On the number of solutions of NLS equations with magnetics fields in expanding domains*

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Abstract

In this paper we look for multiple weak solutions $u : \Omega_\lambda \to \mathbb{C}$ for the complex equation $(-i\nabla - A(\frac{x}{\lambda}))^2 u + u = f(|u|^2)u$ in $\Omega_\lambda = \lambda\Omega$. The set $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $\lambda > 0$ is a parameter, A is a regular magnetic field and f is a superlinear function with subcritical growth. Our main result relates, for large values of λ, the number of solutions with the topology of the set Ω. In the proof we apply minimax methods and Ljusternick-Schnirelmann theory.

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1 Introduction

In this paper we establish the existence of multiple solutions for the following complex equation

$$\left(-i\nabla - A\left(\frac{x}{\lambda}\right)\right)^2 u + u = f(|u|^2)u, \quad x \in \Omega_\lambda, \tag{P_λ}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $N \geq 3$, i is the imaginary unit, $\lambda > 0$ is a parameter and $\Omega_\lambda := \lambda\Omega$ is an expanding domain. The magnetic field $A : \Omega \to \mathbb{R}^N$ belongs to $C^1(\Omega, \mathbb{R}^N)$ and the nonlinearity f satisfies the following conditions:

(f_0) $f \in C^1(\mathbb{R}, \mathbb{R})$;

(f_1) $f(s) = 0$ for $s < 0$ and $f(s) = o(1)$ at the origin;

(f_2) there exists $q \in (2, 2^*)$ such that

$$\lim_{s\to\infty} \frac{f(s)}{s^{(q-2)/2}} = 0,$$

where $2^* := 2N/(N-2)$;

(f_3) there exists $\theta > 2$ such that

$$0 < \frac{\theta}{2} F(s) \leq s f(s), \quad \text{for each } s > 0,$$

where $F(s) := \int_0^s f(t)dt$;

(f_4) there exist $\sigma \in (2, 2^*)$ and $C_\sigma > 0$ such that

$$f'(s) \geq C_\sigma s^{\frac{(\sigma-4)}{2}}, \quad \text{for each } s > 0.$$

This class of problem is related with the existence of solitary waves, namely solutions of the form $\psi(x,t) := e^{-i\frac{E}{h}t}u(x)$, with $E \in \mathbb{R}$, for the nonlinear Schrödinger equation

$$ih\frac{\partial\psi}{\partial t} = \left(\frac{h}{i}\nabla - A(z)\right)^2 \psi + U(z)\psi - f(|\psi|^2)\psi, \quad z \in \Omega, \tag{NLS}$$

where $t > 0$, $N \geq 2$ and h is the Planck constant. The Schrödinger operator is defined by

$$\left(\frac{h}{i}\nabla - A(z)\right)^2 \psi := -h^2\Delta\psi - \frac{2h}{i}A \cdot \nabla\psi + |D|^2\psi - \frac{h}{i}\psi \operatorname{div} A.$$

In the 3-dimensional case the magnetic field B is exactly the curl of A, while for higher dimensions $N \geq 4$ it is the 2-form given by $B_{i,j} := \partial_j A_k - \partial_k A_j$.

The function $U(x)$ is a real electric potential and the nonlinear term f is a superlinear function. A direct computation shows that ψ is a solitary wave for (NLS) if, and only if, u is a solution of the following problem

$$\left(\frac{h}{i}\nabla - A(z)\right)^2 u + V(z)u = f(|u|^2)u, \quad \text{in } \Omega, \tag{1.1}$$

where $V(z) = U(z) - E$. It is important to investigate the existence and the shape of such solutions in the semiclassical limit, namely, as $h \to 0^+$. The importance of this study relies on the fact that the transition from Quantum Mechanics to Classical Mechanics can be formally performed by sending the Planck constant to zero.

There is a vast literature concerning the existence and multiplicity of bound state solutions for (1.1) with no magnetic vector potential, namely $A \equiv 0$. Since the seminal paper of Floer and Weinstein [18], many authors have applied different techniques to obtain existence of solutions in this case (see [25, 28, 15, 4, 29] and references therein). Some of these works have dealt with the asymptotic behavior of the solutions as $h \to 0^+$. Roughly speaking, these solutions concentrate around critical points of the potential V. There are also some papers relating the topology of the set of critical points of V with the number of solutions of the problem (see [12, 24] for example).

If we now consider the magnetic case $A \not\equiv 0$, it appears that the first result was obtained by Esteban and Lions [17]. They have used the concentration-compactness principle and minimization arguments to obtain solution for $h > 0$ fixed and dimensions $N = 2$ or $N = 3$. More recently, Kurata [21] proved that the problem has a least energy solution for any $h > 0$ when a technical condition relating V and A is assumed. Under this technical condition, he proved that the associated functional satisfies the Palais-Smale compactness condition at any level. We also would like to cite the papers [13, 14, 10, 27, 11, 3] for other results related with the problem (1.1) in the presence of magnetic field.

We come back now to the case $A \equiv 0$. If we suppose that $V \equiv 1$, a simple calculation shows that u is a solution of (1.1) if, and only if, the function $v(x) := u(hx)$ solves

$$-\Delta v + v = f(|v|^2)v \ \text{ in } \Omega_\lambda, \quad u \in H_0^1(\Omega_\lambda), \tag{1.2}$$

where $\lambda = h^{-1}$. Notice that λ becomes large as h is small. Benci and Cerami proved in [5] that, for homogeneous nonlinearities $f(s) = s^{(q-2)/2}$ with $2 < q < 2^*$, the number of positive is affected by the topology of Ω. More specifically, they proved that (1.2) has at least $\mathrm{cat}_\Omega(\Omega)$ positive solutions whenever $\lambda > \lambda^*$ (these solutions take values in $\mathbb{R}$). Here, $\mathrm{cat}_X(Y)$ denotes the Ljusternik-Schnirelmann category of Y in X, namely the least number of closed and contractible sets in the topological space X which cover the closed set $Y \subset X$. The results found in [5] were extended in several senses: nonhomogeneous or critical nonlinearities, p-Laplacian operators, exterior domains, domains with symmetry, nodal solutions instead of positive ones, systems, etc. We limit ourselves to citing the papers [26, 7, 9, 22, 16, 2, 8, 1, 19, 20] and the references therein.

