Positive solutions for nonlinear elliptic equations with fast increasing weights

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We find positive rapidly decaying solutions for the equation
$$-\operatorname{div}(K(x)\nabla u) = K(x)u^{2^*-1} + \lambda K(x)|x|^{\alpha-2}u$$
in $\mathbb{R}^N$, where $N \geqslant 3$, the nonlinearity is given by the critical Sobolev exponent $2^* = 2N/(N-2)$, the weight is $K(x) = \exp(\frac{1}{4}|x|^\alpha)$, $\alpha \geqslant 2$ and λ is a parameter.

1. Introduction

We study the equation

$$-\operatorname{div}(K(x)\nabla u) = K(x)u^{2^*-1} + \lambda K(x)|x|^{\alpha-2}u, \quad u > 0 \in \mathbb{R}^N, \qquad (1.1)$$

where $N \geqslant 3$, the nonlinearity is given by the critical Sobolev exponent $2^* = 2N/(N-2)$, the weight is $K(x) = \exp(\frac{1}{4}|x|^\alpha)$, $\alpha \geqslant 2$ and λ is a parameter. According to the function space in which we seek solutions, u is forced to decrease sufficiently fast to infinity.

As in [12], for $\alpha = 2$ and $\lambda = (N-2)/(N+2)$, equation (1.1) occurs when one tries to find self-similar solutions

$$v(x,t) = t^{(2-N)/(N+2)}u(xt^{-1/2})$$

to the parabolic equation

$$v_t - \Delta v = |v|^{4/(N-2)}v, \quad \mathbb{R}^N \times (0, +\infty).$$

Notice that (1.1) is equivalent to

$$-\Delta u - \tfrac{1}{4}\alpha|x|^{\alpha-2}(x \cdot \nabla u) = |u|^{2^*-2}u + \lambda|x|^{\alpha-2}u. \qquad (1.2)$$

Equation (1.1) was treated in [11] in the case $\alpha = 2$. The authors noticed a dichotomy in the existence range of λ for $N = 3$, relative to space dimensions $N \geqslant 4$. More precisely, for $N \geqslant 4$, there is a solution if and only if $\lambda \in (\frac{1}{4}N, \frac{1}{2}N)$. If $N = 3$, there is a solution for $\lambda \in (1, \frac{3}{2})$, there is no solution for $\lambda \leqslant \frac{3}{4}$ and $\lambda \geqslant \frac{3}{2}$. Complementing this result, it was proved in [1] that no radial solution exists for $\lambda \leqslant 1$. Here we present the situations where an analogous dichotomy happens, depending on the parameters $\alpha \geqslant 2$ and λ, thus extending the existence results of [11] for (1.1) and every $\alpha \geqslant 2$. We also show in detail a non-existence result similar to the one in [1], relative to radial solutions of (1.1). We address questions about symmetry breaking and multiplicity of solutions, where the least energy solutions of (1.1) are not radial, and there are at least two positive solutions.

Our main results are stated below in terms of the first eigenvalue,

$$\lambda_1 = \tfrac{1}{4}\alpha(N - 2 + \alpha),$$

of the problem

$$-\operatorname{div}(K(x)\nabla u) = \lambda K(x)|x|^{\alpha - 2}u \quad \text{in } \mathbb{R}^N. \tag{1.3}$$

We obtain solutions of (1.1) by minimizing the expression

$$Q_\lambda(u) = \frac{\int_{\mathbb{R}^N} K(x)|\nabla u|^2 - \lambda \int_{\mathbb{R}^N} K(x)|x|^{\alpha - 2}u^2}{(\int_{\mathbb{R}^N} K(x)|u|^{2^*})^{2/2^*}} \tag{1.4}$$

over the space $H(\alpha)$ defined as the completion of the smooth functions with compact support $C_c^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\| = \left(\int_{\mathbb{R}}^N K(x)|\nabla u|^2\right)^{1/2}.$$

The infima of Q_λ in $H(\alpha)$ are called ground states.

We shall prove the following results.

THEOREM 1.1 (strong weights). *If $2 < \alpha \leqslant N - 2$, problem (1.1) has a solution if and only if $\lambda \in (\frac{1}{2}\lambda_1, \lambda_1)$. Moreover, in this situation the ground state is achieved.*

THEOREM 1.2 (very strong weights). *If $N - 2 < \alpha$ and $\lambda \in (\frac{1}{4}\alpha^2, \lambda_1)$, problem (1.1) has a ground-state solution. Moreover, if $\lambda \leqslant \frac{1}{2}\lambda_1$ or $\lambda \geqslant \lambda_1$, then (1.1) has no solution.*

The next results complement the ones above.

THEOREM 1.3 (no radial solution). *If $N - 2 < \alpha$ and $\lambda \leqslant \frac{1}{4}\alpha^2$, the problem (1.1) has no radial solution.*

THEOREM 1.4 (symmetry breaking). *Let*

$$B^2 = \frac{4^{2/\alpha}}{\alpha^2} \sup_{0 < y < \infty} \int_0^y e^s s^{\mu - 1} \, ds \int_y^\infty e^{-s} s^{-\mu + (2/\alpha) - 1} \, ds,$$

where $\mu = N/\alpha$. If

$$\frac{1}{4B^2}\left(1 - \frac{\lambda}{\lambda_1}\right) > \frac{(N - 1)(N - 2)}{4}$$

when either

$$\frac{1}{2} < \frac{\lambda}{\lambda_1} < 1 \quad if \; 2 < \alpha \leqslant N - 2$$

or

$$\frac{\alpha^2}{4\lambda_1} < \frac{\lambda}{\lambda_1} < 1 \quad if \; \alpha > N - 2$$

holds, then the ground-state solution of (1.1) exists and it is non-radial.

The motivation for studying such problems stems from a phenomenon noted in [6] which led to the notion of critical dimension (see [17]). Let $B_1(0)$ be the unit ball in $\mathbb{R}^N$, with $N \geqslant 3$. Let μ_1 denote the first eigenvalue of the Laplacian with zero Dirichlet boundary conditions on $B_1(0)$. The following problem was studied in [6]:

$$\left. \begin{aligned} -\Delta u = u^{2^*-1} + \lambda u, \quad u > 0 \text{ in } B_1(0), \\ u = 0 \quad \text{on } \partial B_1(0). \end{aligned} \right\} \tag{1.5}$$

The authors proved that, for $N \geqslant 4$, (1.5) has a solution if and only if $\lambda \in (0, \mu_1)$, while, for $N = 3$, problem (1.5) has a solution if and only if $\lambda \in (\frac{1}{4}\mu_1, \mu_1)$. The same phenomenon has been noted for problems with critical exponents on geodesic balls on the sphere [2–4,7], and also for more general equations involving the p-Laplacian on the ball in $\mathbb{R}^N$ [9,10] or on the N-dimensional sphere [5]. A similar situation in $\mathbb{R}^N$ should be in some sense expected when the Dirichlet boundary conditions are replaced by a requirement that solutions have a fast decay at infinity. We impose this requirement by introducing the fast increasing weights $K(x)$. By the method developed in [6], we obtain solutions of (1.1) as critical points of Q_λ.

We note that for our existence results we consistently use radial test functions. We infer that, whenever a ground-state solution is achieved in $H(\alpha)$, the ground state in $H_{\mathrm{rad}}(\alpha)$ (radial functions in $H(\alpha)$) is also achieved. Indeed, this fact follows by repeating the original arguments in $H(\alpha)$ in the radial setting $H_{\mathrm{rad}}(\alpha)$. We do not know *a priori* whether the ground state in $H(\alpha)$ is radial. Theorem 1.4 guarantees cases when the ground state is not radial and, considering the least energy solution in $H_{\mathrm{rad}}(\alpha)$, we are thus led to the existence of at least two positive solutions of (1.1).

One is left with the following open question: are there situations when the ground state is achieved, but no radial solutions exist? According to our result above, if this happens, it has to be in the range $\alpha > N - 2$ and $\frac{1}{2}\lambda_1 < \lambda \leqslant \frac{1}{4}\alpha^2$.

