Multiple solutions for a nonlinear Schrödinger equation with magnetic fields*

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Abstract

We study a nonlinear Schrödinger equation in presence of a magnetic field and relate the number of solutions with the topology of the set where the potential attains its minimum value. In the proofs we apply variational methods, penalization techniques and Ljusternik-Schnirelmann theory.

1 Introduction

In this paper we are concerned with the nonlinear Schrödinger equation

$$ih\frac{\partial\psi}{\partial t} = \left(\frac{h}{i}\nabla - A(z)\right)^2 \psi + U(z)\psi - f(|\psi|^2)\psi, \quad z \in \mathbb{R}^N,$$

where $t \in \mathbb{R}$, $N \geq 2$, the function ψ takes values in $\mathbb{C}$, h is the Planck constant and i is the imaginary unit. The function $A : \mathbb{R}^N \to \mathbb{R}^N$ denotes a magnetic

*C. O. Alves was partially supported by CNPq/Brazil 620150/2008-4 and INCT Matemática. G.M. Figueiredo was partially supported by CNPq/Brazil 300705/2008-5. M.F. Furtado was partially supported by CNPq/Brazil 305984/2007-1.

potential and the Schrödinger operator is defined by

$$\left(\frac{h}{i}\nabla - A(z)\right)^2 \psi := -h^2\Delta\psi - \frac{2h}{i}A\cdot\nabla\psi + |A|^2\psi - \frac{h}{i}\psi \operatorname{div} A.$$

In the 3-dimensional case the magnetic field B is exactly the curl of A, while for higher dimensions $N \geq 4$, it is the 2-form given by $B_{i,j} = \partial_j A_k - \partial_k A_j$. The function $U(x)$ is a real electric potential and the nonlinear term $f(t)$ is a superlinear function.

We are particularly interested in the existence of solitary waves of the form $\psi(x,t) := e^{-i\frac{E}{h}t}u(x)$, with $E \in \mathbb{R}$. It is important to investigate the existence and the shape of such solutions in the semiclassical limit, namely, as $h \to 0$. The importance of this study relies on the fact that the transition to Quantum Mechanics to Classical Mechanics can be formally performed by sending the Planck constant to zero. If we put the solitary wave expression of ψ in the above equation, we are lead to look for solutions of the following problem

$$\begin{cases} \left(\frac{\varepsilon}{i}\nabla - A(z)\right)^2 u + V(z)u = f(|u|^2)u, & z \in \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N, \mathbb{C}) \end{cases} \tag{P_ε}$$

where $V(z) = U(z) - E$ and we replaced h by ε.

There is a vast literature concerning the existence and multiplicity of bound state solutions for (P_ε) with no magnetic vector potential, namely $A \equiv 0$. Since the seminal paper of Floer and Weinstein [15], many authors have applied different techniques to obtain existence of solutions in this case (see [21, 23, 13, 3, 24] and references there in). In the most of these papers the authors also have studied the asymptotic behavior of the solutions as $\varepsilon \to 0$. Roughly speaking, these solutions concentrate around critical points of the potential V. There are also some papers relating the topology of the set of critical points of V with the number of solutions of the problem (see [9, 20, 2] for example).

If we now consider the magnetic case, it appears that the first result was obtained by Esteban and Lions [14]. In this paper the authors have used the concentration-compactness principle and minimization arguments to obtain solution for $\varepsilon > 0$ fixed and $N = 2$ and 3. More recently, Kurata [17] have proved that the problem has a least energy solution for any $\varepsilon > 0$ when a technical condition relating $V(x)$ and $A(x)$ is assumed. Under this technical condition, he have proved that the associated functional satisfies the Palais-Smale compactness condition at any level. We also would like to cite the papers [10, 11, 5, 22, 8] for other results related with the problem (P_ε).

The main motivation of our paper becomes from the aforementioned works, the paper of Cingolani [6], Cingolani and Secchi [12] and the recent work of Cingolani and Clapp [7]. In [6, 7] the authors have used Ljusternik-Schnirelmann theory to obtain multiple solutions for (P_ε). However, the conditions used in [6, 7] are in some sense related with a global condition introduced by Rabinowitz in [21], namely

$$0 < \inf_{z\in\mathbb{R}^N} V(z) < \liminf_{|z|\to\infty} V(z).$$

In [12], the authors have worked with local conditions on V. More precisely, they have supposed that there are bounded sets $\Omega_1, \ldots, \Omega_k \subset \mathbb{R}^N$ which are mutually disjoint and such that

$$\inf_{\Omega_j} V(z) < \inf_{\partial\Omega_j} V(z), \;\; j = 1, \ldots, k.$$

By adapting some arguments found in [13], they have obtained the existence of solution with k peaks for $\varepsilon > 0$ small enough.

The aim of this paper is to complement the studies made in [6, 7, 12]. We shall obtain multiple solution for the problem (P_ε) by combining a local assumption on V as above, the penalization scheme of del Pino and Felmer [13] and the Ljusternik-Schnirelmann theory. It is worthwhile to mention that, in the arguments developed in [13], one of the key points is the existence of estimates involving the L^∞-norm of the solutions of a modified problem. As pointed out in [12], this kind of estimates are more delicated in the magnetic case because A can be unbounded and there is no relation between the natural space to deal with (P_ε) and the limit space $H^1(\mathbb{R}^N, \mathbb{C})$ as $\varepsilon \to 0$. These problems are overcame in [12] by the use of diamagnetic (see Section 2) and Kato's inequalities for magnetic fields. Here, we obtain the desired L^∞ estimates by a different approach, which is based on Moser's iteration method (see [19]) instead of Kato's inequality. As far as we know, this is the first time that local Mountain Pass and topological arguments are combined to get multiple solutions for (P_ε). We believe that the ideas contained here can be applied in other situations to deal with local conditions on the potential V.

Before stating our main result, we need to present the hypotheses on the potential V and the nonlinearity f. We shall assume that

(V_1) $V_0 := \inf_{z \in \mathbb{R}^N} V(z) > 0$,

(V_2) there exists an open bounded set $\Omega \subset \mathbb{R}^N$ such that

$$V_0 < \min_{z \in \partial\Omega} V(z)$$

and $M := \{z \in \Omega : V(z) = V_0\} \neq \varnothing$.

We also suppose that $f \in C^1(\mathbb{R}, \mathbb{R})$ satisfies

(f_1) $f(s) = 0$ for each $s \leq 0$;

(f_2) $\lim_{s \to 0^+} f(s) = 0$;

(f_3) there exists $q \in (2, 2^*)$ such that

$$\lim_{s \to \infty} \frac{f(s)}{s^{(q-2)/2}} = 0,$$

where $2^* := 2N/(N-2)$ if $N \geq 3$, and $2^* := \infty$ if $N = 2$;

(f_4) there exists $\theta > 2$ such that

$$0 < \frac{\theta}{2} F(s) \leq s f(s), \quad \text{for each } s > 0,$$

where $F(s) := \int_0^s f(\tau) \mathrm{d}\tau$;

(f_5) there exist $\sigma \in (2, 2^*)$ and $C_\sigma > 0$ such that

$$f'(s) \geq C_\sigma s^{(\sigma-4)/2}, \quad \text{for each } s > 0.$$

We shall establish a relation between the number of solutions of (P_ε) and the topology of the set M. In order to make a precise statement let us recall that, for any closed subset Y of a topological space X, the Ljusternik-Schnirelmann category of Y in X, $\mathrm{cat}_X(Y)$, stands for the least number of closed and contractible sets in X which cover Y.

We present below the main result of this paper.

Theorem 1.1. *Suppose that V satisfies $(V_1) - (V_2)$ and f satisfies $(f_1) - (f_5)$. Then, for any $\delta > 0$ such that*

$$M_\delta := \{z \in \mathbb{R}^N : \mathrm{dist}(z, M) < \delta\} \subset \Omega,$$

there exists $\varepsilon_\delta > 0$ such that, for any $\varepsilon \in (0, \varepsilon_\delta)$, the problem ($P_\varepsilon$) has at least $\mathrm{cat}_{M_\delta}(M)$ nonzero solutions. Moreover, if $\varepsilon_n \to 0^+$, u_{ε_n} is one of these solutions and $\eta_{\varepsilon_n} \in \mathbb{R}^N$ is a global maximum point of $|u_{\varepsilon_n}|$, we have that

$$\lim_{\varepsilon_n \to 0^+} V(\eta_{\varepsilon_n}) = V_0.$$

The proof will be done by variational techniques. Since we have no information on the behavior of the potential V at the infinity we adapt the argument introduced by del Pino and Felmer in [13]. It consists in making a suitable modification on f, solving a modified problem and then check that, for ε small enough, the solutions of the new problem are indeed solutions of the original one. In order to obtain multiple solutions for the modified problem, we use a technique introduced by Benci and Cerami in [4]. The main idea is to make precise comparisons between the category of some sublevel sets of the modified functional and the category of the set M.

The paper is organized as follows. In the next section we present the variational setting of the problem and we apply the penalization technique to obtain compactness for the modified problem. In Section 3 we prove a version of Theorem 1.1 for the modified problem. In the final Section 4 we prove our main theorem.

2 Variational Framework

In this section we fix some notations and present the variational setting of our problem. Throughout the paper we write only $\int u$ instead of $\int_{\mathbb{R}^N} u(z)\, \mathrm{d}z$. For any $B \subset \mathbb{R}^N$ we denote by $B^c := \mathbb{R}^N \setminus B$ the complement of B.