In view of the results of Benci and Cerami [5], it is natural to ask if the same kind of result holds for the problem with magnetic field. The main goal of this paper is to present a positive answer to this question. So, we relate the number of solution for (P_λ) with topology of the set Ω when the parameter λ is large. We prove that, for largue values of λ, the magnetic field does not play any role on the numbers of solutions of the equation (P_λ) and therefore a result in the same spirit of [5] holds. More specifically, we shall prove the following.

Theorem 1.1 *Suppose that $A \in C^1(\Omega, \mathbb{R}^N)$ is bounded and f satisfies $(f_1)-(f_4)$. Then there exists $\lambda^* > 0$ such that, for each $\lambda > \lambda^*$, the problem (P_λ) has at least $cat_{\Omega_\lambda}(\Omega_\lambda)$ nontrivial weak solutions.*

In the proof we apply variational methods, Ljusternik-Schnirelmann theory and the technique introduced by Benci and Cerami [5]. It consists in making precise comparisons between the category of some sublevel sets of the associated functional and the category of the set Ω. In order to get these comparisons we need to make a carefull study of the behavior of some minimax levels related to the equation in (P_λ) posed in appropriated subsets of $\mathbb{R}^N$. We follow an argument which has already appeared in a previous paper of the first author [1], where the non-magnetic case is handled. Is is worthwhile to mention that, since we deal with different problems with scalar or complex solutions, the calculations are more involved and, in some sense, surprising.

The paper is organized as follows. In the next section we present the variational setting of the problem. In Section 3 we study the various minimax levels associated to the problem. In the final Section 4 we prove our main theorem.

2 Variational framework

From now on we will assume, without loss of generality, that $0 \in \Omega$. Let us fix real numbers $R > r > 0$ such that $B_r(0) \subset \Omega \subset B_R(0)$ and the sets

$$\Omega^+ := \{x \in \mathbb{R}^N : \mathrm{dist}(x, \overline{\Omega}) \le r\}, \quad \Omega^- := \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) \ge r\}$$

are homotopically equivalent to Ω.

For each $\lambda > 0$, we shall denote by E_{A_λ} the Hilbert space obtained by the closure of $C_0^\infty(\Omega_\lambda, \mathbb{C})$ under the scalar product

$$\langle u, v\rangle_{A_\lambda} := \mathrm{Re}\left(\int_{\Omega_\lambda} \nabla_{A_\lambda} u \overline{\nabla_{A_\lambda} v} + u\overline{v}\right),$$

where $\mathrm{Re}(w)$ denotes the real part of $w \in \mathbb{C}$, $\overline{w}$ is its complex conjugated, $\nabla_{A_\lambda} u := (D_1 u, D_2 u, ..., D_N u)$ and $D_j := -i\partial_j - A_j(x/\lambda)$, for $j = 1, \dots, N$. The norm induced by this inner product is given by

$$\|u\|_{A_\lambda} := \left(\int_{\Omega_\lambda} |\nabla_{A_\lambda} u|^2 + |u|^2\right)^{1/2}.$$

As proved by Esteban and Lions in [17, Section II], for any $u \in E_{A_\lambda}$, there holds the *diamagnetic inequality*, namely

$$|\nabla|u|(x)| = \left|\mathrm{Re}\left(\nabla u \frac{\overline{u}}{|u|}\right)\right| = \left|\mathrm{Re}\bigg((\nabla u - iA_\lambda u)\frac{\overline{u}}{|u|}\bigg)\right| \le |\nabla_{A_\lambda} u(x)|. \tag{2.1}$$

Thus, if $u \in E_{A_\lambda}$, then $|u|$ belongs to the usual Sobolev space $H_0^1(\Omega_\lambda, \mathbb{R})$. Moreover, the embedding $E_{A_\lambda} \hookrightarrow L^q(\Omega_\lambda, \mathbb{C})$ is continuous for each $1 \le q \le 2^*$ and it is compact for $1 \le q < 2^*$.

We say that a function $u \in E_{A_\lambda}$ is a weak solution of the problem (P_λ) if

$$\mathrm{Re}\bigg(\int_{\Omega_\lambda} \nabla_{A_\lambda} u \overline{\nabla_{A_\lambda} v} + u\overline{v} - f(|u|^2)u\overline{v}\bigg) = 0, \quad \text{for each } v \in E_{A_\lambda}.$$

In view of $(f_0) - (f_2)$, we have that the functional $I_\lambda : E_{A_\lambda} \to \mathbb{R}$ given by

$$I_\lambda(u) := \frac{1}{2}\int_{\Omega_\lambda} |\nabla_{A_\lambda} u|^2 + \frac{1}{2}\int_{\Omega_\lambda} |u|^2 - \frac{1}{2}\int_{\Omega_\lambda} F(|u|^2) \tag{2.2}$$

is well defined. Moreover, $I_\lambda \in C^1(E_{A_\lambda}, \mathbb{R})$ with the following derivative

$$I'_\lambda(u)v = \mathrm{Re}\bigg(\int_{\Omega_\lambda} \nabla_{A_\lambda} u \overline{\nabla_{A_\lambda} v} + u\overline{v} - f(|u|^2)u\overline{v}\bigg).$$

Thus the weak solutions of (P_λ) are precisely the critical points of I_λ.

Let E be a Banach space and $J \in C^1(E, \mathbb{R})$. We say that $(u_n) \subset E$ is a Palais-Smale sequence ((PS)-sequence for short) if $\sup_{n\in\mathbb{N}} |J(u_n)| < \infty$ and $J'(u_n) \to 0$. We say that J satisfies the Palais-Smale condition if any (PS)-sequence possesses a convergent subsequence.