The paper is structured as follows. After a brief section of preparatory results, we discuss in §3 the non-existence parts of theorems 1.1 and 1.2. Section 4 deals with the existence part of theorem 1.1. We prove theorem 1.2 in §5, and theorem 1.3 in §6. Finally, we present the symmetry-breaking argument that leads to theorem 1.4 in §7.

2. Preliminaries

Hereafter we write only $\int u$ to denote $\int_{\mathbb{R}^N} u(x)\, \mathrm{d}x$. For any $\alpha \geqslant 2$ we define $\theta(x) = \frac{1}{4}|x|^\alpha$, $K(x) = \exp(\theta(x))$, the Hilbert space $H(\alpha)$ as being the completion

of $C_c^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\| = \left(\int K(x)|\nabla u|^2 \right)^{1/2}$$

induced by the inner product

$$(u, v) = \int K(x)\nabla u \cdot \nabla v.$$

We define the weighted spaces

$$L^2(\alpha) = \left\{ u \text{ measurable in } \mathbb{R}^N : \int K(x)|x|^{\alpha-2}u^2 < \infty \right\}$$

and

$$L^{2^*}(\alpha) = \left\{ u \text{ measurable in } \mathbb{R}^N : \int K(x)|u|^{2^*} < \infty \right\}.$$

PROPOSITION 2.1. *The space $H(\alpha)$ is continuously embedded in $L^2(\alpha)$ and $L^{2^*}(\alpha)$.*

Proof. For any $u \in C_c^\infty(\mathbb{R}^N)$ we have

$$\int |\nabla(K(x)^{1/2}u)|^2 = \int |K(x)^{1/2}\nabla u + \nabla(K(x)^{1/2})u|^2$$

$$= \int K(x)|\nabla u|^2 + \int \nabla(K(x)^{1/2}u^2) \cdot \nabla(K(x)^{1/2}).$$

Integrating by parts we get

$$\int |\nabla(K(x)^{1/2}u)|^2 = \int K(x)|\nabla u|^2 - \int K(x)^{1/2}u^2 \Delta(K(x)^{1/2})$$

$$= \int K(x)|\nabla u|^2 - \frac{1}{2} \int K(x)(\Delta\theta(x) + \tfrac{1}{2}|\nabla\theta(x)|^2)u^2.$$

From the Sobolev inequality we have

$$\int |\nabla(K(x)^{1/2}u)|^2 \geqslant S_0 \left(\int (K(x)^{1/2}|u|)^{2^*} \right)^{2/2^*}$$

$$\geqslant S_0 \left(\int K(x)|u|^{2^*} \right)^{2/2^*},$$

where $S_0 > 0$ denotes the best Sobolev constant. Hence,

$$\int K(x)|\nabla u|^2 \geqslant \frac{1}{2} \int K(x)(\Delta\theta(x) + \tfrac{1}{2}|\nabla\theta(x)|^2)u^2 + S_0 \left(\int K(x)|u|^{2^*} \right)^{2/2^*}. \quad (2.1)$$

We therefore obtain

$$\int K(x)|\nabla u|^2 \geqslant \frac{1}{2} \int K(x)(\Delta\theta(x) + \tfrac{1}{2}|\nabla\theta(x)|^2)u^2 \qquad (2.2)$$

and, since

$$\Delta\theta(x) + \tfrac{1}{2}|\nabla\theta(x)|^2 = |x|^{\alpha-2}\left(\frac{\alpha(N-2+\alpha)}{4} + \frac{\alpha^2}{32}|x|^\alpha\right)$$
$$\geqslant \frac{\alpha(N-2+\alpha)}{4}|x|^{\alpha-2},$$

we find from (2.1) that

$$\int K(x)|\nabla u|^2 \geqslant \frac{\alpha(N-2+\alpha)}{8}\int K(x)|x|^{\alpha-2}u^2,$$

which implies that $H(\alpha)$ embeds continuously in $L^2(\alpha)$, and

$$\int K(x)|\nabla u|^2 \geqslant S_0\left(\int K(x)|u|^{2^*}\right)^{2/2^*},$$

which implies that $H(\alpha)$ embeds continuously into $L^{2^*}(\alpha)$. $\qquad\square$

PROPOSITION 2.2. *For any $\alpha \geqslant 2$ the space $H(\alpha)$ is compactly embedded in $L^2(\alpha)$.*

Proof. In order to show that the embedding is compact, we proceed as in [11, proposition 1.1]. Let $(u_n) \subset H(\alpha)$ be such that

$$u_n \rightharpoonup 0 \quad \text{weakly in } H(\alpha) \text{ and } \|u_n\| \leqslant 1.$$

Given $\varepsilon > 0$, we can use the definition of θ to obtain $R = R(\varepsilon) > 0$ such that

$$\Delta\theta(x) + \tfrac{1}{2}|\nabla\theta(x)|^2 = |x|^{\alpha-2}\left(\frac{\alpha(N-2+\alpha)}{4} + \frac{\alpha^2}{32}|x|^\alpha\right) \geqslant \frac{2}{\varepsilon}|x|^{\alpha-2}$$

for any $|x| \geqslant R$. Hence, by (2.2), we obtain

$$\int_{\mathbb{R}^N \setminus B_R(0)} K(x)|x|^{\alpha-2}u_n^2 \leqslant \varepsilon \int K(x)|\nabla u_n|^2 \leqslant \varepsilon. \tag{2.3}$$

On the other hand, arguments based on the Rellich–Kondrachov theorem imply that $u_n \to 0$ in $L^2(B_R(0))$. Since $K(x)|x|^{\alpha-2} \in L^\infty(B_R(0))$, it follows that

$$\int_{B_R(0)} K(x)|x|^{\alpha-2}u_n^2 \leqslant \varepsilon \quad \text{for } n \geqslant n_\varepsilon.$$

The above expression and (2.3) imply that $\int K(x)|x|^{\alpha-2}u_n^2 \leqslant 2\varepsilon$ for $n \geqslant n_\varepsilon$, i.e. $u_n \to 0$ strongly in $L^2(\alpha)$. The proposition is proved. $\qquad\square$

Proposition 2.2 and standard spectral theory for compact operators imply that the eigenvalue problem (1.3) has a sequence of eigenvalues

$$0 < \lambda_1 < \lambda_2 \leqslant \cdots \leqslant \lambda_k \leqslant \cdots$$

such that $\lambda_k \to \infty$ as $k \to \infty$. Moreover, the first eigenvalue can be characterized as the infimum of the Rayleigh quotient

$$\lambda_1 = \inf_{u \in H^1(\alpha)\setminus\{0\}} \frac{\int |\nabla u|^2 K(x)}{\int |u|^2 K(x)|x|^{\alpha-2}}.$$

We note that (1.3) is equivalent to

$$-\Delta u - \tfrac{1}{4}\alpha(x \cdot \nabla u)|x|^{\alpha-2} = \lambda u|x|^{\alpha-2}.$$

A direct calculation shows that $\varphi_1(x) = \exp(-\tfrac{1}{4}|x|^\alpha) > 0$ is an eigenfunction associated with the first eigenvalue $\lambda_1 = \tfrac{1}{4}\alpha(N - 2 + \alpha)$.

3. Non-existence range

In this section we deal with the non-existence parts of theorems 1.1 and 1.2. We use a Pohozaev identity to show that equation (1.1) has no non-trivial solution when $\lambda \leqslant \tfrac{1}{2}\lambda_1$. The result is quite flexible in the sense that it prevents not only positive solutions, but also sign-changing solutions in $H(\alpha)$ (see proposition 3.3). For the reader's convenience we give a proof of the following inequality.

LEMMA 3.1 (Hardy inequality). *Let* $u \in C_c^\infty(\mathbb{R}^N)$, $a > -N$ *and* $R \geqslant 0$. *Then*

$$\int_{\mathbb{R}^N \setminus B_R(0)} |u|^2 |x|^a \leqslant \frac{4}{(N+a)^2} \int_{\mathbb{R}^N \setminus B_R(0)} (x \cdot \nabla u)^2 |x|^a.$$

Moreover, if $R = 0$, *the inequality is strict unless* $u \equiv 0$.

