose that $|(u_\varepsilon(x), v_\varepsilon(x))| \le a$ for each $x \in \mathbb{R}^N \setminus \Lambda$. Then, it follows from the definition of H and $\widehat{Q}$ that $H(\cdot, u_\varepsilon, v_\varepsilon) \equiv Q(u_\varepsilon, v_\varepsilon)$, and therefore $(u_\varepsilon, v_\varepsilon)$ is also a solution for the problem $(\widehat{S}_\varepsilon)$.

In view of this lemma, we deal in the sequel with the modified problem

$$(S_{\varepsilon,a}) \qquad \begin{cases} -\Delta u + W(\varepsilon x)u = H_u(x,u,v) \text{ in } \mathbb{R}^N, \\ -\Delta v + V(\varepsilon x)v = H_v(x,u,v) \text{ in } \mathbb{R}^N, \\ u, v \in H^1(\mathbb{R}^N), \end{cases}$$

and we will look for solutions $(u_\varepsilon, v_\varepsilon)$ verifying

$$|(u_\varepsilon(\varepsilon x), v_\varepsilon(\varepsilon x))| \le a \text{ for each } x \in \mathbb{R}^N \setminus \Lambda_\varepsilon,$$

where $\Lambda_\varepsilon := \{x \in \mathbb{R}^N : \varepsilon x \in \Lambda\}$.

For each $\varepsilon > 0$ we denote by X_ε the Hilbert space

$$X_\varepsilon := \left\{(u,v) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N) : \int (W(\varepsilon x)|u|^2 + V(\varepsilon x)|v|^2) < \infty\right\}$$

endowed with the norm

$$\|(u,v)\|_\varepsilon^2 := \int (|\nabla u|^2 + |\nabla v|^2 + W(\varepsilon x)|u|^2 + V(\varepsilon x)|v|^2).$$

Conditions (H_3) and (Q_1) imply that the critical points of the C^1-functional $J_\varepsilon : X_\varepsilon \to \mathbb{R}$ given by

$$J_\varepsilon(u,v) := \frac{1}{2}\int \left(|\nabla u|^2 + |\nabla v|^2 + W(\varepsilon x)|u|^2 + V(\varepsilon x)|v|^2\right) - \int H(x,u,v)$$

are weak solutions of $(S_{\varepsilon,a})$. We recall that these critical points belong to the Nehari manifold of J_ε, namely on the set

$$\mathcal{N}_\varepsilon := \{(u,v) \in X_\varepsilon \setminus \{(0,0)\} : J'_\varepsilon(u,v)(u,v) = 0\}.$$

It is well known that, for any nontrivial element $(u, v) \in X_\varepsilon$ the function $t \mapsto J_\varepsilon(tu, tv)$, for $t \geq 0$, achieves its maximum value at a unique point $t_u > 0$ such that $t_u(u, v) \in \mathcal{N}_\varepsilon$.

2.1. **The Palais-Smale condition.** Since we are intending to apply critical point theory we need to introduce some compactness property. So, let V be a Banach space, $\mathcal{V}$ be a C^1-manifold of V and $I : V \to \mathbb{R}$ a C^1-functional. We say that $I|_\mathcal{V}$ satisfies the Palais-Smale condition at level c ($(\mathrm{PS})_c$ for short) if any sequence $(u_n) \subset \mathcal{V}$ such that $I(u_n) \to c$ and $\|I'(u_n)\|_* \to 0$ contains a convergent subsequence. Here, we are denoting by $\|I'(u)\|_*$ the norm of the derivative of I restricted to $\mathcal{V}$ at the point u.

It is proved in [1] that the unconstrained functional satisfies $(\mathrm{PS})_c$ for each $c \in \mathbb{R}$. Nevertheless, to get multiple critical points, we need to work with the functional J_ε constrained to $\mathcal{N}_\varepsilon$. In order to prove the desired compactness result we shall first present some properties of $\mathcal{N}_\varepsilon$.