In view of the subcritical growth of f and condition (f_3), it is standard to check that I_λ satisfies the Palais-Smale condition. Moreover, $(f_1) - (f_3)$ imply that I_λ has the mountain pass geometry. Hence, for each $\lambda > 0$, there exists $u_\lambda \in E_{A_\lambda}$ such that $I_\lambda(u_\lambda) = b_\lambda$ and $I'_\lambda(u_\lambda) = 0$, where b_λ denotes the mountain pass level of the functional I_λ. By using (f_4) and arguing as in [30], we can prove that b_λ can also be characterized as

$$b_\lambda = \inf_{u \in \mathcal{M}_\lambda} I_\lambda(u), \tag{2.3}$$

where $\mathcal{M}_\lambda$ is the Nehari manifold associated to I_λ, namely

$$\mathcal{M}_\lambda := \{u \in E_{A_\lambda} \backslash \{0\} : I'_\lambda(u)u = 0\}. \tag{2.4}$$

From (f_1) and (f_2), there exists $r = r(\lambda) > 0$ such that

$$\|u\|_{A_\lambda} \ge r > 0, \tag{2.5}$$

for all $u \in \mathcal{M}_\lambda$.

Since we are intending to consider the functional I_λ constrained to $\mathcal{M}_\lambda$, we shall need the following result.

Proposition 2.1 *The functional I_λ constrained to $\mathcal{M}_\lambda$ satisfies the Palais-Smale condition.*

Proof. Let (u_n) be a (PS)-sequence for I_λ constrained to $\mathcal{M}_\lambda$. Then $I_\lambda(u_n) \to d$ and

$$I'_\lambda(v_n) = \mu_n G'_\lambda(v_n) + o_n(1), \tag{2.6}$$

for some $(\mu_n) \subset \mathbb{R}$, where $G_\lambda : E_{A_\lambda} \to \mathbb{R}$ is given by

$$G_\lambda(u) := \int_{\Omega_\lambda} |\nabla_{A_\lambda} u|^2 + \int_{\Omega_\lambda} |u|^2 - \int_{\Omega_\lambda} f(|u|^2)|u|^2,$$

and $o_n(1)$ denotes a quantity approaching zero as $n \to \infty$. Since $u_n \in \mathcal{M}_\lambda$ the condition (f_4) provides

$$G'_\lambda(u_n)u_n = -2\int_{\Omega_\lambda} f'(|u_n|^2)|u_n|^4 \leq -2C_\sigma \int_{\Omega_\lambda} |u_n|^\sigma. \tag{2.7}$$

Standard arguments show that (u_n) is bounded. Thus, up to a subsequence, $G'_\lambda(u_n)u_n \to l \leq 0$. If $l \neq 0$, we infer from (2.6) that $\mu_n = o_n(1)$. In this case, we can use (2.6) again to conclude that (u_n) is a $(\mathrm{PS})_d$ sequence for I_λ in E_{A_λ} and therefore (u_n) has a strongly convergent subsequence.

If $l = 0$, it follows from (2.7) that $u_n \to 0$ in $L^\sigma(\Omega_\lambda, \mathbb{C})$. The boundedness of (u_n) in E_{A_λ} and the interpolation inequality provides $u_n \to 0$ in $L^s(\Omega_\lambda, \mathbb{C})$ for any $2 \leq s < 2^*$. On the other hand, by (f_1) and (f_2),

$$\|u_n\|^2_{A_\lambda} = \int_{\Omega_\lambda} f(|u_n|^2)|u_n|^2 \leq c_1 \int_{\Omega_\lambda} |u_n|^2 + c_2 \int_{\Omega_\lambda} |u_n|^q,$$

which this contradicts (2.5), since the right-hand side above goes to zero as $n \to \infty$. The proposition is proved. ■

As a byproduct of the above arguments we obtain the following result.

Corollary 2.2 *If u is a critical point of I_λ constrained to $\mathcal{M}_\lambda$, then u is a nontrivial critical point of I_λ on E_{A_λ}.*

We introduce now some kind of limiting functional associated to I_λ. This limiting functional turns out to be defined in the space $H^1(\mathbb{R}^N, \mathbb{R})$. More specifically, we define $J_\infty(v) : H^1(\mathbb{R}^N, \mathbb{R}) \to \mathbb{R}$ by setting

$$J_\infty(v) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 + \frac{1}{2}\int_{\mathbb{R}^N} |v|^2 - \frac{1}{2}\int_{\mathbb{R}^N} F(|v|^2),$$

with Nehari manifold and mountains pass level given by

$$\mathcal{N}_\infty := \{v \in H^1(\mathbb{R}^N, \mathbb{R}) \setminus \{0\} : J'_\infty(v)v = 0\}, \quad c_\infty := \inf_{v \in \mathcal{N}_\infty} J_\infty(v).$$

The following compactness property will be crucial in our arguments. Its proof can be done arguing along the same lines of the proof found in [1, Theorem 3.1]. We omit the details.

Proposition 2.3 *Suppose that $(v_n) \subset \mathcal{N}_\infty$ is such that $J_\infty(v_n) \to c_\infty$ and $v_n \rightharpoonup v$ weakly in $H^1(\mathbb{R}^N, \mathbb{R})$. Then we have either*

(i) *$v \neq 0$ and $v_n \to v$ strongly in $H^1(\mathbb{R}^N, \mathbb{R})$ with $v > 0$ almost everywhere in $\mathbb{R}^N$, $J_\infty(v) = c_\infty$ and $J'_\infty(v) = 0$;*

or

(ii) *there exists $(y_n) \subset \mathbb{R}^N$ with $|y_n| \to \infty$ such that the sequence $\widetilde{v}_n := v_n(\cdot + y_n)$ weakly converges to $\widetilde{v} \neq 0$ in $H^1(\mathbb{R}^N, \mathbb{R})$. Moreover, the function $\widetilde{v}$ has the same properties of the function v of the item (i) above.*