Proof. Integrate the identity

$$\operatorname{div}(u^2|x|^a x) = |x|^a x \cdot \nabla(u^2) + (N+a)|x|^a u^2$$

on the domain $\mathbb{R}^N \setminus B_R(0)$ to get

$$0 \geqslant -R^{a+1} \int_{\partial B_R(0)} u^2 \, \mathrm{d}\sigma$$

$$= 2 \int_{\mathbb{R}^N \setminus B_R(0)} |x|^a u x \cdot \nabla u + (N+a) \int_{\mathbb{R}^N \setminus B_R(0)} |x|^a u^2.$$

The Cauchy–Schwarz inequality then gives

$$2\left(\int_{\mathbb{R}^N \setminus B_R(0)} |x|^a u^2\right)^{1/2} \left(\int_{\mathbb{R}^N \setminus B_R(0)} |x|^a (x \cdot \nabla u)^2\right)^{1/2} \geqslant (N+a) \int_{\mathbb{R}^N \setminus B_R(0)} |x|^a u^2,$$

which is the desired inequality.

When $R = 0$ the only way to have equality is if the functions $|u(x)|$ and $|x \cdot \nabla u(x)|$ are proportional. This can only happen when $u \equiv 0$. $\square$

REMARK 3.2. We note that when $a = \alpha - 2$ lemma 3.1 holds for $u \in H(\alpha)$.

In the following we derive our non-existence result.

PROPOSITION 3.3. *Suppose that* $u \in H(\alpha)$, $\alpha \geqslant 2$, *satisfies (1.2) and* $\lambda \leqslant \tfrac{1}{2}\lambda_1$. *Then* $u \equiv 0$.

Proof. On multiplying (1.2) by u and integrating by parts we obtain

$$\int |\nabla u|^2 - \tfrac{1}{8}\alpha \int |x|^{\alpha-2}(\nabla u^2 \cdot x) = \int |u|^{2^*} + \lambda \int |x|^{\alpha-2} u^2. \qquad (3.1)$$

We now note that

$$\int |x|^{\alpha-2}(\nabla u^2 \cdot x) = -\int u^2 \operatorname{div}(x|x|^{\alpha-2}) = -(N-2+\alpha)\int |x|^{\alpha-2}u^2;$$

hence, from (3.1) we obtain

$$\int |\nabla u|^2 + \tfrac{1}{8}\alpha(N-2+\alpha)\int |x|^{\alpha-2}u^2 = \int |u|^{2^*} + \lambda\int |x|^{\alpha-2}u^2. \qquad (3.2)$$

On the other hand, we can multiply (1.2) by $(x \cdot \nabla u)$ and integrate by parts once more to obtain

$$\frac{N-2}{2}\int |\nabla u|^2 + \frac{\alpha}{4}\int |x|^{\alpha-2}(x \cdot \nabla u)^2 = \frac{N}{2^*}\int |u|^{2^*} + \frac{\lambda(N-2+\alpha)}{2}\int |x|^{\alpha-2}u^2.$$

Suppose, by contradiction, that $u \not\equiv 0$. Combining the above expression with (3.2) we obtain

$$(\tfrac{1}{2}\alpha\lambda + \tfrac{1}{16}\alpha(N-2)(N-2+\alpha))\int |x|^{\alpha-2}u^2 = \tfrac{1}{4}\alpha\int |x|^{\alpha-2}(x \cdot \nabla u)^2$$

$$> \tfrac{1}{16}\alpha(N-2+\alpha)^2\int |x|^{\alpha-2}u^2,$$

where we have used the strict Hardy inequality of lemma 3.1. The above expression implies that

$$\lambda > \tfrac{1}{8}\alpha(N-2+\alpha) = \tfrac{1}{2}\lambda_1,$$

which contradicts the hypothesis. Hence, $u \equiv 0$ and the proposition is proved. $\quad\square$

It is also easy to see that (1.2) does not admit positive solutions when $\lambda \geqslant \lambda_1$. Indeed, since φ_1 satisfies (1.3), we assume that equation (1.1) has a positive solution, we multiply (1.1) by φ_1 and integrate by parts. We obtain

$$(\lambda_1 - \lambda)\int K(x)u\varphi_1 = \int K(x)u^{2^*-1}\varphi_1,$$

which implies that $\lambda < \lambda_1$.

4. Existence range

In this section we prove the existence stated in theorem 1.1. We define

$$S_\lambda = \inf_{u \in H(\alpha)\setminus\{0\}} Q_\lambda(u),$$

and $S_0(K)$ in accordance with $\lambda = 0$. For $\alpha \geqslant 2$ fixed and $\lambda \in \mathbb{R}$, we follow [6] by using the expression (1.4). We first note that $S_0(K) = S_0$, the best Sobolev constant. Indeed, from (2.1) we have $S_0(K) \geqslant S_0$. Using a smooth cut-off function $\varphi(x) \equiv 1$ in $B_1(0)$ and $\varphi(x) \equiv 0$ outside $B_2(0)$, and a sequence $u_n(x) = \varphi(x)(\varepsilon_n + |x^2|)^{(2-N)/2}$ with $\varepsilon_n \to 0$, we obtain $S_0(K) \leqslant \liminf_{n\to\infty} Q_0(u_n) = S_0$.

The existence result will be proved by checking that, under the hypotheses of theorem 1.1, the number S_λ is achieved. This is exactly the content of the next two results.

LEMMA 4.1. *If $0 < S_\lambda(K) < S_0(K)$, then $S_\lambda(K)$ is achieved.*

Proof. It suffices to argue as in [11, lemma 4.11]. $\square$

PROPOSITION 4.2. *If $2 \leqslant \alpha \leqslant N - 2$, for any $\lambda \in (\frac{1}{2}\lambda_1, \lambda_1)$ we have $0 < S_\lambda(K) < S_0(K)$.*

Proof. Let $\varphi \in C^\infty(\mathbb{R}^N, [0,1])$ be such that $\varphi \equiv 1$ on $B_1(0)$ and $\varphi \equiv 0$ outside $B_2(0)$. Given $a \geqslant 2$ and $b \geqslant 0$, we can easily check that

$$\int |x|^a (1 + |x|^2)^{b-N} < \infty \quad \text{provided that } N > a + 2b.$$

In this case, we can compute

$$\int \frac{\varphi^2 |x|^a}{(\varepsilon + |x|^2)^{N-b}} = \int \frac{|x|^a}{(\varepsilon + |x|^2)^{N-b}} + \int \frac{(\varphi^2 - 1)|x|^a}{(\varepsilon + |x|^2)^{N-b}}$$

$$= O(1) + \int \frac{\varepsilon^{a/2} |x/\sqrt{\varepsilon}|^a}{\varepsilon^{N-b}(1 + |x/\sqrt{\varepsilon}|^2)^{N-b}}$$

$$= O(1) + \varepsilon^{b+a/2-N/2} \int \frac{|x|^a}{(1 + |x|^2)^{N-b}}, \tag{4.1}$$

as $\varepsilon \to 0^+$.