Lemma 2.2. *There exist positive constants a_1, δ such that, for each $a \in (0, a_1)$, $(u, v) \in \mathcal{N}_\varepsilon$, there hold*

$$\int_{\Lambda_\varepsilon} Q(u, v) \geq \delta \tag{5}$$

and

$$\int_{\mathbb{R}^N \setminus \Lambda_\varepsilon} (W(\varepsilon x)u^2 + V(\varepsilon x)v^2) \leq 2p \int_{\Lambda_\varepsilon} Q(u, v). \tag{6}$$

Proof. Since H has subcritical growth, it is easy to obtain $\widehat{\delta} > 0$ such that

$$\|(u, v)\|_\varepsilon \geq \widehat{\delta} \text{ for each } (u, v) \in \mathcal{N}_\varepsilon.$$

Thus, we can use (1) and (4) to get

$$\begin{aligned} \delta^2 \leq \|(u, v)\|_\varepsilon^2 &= \int_{\Lambda_\varepsilon} (uQu + vQ_v) + \int_{\mathbb{R}^N \setminus \Lambda_\varepsilon} (uH_u + vH_v) \\ &\leq p \int_{\Lambda_\varepsilon} Q(u, v) + \frac{1}{2} \int_{\mathbb{R}^N \setminus \Lambda_\varepsilon} (W(\varepsilon x)u^2 + V(\varepsilon x)v^2), \end{aligned}$$

and therefore

$$\frac{\widehat{\delta}^2}{2} \leq \frac{1}{2} \|(u, v)\|_\varepsilon^2 \leq p \int_{\Lambda_\varepsilon} Q(u, v),$$

which implies (5) with $\delta = \frac{\widehat{\delta}^2}{2p}$.

Recalling that $(u, v) \in \mathcal{N}_\varepsilon$ and using (4) and (1) again, we obtain

$$\begin{aligned} \int_{\mathbb{R}^N \setminus \Lambda_\varepsilon} (W(\varepsilon x)u^2 + V(\varepsilon x)v^2) &\leq \int_{\mathbb{R}^N \setminus \Lambda_\varepsilon} (uH_u + vH_v) + \int_{\Lambda_\varepsilon} (uQ_u + vQ_v) \\ &\leq \frac{1}{2} \int_{\mathbb{R}^N \setminus \Lambda_\varepsilon} (W(\varepsilon x)u^2 + V(\varepsilon x)v^2) + p \int_{\Lambda_\varepsilon} Q(u, v), \end{aligned}$$

from which follows (6). The lemma is proved. $\square$

The following technical results is the key stone in our compactness result.

Lemma 2.3. *Let $\phi_\varepsilon : X_\varepsilon \to \mathbb{R}$ be given by*

$$\phi_\varepsilon(u, v) := \|(u, v)\|_\varepsilon^2 - \int \Big(uH_u(\varepsilon x, u, v) + vH_v(\varepsilon x, u, v) \Big).$$

Then there exist $a_2, b > 0$ such that, for each $a \in (0, a_2)$,

$$\phi'_\varepsilon(u,v)(u,v) \leq -b < 0 \text{ for each } (u,v) \in \mathcal{N}_\varepsilon. \tag{7}$$

Proof. Given $(u,v) \in \mathcal{N}_\varepsilon$, we can use the definition of H, (1) and (2) to get

$$\begin{aligned}
\phi'_\varepsilon(u,v)(u,v) &= \int_{\Lambda_\varepsilon} (uQ_u + vQ_v) - (u^2Q_{uu} + v^2Q_{vv} + 2uvQ_{uv}) \\
&+ \int_{\mathbb{R}^N\setminus\Lambda_\varepsilon} (uH_u + vH_v) - \int_{\mathbb{R}^N\setminus\Lambda_\varepsilon} \left(u^2H_{uu} + v^2H_{vv} + 2uvH_{uv}\right) \\
&= -p(p-2)\int_{\Lambda_\varepsilon} Q(u,v) + \int_{\mathbb{R}^N\setminus\Lambda_\varepsilon} D_1 - \int_{\mathbb{R}^N\setminus\Lambda_\varepsilon} D_2
\end{aligned} \tag{8}$$

with

$$D_1 := (uH_u + vH_v) \text{ and } D_2 := \left(u^2H_{uu} + v^2H_{vv} + 2uvH_{uv}\right).$$

In what follows we denote $|z| := \sqrt{u^2+v^2}$. By using the definition of $\widehat{Q}$, η and (1) again, we obtain

$$\begin{aligned}
|D_1| &= \left|\eta'\frac{Q}{|z|} + p\eta\frac{Q}{|z|^2} - A\eta'|z| + 2A(1-\eta)\right||z|^2 \\
&\leq \left(\frac{C}{a}A5a + pA + A\frac{C}{a}5a + 4A\right)|z|^2 \\
&\leq C_1A|z|^2.
\end{aligned}$$

Since $A \to 0$ as $a \to 0^+$, the last inequality combined with (H_3) leads to

$$\int_{\mathbb{R}^N\setminus\Lambda_\varepsilon} (uH_u + vH_v) \leq o(1)\int_{\mathbb{R}^n\setminus\Lambda_\varepsilon} (W(\varepsilon x)u^2 + V(\varepsilon x)v^2), \tag{9}$$

where $o(1) \to 0$ as $a \to 0^+$.