By the change of variables $z \mapsto \varepsilon x$ we can see that (P_ε) is equivalent to

$$\begin{cases} \left(\dfrac{1}{i}\nabla - A_\varepsilon(x)\right)^2 u + V_\varepsilon(x)u = f(|u|^2)u, & x \in \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N, \mathbb{C}), \end{cases} \tag{D_ε}$$

with $A_\varepsilon(x) := A(\varepsilon x)$ and $V_\varepsilon(x) := V(\varepsilon x)$. We denote by $H_\varepsilon = H_\varepsilon(\mathbb{R}^N, \mathbb{C})$ the Hilbert space obtained by the closure of $C_0^\infty(\mathbb{R}^N, \mathbb{C})$ under the scalar product

$$\langle u, v\rangle_\varepsilon := \mathrm{Re}\left(\int \nabla_\varepsilon u \overline{\nabla_\varepsilon v} + V_\varepsilon(x) u \overline{v}\right),$$

where $\mathrm{Re}(w)$ denotes the real part of $w \in \mathbb{C}$, $\overline{w}$ is its conjugated, $\nabla_\varepsilon u := (D_1^\varepsilon u, D_2^\varepsilon u, ..., D_N^\varepsilon u)$ and $D_j^\varepsilon := i^{-1}\partial_j - A_j(\varepsilon x)$, for $j = 1, \ldots, N$. The norm induced by this inner product is given by

$$\|u\|_\varepsilon = \left(\int |\nabla_\varepsilon u|^2 + V_\varepsilon(x)|u|^2\right)^{1/2}.$$

As proved by Esteban and Lions in [14, Section II], for any $u \in H_\varepsilon$ there holds

$$|\nabla |u|(x)| = \left|\mathrm{Re}\left(\nabla u \frac{\overline{u}}{|u|}\right)\right| = \left|\mathrm{Re}\left((\nabla u - iA_\varepsilon u)\frac{\overline{u}}{|u|}\right)\right| \le |\nabla_\varepsilon u(x)|. \tag{2.1}$$

The above expression is the so called diamagnetic inequality. It follows from it that, if $u \in H_\varepsilon$, then $|u| \in H^1(\mathbb{R}^N, \mathbb{R})$. Moreover, the embedding $H_\varepsilon \hookrightarrow L^q(\mathbb{R}^N, \mathbb{R})$ is continuous for each $2 \le q \le 2^*$ and, for each bounded set $\Lambda \subset \mathbb{R}^N$ and $2 \le q < 2^*$, the embedding below is compact

$$H_\varepsilon \hookrightarrow L^q(\Lambda, \mathbb{R}). \tag{2.2}$$

We say that a function $u \in H_\varepsilon$ is a weak solution of the problem (P_ε) if

$$\mathrm{Re}\left(\int \nabla_\varepsilon u \overline{\nabla_\varepsilon v} + V_\varepsilon(x) u \overline{v} - f(|u|^2) u \overline{v}\right) = 0, \quad \text{for each } v \in H_\varepsilon.$$

In view of (f_2) and (f_3) we have that the associated functional $I_\varepsilon : H_\varepsilon \to \mathbb{R}$ given by

$$I_\varepsilon(u) := \frac{1}{2}\int |\nabla_\varepsilon u|^2 + \frac{1}{2}\int V_\varepsilon(x)|u|^2 - \frac{1}{2}\int F(|u|^2)$$

is well defined. Moreover, $I_\varepsilon \in C^1(H_\varepsilon, \mathbb{R})$ with the following derivative

$$I_\varepsilon'(u)v = \mathrm{Re}\left(\int \nabla_\varepsilon u \overline{\nabla_\varepsilon v} + V_\varepsilon(x) u \overline{v} - f(|u|^2) u \overline{v}\right).$$

Hence, the weak solutions of (D_ε) are precisely the critical points of I_ε.

Since we are intending to apply critical point theory for the functional I_ε, we need to obtain some compactness property. However, since $\mathbb{R}^N$ is unbounded, we know that the usual Sobolev embeddings are not compact, and so I_ε can not verify the Palais-Smale condition. In order to overcome this problem we make a slightly adaptation of the penalization method introduced by Del Pino and Felmer in [13](see also [2]). We choose $k > \theta/(\theta-2)$, where θ is given by (f_4), and take $a > 0$ to be the unique number such that $f(a)/a = V_0/k$, with V_0 given by (V_1). We set

$$\widehat{f}(s) := \begin{cases} f(s), & \text{if } s \leq a, \\ \dfrac{V_0}{k}, & \text{if } s > a. \end{cases}$$

Let $0 < t_a < a < T_a$ and consider $\vartheta \in C_0^\infty(\mathbb{R}, \mathbb{R})$ such that

(ϑ_1) $\vartheta(s) \leq \widehat{f}(s)$ for all $s \in [t_a, T_a]$,

(ϑ_2) $\vartheta(t_a) = \widehat{f}(t_a)$, $\vartheta(T_a) = \widehat{f}(T_a)$, $\vartheta'(t_a) = \widehat{f}'(t_a)$ and $\vartheta'(T_a) = \widehat{f}'(T_a)$,

(ϑ_3) the map $s \mapsto \vartheta(s)$ is increasing for all $s \in [t_a, T_a]$.

By using the above functions we can define $\widetilde{f} \in C^1(\mathbb{R}, \mathbb{R})$ as follows

$$\widetilde{f}(s) := \begin{cases} \widehat{f}(s), & \text{if } s \notin [t_a, T_a], \\ \vartheta(s), & \text{if } s \in [t_a, T_a]. \end{cases}$$

If χ_Ω denotes the characteristic function of the set Ω, we introduce the penalized nonlinearity $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ by setting

$$g(x, s) := \chi_\Omega(x) f(s) + (1 - \chi_\Omega(x)) \widetilde{f}(s). \tag{2.3}$$

Now, we shall consider the modified problem

$$\begin{cases} \left(\dfrac{1}{i}\nabla - A_\varepsilon(x)\right)^2 u + V_\varepsilon(x) u = g_\varepsilon(x, |u|^2) u, \qquad x \in \mathbb{R}^N, \\ u \in H_\varepsilon, \end{cases} \tag{$(\widetilde{D}_\varepsilon)_a$}$$

where $g_\varepsilon(x, u) := g(\varepsilon x, u)$. Notice that, if

$$\Omega_\varepsilon := \{x \in \mathbb{R}^N : \varepsilon x \in \Omega\}$$

and u is a solution of the above problem such that $|u_\varepsilon(x)| \leq t_a$ in Ω_ε^c then, in view of the definition of g, there holds $g(\varepsilon x, |u|^2)u = f(|u|^2)u$ for each $x \in \mathbb{R}^N$. Thus, the function u is also a solution of the original problem (D_ε).

In view of the above comment, we deal in the sequel with the modified problem $(\widetilde{D}_\varepsilon)_a$. We start the by noticing that, in view of $(f_1) - (f_5)$ and $(\vartheta_1) - (\vartheta_3)$, we can check that $g(x, s)$ is a Carathéodory function satisfying the following properties uniformly in $x \in \mathbb{R}^N$:

(g_1) $g(x,s)=0$ for each $s\leq 0$;

(g_2) $\lim_{s\to 0^+} g(x,s)=0$;

(g_3) $\lim_{s\to\infty} \dfrac{g(x,s)}{s^{(q-2)/2}}=0$;

(g_4) (i) $0\leq \dfrac{\theta}{2}G(x,s)<g(x,s)s$, for each $x\in\Omega$, $s>0$,

(ii) $0\leq G(x,s)\leq \dfrac{V(x)}{k}s$ and $0\leq g(x,s)\leq \dfrac{V(x)}{k}$, for each $x\in\Omega^c$, $s>0$;

(g_5) the function $s\mapsto g(x,s)$ is increasing in $(0,\infty)$.

Let $G(x,s):=\int_0^\tau g(x,\tau)\,\mathrm{d}\tau$ and $G_\varepsilon(x,s):=G(\varepsilon x,s)$. By (g_2) and (g_3), the functional associated to $(\widetilde{D}_\varepsilon)_a$, namely

$$J_\varepsilon(u):=\frac{1}{2}\int|\nabla_\varepsilon u|^2+\frac{1}{2}\int V_\varepsilon(x)|u|^2-\frac{1}{2}\int G_\varepsilon(x,|u|^2),\qquad u\in H_\varepsilon$$

belongs to $C^1(H_\varepsilon,\mathbb{R})$. Moreover, its critical points are the weak solutions of the modified problem $(\widetilde{D}_\varepsilon)_a$.

The main feature of the modified functional is that it satisfies the Palais-Smale condition, as we can see from the next result.

Lemma 2.1. *The functional J_ε satisfies the (PS)$_d$ condition for any level $d\in\mathbb{R}$.*

Proof. We shall adapt the arguments presented in [13, Lemma 1.1]. Suppose that $(u_n)\subset H_\varepsilon$ is a (PS)$_d$ sequence for J_ε, that is, $J_\varepsilon(u_n)\to d$ and $J'_\varepsilon(u_n)\to 0$.

We first prove that (u_n) is bounded in H_ε. Indeed, by using (g_4) we obtain

$$\begin{aligned}
d+o_n(1)\|u_n\|_\varepsilon &\geq J_\varepsilon(u_n)-\frac{1}{\theta}J'_\varepsilon(u_n)u_n\\
&\geq \left(\frac{1}{2}-\frac{1}{\theta}\right)\|u_n\|_\varepsilon^2+\frac{1}{\theta}\int_{\Omega_\varepsilon^c} g_\varepsilon(x,|u_n|^2)|u_n|^2\\
&\quad -\frac{1}{2}\int_{\Omega_\varepsilon^c} G_\varepsilon(x,|u_n|^2)\\
&\geq \frac{1}{2}\left(\frac{\theta-2}{\theta}-\frac{1}{k}\right)\|u_n\|_\varepsilon^2,
\end{aligned}$$

where $o_n(1)$ denotes a quantity approaching zero as $n\to\infty$. Since $k>\theta/(\theta-2)$ we conclude from the above inequality that (u_n) is bounded in H_ε.