3 The behavior of the minimax levels

In the proof of Theorem 1.1, we need to consider the asymptotic behavior of b_λ and other related minimax levels. In what follows we introduce these related minimax. We start by considering the functional $J_\lambda : H_0^1(\Omega_\lambda, \mathbb{R}) \to \mathbb{R}$ given by

$$J_\lambda(v) := \frac{1}{2}\int_{\Omega_\lambda} |\nabla v|^2 + \frac{1}{2}\int_{\Omega_\lambda} |v|^2 - \frac{1}{2}\int_{\Omega_\lambda} F(|v|^2). \tag{3.1}$$

We also define

$$c_\lambda := \inf_{v \in \mathcal{N}_\lambda} J_\lambda(v), \tag{3.2}$$

where $\mathcal{N}_\lambda$ is the Nehari manifold associated to J_λ, that is

$$\mathcal{N}_\lambda := \{v \in H_0^1(\Omega_\lambda, \mathbb{R})\backslash\{0\} : J'_\lambda(v)v = 0\}. \tag{3.3}$$

We recall that $B_{\lambda r}(0) \subset \Omega_\lambda$ and define the triples $(I_{\lambda,r}, b_{\lambda,r}, \mathcal{M}_{\lambda,r})$ and $(J_{\lambda,r}, c_{\lambda,r}, \mathcal{N}_{\lambda,r})$ in a similar way, just replacing Ω_λ by $B_{\lambda r}(0)$ in (2.2)-(2.4) and (3.1)-(3.3), respectively.

In our first result, we present the asymptotic behavior of the minimax $b_{\lambda,r}$ and $c_{\lambda,r}$ as $\lambda \to \infty$.

Lemma 3.1 *We have that*

$$\lim_{\lambda\to\infty} c_{\lambda,r} = c_\infty, \quad \lim_{\lambda\to\infty} b_{\lambda,r} = c_\infty.$$

Proof. The first equality is proved in [1, Proposition 4.2]. In order to check the second one we notice that, by the diamagnetic inequality, $c_{\lambda,r} \leq b_{\lambda,r}$. Thus

$$c_\infty = \liminf_{\lambda\to\infty} c_{\lambda,r} \leq \liminf_{\lambda\to\infty} b_{\lambda,r}. \tag{3.4}$$

Let $(\lambda_n) \subset \mathbb{R}$ be such that $\lambda_n \nearrow \infty$. Since $c_{\lambda_n,r}$ is achieved there exists $v_n \in \mathcal{N}_{\lambda_n,r}$ such that $J_{\lambda_n,r}(v_n) = c_{\lambda_n,r}$ and $J'_{\lambda_n,r}(v_n) = 0$. By using the Schwartz symmetrization process and well known arguments (see [1, Proposition

4.4]) we can prove that the function v_n can be taken radial. If we set $v_n(x) = 0$ for a.e. $x \in \mathbb{R}^N \setminus B_{\lambda_n r}(0)$ and recall that $c_{\lambda_n,r} \to c_\infty$, we obtain

$$\lim_{n\to\infty} J_\infty(v_n) = c_\infty, \quad J'_\infty(v_n)v_n = 0 \quad \text{and} \quad J_\infty(tv_n) \le J_\infty(v_n) = c_{\lambda_n,r}, \tag{3.5}$$

for any $t \ge 0$. A standard calculation shows that (v_n) is bounded and therefore $v_n \rightharpoonup v$ weakly in $H^1(\mathbb{R}^N, \mathbb{R})$.

Claim 1.: the weak limit v is nonzero.

In order to prove the claim we first verify that, for some sequence $(y_n) \subset \mathbb{R}^N$ and constants $L,\ \mu > 0$, there holds

$$\liminf_{n\to\infty} \int_{B_L(y_n)} |v_n|^2 \ge \mu > 0. \tag{3.6}$$

Indeed, if this is not true, if follows from [23, Lemma I.1] that $v_n \to 0$ in $L^s(\mathbb{R}^N, \mathbb{R})$ for any $2 < s < 2^*$. Given $\delta > 0$, we can use (f_1) and (f_2) to get

$$0 \le \left| \int f(|v_n|^2) v_n^2 \right| \le \delta \int |v_n|^2 + C_\delta \int |u_n|^q,$$

for some constant $C_\delta > 0$. Since (v_n) is bounded in $L^2(\mathbb{R}^N, \mathbb{R})$, $v_n \to 0$ in $L^q(\mathbb{R}^N, \mathbb{R})$ and $\delta > 0$ is arbitrary, we conclude that

$$\lim_{n\to\infty} \|v_n\|^2_{H^1(\mathbb{R}^N,\mathbb{R})} = \lim_{n\to\infty} \int f(|v_n|^2)|v_n|^2 = 0,$$

which contradicts $J_\infty(v_n) \to c_\infty > 0$. Thus (3.6) holds. Since each v_n is a radial function we conclude that the sequence (y_n) is bounded in $\mathbb{R}^N$. Hence, the inequality (3.6) combined with the strong convergence of (v_n) in $L^2_{loc}(\mathbb{R}^N, \mathbb{R})$ gives $v \ne 0$.

In view of (3.5) and Claim 1, we can use Proposition 2.3 to conclude that $v_n \to v$ strongly in $H^1(\mathbb{R}^N, \mathbb{R})$. Moreover, this convergence also holds almost everywhere in $\mathbb{R}^N$ and strongly in $L^2(\mathbb{R}^N, \mathbb{R})$. Hence, there exists $\psi_2 \in L^2(\mathbb{R}^N, \mathbb{R})$ such that $|v_n(x)| \le \psi_2(x)$ for a.e. point in $\mathbb{R}^N$.