In order to estimate the last integral in (8), we first compute

$$D_2 = -A\eta'(|z|^2 + 4|z|)|z|^2 + 2A(1-\eta)|z|^2 + \eta''Q|z||z|^2 + D_3 + D_4, \tag{10}$$

with

$$D_3 := \frac{2\eta'}{|z|}\left(u^3Q_u + v^3Q_v + u^2vQ_v + uv^2Q_u\right)$$

and

$$D_4 := \eta(u^2Q_{uu} + v^2Q_{vv} + 2uvQ_{uv}).$$

In view of (3) we have that

$$\left|A\eta'(|z|^2 + 4|z|)|z|^2\right| \leq A\frac{C}{a}(25a^2 + 20a)|z|^2 = o(1)|z|^2.$$

By using the definition of A, we also obtain

$$2A(1-\eta)|z|^2 = o(1)|z|^2 \text{ and } \eta''Q|z||z|^2 = o(1)|z|^2.$$

Moreover, we infer from (1) that

$$|D_3| = |4p\eta'Q||z| \leq 4p\frac{C}{a}A|z|^2 5a = 20pCA|z|^2 = o(1)|z|^2.$$

Finally, (2) implies that

$$D_4 = \eta(u^2Q_{uu} + v^2Q_{vv} + 2uvQ_{uv}) = \eta p(p-1)Q \geq 0.$$

From these estimates, we derive that

$$\int_{\mathbb{R}^N\setminus\Lambda_\varepsilon} \left(u^2H_{uu} + v^2H_{vv} + 2uvH_{uv}\right) \leq o(1)\int_{\mathbb{R}^N\setminus\Lambda_\varepsilon} (W(\varepsilon x)u^2 + V(\varepsilon x)v^2).$$

Thus, it follows from (9) and (8) that

$$\phi_\varepsilon'(u,v)(u,v) \leq -p(p-2)\int_{\Lambda_\varepsilon} Q(u,v) + o(1)\int_{\mathbb{R}^N\setminus\Lambda_\varepsilon} (W(\varepsilon x)u^2 + V(\varepsilon x)v^2).$$

Now we can use Lemma 2.2 to obtain, for a small enough,

$$\phi_\varepsilon'(u,v)(u,v) \leq (-p(p-2)+o(1))\int_{\Lambda_\varepsilon} Q(u,v) \leq -\frac{p(p-2)}{2}\delta = -b < 0.$$

The lemma is proved. $\square$

Proposition 1. *The functional J_ε restricted to $\mathcal{N}_\varepsilon$ satisfies $(\mathrm{PS})_c$ for each $c \in \mathbb{R}$.*

Proof. Let $(u_n, v_n) \subset \mathcal{N}_\varepsilon$ be such that

$$J_\varepsilon(u_n,v_n) \to c \text{ and } \|J_\varepsilon'(u_n,v_n)\|_* = o_n(1),$$

where $o_n(1)$ approaches zero as $n \to \infty$. Then there exists $(\lambda_n) \subset \mathbb{R}$ satisfying

$$J_\varepsilon'(u_n,v_n) = \lambda_n \phi_\varepsilon'(u_n,v_n) + o_n(1), \tag{11}$$

with ϕ_ε as in Lemma 2.3. Since $(u_n,v_n) \in \mathcal{N}_\varepsilon$ we have that

$$0 = J_\varepsilon'(u_n,v_n)(u_n,v_n) = \lambda_n \phi_\varepsilon'(u_n,v_n)(u_n,v_n) + o_n(1)\|(u_n,v_n)\|_\varepsilon.$$

Straightforward calculations show that (u_n,v_n) is bounded. Moreover, in view of Lemma 2.3, we may suppose that $\phi_\varepsilon'(u_n,v_n)(u_n,v_n) \to l < 0$. Hence, the above expression shows that $\lambda_n \to 0$ and therefore we conclude that $J_\varepsilon'(u_n,v_n) \to 0$ in the dual space of X_ε. It follows from [1, Lemma 3.2] that (u_n,v_n) has a convergent subsequence. $\square$

3. Multiplicity of solutions for $(S_{\varepsilon,a})$.

The main result of this section can be stated as follows.

Theorem 3.1. *For any $\delta > 0$ verifying $M_\delta \subset \Lambda$, there exists $\varepsilon_\delta > 0$ such that, for any $\varepsilon \in (0,\varepsilon_\delta)$, the system $(S_{\varepsilon,a})$ has at least $\mathrm{cat}_{M_\delta}(M)$ solutions.*

The proof of the above result is rather long and it will be done by applying the following Ljusternik-Schnirelmann abstract result.