Claim. for any given $\zeta>0$ there exists $R=R(\zeta)>0$ such that $\Omega_\varepsilon\subset B_R(0)$ and

$$\limsup_{n\to\infty}\int_{B_R(0)^c}(|\nabla_\varepsilon u_n|^2+V_\varepsilon(x)|u_n|^2)\leq\zeta.$$

Assuming the claim we can conclude the proof as follows. By going to a subsequence if necessary, we may suppose that $u_n\rightharpoonup u$ weakly in H_ε. The local compactness given in (2.2) and the subcritical growth of g imply that $J'_\varepsilon(u) = 0$, and therefore

$$\|u\|_\varepsilon^2 = \int g_\varepsilon(x, |u|^2)|u|^2.$$

Since $J'_\varepsilon(u_n)u_n \to 0$, we also have that

$$\|u_n\|_\varepsilon^2 = \int g_\varepsilon(x, |u_n|^2)|u_n|^2 + o_n(1).$$

By using the claim, the subcritical growth of g and the local compactness mentioned in (2.2), we can check that

$$\lim_{n\to\infty} \int g_\varepsilon(x, |u_n|^2)|u_n|^2 = \int g_\varepsilon(x, |u|^2)|u|^2.$$

All together, the above informations imply that $\|u_n\|_\varepsilon \to \|u\|_\varepsilon$, and so $u_n \to u$ in H_ε.

In order to check the claim we consider $\eta_R \in C^\infty(\mathbb{R}^N, \mathbb{R})$ such that $0 \le \eta_R \le 1$, $\eta_R \equiv 0$ in $B_{R/2}(0)$, $\eta_R \equiv 1$ in $B_R(0)^c$ and $|\nabla \eta_R| \le C/R$, where $C > 0$ is a constant independent of R. Since the sequence $(\eta_R u_n)$ is bounded in H_ε, we have that $J'_\varepsilon(u_n)(\eta_R u_n) = o_n(1)$, that is,

$$\mathrm{Re}\left(\int \nabla_\varepsilon u_n \overline{\nabla_\varepsilon(u_n \eta_R)}\right) + \int V_\varepsilon(x)|u_n|^2\eta_R = \int g_\varepsilon(x, |u_n|^2)|u_n|^2\eta_R + o_n(1).$$

Since η_R take values in $\mathbb{R}$, a direct calculation shows that

$$\overline{\nabla_\varepsilon(u_n\eta_R)} = i\overline{u_n}\nabla\eta_R + \eta_R\overline{\nabla_\varepsilon u_n}.$$

The two above equalities and (g_4)(ii) imply that

$$\begin{aligned}\int \left(|\nabla_\varepsilon u_n|^2 + V_\varepsilon(x)|u_n|^2\right)\eta_R \quad &\le \quad \frac{1}{k}\int V_\varepsilon(x)|u_n|^2\eta_R \\ &\quad + \mathrm{Re}\left(\int -i\overline{u_n}\nabla_\varepsilon u_n \nabla\eta_R\right) + o_n(1).\end{aligned}$$

By using the definition of η_R, Hölder's inequality and the boundedness of (u_n) we obtain

$$\begin{aligned}\left(1 - \frac{1}{k}\right)\int_{B_R(0)^c} \left(|\nabla_\varepsilon u_n|^2 + V_\varepsilon(x)|u_n|^2\right) \quad &\le \quad \frac{C}{R}\|\overline{u_n}\|_{L^2}\|\nabla_\varepsilon u_n\|_{L^2} + o_n(1) \\ &\le \quad \frac{C_1}{R} + o_n(1).\end{aligned}$$

So, for any fixed $\zeta > 0$, we can choose $R > 0$ large enough in such way that $\Omega_\varepsilon \subset B_R(0)$ and

$$\limsup_{n\to\infty} \int_{B_R(0)^c} (|\nabla_\varepsilon u|^2 + V_\varepsilon(x)|u_n|^2) \le \zeta.$$

This finishes the proof. $\square$

Since we are looking for multiple critical points of the functional J_ε, we shall consider it constrained to an appropriated subset of H_ε. More specifically, let us denote by $\mathcal{N}_\varepsilon$ the Nehari manifold of J_ε, namely

$$\mathcal{N}_\varepsilon := \{u \in H_\varepsilon \setminus \{0\} : J_\varepsilon'(u)u = 0\}.$$

By using the growth conditions of g we can show that there exists $r > 0$, which is independent of $\varepsilon > 0$, such that

$$\|u\|_\varepsilon \geq r > 0, \quad \text{for each } u \in \mathcal{N}_\varepsilon. \tag{2.4}$$

We state and prove below the main result of this section.

Proposition 2.2. *The functional J_ε restricted to $\mathcal{N}_\varepsilon$ satisfies the (PS)$_d$ condition for any level $d \in \mathbb{R}$.*

Proof. Let $(u_n) \subset \mathcal{N}_\varepsilon$ be a (PS)$_d$ sequence of J_ε restricted to $\mathcal{N}_\varepsilon$. Then there exists $(\lambda_n) \subset \mathbb{R}$ such that

$$J_\varepsilon'(u_n) = \lambda_n \phi_\varepsilon'(u_n) + o_n(1), \tag{2.5}$$

where $\phi_\varepsilon : H_\varepsilon \to \mathbb{R}$ is given by

$$\phi_\varepsilon(u) := J_\varepsilon'(u)u = \int \left(|\nabla_\varepsilon u|^2 + V_\varepsilon(x)|u|^2 - g_\varepsilon(x, |u|^2)|u|^2\right).$$

For any fixed $u \in \mathcal{N}_\varepsilon$, since $g(\varepsilon x, |u|^2)$ is constant on $\Omega_\varepsilon^c \cap \{|u|^2 > T_a\}$, we can use the definition of g and the monotonicity of ϑ to get

$$\begin{aligned} \phi_\varepsilon'(u)u &= 2\|u\|_\varepsilon^2 - 2\int g_\varepsilon(x, |u|^2)|u|^2 - 2\int g_\varepsilon'(x, |u|^2)|u|^4 \\ &= -2\int g_\varepsilon'(x, |u|^2)|u|^4 \leq \int_{\Omega_\varepsilon \cup \{|u|^2 < t_a\}} f'(|u|^2)|u|^4. \end{aligned}$$

It follows from (f_5) that

$$\phi_\varepsilon'(u_n)u_n \leq -2C_\sigma \int_{\Omega_\varepsilon \cup \{|u_n|^2 < t_a\}} |u_n|^\sigma \leq -2C_\sigma \int_{\Omega_\varepsilon} |u_n|^\sigma, \tag{2.6}$$

with $\sigma \in (2, 2^*)$.

By the boundedness of (u_n) we may assume that $\phi_\varepsilon'(u_n)u_n \to l \leq 0$. If $l \neq 0$ we infer from (2.5) that $\lambda_n = o_n(1)$. In this case, we can use (2.5) again to conclude that (u_n) is a (PS)$_d$ sequence for the unconstrained functional. So, we can apply Proposition 2.1 to obtain a convergent subsequence.

It remains to prove that $l \neq 0$. Suppose, by contradiction, that $l = 0$. It follows from (2.6) that $u_n \to 0$ in $L^\sigma(\Omega_\varepsilon)$. Hence, we can use $J_\varepsilon'(u_n)u_n = 0$, the subcritical growth of g and (g_4)(ii) to conclude that

$$\|u_n\|_\varepsilon^2 = \int_{\Omega_\varepsilon^c} g_\varepsilon(x, |u_n|^2)|u_n|^2 + o_n(1) \leq \frac{1}{k}\int_{\Omega_\varepsilon^c} V_\varepsilon(x)|u_n|^2 + o_n(1).$$

The above expression implies that $\|u_n\|_\varepsilon^2 \to 0$, which leads to a contradiction with (2.4). This contradiction concludes the proof. $\square$

3 Multiple solutions for the modified problem

Ini this section we shall prove a multiplicity result for the problem $(\widetilde{D}_\varepsilon)_a$. Along all the section we will assume that $\delta > 0$ is small in such way that $M_\delta \subset \Omega$.

We start by considering the limiting problem associated to (D_ε), namely the scalar problem

$$\begin{cases} -\Delta w + V_0 w = f(w^2)w, \text{ in } \mathbb{R}^N, \\ w \in H^1(\mathbb{R}^N, \mathbb{R}) \end{cases} \tag{A_0}$$

which has the following associated functional

$$E_0(w) := \frac{1}{2}\int |\nabla w|^2 + \frac{1}{2}\int V_0|w|^2 - \frac{1}{2}\int F(w^2), \quad w \in H^1(\mathbb{R}^N, \mathbb{R}).$$

We also consider

$$\mathcal{M}_0 := \left\{ w \in H^1(\mathbb{R}^N, \mathbb{R}) \setminus \{0\} : E_0'(w)w = 0 \right\}$$

and

$$c_0 := \inf_{w \in \mathcal{M}_0} E_0(w),$$

or, equivalently,

$$c_0 = \inf_{w H^1(\mathbb{R}^N,\mathbb{R})\setminus\{0\}} \max_{t \geq 0} E_0(tw) > 0. \tag{3.7}$$

By using the hypothesis on f we can prove that problem (A_0) has a positive ground state solution. The next lemma can be found in [1, Theorem 3.1].