For $\varepsilon > 0$, let us define

$$u_\varepsilon = K^{-1/2}\varphi v_\varepsilon,$$

where v_ε is the 'instanton' given by

$$v_\varepsilon(x) = \left(\frac{1}{\varepsilon + |x|^2}\right)^{(N-2)/2}.$$

Our aim is to check that, for small values of ε, we have $Q_\lambda(u_\varepsilon) < S$. In order to do this, we first compute

$$\int |\nabla u_\varepsilon|^2 K = \int \varphi^2 \left(|\nabla v_\varepsilon|^2 - \frac{\alpha}{4} v_\varepsilon(x \cdot \nabla v_\varepsilon)|x|^{\alpha-2} + \frac{\alpha^2}{4 \cdot 16} v_\varepsilon^2 |x|^{2(\alpha-1)}\right)$$

$$+ \int v_\varepsilon^2 |\nabla \varphi|^2 + 2 \int \varphi v_\varepsilon \nabla \varphi \cdot \left(\nabla v_\varepsilon - \frac{\alpha}{8} v_\varepsilon |x|^{\alpha-2} x\right).$$

It is easy to check that the two last terms of the right-hand side are bounded as $\varepsilon \to 0$. Thus, by using the definition of v_ε, we get

$$\int |\nabla u_\varepsilon|^2 K = (2 - N)^2 \int \frac{\varphi^2 |x|^2}{(\varepsilon + |x|^2)^N} + \frac{\alpha(N-2)}{4} \int \frac{\varphi^2 |x|^\alpha}{(\varepsilon + |x|^2)^{N-1}}$$

$$+ \frac{\alpha^2}{4 \cdot 16} \int \frac{\varphi^2 |x|^{2(\alpha-1)}}{(\varepsilon + |x|^2)^{N-2}} + O(1)$$

$$= I_1 + I_2 + I_3 + O(1).$$

By using the estimate in (4.1) we obtain

$$
\left.
\begin{aligned}
I_1 &= \varepsilon^{1-N/2} A_1 + O(1) && \text{(for } N > 2), \\
I_2 &= \varepsilon^{\alpha/2+(1-N/2)} A_2 + O(1) && \text{(for } N > \alpha + 2), \\
I_3 &= \varepsilon^{\alpha+(1-N/2)} A_3 + O(1) && \text{(for } N > 2\alpha + 2),
\end{aligned}
\right\}
\tag{4.2}
$$

with

$$
A_1 := (N-2)^2 \int \frac{|x|^2}{(1+|x|^2)^N} \qquad \text{(for } N > 2),
$$

$$
A_2 := \frac{\alpha(N-2)}{4} \int \frac{|x|^\alpha}{(1+|x|^2)^{N-1}} \qquad \text{(for } N > \alpha + 2),
$$

$$
A_3 := \frac{\alpha^2}{4 \cdot 16} \int \frac{|x|^{2(\alpha-1)}}{(1+|x|^2)^{N-2}} \qquad \text{(for } N > 2\alpha + 2).
$$

In the same way, we have

$$
\begin{aligned}
\lambda \int |u_\varepsilon|^2 K |x|^{\alpha-2} &= \lambda \int \frac{\varphi^2 |x|^{\alpha-2}}{(\varepsilon+|x|^2)^{N-2}} \\
&= \varepsilon^{\alpha/2+(1-N/2)} \lambda A_4 + O(1) \quad \text{(for } N > \alpha + 2),
\end{aligned}
\tag{4.3}
$$

where

$$
A_4 := \int \frac{|x|^{\alpha-2}}{(1+|x|^2)^{N-2}} \qquad \text{(for } N > \alpha + 2).
$$

Arguing as in (4.1), we can compute

$$
\int |u_\varepsilon|^{2^*} K = \int \frac{\varphi^{2^*} K^{2/(2-N)}}{(\varepsilon+|x|^2)^N} = \varepsilon^{-N/2} A_0 + O(1) \quad \text{(for } N > 2),
$$

where

$$
A_0 := \int \frac{1}{(1+|x|^2)^N} \qquad \text{(for } N > 2).
$$

Hence, we get

$$
\left(\int |u_\varepsilon|^{2^*} K \right)^{2/2^*} = (\varepsilon^{-N/2} A_0 + O(1))^{2/2^*}
$$

$$
= (\varepsilon^{-N/2} A_0)^{2/2^*} + \frac{N-2}{N} (\varepsilon^{-N/2} A_0 + O(1))^{-2/N} O(1),
$$

and therefore

$$
\left(\int |u_\varepsilon|^{2^*} K \right)^{2/2^*} = \varepsilon^{1-N/2} A_0^{(N-2)/N} + O(\varepsilon) \quad \text{(for } N > 2).
\tag{4.4}
$$

We now consider several distinct cases depending on the values of N and α.

CASE 1 ($N > 2\alpha + 2$). In this case, all the equalities in (4.2)–(4.4) hold and we have

$$
\begin{aligned}
Q_\lambda(u_\varepsilon) &= \frac{\varepsilon^{1-N/2}(A_1 + \varepsilon^{\alpha/2}(A_2 - \lambda A_4) + \varepsilon^\alpha A_3) + O(1)}{\varepsilon^{1-N/2}A_0^{(N-2)/N} + O(\varepsilon)} \\
&= A_0^{-1+2/N}(A_1 + \varepsilon^{\alpha/2}(A_2 - \lambda A_4) + \varepsilon^\alpha A_3 + O(\varepsilon^{N/2-1})) \\
&< A_0^{-1+2/N}A_1,
\end{aligned}
$$

for $\varepsilon > 0$ small enough, provided that $\lambda > A_2/A_4$. However, it is well known [6] that $A_0^{-1+2/N}A_1 = S_0 = S_0(K)$ and therefore it remains to check that

$$
\frac{A_2}{A_4} = \tfrac{1}{8}\alpha(N + \alpha - 2) \quad \text{(for } N > \alpha + 2\text{)}. \tag{4.5}
$$

In order to do this, we denote by ω_{N-1} the area of the sphere $\mathbb{S}^{N-1} \subset \mathbb{R}^N$ and integrate by parts to get

$$
\begin{aligned}
A_4 &= \int \frac{|x|^{\alpha-2}}{(1+|x|^2)^{N-2}} \\
&= \omega_{N-1}\int_0^\infty r^{(N+\alpha-2)-1}\frac{1}{(1+r^2)^{N-2}}\,\mathrm{d}r \\
&= \frac{2(N-2)}{(N+\alpha-2)}\omega_{N-1}\int_0^\infty \frac{r^{N+\alpha-1}}{(1+r^2)^{N-1}}\,\mathrm{d}r \\
&= \frac{8}{\alpha(N+\alpha-2)}A_2.
\end{aligned}
$$

Thus, (4.5) holds and the proposition is proved in the case when $N > 2\alpha + 2$.

CASE 2 ($N = 2\alpha + 2$). In this case, we need to perform another estimate for I_3. Thus, we first note that

$$
\int_{B_1(0)} \frac{|x|^{2(\alpha-1)}}{(\varepsilon + |x|^2)^{2\alpha}} \leqslant I_3 \leqslant \int_{B_2(0)} \frac{|x|^{2(\alpha-1)}}{(\varepsilon + |x|^2)^{2\alpha}}.
$$

On the other hand, for any $R > 0$, it holds that

$$
\begin{aligned}
\int_{B_R(0)} \frac{|x|^{2(\alpha-1)}}{(\varepsilon + |x|^2)^{2\alpha}} &= \tfrac{1}{2}\omega_{N-1}\int_0^R \frac{r^{2(\alpha-1)}r^{2\alpha}(2r)}{(\varepsilon + r^2)^{2\alpha}}\,\mathrm{d}r \\
&= \tfrac{1}{2}\omega_{N-1}\int_\varepsilon^{\varepsilon+R^2} \frac{(s-\varepsilon)^{2\alpha-1}}{s^{2\alpha}}\,\mathrm{d}s \\
&= \tfrac{1}{2}\omega_{N-1}\left(\int_\varepsilon^{\varepsilon+R^2} \frac{1}{s}\,\mathrm{d}s + \int_\varepsilon^{\varepsilon+R^2} \sum_{i=1}^{2\alpha-1} C_i \frac{s^{(2\alpha-1)-i}}{s^{2\alpha}}\varepsilon^i\,\mathrm{d}s\right) \\
&= \tfrac{1}{2}\omega_{N-1}\log|\varepsilon + R^2| - \tfrac{1}{2}\omega_{N-1}\log|\varepsilon| + O(1) \\
&= \tfrac{1}{2}\omega_{N-1}|\log\varepsilon| + O(1).
\end{aligned}
$$

Hence, as in case 1, we get

$$\int |\nabla u_\varepsilon|^2 K = \varepsilon^{1-N/2}(A_1 + \varepsilon^{\alpha/2}A_2 + \tfrac{1}{2}\varepsilon^{-1+N/2}\omega_{N-1}|\log \varepsilon|) + O(1)$$

$$= \varepsilon^{1-N/2}(A_1 + \varepsilon^{\alpha/2}A_2 + \tfrac{1}{2}\varepsilon^{\alpha}\omega_{N-1}|\log \varepsilon|) + O(1),$$

and therefore

$$Q_\lambda(u_\varepsilon) = A_0^{-1+2/N}(A_1 + \varepsilon^{\alpha/2}(A_2 - \lambda A_4) + \tfrac{1}{2}\varepsilon^{\alpha}\omega_{N-1}|\log \varepsilon| + O(\varepsilon^{N/2-1})) < S_0,$$

for ε sufficiently small.