Theorem 3.2. *Let I be a C^1-functional defined on a C^1-Finsler manifold $\mathcal{V}$. If I is bounded from below and satisfies the Palais-Smale condition, then I has at least $\mathrm{cat}_{\mathcal{V}}(\mathcal{V})$ distinct critical points.*

We shall apply the above theorem for the functional J_ε constrained to $\mathcal{N}_\varepsilon$. By Proposition 1 the Palais-Smale condition is satisfied. So, we need only to relate the category of $\mathcal{N}_\varepsilon$ with that of M. This is exactly the content of the next two subsections. The following result, whose the proof is similar to that presented in [9, Lemma 4.3], will be used.

Lemma 3.3. *Let Γ, Ω^+, Ω^- be closed sets with $\Omega^- \subset \Omega^+$. Let $\Phi : \Omega^- \to \Gamma$, $\beta : \Gamma \to \Omega^+$ be two continuous maps such that $\beta \circ \Phi$ is homotopically equivalent to the embedding $\iota : \Omega^- \to \Omega^+$. Then $\mathrm{cat}_\Gamma(\Gamma) \geq \mathrm{cat}_{\Omega^+}(\Omega^-)$.*

3.1. **The map Φ_ε.** In order to construct the map Φ_ε we start by noticing that, by [2, Proposition 2.1], there exists $(w_1, w_2) \in E$ such that w_1, w_2 are positive on $\mathbb{R}^N$ and

$$I'_{x_0}(w_1, w_2) = 0 \text{ and } I_{x_0}(w_1, w_2) = C(x_0) = C^*.$$

We recall that E and I_{x_0} were defined in the introduction and we shall use the following norm on the space E

$$\|(u,v)\|^2_{x_0} := \int (|\nabla u|^2 + |\nabla v|^2 + W(x_0)|u|^2 + V(x_0)|v|^2),$$

for any $(u, v) \in E$.

Let us consider $\delta > 0$ such that $M_\delta \subset \Lambda$ and $\psi \in C^\infty(\mathbb{R}^+, [0, 1])$ a non-increasing function such that $\psi \equiv 1$ on $[0, \delta/2]$ and $\psi \equiv 0$ on $[\delta, \infty)$. For any $y \in M$, we define the function $\Psi_{i,\varepsilon,y} \in X_\varepsilon$ by setting

$$\Psi_{i,\varepsilon,y}(x) := \psi(|\varepsilon x - y|) w_i\left(\frac{\varepsilon x - y}{\varepsilon}\right),\ i = 1, 2,$$

and denote by $t_\varepsilon > 0$ the unique positive number verifying

$$J_\varepsilon(t_\varepsilon(\Psi_{1,\varepsilon,y}, \Psi_{2,\varepsilon,y})) = \max_{t\geq 0} J_\varepsilon(t(\Psi_{1,\varepsilon,y}, \Psi_{2,\varepsilon,y})).$$

In view of the above remarks, it is well defined the function $\Phi_\varepsilon : M \to \mathcal{N}_\varepsilon$ given by

$$\Phi_\varepsilon(y) := t_\varepsilon(\Psi_{1,\varepsilon,y}, \Psi_{2,\varepsilon,y}).$$

In next lemma we prove an important relationship between Φ_ε and the set M.

Lemma 3.4. *Uniformly for $y \in M$, we have*

$$\lim_{\varepsilon\to 0^+} J_\varepsilon(\Phi_\varepsilon(y)) = C^*.$$

Proof. Suppose, by contradiction, that the lemma is false. Then there exist $\delta_0 > 0$, $(y_n) \subset M$ and $\varepsilon_n \to 0^+$ such that

$$|J_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - C^*| \geq \delta_0 > 0. \tag{12}$$

We notice that, if $z \in B_{\delta/\varepsilon_n}(0)$ then $\varepsilon_n z + y_n \in B_\delta(y_n) \subset M_\delta \subset \Lambda$. Thus, recalling that $H \equiv Q$ in Λ and $\psi(s) = 0$ for $s \geq \delta$, we can use the change of variables $z \mapsto (\varepsilon_n x - y_n)/\varepsilon_n$ to write

$$\begin{aligned} J_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) \quad = \quad & \frac{t^2_{\varepsilon_n}}{2} \int_{\mathbb{R}^N} \left(|\nabla(\psi(|\varepsilon z|) w_1(z))|^2 + |\nabla(\psi(|\varepsilon z|) w_2(z))|^2\right) \mathrm{d}z \\ & + \frac{t^2_{\varepsilon_n}}{2} \int_{\mathbb{R}^N} W(\varepsilon_n z + y_n) |\psi(|\varepsilon_n z|) w_1(z)|^2 \mathrm{d}z \\ & + \frac{t^2_{\varepsilon_n}}{2} \int_{\mathbb{R}^N} V(\varepsilon_n z + y_n) |\psi(|\varepsilon_n z|) w_2(z)|^2 \mathrm{d}z \\ & - \int_{\mathbb{R}^N} Q(t_{\varepsilon_n} \psi(|\varepsilon_n z|) w_1(z), t_{\varepsilon_n} \psi(|\varepsilon_n z|) w_2(z))\, \mathrm{d}z. \end{aligned}$$