Lemma 3.1. *Let $(w_n) \subset \mathcal{M}_0$ be such that $E_0(u_n) \to c_0$ and $w_n \rightharpoonup w$ weakly in $H^1(\mathbb{R}^N, \mathbb{R})$. Then there exists a sequence $(\widetilde{y}_n) \subset \mathbb{R}^N$ such that $w_n(\cdot + \widetilde{y}_n) \to w \in \mathcal{M}_0$ with $E_0(w) = c_0$. Moreover, if $w \neq 0$, then $(\widetilde{y}_n)$ can be taken identically zero and therefore, in this case, $w_n \to w$ strongly in $H^1(\mathbb{R}^N, \mathbb{R})$.*

By using the above lemma, a result due to Lions [18, Lemma I.1] and the invariance of $\mathbb{R}^N$ under translations, we can obtain a positive ground state solution of (A_0), that is, a positive function $\omega \in H^1(\mathbb{R}^N, \mathbb{R})$ such that $E_0(\omega) = c_0$ and $E_0'(\omega) = 0$. From now on we will denote by ω such solution.

Let $\psi \in C^\infty(\mathbb{R}^+, [0, 1])$ be such that $\psi \equiv 1$ in $[0, \delta/2]$ and $\psi \equiv 0$ in $[\delta, \infty)$. We define for each $y \in M$ the function

$$\Psi_{\varepsilon,y}(x) := \psi(|\varepsilon x - y|)\omega\left(\frac{\varepsilon x - y}{\varepsilon}\right) \exp\left(i\tau_y\left(\frac{\varepsilon x - y}{\varepsilon}\right)\right),$$

where $\tau_y(x) := \sum_{j=1}^N A_j(x)x_j$. Let $t_\varepsilon > 0$ be the unique positive number such that

$$\max_{t \geq 0} J_\varepsilon(t\Psi_{\varepsilon,y}) = J_\varepsilon(t_\varepsilon \Psi_{\varepsilon,y}).$$

By noticing that $t_\varepsilon \Psi_{\varepsilon,y} \in \mathcal{N}_\varepsilon$, we can now define $\Phi_\varepsilon : M \to \mathcal{N}_\varepsilon$ as

$$\Phi_\varepsilon(y) := t_\varepsilon \Psi_{\varepsilon,y}.$$

The energy of the above functions has the following behavior as ε becomes small.

Lemma 3.2. *Uniformly for $y \in M$, we have*

$$\lim_{\varepsilon \to 0^+} J_\varepsilon(\Phi_\varepsilon(y)) = c_0.$$

Proof. Arguing by contradiction, we suppose that there exist $\gamma > 0$, $(y_n) \subset M$ and $\varepsilon_n \to 0^+$ such that

$$|J_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - c_0| \geq \gamma > 0. \tag{3.8}$$

In order to simplify the notation, we write only Φ_n, Ψ_n and t_n to denote $\Phi_{\varepsilon_n}(y_n)$, Ψ_{ε_n,y_n} and t_{ε_n}, respectively.

We begin observing that, arguing as in [6, Lemma 3.2], we can check that

$$\|\Psi_n\|_{\varepsilon_n}^2 \to \int (|\nabla \omega|^2 + V_0|\omega|^2), \tag{3.9}$$

On the other hand, since $J'_{\varepsilon_n}(t_n\Psi_n)(t_n\Psi_n) = 0$, the change of variables $z := (\varepsilon_n x - y_n)/\varepsilon_n$ provides

$$\begin{aligned} \|t_n\Psi_n\|_{\varepsilon_n}^2 &= \int g\left(\varepsilon_n x, |t_n\Psi_n(x)|^2\right) |t_n\Psi_n(x)|^2 \, \mathrm{d}x \\ &= \int g\left(\varepsilon_n z + y_n, |t_n\psi(|\varepsilon_n z|)\omega(z)|^2\right) |t_n\psi(|\varepsilon_n z|)\omega(z)|^2 \, \mathrm{d}z. \end{aligned}$$

If $z \in B_{\delta/\varepsilon_n}(0)$, then $\varepsilon_n z + y_n \in B_\delta(y_n) \subset M_\delta \subset \Omega$. Thus, since $g(x,s) = f(s)$ for any $x \in \Omega$ and $\psi(s) = 0$ for $s \geq \delta$, the above expression yields

$$\|\Psi_n\|_{\varepsilon_n}^2 = \int f(|t_n\psi(|\varepsilon_n z|)\omega(z)|^2) \, |\psi(|\varepsilon_n z|)\omega(z)|^2 \, \mathrm{d}z. \tag{3.10}$$

Let $\alpha := \min\{w(z) : |z| \leq \delta/2\}$. If $n_0 \in \mathbb{N}$ is such that $B_{\delta/2}(0) \subset B_{\delta/(2\varepsilon_n)}(0)$ for all $n \geq n_0$, we obtain

$$\|\Psi_n\|_{\varepsilon_n}^2 \geq \int_{B_{\delta/2}(0)} f(|t_n\omega(z)|^2)|\omega(z)|^2 \, \mathrm{d}z \geq f(|t_n\alpha|^2) \int_{B_{\delta/2}(0)} |\omega(z)|^2 \, \mathrm{d}z. \tag{3.11}$$

for all $n \geq n_0$, where we have used that f is increasing.

If $|t_n| \to \infty$, we can use (3.11) and (f_3) to conclude that $\|\Psi_n\|_{\varepsilon_n}^2 \to +\infty$, contradicting (3.9). Thus, up to a subsequence, $t_n \to t_0 \geq 0$.

Since g has subcritical growth and $t_n\Psi_n \in \mathcal{N}_{\varepsilon_n}$, it follows that $t_0 > 0$. Thus, we can take the limit in (3.10) to obtain

$$\int |\nabla(t_0\omega)|^2 + |(t_0\omega)|^2 = \int f(|t_0\omega|^2)|t_0\omega|^2,$$

from which follows that $t_0\omega \in \mathcal{M}_0$. Since ω also belongs to $\mathcal{M}_0$, we conclude that $t_0 = 1$. This and Lebesgue's theorem imply that

$$\int F(|t_n\Psi_n|^2) \to \int F(|\omega|^2).$$

Hence, letting $n \to \infty$ in

$$J_{\varepsilon_n}(\Phi_n) = \frac{t_n^2}{2}\|\Psi_n\|_\varepsilon^2 - \frac{1}{2}\int F(|t_n\Psi_n|^2)$$

and using (3.9), we conclude that

$$\lim_{n\to\infty} J_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) = E_0(\omega) = c_0,$$

which contradicts (3.8) and proves the lemma. □

Let us consider $\rho = \rho_\delta > 0$ in such way that $M_\delta \subset B_\rho(0)$ and define $\Upsilon : \mathbb{R}^N \to \mathbb{R}^N$ by setting $\Upsilon(x) := x$ for $|x| < \rho$ and $\Upsilon(x) := \rho x/|x|$ for $|x| \geq \rho$. We also consider the barycenter map $\beta_\varepsilon : \mathcal{N}_\varepsilon \to \mathbb{R}^N$ given by

$$\beta_\varepsilon(u) := \frac{\displaystyle\int \Upsilon(\varepsilon x)|u(x)|^2 \, \mathrm{d}x}{\displaystyle\int |u(x)|^2 \, \mathrm{d}x}.$$

Since $M \subset B_\rho(0)$, the definition of Υ and Lebesgue's theorem imply that

$$\lim_{\varepsilon\to 0} \beta_\varepsilon(\Phi_\varepsilon(y)) = y \quad \text{uniformly for } y \in M. \tag{3.12}$$

We now consider the following subset of the Nehari manifold

$$\widetilde{\mathcal{N}}_\varepsilon := \{u \in \mathcal{N}_\varepsilon : J_\varepsilon(u) \leq c_0 + h(\varepsilon)\}, \tag{3.13}$$

where $h : \mathbb{R}^+ \to \mathbb{R}^+$ is such that $h(\varepsilon) \to 0$ as $\varepsilon \to 0^+$. Given $y \in M$, we can use Lemma 3.2 to conclude that $h(\varepsilon) = |J_\varepsilon(\Phi_\varepsilon(y)) - c_0|$ is such that $h(\varepsilon) \to 0$ as $\varepsilon \to 0^+$. Thus, $\Phi_\varepsilon(y) \in \widetilde{\mathcal{N}}_\varepsilon$ and therefore $\widetilde{\mathcal{N}}_\varepsilon \neq \varnothing$ for any $\varepsilon > 0$.

We present below an interesting relation between $\widetilde{\mathcal{N}}_\varepsilon$ and the barycenter map.

Proposition 3.3. *For any $\delta > 0$ we have that*

$$\lim_{\varepsilon\to 0^+} \sup_{u\in\widetilde{\mathcal{N}}_\varepsilon} \operatorname{dist}(\beta_\varepsilon(u), M_\delta) = 0.$$

The above result is a version of [6, Lemma 4.1]. Since here we deal with nonhomogeneous nonlinearities, the arguments used there does not apply in our situation. Hence, we need another approach. We are going to use the following compactness result.

Lemma 3.4. *Let $\varepsilon_n \to 0^+$ and $(u_n) \subset \mathcal{N}_{\varepsilon_n}$ be such that $J_{\varepsilon_n}(u_n) \to c_0$. Then there exists a sequence $(\widetilde{y}_n) \subset \mathbb{R}^N$ such that $v_n := |u_n|(\cdot + \widetilde{y}_n)$ has a convergent subsequence in $H^1(\mathbb{R}^N, \mathbb{R})$. Moreover, up to a subsequence, $\varepsilon_n \widetilde{y}_n \to y_0 \in M$.*

By assuming the above result we can prove Proposition 3.3 as follows.

