Positive solutions for a quasilinear Schrödinger equation with critical growth *

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Abstract

We consider the quasilinear problem

$$-\varepsilon^p \mathrm{div}(|\nabla u|^{p-2}\nabla u) + V(z)u^{p-1} = f(u) + u^{p^*-1}, \quad u \in W^{1,p}(\mathbb{R}^N),$$

where $\varepsilon > 0$ is a small parameter, $1 < p < N$, $p^* = Np/(N-p)$, V is a positive potential and f is a superlinear function. Under a local condition for V we relate the number of positive solutions with the topology of the set where V attains its minimum. In the proof we apply Ljusternik-Schnirelmann theory.

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1 Introduction

The main purpose of this paper is to establish a multiplicity result for the following quasilinear critical problem

$$(P_\varepsilon) \qquad \begin{cases} -\varepsilon^p \Delta_p u + V(z)u^{p-1} = f(u) + u^{p^*-1} \ \text{ in } \mathbb{R}^N, \\ u \in C^{1,\alpha}_{loc}(\mathbb{R}^N) \cap W^{1,p}(\mathbb{R}^N), \quad u(z) > 0 \ \text{ for all } z \in \mathbb{R}^N, \end{cases}$$

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where $\varepsilon > 0$, $1 < p < N$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator, $p^* := Np/(N-p)$, $0 < \alpha < 1$ and $V \in C(\mathbb{R}^N, \mathbb{R})$ is such that

(V_1) $V_0 := \inf_{x \in \mathbb{R}^N} V(x) > 0$,

(V_2) there exists an open bounded set $\Lambda \subset \mathbb{R}^N$ such that

$$V_0 < \min_{\partial \Lambda} V$$

and $M := \{x \in \Lambda : V(x) = V_0\} \neq \varnothing$.

We also suppose that $f \in C^1(\mathbb{R}^+, \mathbb{R})$ satisfies

(f_1) $f(s) = o(s^{p-1})$ as $s \to 0^+$,

(f_2) there exists $p < q < p^*$ such that $f(s) = o(s^{q-1})$ as $s \to \infty$,

(f_3) there exists $p < \theta < q$ such that

$$0 < \theta F(s) := \theta \int_0^s f(\tau) \mathrm{d}\tau \leq s f(s) \quad \text{for all } s > 0,$$

(f_4) the function $s \mapsto f(s)/s^{p-1}$ is increasing,

(f_5) $f(s) \geq \lambda s^{q_1 - 1}$ for all $s > 0$, with $q_1 \in (p, p^*)$ and λ satisfying

$(f_5 a)$ $\lambda > 0$ if either $N \geq p^2$, or $p < N < p^2$ and $p^* - p/(p-1) < q_1 < p^*$,

$(f_5 b)$ λ is sufficiently large if $p < N < p^2$ and $p < q_1 \leq p^* - p/(p-1)$.

We are interested in relating the number of positive solutions with the topology of the set M. If Y is a closed set of a topological space X, we denote by $\operatorname{cat}_X(Y)$ the Ljusternik-Schnirelmann category of Y in X, namely the least number of closed and contractible sets in X which cover Y. We shall prove the following result.

Theorem 1.1. *Suppose that the potential V satisfies $(V_1)-(V_2)$ and the function f satisfies $(f_1)-(f_5)$. Then, for any $\delta > 0$ such that*

$$M_\delta := \{x \in \mathbb{R}^N : \operatorname{dist}(x, M) < \delta\} \subset \Lambda,$$

there exists $\varepsilon_\delta > 0$ such that, for any $\varepsilon \in (0, \varepsilon_\delta)$, the problem (P_ε) has at least $\operatorname{cat}_{M_\delta}(M)$ solutions.

In the proof we will apply critical point theory. Unfortunately, since we have no information about the behavior of the potential V at the infinity, we are not able to show that the functional associated to (P_ε) satisfies any compactness condition. Hence, we follow an argument introduced by del Pino and Felmer in [10], which consists in making a suitable modification on f, solve a modified problem and then check that, for ε small enough, the solutions of the new problem are indeed solutions of the original one.

We notice that, even for the modified problem, it is not easy to obtain compactness in view of the critical growth of the nonlinearity. To overcome this problem we use some calculations from [19] (see also [18]), where the author used the ideas of the paper of Brezis and Nirenberg [7]. We emphasize that, in [19], the author showed that the weak limit of a Palais-Smale sequence is a nontrivial solution of the modified problem. Here, since we want to apply Ljusternik-Schnirelmann theory, we need effectively check that the modified functional satisfies Palais-Smale below a suitable level (see Section 3). The concentration-compactness principle due to Lions [17] plays a fundamental role in this setting.