CASE 3 ($\alpha + 2 < N < 2\alpha + 2$). In this case, since $N < 2\alpha + 2$, we can estimate I_3 as follows:

$$I_3 = \frac{\alpha^2}{4 \cdot 16} \int_{B_2(0)} \frac{\varphi^2 |x|^{2(\alpha-1)}}{(\varepsilon + |x|^2)^{N-2}} \leqslant \int_{B_2(0)} \frac{|x|^{2(\alpha-1)}}{|x|^{2(N-2)}} = O(1)$$

and therefore

$$Q_\lambda(u_\varepsilon) = A_0^{-1+2/N}(A_1 + \varepsilon^{\alpha/2}(A_2 - \lambda A_4) + O(\varepsilon^{N/2-1})) < S_0$$

for ε small enough.

CASE 4 ($N = \alpha + 2$). In this case $I_3 = O(1)$. However, (4.3) does not hold and we also need to estimate I_2. But this can be made as in case 2, and we can check that

$$I_2 = \tfrac{1}{2}\omega_{N-1}\tfrac{1}{4}\alpha(N-2)|\log \varepsilon| + O(1)$$

and

$$\lambda \int |u_\varepsilon|^2 K|x|^{\alpha-2} = \tfrac{1}{2}\omega_{N-1}|\log \varepsilon| + O(1).$$

Hence,

$$Q_\lambda(u_\varepsilon) = A_0^{-1+2/N}(A_1 + \tfrac{1}{2}\omega_{N-1}(\tfrac{1}{4}\alpha(N-2) - \lambda) + O(\varepsilon^{N/2-1})) < S_0,$$

provided ε is small and

$$\lambda > \tfrac{1}{4}\alpha(N-2) = \tfrac{1}{4}\alpha^2 = \tfrac{1}{8}\alpha(N+\alpha-2).$$

This concludes the proof of the proposition. $\qquad \square$

5. Existence when $N < \alpha + 2$ and $\frac{1}{4}\alpha^2 < \lambda < \frac{1}{4}\alpha(N - 2 + \alpha)$

In this section we prove the existence statement of theorem 1.2. For this we work with test functions using the solutions

$$U(x) = c_N \left(\frac{\varepsilon}{\varepsilon^2 + |x|^2}\right)^{(N-2)/2}$$

of $-\Delta U = U^{2^*-1}$. For convenience, define $\varphi(x) = K^{-1/2}(x) = \exp(-\frac{1}{8}|x|^\alpha)$ and let $u = \varphi^2 U$. We calculate

$$
\begin{aligned}
\int K|\nabla u|^2 &= \int \varphi^{-2}|\nabla(\varphi^2 U)|^2 \\
&= \int |2U\nabla\varphi + \varphi\nabla U|^2 \\
&= \int 4U^2|\nabla\varphi|^2 + 2\varphi U(\nabla\varphi \cdot \nabla U) + \mathrm{div}(\varphi^2 U\nabla U) - \varphi^2 U\Delta U \\
&= \int 4U^2|\nabla\varphi|^2 + \tfrac{1}{2}(\nabla\varphi^2 \cdot \nabla U^2) + \varphi^2 U^{2^*} + \mathrm{div}(\varphi^2 U\nabla U) \\
&= \int 4U^2|\nabla\varphi|^2 - \tfrac{1}{2}U^2\Delta\varphi^2 + \varphi^2 U^{2^*} + \mathrm{div}(\varphi^2 U\nabla U + \tfrac{1}{2}U^2\nabla\varphi^2).
\end{aligned}
$$

Since the divergences integrate to zero, we get

$$
\int K|\nabla u|^2 = \int \varphi^2 U^{2^*} + \int U^2(4|\nabla\varphi|^2 - \tfrac{1}{2}\Delta\varphi^2).
$$

Using the explicit form of φ, a direct calculation yields

$$
4|\nabla\varphi|^2 - \tfrac{1}{2}\Delta\varphi^2 = \tfrac{1}{8}\alpha(N + \alpha - 2)|x|^{\alpha-2}\varphi^2 + \tfrac{1}{32}\alpha^2|x|^{2\alpha-2}\varphi^2
$$

so we can write

$$
\int U^2(4|\nabla\varphi|^2 - \tfrac{1}{2}\Delta\varphi^2) = \int U^2|x|^{\alpha-2}\Psi(x),
$$

where $\Psi(x) = \tfrac{1}{8}\alpha(N + \alpha - 2)\varphi^2 + \tfrac{1}{32}\alpha^2|x|^\alpha\varphi^2$.

We now show that

$$
\int U^2|x|^{\alpha-2}\Psi(x) = c_N^2 \varepsilon^{N-2} \int |x|^{\alpha+2-2N}\Psi(x) + o(\varepsilon^{N-2}); \tag{5.1}
$$

therefore,

$$
\int K|\nabla u|^2 = \int \varphi^2 U^{2^*} + c_N^2 \varepsilon^{N-2}\tfrac{1}{8}\alpha(N + \alpha - 2) \int \frac{|x|^{\alpha-2}}{|x|^{2(N-2)}}\varphi^2
$$

$$
+ c_N^2 \varepsilon^{N-2}\tfrac{1}{32}\alpha^2 \int \frac{|x|^{2\alpha-2}}{|x|^{2(N-2)}}\varphi^2 + o(\varepsilon^{N-2}). \tag{5.2}
$$

Indeed,

$$
\int |x|^{\alpha+2-2N}\Psi(x) - \int \frac{|x|^{\alpha-2}}{(|x|^2 + \varepsilon^2)^{N-2}}\Psi(x)
$$

$$
= \int \frac{|x|^{\alpha+2-2N}((|x|^2 + \varepsilon^2)^{N-2} - |x|^{2(N-2)})}{(|x|^2 + \varepsilon^2)^{N-2}}\Psi(x)
$$

$$= \sum_{k=1}^{N-2} \binom{N-2}{k} \int \frac{|x|^{\alpha+2-2N} |x|^{2(N-2-k)} \varepsilon^{2k}}{(|x|^2 + \varepsilon^2)^{N-2}} \Psi(x)$$

$$\leqslant \sum_{k=1}^{N-2} \binom{N-2}{k} \int \frac{|x|^{\alpha+2-2N} \varepsilon^{2k}}{(|x|^2 + \varepsilon^2)^k} \Psi(x).$$

Each of the integrals

$$\int \frac{|x|^{\alpha+2-2N} \varepsilon^{2k}}{(|x|^2 + \varepsilon^2)^k} \Psi(x)$$

is estimated as follows. Pick

$$0 < \mu < \min\left\{\frac{\alpha}{N-2} - 1, 2\right\}, \quad p = \frac{2}{\mu}, \quad q = \frac{2}{2-\mu}.$$

We use Young's inequality,

$$\frac{a^p}{p} + \frac{b^q}{q} \geqslant ab, \quad \text{with } a = (p|x|^2)^{1/p} \text{ and } b = (q\varepsilon^2)^{1/q}.$$

Then

$$|x|^2 + \varepsilon^2 \geqslant \frac{2}{\mu^{\mu/2}(2-\mu)^{(2-\mu)/2}} |x|^\mu \varepsilon^{2-\mu}; \tag{5.3}$$

hence,

$$\int \frac{|x|^{\alpha+2-2N} \varepsilon^{2k}}{(|x|^2 + \varepsilon^2)^k} \Psi(x) \leqslant \frac{\mu^{k\mu/2}(2-\mu)^{k(2-\mu)/2}}{2^k} \int |x|^{\alpha+2-2N-k\mu} \varepsilon^{k\mu} \Psi(x).$$

From $\alpha + 2 - 2N - k\mu > -N$ we find that every integral on the right-hand side is convergent. Hence,

$$\int \frac{|x|^{\alpha+2-2N} \varepsilon^{2k}}{(|x|^2 + \varepsilon^2)^k} \Psi(x) \leqslant C(k,\mu)\varepsilon^{k\mu}$$

and this proves (5.1).