Proof of Proposition 3.3. Let $(\varepsilon_n) \subset \mathbb{R}$ be such that $\varepsilon_n \to 0^+$. By definition, there exists $(u_n) \subset \widetilde{\mathcal{N}}_{\varepsilon_n}$ such that

$$\mathrm{dist}(\beta_{\varepsilon_n}(u_n), M_\delta) = \sup_{u \in \widetilde{\mathcal{N}}_{\varepsilon_n}} \mathrm{dist}(\beta_{\varepsilon_n}(u), M_\delta) + o_n(1).$$

Thus, it suffices to find a sequence $(y_n) \subset M_\delta$ such that

$$|\beta_{\varepsilon_n}(u_n) - y_n| = o_n(1). \tag{3.14}$$

It follows from the diamagnetic inequality (2.1) that $E_0(tu_n) \leq J_{\varepsilon_n}(tu_n)$. Thus, recalling that $(u_n) \subset \widetilde{\mathcal{N}}_{\varepsilon_n} \subset \mathcal{N}_{\varepsilon_n}$, we can use (3.7) to obtain

$$c_0 \leq \max_{t \geq 0} E_0(tu_n) \leq \max_{t \geq 0} J_{\varepsilon_n}(tu_n) = J_{\varepsilon_n}(u_n) \leq c_0 + h(\varepsilon_n),$$

from which follows that $J_{\varepsilon_n}(u_n) \to c_0$. Thus, we may invoke Lemma 3.4 to obtain a sequence $(\widetilde{y}_n) \subset \mathbb{R}^N$ such that $(y_n) := (\varepsilon_n \widetilde{y}_n) \subset M_\delta$, for n large. The strong convergence of $|u_n|(\cdot + \widetilde{y}_n)$ implies that

$$\begin{aligned}
\beta_{\varepsilon_n}(u_n) &= \frac{\displaystyle\int \Upsilon(\varepsilon_n x)|u_n|^2 \mathrm{d}x}{\displaystyle\int |u_n|^2 \mathrm{d}x} = \frac{\displaystyle\int \Upsilon(\varepsilon_n z + y_n)|u_n(z + \widetilde{y}_n)|^2 \mathrm{d}z}{\displaystyle\int |u_n(z + \widetilde{y}_n)|^2 \mathrm{d}z} \\
&= y_n + \frac{\displaystyle\int \left(\Upsilon(\varepsilon_n z + y_n) - y_n\right)|u_n(z + \widetilde{y}_n)|^2 \mathrm{d}z}{\displaystyle\int |u_n(z + \widetilde{y}_n)|^2 \mathrm{d}z}.
\end{aligned}$$

Since $\varepsilon_n z + y_n \to y_0 \in M$, we have that $\beta_{\varepsilon_n}(u_n) = y_n + o_n(1)$ and therefore the sequence (y_n) satisfies (3.14). The lemma is proved. $\square$

We proceed now with the proof of Lemma 3.4.

Proof of Lemma 3.4. As in Lemma 2.1 we have that (u_n) is bounded in H_ε. We start by proving that there exists a sequence $(\widetilde{y}_n) \subset \mathbb{R}^N$ and constants $R, \gamma > 0$ such that

$$\liminf_{n \to \infty} \int_{B_R(\tilde{y}_n)} |u_n|^2 \geq \gamma > 0. \tag{3.15}$$

Indeed, if this is not true, then the boundedness of $(|u_n|)$ in $H^1(\mathbb{R}^N, \mathbb{R})$ and a lemma due to Lions [18, Lemma I.1] imply that $|u_n| \to 0$ in $L^s(\mathbb{R}^N)$ for all $2 < s < 2^*$. Given $\xi > 0$, we can use (g_2), (g_3) and $u_n \in \mathcal{N}_{\varepsilon_n}$ to get

$$\|u_n\|^2_{\varepsilon_n} = \int g(\varepsilon x, |u_n|^2)|u_n|^2 \leq \xi \int |u_n|^2 + C_\xi \int |u_n|^q.$$

Since $u_n \to 0$ in $L^q(\mathbb{R}^N)$ and ξ is arbitrary, we conclude that $\|u_n\|_{\varepsilon_n} \to 0$. Moreover, since $\int g(\varepsilon x, |u_n|^2)|u_n|^2 \to 0$, it follows from (g_4) that $\int G(\varepsilon x, |u_n|^2) \to 0$. Hence, $J_{\varepsilon_n}(u_n) \to 0$, contradicting $c_0 > 0$. Thus, (3.15) holds and, along a subsequence,

$$v_n := |u_n|(\cdot + \tilde{y}_n) \rightharpoonup v \neq 0 \quad \text{weakly in } H^1(\mathbb{R}^N, \mathbb{R}).$$

We now consider $t_n > 0$ such that $w_n := t_n v_n \in \mathcal{M}_0$. It follows from the diamagnetic inequality (2.1) that

$$c_0 \leq E_0(w_n) \leq \max_{t \geq 0} J_{\varepsilon_n}(t v_n) = J_{\varepsilon_n}(u_n) = c_0 + o_n(1). \tag{3.16}$$

Hence $E_0(w_n) \to c_0$ from which follows that $w_n \not\to 0$ in $H^1(\mathbb{R}^N, \mathbb{R})$.

Since (v_n) and (w_n) are bounded in $H^1(\mathbb{R}^N, \mathbb{R})$ and $v_n \not\to 0$ in $H^1(\mathbb{R}^N, \mathbb{R})$, the sequence (t_n) is bounded. Thus, up to a subsequence, $t_n \to t_0 \geq 0$. If $t_0 = 0$ then $\|w_n\|_{H^1(\mathbb{R}^N,\mathbb{R})} \to 0$, which does not occurs. Hence $t_0 > 0$, and therefore the sequence (w_n) satisfies

$$E_0(w_n) \to c_0, \quad w_n \rightharpoonup w := t_0 v \neq 0 \ \text{ weakly in } H^1(\mathbb{R}^N, \mathbb{R}).$$

It follows from Lemma 3.1 that $w_n \to w$, or equivalently, $v_n \to v$ in $H^1(\mathbb{R}^N, \mathbb{R})$. This proves the first part of the lemma.

In order to finish the proof we set $y_n := \varepsilon_n \widetilde{y}_n$ and claim that (y_n) has a bounded subsequence. Indeed, if this is not the case, then $|y_n| \to \infty$. Consider $R > 0$ such that $\Omega \subset B_R(0)$. Since we may suppose that $|y_n| > 2R$ we have that for any $z \in B_{R/\varepsilon_n}(0)$

$$|\varepsilon_n z + y_n| \geq |y_n| - |\varepsilon_n z| > R.$$

If we now set $\Gamma_n := B_{R/\varepsilon_n}(0)$, we can use $(u_n) \subset \mathcal{N}_{\varepsilon_n}$, (V_1), (2.1), the change of variables $x \mapsto z + \tilde{y}_n$, the above expression and (2.3) to get

$$\begin{aligned} \int |\nabla v_n|^2 + V_0 \int |v_n|^2 &\leq \int g(\varepsilon_n z + y_n, |v_n|^2)|v_n|^2 \,\mathrm{d}z \\ &= \int_{\Gamma_n} \widetilde{f}(|v_n|^2)|v_n|^2 + \int_{\Gamma_n^c} g(\varepsilon_n z + y_n, |v_n|^2)|v_n|^2 \,\mathrm{d}z \\ &\leq \int_{\Gamma_n} \widetilde{f}(|v_n|^2)|v_n|^2 + \int_{\Gamma_n^c} f(|v_n|^2)|v_n|^2 \end{aligned}$$

Since $v_n \to v$ in $H^1(\mathbb{R}^N, \mathbb{R})$ and $\widetilde{f}(s) \leq V_0/k$, we obtain

$$\min\left\{1, V_0\left(1 - \frac{1}{k}\right)\right\}\left(\int |\nabla v_n|^2 + |v_n|^2\right) = o_n(1),$$

which contradicts $v \not\equiv 0$. This contradiction shows that (y_n) has a bounded subsequence. Thus, up to a subsequence, we have that

$$y_n \to y_0 \in \mathbb{R}^N.$$

If $y_0 \notin \overline{\Omega}$ we can proceed as above and conclude that $v_n \to 0$. Thus, we have that $y_0 \in \overline{\Omega}$.