In order to obtain multiple solutions for the modified problem, we use a technique introduced by Benci and Cerami in [5]. The main idea is to make precisely comparisons between the category of some sublevel sets of the modified functional and the category of the set M. This kind of argument for the Schrödinger equation has already appeared in [8] (see also [2]), where subcritical problems were considered.

The main motivation for the study of (P_ε) in the semilinear case $p = 2$ arise from seeking standing waves solutions for the nonlinear Schrödinger equation

$$i\varepsilon\frac{\partial\psi}{\partial t} = -\frac{\varepsilon^2}{2m}\Delta\psi + V(z)\psi - \gamma|\psi|^{r-2}\psi \quad \text{in } \mathbb{R}^N,$$

namely solutions of the form $\psi(z,t) = \exp(-i\varepsilon^{-1}t)u(z)$, where ε, m and γ are positive constants and $r > 1$. Indeed, this is equivalent to solve the semilinear elliptic equation

$$-\frac{\varepsilon^2}{2m}\Delta u + V(z)u = \gamma|u|^{r-2}u \quad \text{in } \mathbb{R}^N. \tag{1.1}$$

The case $N = 1$ and $r = 3$ was considered by Floer and Weinstein in [13], where the authors used Lyapunov-Schmidt reduction to prove the existence of standing waves solutions concentrating at each given nondegenerated critical point of the bounded potential V. Their results were extend to higher dimension by Oh [20, 21] in the subcritical case $2 < r < 2^*$. The first author to apply critical point theory to this kind of problem was Rabinowitz in [23]. He proved the existence of solutions for more general subcritical nonlinearities $f(u)$ by supposing

$$0 < \inf_{z\in\mathbb{R}^N} V(z) < \liminf_{|z|\to\infty} V(z). \tag{1.2}$$

Wang [25] complemented the work of Rabinowitz by obtaining the concentration behavior of the solutions.

Note that the above condition for V is stronger than (V_2). The local condition (V_2) was first considered by del Pino and Felmer in [10], where the authors also dealt with a subcritical nonlinearity $f(u)$ and introduced the local mountain pass argument which is used here. The result in [10] is related with existence and concentration behavior of solutions. It was complemented in [3] to the critical case. Recently, this last result was extended to the quasilinear case $1 < p < N$ in a paper of do Ó [19].

There is also a quite extensive literature about multiplicity of solutions for the Schrödinger equation. We cite here some works which are closely related with our result. We begin by quoting the paper of Cingolani and Lazzo [8], which related the topology of the set of minima of V with the number of positive solutions of (1.1) by assuming that V satisfies (1.2). In [2] Alves and Figueiredo extended this last result to the quasilinear case with nonhomogeneous subcritical nonlinearity $f(u)$ under the local condition (V_2). The critical case was considered by Figueiredo in [12]. We also would like to cite the papers [9, 4, 15, 22] where similar results were obtained for some related semilinear equations.

In view of the existence results presented in [3], it is natural to ask if we can obtain multiplicity results for the quasilinear problem under the local condition (V_2). In this paper, we present a positive answer for this question. Our result extend those presented in [2, 12] and complement those of [8, 3, 19]. Finally we emphasize that, although we deal with quasilinear case, our result seem to be new even in the semilinear case $p = 2$.

The paper is organized as follows: in Section 2 we modify the original problem. The Palais-Smale condition for the modified functional is proved in Section 3. In Section 4, we obtain a multiplicity result for the modified problem. Theorem 1.1 is proved in Section 5.

2 The modified functional

Throughout the paper the conditions $(V_1) - (V_2)$ and $(f_1) - (f_5)$ will be assumed. For save notation, we write only $\int u$ instead of $\int_{\mathbb{R}^N} u(x)\mathrm{d}x$.