Since Q is homogeneous, we have that $t_{\varepsilon_n} \to 1$. This and Lebesgue's theorem imply that

$$\lim_{n\to\infty} \|(\Psi_{1,\varepsilon_n,y_n}, \Psi_{1,\varepsilon_n,y_n})\|^2_\varepsilon = \|(w_1, w_2)\|^2_{x_0}$$

and

$$\lim_{n\to\infty}\int Q(\Psi_{1,\varepsilon_n,y_n},\Psi_{2,\varepsilon_n,y_n}) = \int Q(w_1,w_2).$$

Therefore

$$\lim_{n\to\infty} J_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) = I_{x_0}(w_1,w_2) = C^*$$

which contradicts (12). The lemma is proved. □

3.2. **The map β_ε.** Consider $\delta>0$ such that $M_\delta\subset\Lambda$ and choose $\rho=\rho(\delta)>0$ satisfying $M_\delta\subset B_\rho(0)$. Let $\Upsilon:\mathbb{R}^N\to\mathbb{R}^N$ be defined as $\Upsilon(x):=x$ for $|x|\le\rho$ and $\Upsilon(x):=\rho x/|x|$ for $|x|\ge\rho$, and consider the map $\beta_\varepsilon:\mathcal{N}_\varepsilon\to\mathbb{R}^N$ given by

$$\beta_\varepsilon(u,v) := \frac{\displaystyle\int_{\mathbb{R}^N}\Upsilon(\varepsilon x)|u(x)|^2}{\displaystyle\int_{\mathbb{R}^N}|u(x)|^2} + \frac{\displaystyle\int_{\mathbb{R}^N}\Upsilon(\varepsilon x)|v(x)|^2}{\displaystyle\int_{\mathbb{R}^N}|v(x)|^2}.$$

Since $M\subset B_\rho(0)$, we can use the definition of Υ and Lebegue's theorem to conclude that

$$\lim_{\varepsilon\to0^+}\beta_\varepsilon(\Phi_\varepsilon(y)) = y \text{ uniformly for } y\in M. \tag{13}$$

We also have the following technical result.

Lemma 3.5. *Let $\varepsilon_n\to0^+$ and $(u_n,v_n)\subset\mathcal{N}_{\varepsilon_n}$ be such that $J_{\varepsilon_n}(u_n,v_n)\to C^*$. Then there exists $(\widetilde{y}_n)\subset\mathbb{R}^N$ such that the sequence $(\widetilde{u}_n,\widetilde{v}_n):=(u_n(\cdot+\widetilde{y}_n),v_n(\cdot+\widetilde{y}_n))$ has a convergent subsequence in E. Moreover, up to a subsequence, $(y_n):=(\varepsilon_n\widetilde{y}_n)$ is such that $y_n\to y\in M$.*

For the proof of the above lemma we shall use the following property of minimizing sequences of the autonomous system. The proof is similar to that presented in [2, Proposition 2.1] and it will be omitted.

Lemma 3.6. *Let $(u_n,v_n)\subset\mathcal{M}_{x_0}$ be such that $I_0(u_n,v_n)\to C^*$ and $(u_n,v_n)\rightharpoonup(u,v)$ weakly in E. Then there exists $(\widetilde{y}_n)\subset\mathbb{R}^N$ such that the sequence $(\widetilde{u}_n,\widetilde{v}_n):=(u_n(\cdot+\tilde{y}_n),v_n(\cdot+\tilde{y}_n))$ strongly converges to $(\widetilde{u},\widetilde{v})\in\mathcal{M}_{x_0}$ with $I_0(\widetilde{u},\widetilde{v})=C^*$. Moreover, if $(u,v)\neq(0,0)$, then $(\widetilde{y}_n)$ can be taken identically zero and therefore $(u_n,v_n)\to(u,v)$ in E.*