In order to prove that $V(y_0) = V_0$ we suppose, by contradiction, that $V(y_0) > V_0$. We can use (2.1), the strong convergence of w_n in $H^1(\mathbb{R}^N, \mathbb{R})$, Fatou's lemma and the invariance of $\mathbb{R}^N$ by translations, to obtain

$$\begin{aligned} c_0 &= E_0(w) < \frac{1}{2}\int \big(|\nabla w|^2 + V(y_0)|w|^2\big) - \frac{1}{2}F(|w|^2) \\ &\le \liminf_{n\to\infty} \big[\frac{1}{2}\int \big(|\nabla w_n|^2 + V(\varepsilon_n z + \widetilde{y}_n)|w_n|^2\big) - \frac{1}{2}\int F(|w_n|^2)\big] \\ &\le \liminf_{n\to\infty} \big[\frac{1}{2}\int \big(|\nabla |w_n||^2 + V(\varepsilon_n z + \widetilde{y}_n)|w_n|^2\big) - \frac{1}{2}\int F(|w_n|^2)\big]. \end{aligned}$$

By diamagnetic inequality (2.1) we conclude

$$\begin{aligned} c_0 &< \liminf_{n\to\infty} \big[\frac{1}{2}\int \big(|\nabla_{\varepsilon_n} w_n|^2 + V(\varepsilon_n z + \widetilde{y}_n)|w_n|^2\big) - \frac{1}{2}\int F(|w_n|^2)\big]. \\ &\le \liminf_{n\to\infty} J_{\varepsilon_n}(t_n u_n) \le \liminf_{n\to\infty} J_{\varepsilon_n}(u_n) = c_0, \end{aligned}$$

which does not make sense. Hence $V(y_0) = V_0$ and $y_0 \in \overline{\Omega}$. The condition (V_2) implies that $y_0 \notin \partial\Omega$, that is, $y_0 \in M$. The proof is finished. ☐

Corollary 3.5. *Assume the same hypotheses of Lemma 3.4. Then, for any given $\gamma > 0$, there exists $R > 0$ and $n_0 \in \mathbb{N}$ such that*

$$\int_{B_R(\widetilde{y}_n)^c} \big(|\nabla |u_n||^2 + |u_n|^2\big) < \gamma, \quad \textit{for all } n \ge n_0.$$

Proof. By using the same notation of the proof of Lemma 3.4, we have for any $R > 0$

$$\int_{B_R(\widetilde{y}_n)^c} \big(|\nabla |u_n||^2 + |u_n|^2\big) = \int_{B_R(0)^c} (|\nabla v_n|^2 + |v_n|^2).$$

Since v_n strongly converges in $H^1(\mathbb{R}^N, \mathbb{R})$ the result follows. ☐

We finalize the section presenting a relation between the topology of M and the number of solutions of the modified problem $(\widetilde{D}_\varepsilon)_a$.

Theorem 3.6. *For any $\delta > 0$ verifying $M_\delta \subset \Omega$, there exists $\widehat{\varepsilon}_\delta > 0$ such that, for any $0 < \varepsilon < \widehat{\varepsilon}_\delta$, the problem $(\widetilde{D}_\varepsilon)_a$ has at least $cat_{M_\delta}(M)$ nontrivial solutions.*

Proof. Given $\delta > 0$ such that $M_\delta \subset \Omega$, we can use (3.12), Lemma 3.2, Proposition 3.3, and argue as in [9, Section 6] to obtain $\widehat{\varepsilon}_\delta > 0$ such that, for any $\varepsilon \in (0, \widehat{\varepsilon}_\delta)$, the diagram

$$M \xrightarrow{\Phi_\varepsilon} \widetilde{\mathcal{N}}_\varepsilon \xrightarrow{\beta_\varepsilon} M_\delta$$

is well defined and $\beta_\varepsilon \circ \Phi_\varepsilon$ is homotopically equivalent to the embedding $\iota : M \to M_\delta$. Thus

$$\mathrm{cat}_{\widetilde{\mathcal{N}}_\varepsilon}(\widetilde{\mathcal{N}}_\varepsilon) \geq \mathrm{cat}_{M_\delta}(M).$$

It follows from Proposition 2.2 and standard Ljusternik-Schnirelmann theory that J_ε possesses at least $\mathrm{cat}_{\widetilde{\mathcal{N}}_\varepsilon}(\widetilde{\mathcal{N}}_\varepsilon)$ critical points on $\mathcal{N}_\varepsilon$. The same argument employed in the proof of Proposition 2.2 shows that each of these critical points is also a critical point of the unconstrained functional J_ε. Thus, we obtain $\mathrm{cat}_{M_\delta}(M)$ nontrivial solutions for $(\widetilde{D}_\varepsilon)_a$. □

4 Proof of Theorem 1.1

In this section we prove our main theorem. The idea is to show that the solutions obtained in Theorem 3.6 verify the following estimate $|u_\epsilon(x) \leq t_a \; \forall x \in \Omega_\epsilon^c$ as ϵ is small enough. This fact implies that these solutions are in fact solutions of the original problem (D_ε). The key ingredient is the following result, whose proof uses an adaptation of the arguments found in [5] and [16], which are related with the Moser iteration method [19] .

Lemma 4.1. *Let $\varepsilon_n \to 0^+$ and $u_n \in \widetilde{\mathcal{N}}_{\varepsilon_n}$ be a solution of $(\widetilde{D}_{\varepsilon_n})_a$. Then $J_{\varepsilon_n}(u_n) \to c_0$ and $|u_n| \in L^\infty(\mathbb{R}^N)$. Moreover, for any given $\gamma > 0$, there exists $R > 0$ and $n_0 \in \mathbb{N}$ such that*

$$\|u_n\|_{L^\infty(B_R(\tilde{y}_n)^c)} < \gamma, \quad \textit{for all } n \geq n_0, \tag{4.17}$$

where $\tilde{y}_n$ is given by Lemma 3.4.

Proof. Since $J_{\varepsilon_n}(u_n) \leq c_0 + h(\varepsilon_n)$ with $\lim_{n\to\infty} h(\varepsilon_n) = 0$, we can argue as in the proof of equation (3.16) to conclude that $J_{\varepsilon_n}(u_n) \to c_0$. Thus, we may invoke Lemma 3.4 to obtain a sequence $(\widetilde{y}_n) \subset \mathbb{R}^N$ satisfying the conclusions of that lemma.

Fix $R > 1$ and consider $\eta_R \in C^\infty(\mathbb{R}^N, \mathbb{R})$ such that $0 \leq \eta_R \leq 1$, $\eta_R \equiv 0$ in $B_{R/2}(0)$, $\eta_R \equiv 1$ in $B_R(0)^c$ and $|\nabla \eta_R| \leq C/R$. For each $n \in \mathbb{N}$ and $L > 0$, we define $\eta_n(x) := \eta_R(x - \tilde{y}_n)$, $u_{L,n} \in H^1(\mathbb{R}^N, \mathbb{R})$ and $z_{L,n} \in H_\varepsilon$ by setting

$$u_{L,n}(x) := \min\{|u_n(x)|, L\}, \qquad z_{L,n} := \eta_n^2 u_{L,n}^{2(\beta-1)} u_n,$$

with $\beta > 1$ to be determined later.

By using the calculation performed in [5, equation (2.2)] and the diamagnetic inequality we obtain

$$\begin{aligned} \mathrm{Re}\left(\nabla_\varepsilon u_n \overline{\nabla_\varepsilon z_{L,n}}\right) &\geq \eta_n^2 u_{L,n}^{2(\beta-1)} |\nabla_\varepsilon u_n|^2 + 2\eta_n |u_n| u_{L,n}^{2(\beta-1)} \nabla \eta_n \cdot \nabla |u_n| \\ &\geq \eta_n^2 u_{L,n}^{2(\beta-1)} |\nabla |u_n||^2 + 2\eta_n |u_n| u_{L,n}^{2(\beta-1)} \nabla \eta_n \cdot \nabla |u_n|. \end{aligned}$$

This inequality, the definition of $z_{L,n}$ and $J'_{\varepsilon_n}(u_n)z_{L,n} = 0$ imply that

$$\begin{aligned}&\int \eta_n^2 u_{L,n}^{2(\beta-1)}|\nabla|u_n||^2 + 2\int \eta_n|u_n|u_{L,n}^{2(\beta-1)}\nabla\eta_n\cdot\nabla|u_n| \\ &\quad\le \int \left(g_{\varepsilon_n}(x,|u_n|^2) - V_{\varepsilon_n}(x)\right)\eta_n^2|u_n|^2u_{L,n}^{2(\beta-1)}.\end{aligned} \tag{4.18}$$

In view of (g_2) and (g_3) we can obtain $C_1 > 0$ such that

$$g(x,s) \le \frac{V_0}{2} + C_1|s|^{(2^*-2)/2}, \quad \text{for any } (x,s) \in \mathbb{R}^N \times \mathbb{R}.$$

This, (4.18) and $V_\varepsilon(x) \ge V_0$ provide

$$\begin{aligned}\int \eta_n^2 u_{L,n}^{2(\beta-1)}|\nabla|u_n||^2 &\le 2\int \eta_n|u_n|u_{L,n}^{2(\beta-1)}|\nabla\eta_n\,||\nabla|u_n|| \\ &\quad + \int\left(\frac{V_0}{2} + C_1|u_n|^{2^*-2} - V_{\varepsilon_n}(x)\right)\eta_n^2|u_n|^2u_{L,n}^{2(\beta-1)} \\ &\le 2\int \eta_n|u_n|u_{L,n}^{2(\beta-1)}|\nabla\eta_n||\nabla|u_n|| + C_1\int \eta_n^2|u_n|^{2^*}u_{L,n}^{2(\beta-1)}.\end{aligned}$$

For any $\tilde\gamma > 0$ we can use Young's inequality to obtain

$$\begin{aligned}\int \eta_n^2 u_{L,n}^{2(\beta-1)}|\nabla|u_n||^2 &\le 2\int\left(\tilde\gamma\eta_n^2|\nabla|u_n||^2 + C_{\tilde\gamma}|u_n|^2|\nabla\eta_n|^2\right)u_{L,n}^{2(\beta-1)} \\ &\quad + C_1\int \eta_n^2|u_n|^{2^*}u_{L,n}^{2(\beta-1)}.\end{aligned}$$

By choosing $\tilde\gamma \le 1/4$ we get

$$\int \eta_n^2 u_{L,n}^{2(\beta-1)}|\nabla|u_n||^2 \le C_2\left(\int |u_n|^2u_{L,n}^{2(\beta-1)}|\nabla\eta_n|^2 + \eta_n^2|u_n|^{2^*}u_{L,n}^{2(\beta-1)}\right). \tag{4.19}$$