Let $t_n > 0$ be such that

$$u_n := t_n \mathrm{e}^{i\tau(x)} v_n \in \mathcal{M}_{\lambda_n,r},$$

where $\tau(x) := \sum_{j=1}^N A_j(0)x^j$. Since $D_j^A = i^{-1}\partial_j i - A_j(x/\lambda)$, a straightforward calculation provides

$$D_j^A(\mathrm{e}^{i\tau(x)} v_n) = \left(\left(A_j(0) - A_j\left(\frac{x}{\lambda_n}\right) \right) v_n + i\partial_j v_n \right) \mathrm{e}^{i\tau(x)},$$

from where it follows that

$$\|\mathrm{e}^{i\tau(x)} v_n\|^2_{A_{\lambda_n}} = \|v_n\|^2_{H^1(\mathbb{R}^N,\mathbb{R})} + \int (A(0) - A(x/\lambda_n))|v_n|^2. \tag{3.7}$$

Thus, recalling that $|\mathrm{e}^{i\tau(x)}| = 1$ and $u_n \in \mathcal{M}_{\lambda_n,r}$, we obtain

$$
\begin{aligned}
b_{\lambda_n,r} &\leq I_{\lambda_n,r}(t_n \mathrm{e}^{i\tau(x)} v_n) \\
&= \frac{t_n^2}{2}\|\mathrm{e}^{i\tau(x)} v_n\|^2_{A_{\lambda_n}} - \frac{1}{2}\int_{B_{\lambda_n r}(0)} F(t_n^2|v_n|^2) \\
&= J_{\lambda_n,r}(t_n v_n) + \frac{t_n^2}{2}\int_{B_{\lambda_n r}(0)} |A(0) - A(x/\lambda_n)||v_n|^2 \\
&\leq c_{\lambda_n,r} + \frac{t_n^2}{2}\int_{B_{\lambda_n r}(0)} |A(0) - A(x/\lambda_n)||v_n|^2.
\end{aligned}
\tag{3.8}
$$

Notice that, for almost everywhere $x \in \Omega_{\lambda_n}$, we have that

$$
|A(0) - A(x/\lambda_n)||v_n|^2 \leq 2|\psi_2(x)|^2 \sup_{x\in\Omega} |A(x)|,
$$

and therefore we can use the Lebesgue's Theorem to conclude that

$$
\lim_{n\to\infty} \int_{B_{\lambda_n r}(0)} |A(0) - A(x/\lambda_n)||v_n|^2 = 0. \tag{3.9}
$$

In view of the above equation, it suffices to prove that (t_n) is bounded. If this is true, we can use the above equation and (3.8) to get

$$
\limsup_{n\to\infty} b_{\lambda_n,r} \leq \limsup_{n\to\infty} c_{\lambda_n,r} = c_\infty.
$$

This and (3.4) complete the proof.

It remains to check that $(t_n) \subset \mathbb{R}$ is bounded. Arguing by contradiction, we suppose that some subsequence of (t_n), still denoted by (t_n), goes to infinity as $n \to \infty$. Recalling that

$$
\|t_n \mathrm{e}^{i\tau(x)} v_n\|^2_{A_{\lambda_n}} = \int_{B_{\lambda_n r}(0)} f(t_n|v_n|^2) t_n^2 |v_n|^2
$$

and using (3.7), we obtain

$$
\begin{aligned}
\int_{B_1(0)} f(t_n^2|v_n|^2)|v_n|^2 &\leq \int_{B_{\lambda_n r}(0)} f(t_n^2|v_n|^2)|v_n|^2 \\
&= \|\mathrm{e}^{i\tau(x)} v_n\|^2_{A_{\lambda_n}} \\
&= \|v_n\|^2_{H^1(\mathbb{R}^N,\mathbb{R})} + \int_{B_{\lambda_n r}(0)} (A(0) - A(x/\lambda_n))|v_n|^2.
\end{aligned}
$$

The boundedness of (v_n) in $H^1(\mathbb{R}^N,\mathbb{R})$ and (3.9) imply that the right-hand side above is also bounded. On the other hand, the condition (f_3) implies that $\lim_{s\to+\infty} f(s^2) = +\infty$. Since $v_n \to v > 0$ strongly in $H^1(\mathbb{R}^N,\mathbb{R})$, we obtain a contradiction taking $n \to \infty$ in the above expression and using Fatou's Lemma. This finishes the proof. ■

For each $x \in \mathbb{R}^N$, let us denote by $\Sigma_{\lambda,x}$ the following set

$$\Sigma_{\lambda,x} := B_{\lambda R}(x) \setminus \overline{B_{\lambda r}(x)}$$

and define the functional $\widehat{J}_{\lambda,x} : H_0^1(\Sigma_{\lambda,x}, \mathbb{R}) \to \mathbb{R}$ by

$$\widehat{J}_{\lambda,x}(v) := \frac{1}{2}\int_{\Sigma_{\lambda,x}} |\nabla v|^2 + \frac{1}{2}\int_{\Sigma_{\lambda,x}} |v|^2 - \int_{\Sigma_{\lambda,x}} F(|v|^2). \tag{3.10}$$

as well as its Nehari manifold

$$\widehat{\mathcal{N}}_{\lambda,x} := \{v \in H_0^1(\Sigma_{\lambda,x}, \mathbb{R})\backslash\{0\} : \widehat{J}'_{\lambda,x}(v)v = 0\}.$$

For $v \in H^1(\mathbb{R}^N, \mathbb{C})$ with compact support, we consider the barycenter map

$$\beta(v) := \frac{\displaystyle\int_{\mathbb{R}^N} x|v|^2}{\displaystyle\int_{\mathbb{R}^N} |v|^2}.$$

and introduce the following quantity

$$a_{\lambda,x} := \inf\left\{\widehat{J}_{\lambda,x}(v) : v \in \widehat{\mathcal{N}}_{\lambda,x} \text{ and } \beta(v) = x\right\}.$$

We present below an important property of the asymptotic behavior of the numbers $a_{\lambda,0}$.