Proof of Lemma 3.5. Straightforward calculations show that $(\widetilde{u}_n,\widetilde{v}_n)$ is bounded. The same argument employed in [2, Lemma 2.1] provides a sequence $(\widetilde{y}_n)\subset\mathbb{R}^N$ and positive constants R,γ such that

$$\liminf_{n\to\infty}\int_{B_R(\tilde{y}_n)}\left(|u_n|^2+|v_n|^2\right)\ge\gamma>0.$$

Thus, by setting $(\widetilde{u}_n(x),\widetilde{v}_n(x)):=(u_n(x+\widetilde{y}_n),v_n(x+\widetilde{y}_n))$ and going to a subsequence if necessary, we can assume that

$$(\widetilde{u}_n,\widetilde{v}_n)\rightharpoonup(\widetilde{u},\widetilde{v})\neq(0,0) \text{ weakly in } E. \tag{14}$$

Let $(t_n)\subset\mathbb{R}^+$ be such that $(\widehat{u}_n,\widehat{v}_n):=t_n(\widetilde{u}_n,\widetilde{v}_n)\in\mathcal{M}_{x_0}$. By using the definition of I_{x_0} and H, and condition (H_3) we obtain

$$\begin{aligned} C^*\le I_{x_0}(\widehat{u}_n,\widehat{v}_n) &= \frac{1}{2}\|(t_nu_n,t_nv_n)\|_{x_0}^2-\int Q(t_nu_n,t_nv_n)\\ &\le \frac{1}{2}\|(t_nu_n,t_nv_n)\|_{\varepsilon}^2-\int H(\varepsilon x,t_nu_n,t_nv_n)\\ &= J_\varepsilon(t_nu_n,t_nv_n)\le J_\varepsilon(u_n,v_n)=C^*+o_n(1), \end{aligned}$$

from which follows that $I_{x_0}(\widehat{u}_n, \widehat{v}_n) \to C^*$. Hence$(\widehat{u}_n, \widehat{v}_n) \not\to (0,0)$ in E. Since $(\widetilde{u}_n, \widetilde{v}_n)$ and $(\widehat{u}_n, \widehat{v}_n)$ are bounded we conclude that $(t_n) \subset \mathbb{R}$ is bounded and, up to a subsequence, $t_n \to t_0 > 0$. Summarizing, we get

$$I_{x_0}(\widehat{u}_n, \widehat{v}_n) \to C^* \quad \text{and} \quad (\widehat{u}_n, \widehat{v}_n) \rightharpoonup t_0(\widetilde{u}, \widetilde{v}) = (\widehat{u}, \widehat{v}) \neq (0,0).$$

It follows from Lemma 3.6 that $(\widehat{u}_n, \widehat{v}_n) \to (\widehat{u}, \widehat{v})$, or equivalently, $(\widetilde{u}_n, \widetilde{v}_n) \to (\widetilde{u}, \widetilde{v})$. This is the first part of the lemma.

We now define $y_n := \varepsilon_n \widetilde{y}_n$ and we shall prove that $y_n \to y \in M$. We begin with the following.

Claim 1. up to a subsequence, $y_n \to y \in \overline{\Lambda}$.

First we prove that (y_n) is bounded. Indeed, suppose by contradiction that, for some subsequence still denoted by (y_n), we have that $|y_n| \to \infty$. If we define

$$I_n := \int \left(|\nabla \widetilde{u}_n|^2 + |\nabla \widetilde{v}_n|^2 + W(\varepsilon_n x + y_n)\widetilde{u}_n^2 + V(\varepsilon_n x + y_n)\widetilde{v}_n^2 \right)$$

and recall that $(u_n, v_n) \in \mathcal{N}_{\varepsilon_n}$, a change of variables shows that

$$I_n = \int \left(\widetilde{u}_n H_u(\varepsilon_n x + y_n, \widetilde{u}_n, \widetilde{v}_n) + \widetilde{v}_n H_v(\varepsilon_n x + y_n, \widetilde{u}_n, \widetilde{v}_n) \right).$$

Consider $R > 0$ such that $\Lambda \subset B_R(0)$. Since we may suppose that $|y_n| > 2R$, for any $x \in B_{R/\varepsilon_n}(0)$ we have

$$|\varepsilon_n x + y_n| \geq |y_n| - |\varepsilon_n x| > R.$$

Thus, by setting $\Gamma_n := B_{R/\varepsilon_n}(0)$, it follows from (4) that

$$\begin{aligned} I_n &\leq \frac{1}{2} \int_{\Gamma_n} \Big(W(\varepsilon_n x + y_n)\widetilde{u}_n^2 + V(\varepsilon_n x + y_n)\widetilde{v}_n^2 \Big) \\ &\quad + \int_{\mathbb{R}^N \setminus \Gamma_n} \Big(\widetilde{u}_n H_u(\varepsilon_n x + y_n, \widetilde{u}_n, \widetilde{v}_n) + \widetilde{v}_n H_v(\varepsilon_n x + y_n, \widetilde{u}_n, \widetilde{v}_n) \Big) \\ &= \frac{1}{2} \int_{\Gamma_n} \Big(W(\varepsilon_n x + y_n)\widetilde{u}_n^2 + V(\varepsilon_n x + y_n)\widetilde{v}_n^2 \Big) + o_n(1), \end{aligned}$$