Let S be the best constant of the embedding $\mathcal{D}^{1,2}(\mathbb{R}^N,\mathbb{R}) \hookrightarrow L^{2^*}(\mathbb{R}^N,\mathbb{R})$ and define $\widehat{u}_{L,n} := \eta_n|u_n|u_{L,n}^{\beta-1}$. We have that

$$\begin{aligned}S^{-1}\|\widehat{u}_{L,n}\|_{L^{2^*}}^2 &\le \int\left|\nabla\left(\eta_n|u_n|u_{L,n}^{\beta-1}\right)\right|^2 \\ &\le 2\int |u_n|^2u_{L,n}^{2(\beta-1)}|\nabla\eta_n|^2 + 2\int\eta_n^2\left|\nabla\left(|u_n|u_{L,n}^{\beta-1}\right)\right|^2.\end{aligned}$$

But

$$\begin{aligned}\int \eta_n^2\left|\nabla\left(|u_n|u_{L,n}^{\beta-1}\right)\right|^2 &= \int_{\{|u_n|\le L\}}\eta_n^2\left|\nabla\left(|u_n|u_{L,n}^{\beta-1}\right)\right|^2 + \int_{\{|u_n|>L\}}\eta_n^2\left|\nabla\left(|u_n|u_{L,n}^{\beta-1}\right)\right|^2 \\ &= \int_{\{|u_n|\le L\}}\eta_n^2\left|\nabla|u_n|^\beta\right|^2 + \int_{\{|u_n|>L\}}\eta_n^2L^{2(\beta-1)}\,|\nabla|u_n||^2 \\ &\le \beta^2\int \eta_n^2u_{L,n}^{2(\beta-1)}|\nabla|u_n||^2,\end{aligned}$$

and therefore

$$\|\widehat{u}_{L,n}\|_{L^{2^*}}^2 \le C_3\beta^2\left(\int |u_n|^2 u_{L,n}^{2(\beta-1)}|\nabla\eta_n|^2 + \int \eta_n^2 u_{L,n}^{2(\beta-1)}|\nabla|u_n||^2\right).$$

This and (4.19) provide

$$\|\widehat{u}_{L,n}\|_{L^{2^*}}^2 \le C_4\beta^2\left(\int |u_n|^2 u_{L,n}^{2(\beta-1)}|\nabla\eta_n|^2 + \int \eta_n^2 |u_n|^{2^*} u_{L,n}^{2(\beta-1)}\right), \qquad (4.20)$$

for all $\beta > 1$. The above expression, the properties of η_n and $u_{L,n} \le |u_n|$, imply that

$$\|\widehat{u}_{L,n}\|_{L^{2^*}}^2 \le C_4\beta^2 \int_{B_{R/2}(\widetilde{y}_n)^c}\left(|u_n|^{2\beta}|\nabla\eta_n|^2 + |u_n|^{2^*-2}|u_n|^{2\beta}\right). \qquad (4.21)$$

If we now set

$$t := \frac{2^*2^*}{2(2^*-2)} > 1, \qquad \alpha := \frac{2t}{t-1} < 2^*, \qquad (4.22)$$

we can apply Hölder's inequality with exponents $t/(t-1)$ and t in (4.21), to get

$$\begin{aligned} \|\widehat{u}_{L,n}\|_{L^{2^*}}^2 \quad \le \quad & C_4\beta^2\|u_n\|_{L^{\beta\alpha}(B_{R/2}(\widetilde{y}_n)^c)}^{2\beta}\left(\int_{B_{R/2}(\widetilde{y}_n)^c}|\nabla\eta_n|^{2t}\right)^{1/t}. \\ & + C_4\beta^2\|u_n\|_{L^{\beta\alpha}(B_{R/2}(\widetilde{y}_n)^c)}^{2\beta}\left(\int_{B_{R/2}(\widetilde{y}_n)^c}|u_n|^{2^*(2^*/2)}\right)^{1/t}. \end{aligned} \qquad (4.23)$$

Since η_n is constant on $B_{R/2}(\widetilde{y}_n)\cup B_R(\widetilde{y}_n)^c$ and $|\nabla\eta_n| \le C/R$, we have that

$$\int_{B_{R/2}(\widetilde{y}_n)^c}|\nabla\eta_n|^{2t} = \int_{R/2\le|x-\widetilde{y}_n|\le R}|\nabla\eta_n|^{2t} \le \frac{C_5}{R^{2t-N}} \le C_5, \qquad (4.24)$$

where we have used $R > 1$ and $2t = \frac{2^*}{2}N > N$ in the last inequality.

Claim. There exists $n_0 \in \mathbb{N}$ and $K > 0$ such that , for any $n \ge n_0$, there holds

$$\int_{B_{R/2}(\widetilde{y}_n)^c}|u_n|^{2^*(2^*/2)} \le K.$$

Assuming the claim, we can use (4.23) and (4.24) to conclude that

$$\|\widehat{u}_{L,n}\|_{L^{2^*}}^2 \le C_6\beta^2\|u_n\|_{L^{\beta\alpha}(B_{R/2}(\widetilde{y}_n)^c)}^{2\beta}.$$

Since

$$\begin{aligned} \|u_{L,n}\|_{L^{\beta 2^*}(B_R(\widetilde{y}_n)^c)}^{2\beta} \quad &= \quad \left(\int_{B_R(\widetilde{y}_n)^c} u_{L,n}^{\beta 2^*}\right)^{2/2^*} \\ &\le \quad \left(\int \eta_n^{2^*}|u_n|^{2^*}u_{L,n}^{2^*(\beta-1)}\right)^{2/2^*} \\ &= \quad \|\widehat{u}_{L,n}\|_{L^{2^*}}^2 \le C_6\beta^2\|u_n\|_{L^{\beta\alpha}(B_{R/2}(\widetilde{y}_n)^c)}^{2\beta}, \end{aligned}$$

we can apply Fatou's lemma in the variable L to obtain

$$\||u_n|\|_{L^{\beta 2^*}(B_R(\widetilde{y}_n)^c)} \leq C_7^{1/\beta}\beta^{1/\beta}\||u_n|\|_{L^{\beta\alpha}(B_{R/2}(\widetilde{y}_n)^c)},$$

whenever $\||u_n|\|^{\beta\alpha} \in L^1(B_{R/2}(\widetilde{y}_n)^c)$.

We now set $\beta := 2^*/\alpha > 1$ and note that, since $|u_n| \in L^{2^*}(\mathbb{R}^N)$, the above inequality holds for this choice of β. Moreover, since $\beta^2\alpha = \beta 2^*$, it follows that the inequality also holds with β replaced by β^2. Hence,

$$\||u_n|\|_{L^{\beta^2 2^*}(B_R(\widetilde{y}_n)^c)} \leq C_7^{1/\beta^2}\beta^{2/\beta^2}\||u_n|\|_{L^{\beta^2\alpha}(B_{R/2}(\widetilde{y}_n)^c)}.$$

By iterating this process and recalling that $\beta\alpha = 2^*$ we obtain, for $k \in \mathbb{N}$,

$$\||u_n|\|_{L^{\beta^k 2^*}(B_R(\widetilde{y}_n)^c)} \leq C_7^{\sum_{i=1}^k \beta^{-i}}\beta^{\sum_{i=1}^m i\beta^{-i}}\||u_n|\|_{L^{2^*}(B_{R/2}(\widetilde{y}_n)^c)}.$$

Since $\beta > 1$ we can take the limit as $k \to \infty$ to get

$$\||u_n|\|_{L^\infty(B_R(\widetilde{y}_n)^c)} \leq C_8\||u_n|\|_{L^{2^*}(B_{R/2}(\widetilde{y}_n)^c)}.$$

By using the change of variables $z \mapsto x - \tilde{y}_n$ we obtain

$$\||u_n|\|_{L^\infty(B_R(\widetilde{y}_n)^c)} \leq C_8\Big(\int_{B_{R/2}(0)^c}|u_n(z+\tilde{y}_n)|^{2^*}\mathrm{d}z\Big)^{\frac{1}{2^*}} = C_8\Big(\int_{B_{R/2}(0)^c}|v_n|^{2^*}\Big)^{\frac{1}{2^*}},$$

where $v_n(x) = |u_n|(x+\widetilde{y}_n)$. By Lemma 3.4 we have that v_n strongly converges in $L^{2^*}(\mathbb{R}^N)$. Thus, for $R > 0$ sufficiently large, there holds

$$\||u_n|\|_{L^\infty(B_R(\widetilde{y}_n)^c)} < \gamma,$$

for large n. This establishes (4.17).