Lemma 3.2 *The following holds*

$$c_\infty < \liminf_{\lambda\to\infty} a_{\lambda,0}.$$

Proof. Since $c_\infty \le a_{\lambda,0}$ for any $\lambda > 0$, we have that $c_\infty \le \liminf_{\lambda\to\infty} a_{\lambda,0}$. Suppose, by contradiction, that for some sequence $\lambda_n \nearrow \infty$ we have that $a_{\lambda_n,0} \to c_\infty$. Then, since the infimum $a_{\lambda_n,0}$ is achieved, we can obtain $v_n \in \widehat{\mathcal{N}}_{\lambda_n,0} \subset \mathcal{N}_\infty$ satisfying $\widehat{J}_{\lambda_n,0}(v_n) = J_\infty(v_n) \to c_\infty$ and $\beta(v_n) = 0$, where we are understanding that the function v_n is extended to the whole space by setting $v_n(x) := 0$ for a.e. $x \in \mathbb{R}^N \setminus \Sigma_{\lambda_n,0}$.

Since the support of v_n is contained in $B_{\lambda_n R}(0) \setminus B_{\lambda_n r}(0)$ we have that $v_n \rightharpoonup 0$ weakly in $H^1(\mathbb{R}^N, \mathbb{R})$. Recalling that $c_\infty > 0$, we have that (v_n) is not strongly convergent. Thus, it follows from item (ii) of Proposition 2.3 that

$$v_n(x) = w_n(x - y_n) + \widetilde{v}(x - y_n) \tag{3.11}$$

with $(w_n) \subset H^1(\mathbb{R}^N, \mathbb{R})$ satisfying $w_n \to 0$ strongly in $H^1(\mathbb{R}^N, \mathbb{R})$, $(y_n) \subset \mathbb{R}^N$ being such that $|y_n| \to \infty$, and $\widetilde{v} \in H^1(\mathbb{R}^N, \mathbb{R})$ verifying

$$J_\infty(\widetilde{v}) = c_\infty, \quad J'_\infty(\widetilde{v}) = 0. \tag{3.12}$$

The functional J_∞ is rotationally invariant. Thus we may assume that $y_n = (y_n^1, 0, \dots, 0)$ and its first coordinates satisfies $y_n^1 < 0$.

Recalling that $\operatorname{supp} v_n \subset \Sigma_{\lambda_n,0}$, we can use (3.11), $w_n \to 0$ and the Lebesgue's Theorem to get

$$\begin{aligned} \lim_{n\to\infty} \int_{\Sigma_{\lambda_n,0}\cap B_{\lambda_n r/2}(y_n)} |v_n|^p &= \lim_{n\to\infty} \int_{B_{\lambda_n r/2}(y_n)} |v_n|^p \\ &= \lim_{n\to\infty} \int_{B_{\lambda_n r/2}(0)} |w_n - \widetilde{v}|^p \qquad (3.13) \\ &= \int_{\mathbb{R}^N} |\widetilde{v}|^p = M > 0. \end{aligned}$$

Moreover, the invariance of the Lebesgue measure, (3.11) and the Lebesgue's Theorem again provide

$$\lim_{n\to\infty} \int_{\Sigma_{\lambda_n,0}} |v_n|^p = \lim_{n\to\infty} \int_{\mathbb{R}^N} |v_n|^p = \lim_{n\to\infty} \int_{\mathbb{R}^N} |w_n - \widetilde{v}|^p = M.$$

This and (3.13) imply that

$$\lim_{n\to\infty} \int_{\Sigma_{\lambda_n,0}\setminus B_{\lambda_n r/2}(y_n)} |v_n|^p = 0. \qquad (3.14)$$

We now claim that

$$\int_{\Sigma_{\lambda_n,0}\cap B_{\lambda_n r/2}(y_n)} x^1 |v_n|^p \mathrm{d}x \le -\frac{\lambda_n r}{2}(M + o_n(1)).$$

Assuming the claim, we obtain

$$\begin{aligned} 0 &= \beta(v_n) = \int_{\Sigma_{\lambda_n,0}\cap B_{\lambda_n r/2}(y_n)} x^1 |v_n|^p + \int_{\Sigma_{\lambda_n,0}\setminus B_{\lambda_n r/2}(y_n)} x^1 |v_n|^p \\ &\le -\frac{\lambda_n r}{2}(M + o_n(1)) + \lambda_n R \int_{\Sigma_{\lambda_n,0}\setminus B_{\lambda_n r/2}(y_n)} |v_n|^p, \end{aligned}$$

and therefore

$$\int_{\Sigma_{\lambda_n,0}\setminus B_{\lambda_n r/2}(y_n)} |v_n|^p \ge \frac{r}{2R}(M + o_n(1)),$$

which contradicts (3.14).

It remains to prove the claim. Given $x = (x^1, \dots, x^N) \in \Sigma_{\lambda_n,0} \cap B_{\lambda_n r/2}(y_n)$ it suffices to check that $x^1 < -\lambda_n r/2$. Since $|x - y_n| \le \lambda_n r/2$ and $y_n = (y_n^1, 0, \dots, 0)$ we have that

$$|x^1 - y_n^1| \le \frac{\lambda_n r}{2}, \qquad \sum_{j=2}^N |x^j|^2 \le \left(\frac{\lambda_n r}{2}\right)^2. \qquad (3.15)$$

On the other hand,

$$|x^1|^2 + \sum_{j=2}^{N} |x^j|^2 = |x|^2 \geq (\lambda_n r)^2$$

and therefore it follows from the second inequality in (3.15) that $|x^1| > \lambda_n r/2$. This, the first inequality in (3.15) and $y_n^1 < 0$ imply that $x^1 < -\lambda_n r/2$, as claimed. This finishes the proof. ■

4 Proof of Theorem 1.1

Let us denote by $u_{\lambda,r} \in \mathcal{N}_{\lambda,r}$ a positive and radial function satisfying $J_{\lambda,r}(u_{\lambda,r}) = c_{\lambda,r}$. We define the map $\Psi_\lambda : \Omega_\lambda^- \to \mathcal{M}_\lambda$ as

$$(\Psi_\lambda(y))(x) := \begin{cases} t_{\lambda,y} \mathrm{e}^{i\tau(x)} u_{\lambda,r}(|x-y|), & \text{if } x \in B_{\lambda r}(y), \\ 0, & \text{otherwise}, \end{cases}$$

where $t_{\lambda,y} \in (0,+\infty)$ is such that $t_{\lambda,y}\mathrm{e}^{i\tau(x)} u_{\lambda,r}(\cdot - y) \in \mathcal{M}_{\lambda,r} \subset \mathcal{M}_\lambda$. Since $u_{\lambda,r}$ is radial, it follows that

$$\beta(\Psi_\lambda(y)) = y \quad \text{for any} \quad y \in \Omega_\lambda^-.$$

Moreover, the function Ψ_λ has the following property.