Returning to (5.2), we use

$$0 = \int \operatorname{div}\left(\frac{|x|^{\alpha-2} x}{|x|^{2(N-2)}} \varphi^2\right) = (\alpha - N + 2) \int \frac{|x|^{\alpha-2}}{|x|^{2(N-2)}} \varphi^2 - \tfrac{1}{4}\alpha \int \frac{|x|^{2\alpha-2}}{|x|^{2(N-2)}} \varphi^2$$

from which we may infer that

$$\tfrac{1}{32}\alpha^2 \int \frac{|x|^{2\alpha-2}}{|x|^{2(N-2)}} \varphi^2 = \tfrac{1}{8}\alpha(\alpha - N + 2) \int \frac{|x|^{\alpha-2}}{|x|^{2(N-2)}} \varphi^2.$$

Therefore,

$$\int K|\nabla u|^2 = \int \varphi^2 U^{2^*} + c_N^2 \varepsilon^{N-2} \tfrac{1}{4}\alpha^2 \int \frac{|x|^{\alpha-2}}{|x|^{2(N-2)}} \varphi^2 + o(\varepsilon^{N-2}).$$

The same estimates used for (5.1) give

$$\lambda \int K|x|^{\alpha-2} u^2 = \lambda c_N^2 \varepsilon^{N-2} \int \frac{|x|^{\alpha-2}}{|x|^{2(N-2)}} \varphi^2 + o(\varepsilon^{N-2})$$

and so

$$\int K|\nabla u|^2 - \lambda \int K|x|^{\alpha-2}u^2$$
$$= \int \varphi^2 U^{2^*} + c_N^2 \varepsilon^{N-2}(\tfrac{1}{4}\alpha^2 - \lambda) \int \frac{|x|^{\alpha-2}}{|x|^{2(N-2)}}\varphi^2 + o(\varepsilon^{N-2}).$$

Also

$$\int K u^{2^*} = \int \varphi^{2(N+2)/(N-2)} U^{2^*}.$$

We show that, for $p > 0$, we can write estimates of the type

$$\int \varphi^p U^{2^*} = \int U^{2^*} + o(\varepsilon^{N-2}). \tag{5.4}$$

This is because

$$\int \varphi^p U^{2^*} = \int U^{2^*} + \int (1 - \varphi^p)c_N^{2^*} \frac{\varepsilon^N}{(|x|^2 + \varepsilon^2)^N}.$$

Using Young's inequality (5.3) again, with

$$\min\left\{\frac{\alpha + N}{N}, 2\right\} > \mu > 2\frac{N-1}{N},$$

we obtain

$$\int (1 - \varphi^p)c_N^{2^*} \frac{\varepsilon^N}{(|x|^2 + \varepsilon^2)^N} \leqslant c_N^{2^*} \frac{\mu^{N\mu/2}(2-\mu)^{N(2-\mu)/2}}{2^N} \int (1 - \varphi^p)\frac{\varepsilon^{N(\mu-1)}}{|x|^{N\mu}},$$

which proves (5.4) because the exponent of ε is

$$N(\mu - 1) > N - 2$$

and the integral is convergent since, for $|x|$ small,

$$1 - \varphi^p(x) \leqslant \tfrac{1}{4}p|x|^\alpha \quad \text{and} \quad \alpha + N - 1 - N\mu > -1.$$

Then, the definition of $Q_\lambda(u)$ gives

$$Q_\lambda(u) = \frac{\int U^{2^*} + c_N^2\varepsilon^{N-2}(\tfrac{1}{4}\alpha^2 - \lambda)\int(|x|^{\alpha-2}/|x|^{2(N-2)})\varphi^2 + o(\varepsilon^{N-2})}{(\int U^{2^*} + o(\varepsilon^{N-2}))^{2/2^*}} < S_0$$

for small $\varepsilon > 0$, provided that $\lambda > \tfrac{1}{4}\alpha^2$.

6. The radial case for $N < \alpha + 2$ and $0 < \lambda \leqslant \tfrac{1}{4}\alpha^2$

In this section we prove theorem 1.3, so we show non-existence of positive radial solutions in the case $N < \alpha + 2$ and $0 < \lambda \leqslant \tfrac{1}{4}\alpha^2$. The argument is more general than that in [1] for the case $\alpha = 2$. We use the method in [8] to obtain a Pohozaev-type identity.

Positive radial solutions $u = u(r)$ of equation (1.2) satisfy

$$-u_{rr} - \left(\frac{N-1}{r} + \frac{\alpha r^{\alpha - 1}}{4} \right) u_r = \lambda r^{\alpha - 2} u + u^{2^* - 1}. \tag{6.1}$$

We set

$$h(r) = \frac{N-1}{r} + \frac{\alpha r^{\alpha - 1}}{4} \quad \text{and} \quad H(r) = r^{N-1} e^{r^\alpha / 4}.$$

Note that $H_r = hH$. After multiplication by H, (6.1) can be written as

$$-(Hu_r)_r = \lambda r^{\alpha - 2} Hu + Hu^{2^* - 1}. \tag{6.2}$$

Let ξ and ζ be solutions of the linearized (6.2) about $u \equiv 0$, i.e. solutions of

$$-(H\xi_r)_r = \lambda r^{\alpha - 2} H\xi. \tag{6.3}$$

We define the Wronskian of u and ξ as

$$W[u, \xi](r) = H(r)(u(r)\xi_r(r) - u_r(r)\xi(r)).$$

After multiplying equation (6.2) by ξ and equation (6.3) by $-u$, we add the two equalities to obtain

$$\frac{\mathrm{d}}{\mathrm{d}r} W[u, \xi] = Hu^{2^* - 1} \xi. \tag{6.4}$$

Similarly, by using the solution ζ instead of ξ we obtain

$$\frac{\mathrm{d}}{\mathrm{d}r} W[u, \zeta] = Hu^{2^* - 1} \zeta. \tag{6.5}$$

We multiply equality (6.4) by $W[u, \zeta]$ and equality (6.5) by $W[u, \xi]$ and add the results. We therefore obtain

$$\frac{\mathrm{d}}{\mathrm{d}r} (W[u, \xi] W[u, \zeta]) = Hu^{2^* - 1} (\xi W[u, \zeta] + \zeta W[u, \xi]).$$

A straightforward calculation yields

$$\frac{\mathrm{d}}{\mathrm{d}r} \left(W[u, \xi] W[u, \zeta] + \frac{2}{2^*} H^2 u^{2^*} \xi \zeta \right) = \frac{2 + 2^*}{2^*} H^{(2 \cdot 2^*)/(2 + 2^*)} u^{2^*} (H^{4/(2 + 2^*)} \xi \zeta)_r. \tag{6.6}$$

We now show how to pick suitable ξ and ζ, positive solutions of (6.3), so that if $N < \alpha + 2$ and $0 < \lambda < \frac{1}{4}\alpha^2$, the right-hand side of (6.6) is negative for all $r > 0$. That is

$$(H^{4/(2 + 2^*)} \xi \zeta)_r < 0,$$

which—because ξ and ζ will be positive—is equivalent to

$$\frac{N-2}{r} + \frac{N-2}{N-1} \frac{\alpha r^{\alpha - 1}}{4} + \frac{\xi_r}{\xi} + \frac{\zeta_r}{\zeta} < 0. \tag{6.7}$$

Making the change of variables

$$s = \tfrac{1}{4} r^\alpha, \quad \xi(r) = e^{-s} f(s), \quad \zeta(r) = e^{-s} g(s),$$

equation (6.3) for ξ and ζ transforms into the confluent hypergeometric equation, also known as Kummer's equation, for f and g:

$$sf_{ss} + (b-s)f_s - af = 0, \tag{6.8}$$

where $b = (N - 2 + \alpha)/\alpha \in (1,2)$ and $a = (N - 2 + \alpha)/\alpha - (4\lambda/\alpha^2) \in (0,b)$. Let $\Phi(a,b;s)$ and $\Psi(a,b;s)$ be the confluent hypergeometric function and confluent hypergeometric function of the second kind, respectively (see [15, §§ 9.9–9.12]).