In order to overcome the lack of compactness of the problem (P_ε) we make a slightly adaptation of the penalization method introduced by del Pino and Felmer in [10]. So, we choose $k > \theta(\theta - p)^{-1}$, where θ is given by (f_3), and set

$$\widehat{f}(s) := \begin{cases} 0 & \text{if } s < 0, \\ f(s) + s^{p^*-1} & \text{if } 0 \le s \le a, \\ \frac{V_0}{k} s^{p-1} & \text{if } s > a, \end{cases}$$

where $a > 0$ is such that $f(a) + a^{p^*-1} = k^{-1}V_0 a^{p-1}$. Let $0 < t_a < a < T_a$ and take a function $\eta \in C_0^\infty(\mathbb{R}, \mathbb{R})$ such that

(η_1) $\eta(s) \le \widehat{f}(s)$ for all $s \in [t_a, T_a]$,

(η_2) $\eta(t_a) = \widehat{f}(t_a)$, $\eta(T_a) = \widehat{f}(T_a)$, $\eta'(t_a) = \widehat{f}'(t_a)$ and $\eta'(T_a) = \widehat{f}'(T_a)$,

(η_3) the map $s \mapsto \eta(s)/s^{p-1}$ is increasing for all $s \in [t_a, T_a]$.

By using the above functions we can define $\widetilde{f} \in C^1(\mathbb{R}, \mathbb{R})$ as follows

$$\widetilde{f}(s) := \begin{cases} \widehat{f}(s) & \text{if } s \notin [t_a, T_a], \\ \eta(s) & \text{if } s \in [t_a, T_a]. \end{cases}$$

If χ_Λ denotes the characteristic function of the set Λ, we introduce the penalized nonlinearity $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ by setting

$$g(z,s) := \chi_\Lambda(z)(f(s) + s^{p^*-1}) + (1 - \chi_\Lambda(z))\widetilde{f}(s). \tag{2.1}$$

Conditions $(f_1) - (f_4)$ and $(\eta_1) - (\eta_3)$ imply that $g(z,s)$ satisfies

(g_1) $g(z,s) = o(s^{p-1})$ as $s \to 0$, uniformly in $z \in \mathbb{R}^N$,

(g_2) $g(z,s) \le f(s) + s^{p^*-1}$ for all $z \in \mathbb{R}^N,\ s > 0$,

(g_3) for $\theta \in (p,q)$ given by (f_3) there hold

(i) $0 < \theta G(z,s) := \theta \int_0^s g(z,\tau)\mathrm{d}\tau < g(z,s)s$ for all $z \in \Lambda$, $s > 0$,

(ii) $0 \le pG(z,s) \le g(z,s)s \le \frac{1}{k}V(z)s^p$ for all $z \in \mathbb{R}^N \setminus \Lambda$, $s > 0$,

(g_4) the function $s \mapsto g(z,s)/s^{p-1}$ is increasing for all $z \in \Lambda$, $s > 0$.

We now note that, if u_ε is a positive solution of the equation

$$-\varepsilon^p \Delta_p u + V(z)u^{p-1} = g(z,u) \quad \text{in } \mathbb{R}^N$$

such that $u_\varepsilon(z) \le t_a$ for all $z \in \mathbb{R}^N \setminus \Lambda$, then $g(z,u_\varepsilon) = f(u_\varepsilon) + u_\varepsilon^{p^*-1}$ and therefore u_ε is also a solution of (P_ε). Hence, we deal in the sequel with the penalized problem

$$(\widetilde{P}_\varepsilon) \qquad \begin{cases} -\Delta_p u + V(\varepsilon x)u^{p-1} = g(\varepsilon x, u) \ \text{ in } \mathbb{R}^N, \\ u \in C^{1,\alpha}_{loc}(\mathbb{R}^N) \cap W^{1,p}(\mathbb{R}^N), \quad u(x) > 0 \ \text{ for all } x \in \mathbb{R}^N, \end{cases}$$

and we will look for solutions u_ε of $(\widetilde{P}_\varepsilon)$ verifying

$$u_\varepsilon(x) \le t_a \quad \text{for all } x \in \mathbb{R}^N \setminus \Lambda_\varepsilon, \tag{2.2}$$

where

$$\Lambda_\varepsilon := \{x \in \mathbb{R}^N : \varepsilon x \in \Lambda\}.$$

For any $\varepsilon > 0$, let us consider the Banach space

$$X_\varepsilon := \left\{ u \in W^{1,p}(\mathbb{R}^N) : \int V(\varepsilon x)|u|^p < \infty \right\}$$

endowed with the norm

$$\|u\|_\varepsilon := \left\{ \int (|\nabla u|^p + V(\varepsilon x)|u|^p) \right\}^{1/p}.$$