Lemma 4.1 *Uniformly for $y \in \Omega_\lambda^-$, there holds*

$$\lim_{\lambda \to +\infty} I_\lambda(\Psi_\lambda(y)) = c_\infty.$$

Proof. Given a sequence $(\lambda_n) \subset \mathbb{R}$ such that $\lambda_n \to \infty$ and $(y_n) \subset \Omega_{\lambda_n}^-$, we shall prove that $I_{\lambda_n}(\Psi_{\lambda_n}(y_n)) \to c_\infty$.

Let $v_n \in \mathcal{N}_\infty$ be defined as $v_n(x) := u_{\lambda_n,r}(x - y_n)$ if $x \in B_{\lambda_n r}(y_n)$, $v_n(x) := 0$ otherwise. Since $u_{\lambda_n,r}$ is such that $J_{\lambda_n,r}(u_{\lambda_n,r}) = c_{\lambda_n,r}$, we can use the invariance of the Lebesgue measure and Lemma 3.1 to conclude that

$$\lim_{n\to\infty} J_\infty(v_n) = c_\infty, \quad J'_\infty(v_n)v_n = 0 \quad \text{and} \quad J_\infty(tv_n) \leq J_\infty(v_n) = c_{\lambda_n,r}, \qquad (4.1)$$

for any $t \geq 0$. Arguing as in the proof of Lemma 3.1 we conclude that, for some function $v \neq 0$, there holds $v_n \rightharpoonup v$ weakly in $H^1(\mathbb{R}^N, \mathbb{R})$. The first equality in (4.1) and Proposition 2.3 imply that $v_n \to v$ strongly in $H^1(\mathbb{R}^N, \mathbb{R})$. Moreover, this convergence also holds almost everywhere in $\mathbb{R}^N$ and strongly in $L^2(\mathbb{R}^N, \mathbb{R})$.

Let $t_n := t_{\lambda_n,y_n}$ be as in the definition of Ψ_λ. Repeating the arguments of Lemma 3.1 we get

$$\begin{aligned} b_{\lambda_n,r} &\leq I_{\lambda_n,r}(\Psi_{\lambda_n}(y_n)) \\ &\leq c_{\lambda_n,r} + \frac{t_n^2}{2} \int_{B_{\lambda_n r}(0)} |A(0) - A(x/\lambda_n)||v_n|^2 \\ &= c_{\lambda_n,r} + o_n(1). \end{aligned}$$

Since $I_{\lambda_n}(\Psi_{\lambda_n}(y_n)) = I_{\lambda_n,r}(\Psi_{\lambda_n}(y_n))$, it suffices to take the limit in the above expression and recall that $\lim_{n\to\infty} b_{\lambda_n,r} = \lim_{n\to\infty} c_{\lambda_n,r} = c_\infty$. The lemma is proved. ■

Given $y \in \Omega_\lambda^-$, we have that $\Psi_\lambda(y) \in \mathcal{M}_\lambda$. Moreover, if we set

$$g(\lambda) := |I_\lambda(\Psi_\lambda(y)) - c_\infty|, \tag{4.2}$$

we have that $I_\lambda(\Psi_\lambda(y)) - c_\infty \le g(\lambda)$. Hence, the set

$$\mathcal{O}_\lambda := \{u \in \mathcal{M}_\lambda : I_\lambda(u) \le c_\infty + g(\lambda)\}$$

contains the function $\Psi_\lambda(y)$, from which it follows that $\mathcal{O}_\lambda \neq \varnothing$.

Before presenting our next result we notice that, for any given $u \in \mathcal{M}_\lambda$, there exists $t_u > 0$ such that $t_u|u| \in \mathcal{N}_\lambda$. By the diamagnetic inequality,

$$\| |u| \|^2_{H^1(\mathbb{R}^N,\mathbb{R})} \le \|u\|^2_{A_\lambda} = \int_{\Omega_\lambda} f(|u|^2)|u|^2.$$

Let us define, for $t > 0$, the function $h_u(t) := J_\lambda(tu)$, for $t > 0$. Since $t_u u \in \mathcal{N}_\lambda$, we have that $h_u'(t_u) = 0$. The above inequality implies that $h_u'(1) \le 0$ and therefore it follows from (f_4) that $t_u \in [0,1]$.

The following result is the key point in the comparison of the category of Ω with that of the sublevel sets of the functional I_λ.