where we used the strong convergence of $(\widetilde{u}_n, \widetilde{v}_n)$ and the fact that the Lebesgue's measure of the set $\mathbb{R}^N \setminus \Gamma_n$ goes to zero as $n \to \infty$. Recalling the condition (H_3) we get

$$\left(1 - \frac{1}{2}\right) \|(\widetilde{u}_n, \widetilde{v}_n)\|_{x_0}^2 = o_n(1),$$

which contradicts (14).

Since $(y_n) \subset \mathbb{R}^N$ is bounded we may suppose that $y_n \to y$. In order to verify that $y \in \overline{\Lambda}$ we suppose, by contradiction, that $y \in \mathbb{R}^N \setminus \overline{\Lambda}$. Thus there exists $r > 0$ such that,

$$y_n \in B_{r/2}(y) \subset \mathbb{R}^N \setminus \overline{\Lambda}$$

for all n large enough. The same argument employed above provides a contradiction. Hence, $y \in \overline{\Lambda}$.

Claim 2. $y \in M$.

It suffices to show that $C(y) = C^*$. Indeed, if this is the case, the property (C_0) stated in the introduction yields $y \in M$. Arguing by contradiction again, we suppose that $C^* < C(y)$. So, recalling that $(\widehat{u}_n, \widehat{v}_n) \to (\widehat{u}, \widehat{v})$ and using Fatous's lemma we get

$$\begin{aligned} C^* &< C(y) = I_y(\widehat{u}, \widehat{v}) \\ &= \liminf_{n\to\infty} \Big[\frac{1}{2} \int \Big(|\nabla \widehat{u}_n|^2 + |\nabla \widehat{v}_n|^2 + W(\varepsilon_n x + y_n)\widehat{u}_n^2 + V(\varepsilon_n x + y_n)\widehat{v}_n^2 \Big) \\ &\qquad - \int_{\mathbb{R}^N} Q(\widehat{u}_n, \widehat{v}_n) \Big] \\ &\le \liminf_{n\to\infty} J_{\varepsilon_n}(t_n u_n, t_n v_n) \le \liminf_{n\to\infty} J_{\varepsilon_n}(u_n, v_n) = C^*, \end{aligned}$$

which does not make sense. Thus, $C(y) = C^*$ and the proof is concluded. □

3.3. **Proof of Theorem 3.1.** Following [11], we introduce the set

$$\Sigma_\varepsilon := \{(u,v) \in \mathcal{N}_\varepsilon : J_\varepsilon(u,v) \le C^* + h(\varepsilon)\},$$

where $h : \mathbb{R}^+ \to \mathbb{R}^+$ is such that $h(\varepsilon) \to 0$ as $\varepsilon \to 0^+$. Given $y \in M$, we can use Lemma 3.4 to conclude that $h(\varepsilon) = |J_\varepsilon(\Phi_\varepsilon(y)) - C^*|$ satisfies $h(\varepsilon) \to 0$ as $\varepsilon \to 0^+$. Thus, $\Phi_\varepsilon(y) \in \Sigma_\varepsilon$ and therefore $\Sigma_\varepsilon \neq \emptyset$ for any $\varepsilon > 0$ small.

The proof of the next lemma can be done as in [14, Lemma 3.3] using Lemma 3.5 instead of [14, Lemma 3.1]. We omit the details.

Lemma 3.7. *For any $\delta > 0$ we have*

$$\lim_{\varepsilon\to 0^+} \sup_{u\in\Sigma_\varepsilon} \operatorname{dist}(\beta_\varepsilon(u,v), M_\delta) = 0. \tag{15}$$

We are now ready to obtain multiple solutions for the modified problem.