We define

$$f(s) = \frac{\Gamma(a)\Gamma(b-a)}{\Gamma(b)}\Phi(a,b;s) \quad \text{and} \quad g(s) = \Gamma(a)\Psi(a,b;s)$$

so that f and g solve (6.8) and have the integral representations

$$f(s) = e^s \int_0^1 e^{-st}t^{b-a-1}(1-t)^{a-1}\,\mathrm{d}t$$

and

$$g(s) = e^s \int_1^\infty e^{-st}t^{b-a-1}(t-1)^{a-1}\,\mathrm{d}t.$$

Using the differentiation formula (9.10.12) and the recurrence relations (9.10.14) and (9.10.13) in [15] we have

$$\frac{\mathrm{d}}{\mathrm{d}s}\Psi(a,b;s) = -a\Psi(a+1,b+1;s),$$

$$\Psi(a+1,b+1;s) = \frac{b-a-1}{s}\Psi(a+1,b;s) + \frac{1}{s}\Psi(a,b;s),$$

$$\Psi(a+1,b;s) = \frac{1}{a}\Psi(a,b;s) - \frac{1}{a}\Psi(a,b-1;s).$$

Combining the three equalities above we get

$$\frac{\mathrm{d}}{\mathrm{d}s}\Psi(a,b;s) = -\frac{b-1}{s}\Psi(a,b;s) + \frac{b-a-1}{s}\Psi(a,b-1;s).$$

After multiplication by $\Gamma(a)$ this becomes

$$\frac{\mathrm{d}}{\mathrm{d}s}g + \frac{b-1}{s}g = \frac{b-a-1}{s}e^s \int_1^\infty e^{-st}t^{b-a-2}(t-1)^{a-1}\,\mathrm{d}t.$$

Changing the variables back to r and ζ, we obtain

$$\zeta_r + \left(\frac{\alpha r^{\alpha-1}}{4} + \frac{(b-1)\alpha}{r}\right)\zeta = \frac{\alpha(b-a-1)}{r}\int_1^\infty e^{-r^\alpha t/4}t^{b-a-2}(t-1)^{a-1}\,\mathrm{d}t. \tag{6.9}$$

When

$$b - a - 1 < 0, \quad \text{i.e. } \lambda < \tfrac{1}{4}\alpha^2,$$

we find that the left-hand side of (6.9) is negative, and, hence,

$$\frac{\zeta_r}{\zeta} < -\frac{\alpha r^{\alpha-1}}{4} - \frac{N-2}{r}.$$

Since ξ_r/ξ is also negative, inequality (6.7) follows from the inequality above.

Assuming that (6.2) has a positive solution, we get a contradiction, as follows: let

$$\mathcal{H} = W[u,\xi]W[u,\zeta] + \frac{2}{2^*}H^2u^{2^*}\xi\zeta$$

be the expression differentiated on the left-hand side of (6.6). Then

$$\mathcal{H} = H^2\left(u_r^2\xi\zeta - u_ru(\xi\zeta)_r + u^2\xi_r\zeta_r + \frac{2}{2^*}u^{2^*}\xi\zeta\right).$$

Since we have

$$\xi(r) = O(1) \quad \text{and} \quad \zeta(r) = O(r^{\alpha(1-b)}) = O(r^{-(N-2)})$$

near $r = 0$, we obtain

$$\lim_{r\to 0^+}\mathcal{H}(r) = 0.$$

On the other hand, for large r we have

$$\xi(r) = O(r^{-\alpha(b-a)}) = O(r^{-4\lambda/\alpha})$$

and

$$\zeta(r) = O(e^{-r^\alpha/4}r^{-\alpha a}) = O(e^{-r^\alpha/4}r^{-(N-2+\alpha-(4\lambda/\alpha))}).$$

These estimates, together with the requirements (which follow from $u \in H(\alpha)$) that

$$\int_0^\infty Hu_r^2 < \infty, \quad \int_0^\infty Hu^2 < \infty \quad \text{and} \quad \int_0^\infty Hu^{2^*} < \infty,$$

imply that

$$\lim_{r\to\infty}\mathcal{H}(r) = 0.$$

But this contradicts $d\mathcal{H}/dr < 0$ for all $r > 0$. When $\lambda = \frac{1}{4}\alpha^2$, (6.9) can be integrated explicitly.

7. Non-radial solutions

In this section we show that there are cases when the least energy solutions are non-radial, proving theorem 1.4. For this, we look at the second derivative of Q_λ calculated at a radial solution u, and we argue that there are situations when this cannot be positive semi-definite. This implies that u is not a ground state. This reasoning appeared in [13, 18].

Proof of theorem 1.4. Let $u = u(r)$ be a radial solution of (1.2), $s \in \mathbb{R}$ and $h \in H(\alpha)$. We write

$$Q_\lambda(u) = \frac{\mathcal{N}(u)}{\mathcal{D}(u)},$$

where

$$\mathcal{N}(u) = \int K|\nabla u|^2 - \lambda\int K|x|^{\alpha-2}u^2$$

and

$$\mathcal{D}(u) = \left(\int K u^{2^*} \right)^{2/2^*}$$

(see (1.4)). Therefore,

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2} Q_\lambda(u + sh) \bigg|_{s=0} = \frac{\langle \mathcal{N}''(u)h, h \rangle}{\mathcal{D}(u)} - 2 \frac{\mathcal{N}'(u)h}{\mathcal{D}^2(u)} \mathcal{D}'(u)h$$
$$+ 2 \frac{\mathcal{N}(u)}{\mathcal{D}^3(u)} (\mathcal{D}'(u)h)^2 - \frac{\mathcal{N}(u)}{\mathcal{D}^2(u)} \langle \mathcal{D}''(u)h, h \rangle.$$

Since by assumption u is a critical point of Q_λ, it follows that the second and third terms cancel each other out. Therefore,

$$\langle Q_\lambda''(u)h, h \rangle = \frac{\langle \mathcal{N}''(u)h, h \rangle - Q_\lambda(u) \langle \mathcal{D}''(u)h, h \rangle}{\mathcal{D}(u)}.$$

Direct calculations show that

$$\langle \mathcal{N}''(u)h, h \rangle = 2 \int K |\nabla h|^2 - 2\lambda \int K |x|^{\alpha-2} h^2$$

and

$$\langle \mathcal{D}''(u)h, h \rangle = 2(2 - 2^*) \left(\int K u^{2^*} \right)^{(2/2^*)-2} \left(\int K u^{2^*-1} h \right)^2$$
$$+ 2(2^* - 1) \left(\int K u^{2^*} \right)^{(2/2^*)-1} \int K u^{2^*-2} h^2.$$

Again, since u is solution of (1.2), we have

$$\mathcal{N}(u) = \int K |\nabla u|^2 - \lambda \int K |x|^{\alpha-2} u^2 = \int K u^{2^*} = \mathcal{D}(u)^{2^*/2} = Q_\lambda(u)^{1/2} N.$$

If u is a local minimum of Q_λ, then

$$\langle Q_\lambda''(u)h, h \rangle \geqslant 0$$

for all $h \in H(\alpha)$, that is

$$\int K |\nabla h|^2 - \lambda \int K |x|^{\alpha-2} h^2$$
$$+ \frac{4}{N-2} Q_\lambda(u)^{-N/2} \left(\int K u^{2^*-1} h \right)^2 - \frac{N+2}{N-2} \int K u^{2^*-2} h^2 \geqslant 0. \quad (7.1)$$

Of course, if we substitute u for h in the inequality above, we obtain an equality because Q_λ is constant in the direction of u. We are therefore led to test an h orthogonal to u. We will choose $h(x) = u(r)Y(\omega)$, where $\omega \in \mathbb{S}^{N-1}$, $x = r\omega$ and Y is a first spherical harmonic function. That is, Y is not identically zero: it satisfies

$$-\Delta_\omega Y = (N-1)Y, \qquad \int_{\mathbb{S}^{N-1}} Y \, \mathrm{d}\omega = 0;$$

we will also assume that Y is normalized so that

$$\|Y\|_{L^2(\mathbb{S}^{N-1})}^2 = 1 \quad \text{and} \quad \|\nabla_\omega Y\|_{L^2(\mathbb{S}^{N-1})}^2 = N - 1.$$