Proposition 4.2 *There exists $\widehat{\lambda} > 0$ such that $\beta(u) \in \Omega_\lambda^+$, whenever $u \in \mathcal{O}_\lambda$ and $\lambda \ge \widehat{\lambda}$.*

Proof. Suppose, by contradiction, that the result is false. Then there exist $\lambda_n \nearrow \infty$ and $u_n \in \mathcal{M}_{\lambda_n}$ such that $I_{\lambda_n}(u_n) \le c_\infty + g(\lambda_n)$, but $x_n := \beta(u_n) \notin \Omega^+_{\lambda_n}$. Let $t_n \in [0,1]$ such that $v_n := t_n|u_n| \in \mathcal{N}_{\lambda_n}$. It follows from diamagnetic inequality and $u_n \in \mathcal{O}_{\lambda_n}$ that

$$J_{\lambda_n}(t_n|u_n|) \le I_{\lambda_n}(t_n u_n) \le I_{\lambda_n}(u_n) \le c_\infty + g(\lambda_n).$$

Hence, the sequence (v_n) has the following properties

$$v_n \in \mathcal{N}_{\lambda_n}, \quad \beta(v_n) = x_n \notin \Omega^+_{\lambda_n}, \quad J_{\lambda_n}(v_n) \le c_\infty + g(\lambda_n).$$

Claim. $\Omega_{\lambda_n} \subset \Sigma_{\lambda_n,x_n}$

Assuming the claim we have that $v_n \in H_0^1(\Sigma_{\lambda_n,x_n},\mathbb{R})$ and we can prove the proposition as follows. Since $\beta(v_n) = x_n$ we have that

$$a_{\lambda_n,0} = a_{\lambda_n,x_n} \le \widehat{J}_{\lambda_n,x_n}(v_n) = J_{\lambda_n}(v_n) \le c_\infty + g(\lambda_n).$$

The definition of g (see (4.2)) and Lemma 4.1 imply that $\lim_{\lambda\to\infty} g(\lambda) = 0$. Thus, we can use the above expression to get

$$\limsup_{n\to\infty} a_{\lambda_n,0} \le \limsup_{n\to\infty} (c_\infty + g(\lambda_n)) = c_\infty,$$

which contradicts Lemma 3.2.

It remains to prove the claim. So, we fix $x \in \Omega_{\lambda_n}$ and recall that $x_n \notin \Omega^+_{\lambda_n}$. Thus, $\lambda_n^{-1}x \in \Omega$ and $\lambda_n^{-1}x_n \notin \Omega^+$. It follows from the definition of Ω^+ that $|\lambda_n^{-1}x - \lambda_n^{-1}x_n| > r$, or equivalently,

$$|x - x_n| > \lambda_n r. \tag{4.3}$$

On the other hand, since $x = \lambda_n y$ for some $y \in \Omega$, we have that

$$\begin{aligned}
|x - x_n| &= \left| x - \frac{\int_{\Omega_{\lambda_n}} z|v_n|^2 \mathrm{d}z}{\int_{\Omega_{\lambda_n}} |v_n|^2} \right| \\
&= \left| \frac{\int_{\Omega_{\lambda_n}} (\lambda_n y - z)|v_n|^2 \mathrm{d}z}{\int_{\Omega_{\lambda_n}} |v_n|^2} \right| = \lambda_n \left| \frac{\int_{\Omega_{\lambda_n}} \left(y - \frac{z}{\lambda_n}\right) |v_n|^2 \mathrm{d}z}{\int_{\Omega_{\lambda_n}} |v_n|^2} \right|.
\end{aligned}$$

But $y \in \Omega$ and $\lambda_n^{-1}z \in \Omega$ for any $z \in \Omega_{\lambda_n}$. Thus, the above expression implies that

$$|x - x_n| \le \lambda_n \operatorname{diam}(\Omega) < \lambda_n R.$$

This and (4.3) provides $x \in B_{\lambda_n R}(x_n) \setminus B_{\lambda_n r}(x_n) = \Sigma_{\lambda_n, x_n}$ and the proposition is proved. ■

Proposition 4.3 *If $\widehat{\lambda} > 0$ is given by Proposition 4.2 then, for each $\lambda \ge \widehat{\lambda}$, there holds*

$$\operatorname{cat}_{\mathcal{O}_\lambda} \mathcal{O}_\lambda \ge \operatorname{cat}_{\Omega_\lambda}(\Omega_\lambda).$$

Proof. Suppose that

$$\mathcal{O}_\lambda = \Upsilon_1 \cup ... \cup \Upsilon_n,$$

where Υ_j, $j = 1, \dots, n$, is closed and contractible in $\mathcal{O}_\lambda$. This means that there exists $h_j \in C([0,1] \times \Upsilon_j, \mathcal{O}_\lambda)$ such that

$$h_j(0, u) = u, \;\; h_j(1, u) = u_j, \;\; \text{for each } u \in \Upsilon_j,$$

and some $u_j \in \Upsilon_j$ fixed. Consider the sets $B_j := \Psi_\lambda^{-1}(\Upsilon_j)$, $j = 1, \dots, n$, which are closed in Ω^-_λ and satisfy

$$\Omega^-_\lambda = B_1 \cup \cdots \cup B_n.$$

By using Proposition 4.2 we conclude that the maps $g_j : [0,1] \times B_j \to \Omega^+_\lambda$ given by

$$g_j(t, y) = \beta(h_j(t, \Psi_r(y)))$$

are well defined. A standard calculation show that these maps are contractions of the sets B_j in Ω^+_λ. Hence that

$$\operatorname{cat}_{\Omega_\lambda}(\Omega_\lambda) = \operatorname{cat}_{\Omega^+_\lambda}(\Omega^-_\lambda) \le n,$$

and the proposition is proved. ■

We are now ready to prove our main result.

Proof of Theorem 1.1. Let $\widehat{\lambda} > 0$ be given by Proposition 4.2 and suppose that $\lambda \geq \widehat{\lambda}$. By using condition (f_3) and arguing as in the proof of Proposition 2.1, we can check that I_λ satisfies the Palais-Smale condition on $\mathcal{O}_\lambda$. Thus, we can apply standard Ljusternik-Scrnirelmann theory and Proposition 4.3 to obtain $\mathrm{cat}_{\mathcal{O}_\lambda}\mathcal{O}_\lambda \geq \mathrm{cat}_{\Omega_\lambda}(\Omega_\lambda)$ critical points of I_λ restricted to $\mathcal{O}_\lambda$. As in Corollary 2.2, each of these critical points is a critical point of the unconstrained functional I_λ, and therefore a nonzero weak solution of the problem (P_λ). ■

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