Proof of Theorem 3.1. Given $\delta > 0$ such that $M_\delta \subset \Lambda$, we can use (13), (15), Lemma 3.4 and argue as in [11, Section 6] to obtain $\varepsilon_\delta > 0$ such that, for any $\varepsilon \in (0, \varepsilon_\delta)$, the diagram

$$M \xrightarrow{\Phi_\varepsilon} \Sigma_\varepsilon \xrightarrow{\beta_\varepsilon} M_\delta$$

is well defined and $\beta_\varepsilon \circ \Phi_\varepsilon$ is homotopically equivalent to the embedding $\iota : M \to M_\delta$. It follows from Proposition 1, Lemma 3.3 and Theorem 3.2 that J_ε restricted to $\mathcal{N}_\varepsilon$ possesses at least $\operatorname{cat}_{M_\delta}(M)$ critical points (u_i, v_i). The same argument employed in the proof of Proposition 1 shows that (u_i, v_i) is also a critical point of the unconstrained functional and therefore a solution for the problem $(S_{\varepsilon,a})$. The theorem is proved. □

4. **Proof of Theorem 1.1.** Once we have obtained multiple solutions for the modified problem $(S_{\varepsilon,a})$ the proof follows by using the same arguments employed in [1, Theorem 1.1]. For the sake of completeness, we sketch them here.

For $\varepsilon > 0$, we define

$$m_\varepsilon := \sup \Big\{ \max_{\partial\Lambda_\varepsilon} |(u_\varepsilon, v_\varepsilon)| : (u_\varepsilon, v_\varepsilon) \in \mathcal{N}_\varepsilon \text{ is a solution for } (S_{\varepsilon,a}) \Big\}.$$

We claim that, for $\varepsilon > 0$ small, the number m_ε is finite. Indeed, suppose by contradiction that for some sequence $\varepsilon_n \to 0^+$ we have that $m_\varepsilon = \infty$. So, there exist $b > 0$ and a sequence $(x_n) \subset \partial\Lambda_{\varepsilon_n}$ such that

$$\min\{u_{\varepsilon_n}(x_n), v_{\varepsilon_n}(x_n)\} \ge b > 0.$$

It follows from [1, Proposition 3.1] that

$$\lim_{n\to\infty} C(x_n) = C^*,$$

which contradicts the statement (C_0) in the introduction. Hence, there exists $\overline{\varepsilon} > 0$ such that $m_\varepsilon < \infty$, for all $\varepsilon \in (0, \overline{\varepsilon})$.

We claim that

$$\lim_{\varepsilon \to 0^+} m_\varepsilon = 0. \tag{16}$$

Indeed, if this is not the case, we can obtain $b > 0$ and a sequence $\varepsilon_n \to 0^+$ satisfying

$$m_{\varepsilon_n} \geq b > 0.$$

Hence, there exists a solution $(u_{\varepsilon_n}, v_{\varepsilon_n}) \in \mathcal{N}_{\varepsilon_n}$ of the problem $(S_{\varepsilon_n,a})$ such that

$$\frac{b}{2} = b - \frac{b}{2} \leq m_{\varepsilon_n} - \frac{b}{2} < \max_{\partial\Lambda_\varepsilon} |(u_{\varepsilon_n}, v_{\varepsilon_n})|$$

and therefore we can obtain a sequence $(x_n) \subset \partial\Lambda_{\varepsilon_n}$, such that

$$\min\{u_{\varepsilon_n}(x_n), v_{\varepsilon_n}(x_n)\} \geq \frac{b}{2} > 0.$$

The same argument employed before provides a contradiction.

Let $\delta > 0$ such that $M_\delta \subset \Lambda$. In view of Theorem 3.1 and (16) we can choose $\varepsilon_\delta > 0$ such that, for any $\varepsilon \in (0, \varepsilon_\delta)$, the problem $(S_{\varepsilon,a})$ has at least $\mathrm{cat}_{M_\delta}(M)$ solutions and

$$m_\varepsilon < \frac{a}{2}.$$

If we denote by $(u_\varepsilon, v_\varepsilon)$ one of these solutions, the above inequality and the calculations performed in [1, Section 4] show that

$$|(u_\varepsilon(\varepsilon x), v_\varepsilon(\varepsilon x))| \leq a \text{ for each } x \in \mathbb{R}^N \setminus \Lambda_\varepsilon.$$

Hence, it follows from Lemma 2.1 that $(u_\varepsilon, v_\varepsilon)$ is a solution for the original system (S_ε).

By denoting $u_\varepsilon^- = \max\{-u_\varepsilon, 0\}$ and $v_\varepsilon^- = \max\{-v_\varepsilon, 0\}$ the negative part of u_ε and v_ε, respectively, we can use $(u_\varepsilon^-, v_\varepsilon^-)$ as a test function in the weak formulation of the system (S_ε) to conclude that u_ε and v_ε are nonnegative functions. Combining (Q_5) with the maximum principle, it follows that both u_ε and v_ε are positive on $\mathbb{R}^N$ and the proof of theorem is complete. ∎

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