With this h we obtain

$$\int_{\mathbb{R}}^N K|\nabla h|^2 = \int_{\mathbb{R}}^N K u_r^2 Y^2 + \int_{\mathbb{R}}^N K u^2 |\nabla_\omega Y|^2$$
$$= \int_0^\infty K(r) r^{N-1} u_r^2 \, dr + (N-1) \int_0^\infty K(r) r^{N-1} u^2 \, dr.$$

Substituting this in (7.1), and since h is orthogonal to all radial functions, we get

$$\int_0^\infty K(r) r^{N-1} u_r^2 \, dr + (N-1) \int_0^\infty K(r) r^{N-1} u^2 \, dr$$
$$- \lambda \int_0^\infty K(r) r^{N+\alpha-3} u^2 \, dr - \frac{N+2}{N-2} \int_0^\infty K(r) r^{N-1} u^{2^*} \, dr \geqslant 0.$$

Since u is a radial solution of (1.2) we have

$$\int_0^\infty K(r) r^{N-1} u_r^2 \, dr = \lambda \int_0^\infty K(r) r^{N+\alpha-3} u^2 + \int_0^\infty K(r) r^{N-1} u^{2^*}.$$

Therefore, (7.1) implies that

$$(N-1) \int_0^\infty K(r) r^{N-1} u^2 \, dr \geqslant \frac{4}{N-2} \int_0^\infty K(r) r^{N-1} u^{2^*}$$

or, equivalently,

$$\tfrac{1}{4}(N-1)(N-2) \int_0^\infty K(r) r^{N-1} u^2 \, dr + \lambda \int_0^\infty K(r) r^{N+\alpha-3} u^2 \, dr$$
$$\geqslant \int_0^\infty K(r) r^{N-1} u_r^2 \, dr. \quad (7.2)$$

We will show that there exist values of the three parameters N, α and λ such that the ground state is achieved and the opposite inequality to (7.2) holds. This implies that the ground state is non-radial. We already have

$$\int_0^\infty K(r) r^{N-1} u_r^2 \, dr \geqslant \lambda_1 \int_0^\infty K(r) r^{N+\alpha-3} u^2 \, dr. \quad (7.3)$$

Since u is assumed to be a radial solution of (1.2), it satisfies

$$-(K r^{N-1} u_r)_r = \lambda K r^{N+\alpha-3} u + K r^{N-1} u^{2^*-1}, \quad u_r(0) = 0;$$

hence, $u_r(r) < 0$ for all $r > 0$. We can apply the Hardy-type inequality of lemma 7.1, below, to obtain

$$\int_0^\infty K(r) r^{N-1} u_r^2 \, dr \geqslant C^{-2} \int_0^\infty K(r) r^{N-1} u^2 \, dr, \quad (7.4)$$

where $B \leqslant C \leqslant 2B$ and B is defined by

$$B^2 = \sup_{0 < x < \infty} \int_0^x K(r) r^{N-1} \, \mathrm{d}r \int_x^\infty \frac{1}{K(r) r^{N-1}} \, \mathrm{d}r. \tag{7.5}$$

Making the change of variables $s = \frac{1}{4} r^\alpha$, we can write

$$B^2 = \frac{4^{2/\alpha}}{\alpha^2} \sup_{0 < y < \infty} \int_0^y \mathrm{e}^s s^{\mu-1} \, \mathrm{d}s \int_y^\infty \mathrm{e}^{-s} s^{-\mu+(2/\alpha)-1} \, \mathrm{d}s,$$

where $\mu = N/\alpha$. We see that, by keeping μ relatively constant, we get $B \approx 1/\alpha$, which is small for α large.

From

$$\int_0^\infty K r^{N-1} u_r^2 = \left(1 - \frac{\lambda}{\lambda_1}\right) \int_0^\infty K r^{N-1} u_r^2 + \frac{\lambda}{\lambda_1} \int_0^\infty K r^{N-1} u_r^2$$

and (7.3) and (7.4) we get

$$\int_0^\infty K r^{N-1} u_r^2 \geqslant \frac{1}{4B^2} \left(1 - \frac{\lambda}{\lambda_1}\right) \int_0^\infty K r^{N-1} u^2 + \lambda \int_0^\infty K r^{N+\alpha-3} u^2.$$

Therefore, whenever

$$\frac{1}{4B^2} \left(1 - \frac{\lambda}{\lambda_1}\right) > \frac{(N-1)(N-2)}{4},$$

the inequality (7.2) cannot hold and so the ground state cannot be radial. Note that, in order for the ground state to exist, we have the sufficient conditions

$$1 > \frac{\lambda}{\lambda_1} > \begin{cases} \dfrac{1}{2} & \text{if } 2 < \alpha \leqslant N-2, \\[2mm] \dfrac{\alpha^2}{4\lambda_1} & \text{if } \alpha > N-2. \end{cases}$$

$\square$

The lemma below is adapted from [14, 16].

LEMMA 7.1. *For every decreasing function $u = u(r)$ in $H_{\mathrm{rad}}(\alpha)$, (7.4) holds where $B \leqslant C \leqslant 2B$, and B is defined by (7.5).*

Proof. Let

$$w(r) = K(r) r^{N-1} \quad \text{and} \quad h(t) = \left(\int_t^\infty \frac{1}{w(y)} \, \mathrm{d}y\right)^{1/4}.$$

Since u is assumed decreasing, positive function, we have

$$u(x) = \int_x^\infty |u'(t)| \, \mathrm{d}t = \int_x^\infty |u'(t)| w^{1/2}(t) h(t) \frac{1}{w^{1/2}(t) h(t)} \, \mathrm{d}t.$$

The Cauchy–Schwarz inequality implies that

$$u(x) \leqslant \left(\int_x^\infty |u'(t)|^2 w(t) h^2(t) \, \mathrm{d}t\right)^{1/2} \left(\int_x^\infty \frac{1}{w(t) h^2(t)} \, \mathrm{d}t\right)^{1/2}.$$

But

$$\int_x^\infty \frac{1}{w(t)h^2(t)}\,\mathrm{d}t = -2\int_x^\infty \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_t^\infty \frac{1}{w(y)}\,\mathrm{d}y\right)^{1/2}\mathrm{d}t$$

$$= 2\left(\int_x^\infty \frac{1}{w(y)}\,\mathrm{d}y\right)^{1/2}$$

$$= 2h^2(x).$$

Therefore,

$$u(x) \leqslant \left(\int_x^\infty w(t)|u'(t)|^2 h^2(t)\,\mathrm{d}t\right)^{1/2} 2^{1/2}h(x)$$

and

$$\int_0^\infty w(x)u^2(x) \leqslant 2\int_0^\infty w(x)h^2(x)\left(\int_x^\infty w(t)|u'(t)|^2 h^2(t)\,\mathrm{d}t\right).$$

Changing the order of integration yields

$$\int_0^\infty w(x)u^2(x) \leqslant 2\int_0^\infty \left(\int_0^t w(x)h^2(x)\right)w(t)|u'(t)|^2 h^2(t)\,\mathrm{d}t.$$

From the definition of B, we estimate that

$$\int_0^t w(x)h^2(x) \leqslant B\int_0^t w(x)\left(\int_0^x w(y)\,\mathrm{d}y\right)^{-1/2}$$

$$= 2B\int_0^t \frac{\mathrm{d}}{\mathrm{d}x}\left(\int_0^x w(y)\,\mathrm{d}y\right)^{1/2}$$

$$= 2B\left(\int_0^t w(y)\,\mathrm{d}y\right)^{1/2}$$

$$\leqslant 2B^2\left(\int_t^\infty \frac{1}{w(y)}\,\mathrm{d}y\right)^{-1/2}$$

$$= 2B^2 h^{-2}(t).$$

Substituting in the inequality above, we get

$$\int_0^\infty w(x)u^2(x) \leqslant 4B^2 \int_0^\infty w(t)|u'(t)|^2\,\mathrm{d}t,$$

which is the inequality (7.4).   □

Acknowledgments

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