It remains to prove the claim. For that purpose we consider a new cut-off function given by $\widetilde{\eta}_n(x) := \eta_n(2x)$, in such way that $\widetilde{\eta}_n \equiv 0$ on $B_{R/4}(\widetilde{y}_n)$ and $\widetilde{\eta}_n \equiv 1$ on $B_{R/2}(\widetilde{y}_n)^c$. If $\widetilde{u}_{L,n} := \widetilde{\eta}_n|u_n|u_{L,n}^{\beta-1}$, we can proceed as before to prove the following version of (4.20)

$$\|\widetilde{u}_{L,n}\|_{L^{2^*}}^2 \leq C_9\beta^2\left(\int |u_n|^2 u_{L,n}^{2(\beta-1)}|\nabla\widetilde{\eta}_n|^2 + \int \widetilde{\eta}_n^2|u_n|^{2^*}u_{L,n}^{2(\beta-1)}\right), \qquad (4.25)$$

We set $\beta := 2^*/2$ to obtain

$$\|\widetilde{u}_{L,n}\|_{L^{2^*}}^2 \leq C_{10}\left(\int |u_n|^2 u_{L,n}^{(2^*-2)}|\nabla\widetilde{\eta}_n|^2 + \int_{B_{R/4}(\widetilde{y}_n)^c}\widetilde{\eta}_n^2|u_n|^2 u_{L,n}^{(2^*-2)}|u_n|^{(2^*-2)}\right).$$

By Hölder's inequality with exponents $2^*/2$ and $2^*/(2^*-2)$ we get

$$\begin{aligned}\|\widetilde{u}_{L,n}\|_{L^{2^*}}^2 &\leq C_{10}\int |u_n|^2 u_{L,n}^{(2^*-2)}|\nabla\widetilde{\eta}_n|^2 \\ &+ C_{10}\left(\int_{B_{R/4}(\widetilde{y}_n)^c}\left(\widetilde{\eta}_n|u_n|u_{L,n}^{(2^*-2)/2}\right)^{2^*}\right)^{2/2^*}\|u_n\|_{L^{2^*}(B_R(\widetilde{y}_n)^c)}^{2^*-2}.\end{aligned}$$

From Corollary 3.5 we obtain $n_0 \in \mathbb{N}$ and $R > 1$ such that

$$\int_{B_R(\widetilde{y}_n)^c} |u_n|^{2^*} \leq \left(\frac{1}{2C_{10}}\right)^{2^*/(2^*-2)},$$

for all $n \geq n_0$. Thus, recalling that $\widetilde{\eta}_n |u_n| u_{L,n}^{(2^*-2)/2} = \widetilde{u}_{L,n}$, $u_{L,n} \leq |u_n|$ and $\nabla \eta_n$ is bounded, we obtain

$$\|\widetilde{u}_{L,n}\|_{L^{2^*}}^2 \leq C_{11} \int |u_n|^2 u_{L,n}^{(2^*-2)} |\nabla \widetilde{\eta}_n|^2 \leq C_{11} \int |u_n|^{2^*} \leq C_{12}.$$

The definition of $\widetilde{\eta}_n$ and the above inequality imply that

$$\left(\int_{B_{R/2}(\widetilde{y}_n)^c} |u_n|^{2^*} u_{L,n}^{2^*(2^*-2)/2}\right)^{2/2^*} \leq \|\widetilde{u}_{L,n}\|_{L^{2^*}}^2 \leq C_{12},$$

for all $n \geq n_0$. Using Fatou's lemma in the variable L, we have

$$\int_{B_{R/2}(\widetilde{y}_n)^c} |u_n|^{2^*(2^*/2)} \leq K := C_{12}^{2^*/2},$$

for all $n \geq n_0$, and therefore the claim holds. □

We are now ready to prove the main result of the paper.

Proof of Theorem 1.1. Suppose that $\delta > 0$ is such that $M_\delta \subset \Omega$. We first claim that there exists $\widetilde{\varepsilon}_\delta > 0$ such that, for any $0 < \varepsilon < \widetilde{\varepsilon}_\delta$ and any solution solution $u \in \widetilde{\mathcal{N}}_\varepsilon$ of the problem $(\widetilde{D}_\varepsilon)_a$, there holds

$$\|u\|_{L^\infty(\mathbb{R}^N \setminus \Omega_\varepsilon)} < t_a. \tag{4.26}$$

In order to prove the claim we argue by contradiction. So, suppose that for some sequence $\varepsilon_n \to 0^+$ we can obtain $u_n \in \widetilde{\mathcal{N}}_{\varepsilon_n}$ such that $J'_{\varepsilon_n}(u_n) = 0$ and

$$\|u_n\|_{L^\infty(\mathbb{R}^N \setminus \Omega_{\varepsilon_n})} \geq t_a. \tag{4.27}$$

As in Lemma 4.1, we have that $J_{\varepsilon_n}(u_n) \to c_0$ and therefore we can use Lemma 3.4 to obtain a sequence $(\widetilde{y}_n) \subset \mathbb{R}^N$ such that $\varepsilon_n \widetilde{y}_n \to y_0 \in M$.

If we take $r > 0$ such that $B_r(y_0) \subset B_{2r}(y_0) \subset \Omega$ we have that

$$B_{r/\varepsilon_n}(y_0/\varepsilon_n) = \frac{1}{\varepsilon_n} B_r(y_0) \subset \Omega_{\varepsilon_n}.$$

Moreover, for any $z \in B_{r/\varepsilon_n}(\widetilde{y}_n)$, there holds

$$\left|z - \frac{y_0}{\varepsilon_n}\right| \leq |z - \widetilde{y}_n| + \left|\widetilde{y}_n - \frac{y_0}{\varepsilon_n}\right| < \frac{1}{\varepsilon_n}(r + o_n(1)) < \frac{2r}{\varepsilon_n},$$

for n large. For this values of n we have that $B_{r/\varepsilon_n}(\widetilde{y}_n) \subset \Omega_{\varepsilon_n}$ or, equivalently, $\mathbb{R}^N \setminus \Omega_{\varepsilon_n} \subset \mathbb{R}^N \setminus B_{r/\varepsilon_n}(\widetilde{y}_n)$. On the other hand, it follows from Lemma 4.1 with $\gamma = t_a$ that, for any $n \geq n_0$ such that $r/\varepsilon_n > R$, there holds

$$\|u_n\|_{L^\infty(\mathbb{R}^N \setminus \Omega_{\varepsilon_n})} \leq \|u_n\|_{L^\infty(\mathbb{R}^N \setminus B_{r/\varepsilon_n}(\widetilde{y}_n))} \leq \|u_n\|_{L^\infty(\mathbb{R}^N \setminus B_R(\widetilde{y}_n))} < t_a,$$

which contradicts (4.27) and proves the claim.

Let $\widehat{\varepsilon}_\delta > 0$ given by Theorem 3.6 and set $\varepsilon_\delta := \min\{\widehat{\varepsilon}_\delta, \widetilde{\varepsilon}_\delta\}$. We shall prove the theorem for this choice of ε_δ. Let $0 < \varepsilon < \varepsilon_\delta$ be fixed. By applying Theorem 3.6 we obtain $\mathrm{cat}_{M_\delta}(M)$ nontrivial solutions of the problem $(\widetilde{D}_\varepsilon)_a$. If $u \in H_\varepsilon$ is one of these solutions we have that $u \in \widetilde{\mathcal{N}}_\varepsilon$, and therefore we can use (4.26) and the definition of g to conclude that $g_\varepsilon(\cdot, |u|^2) \equiv f(|u|^2)$. Hence, u is also a solution of the problem (D_ε). An easy calculation shows that $\widehat{u}(x) := u(x/\varepsilon)$ is a solution of the original problem (P_ε). Then, (P_ε) has at least $\mathrm{cat}_{M_\delta}(M)$ nontrivial solutions.

We now consider $\varepsilon_n \to 0^+$ and take a sequence $u_n \in H_{\varepsilon_n}$ of solutions of the problem (D_{ε_n}) as above. In order to study the behavior of the maximum points of $|u_n|$, we first notice that, by (g_2), there exists $\gamma > 0$ such that

$$g(\varepsilon x, s^2)s^2 \le \frac{V_0}{2}s^2, \quad \text{for all } x \in \mathbb{R}^N,\ |s| \le \gamma. \tag{4.28}$$

By applying Lemma 4.1 we obtain $R > 0$ and $(\widetilde{y}_n) \subset \mathbb{R}^N$ such that

$$\|u_n\|_{L^\infty(B_R(\widetilde{y}_n))^c} < \gamma, \tag{4.29}$$

Up to a subsequence, we may also assume that

$$\|u_n\|_{L^\infty(B_R(\widetilde{y}_n))} \ge \gamma. \tag{4.30}$$

Indeed, if this is not the case, we have $\|u_n\|_{L^\infty(\mathbb{R}^N)} < \gamma$, and therefore it follows from $J'_{\varepsilon_n}(u_n) = 0$, (4.28) and the diamagnetic inequality that

$$\int |\nabla |u_n||^2 + V_0|u_n|^2 \le \|u_n\|^2_{\varepsilon_n} = \int g_{\varepsilon_n}(x, |u_n|^2)|u_n|^2 \le \frac{V_0}{2}\int |u_n|^2.$$

The above expression implies that $\||u_n|\|_{H^1(\mathbb{R}^N,\mathbb{R})} = 0$, which does not make sense. Thus, (4.30) holds.

By using (4.29) and (4.30) we conclude that the maximum point $p_n \in \mathbb{R}^N$ of $|u_n|$ belongs to $B_R(\widetilde{y}_n)$. Hence $p_n = \widetilde{y}_n + q_n$, for some $q_n \in B_R(0)$. Recalling that the associated solution of (P_{ε_n}) is of the form $\widehat{u}_n(x) = u_n(x/\varepsilon_n)$, we conclude that the maximum point η_n of $|\widehat{u}_n|$ is $\eta_n := \varepsilon_n\widetilde{y}_n + \varepsilon_n q_n$. Since $(q_n) \subset B_R(0)$ is bounded and $\varepsilon_n\widetilde{y}_n \to y_0 \in M$ (according to Lemma 3.4), we obtain

$$\lim_{n\to\infty} V(\eta_{\varepsilon_n}) = V(y_0) = V_0,$$

which concludes the proof of the theorem. □

Acknowledgment The second author would like to thank UAME/UFCG - Campina Grande, and specially Professor Claudianor Oliveira Alves for his attention and friendship. This work was done while he was visiting that institution.

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