The weak solutions of $(\widetilde{P}_\varepsilon)$ are the positive critical points of the C^1-functional $I_\varepsilon : X_\varepsilon \to \mathbb{R}$ given by

$$I_\varepsilon(u) := \frac{1}{p}\int (|\nabla u|^p + V(\varepsilon x)|u|^p) - \int G(\varepsilon x, u).$$

We denote by $\mathcal{N}_\varepsilon$ the Nehari manifold of I_ε, that is,

$$\mathcal{N}_\varepsilon := \{u \in X_\varepsilon \setminus \{0\} : \langle I'_\varepsilon(u), u\rangle = 0\}$$

and define the number b_ε by setting

$$b_\varepsilon := \inf_{u \in \mathcal{N}_\varepsilon} I_\varepsilon(u).$$

Note that, for any given $\xi > 0$, we can use (g_2), (f_1) and (f_2) to obtain $C_\xi > 0$ verifying

$$|g(\varepsilon x, s)| \le \xi |s|^{p-1} + C_\xi |s|^{q-1} + |s|^{p^*-1} \text{ for all } x \in \mathbb{R}^N, \ s \in \mathbb{R}. \tag{2.3}$$

This, (2.1) and (g_3) provide $r_\varepsilon > 0$ such that

$$\|u\|_\varepsilon \ge r_\varepsilon > 0 \text{ for all } u \in \mathcal{N}_\varepsilon. \tag{2.4}$$

In what follows, $\operatorname{supp} u$ denotes the support of a function $u \in X_\varepsilon$.

Lemma 2.1. *Let $u \in X_\varepsilon$ be a nonnegative function such that $\operatorname{supp} u \cap \Lambda_\varepsilon$ has positive measure. Then there exists a unique $t_u > 0$ such that $t_u u \in \mathcal{N}_\varepsilon$.*

Proof. If $\psi(t) := I_\varepsilon(tu)$ for $t \ge 0$, inequality (2.3) and the Sobolev embeddings imply that ψ is positive near the $t = 0$. Moreover,

$$\psi(t) \le \frac{t^p}{p}\|u\|_\varepsilon^p - \int_{\Lambda_\varepsilon} G(\varepsilon x, tu).$$

Since the set $\{x \in \Lambda_\varepsilon : u(x) > 0\}$ has positive measure, the above expression and (g_3)(i) imply that $\psi(t) \to -\infty$ as $t \to \infty$. Hence, there exists $t_u > 0$ such that $\psi'(t_u) = 0$, namely the point where ψ attains its maximum. A direct computation shows that $t_u u \in \mathcal{N}_\varepsilon$. The uniqueness follows from the monotonicity condition (g_4). □

Remark 2.2. *If $u \in \mathcal{N}_\varepsilon$, then the last inequality in (g_3)(ii) implies that $\operatorname{supp} u \cap \Lambda_\varepsilon$ has positive measure. Thus, we can argue as above to conclude that $I_\varepsilon(tu) \le I_\varepsilon(u)$ for all $t \ge 0$.*

3 The Palais-Smale condition

Let V be a Banach space, $\mathcal{V}$ be a C^1-manifold of V and $I : V \to \mathbb{R}$ a C^1-functional. We say that I restricted to $\mathcal{V}$ satisfies the Palais-Smale condition at level c if any sequence $(u_n) \subset \mathcal{V}$ such that $I(u_n) \to c$ and $\|I'(u_n)\|_* \to 0$ contains a convergent subsequence. Here, we are denoting by $\|I'(u)\|_*$ the norm of the derivative of I restricted to $\mathcal{V}$ at the point u (see [26, Section 5.3]).

From now on we denote by S the best constant of the Sobolev embedding $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N)$, namely

$$S := \inf\left\{ \int_{\mathbb{R}^N} |\nabla u|^p : u \in W^{1,p}(\mathbb{R}^N), \ \int |u|^{p^*} = 1 \right\}. \tag{3.1}$$

The objective of this section is to establish the following local compactness result for I_ε.

Proposition 3.1. *The functional I_ε restricted to $\mathcal{N}_\varepsilon$ satisfies the Palais-Smale condition at any level $c < \frac{1}{N} S^{N/p}$.*

For the proof, we need first to consider the unconstrained functional.

Lemma 3.2. *The functional I_ε satisfies the Palais-Smale condition at any level $c < \frac{1}{N} S^{N/p}$.*

Proof. Let $(u_n) \subset X_\varepsilon$ be such that $I_\varepsilon(u_n) \to c < \frac{1}{N} S^{N/p}$ and $I'_\varepsilon(u_n) \to 0$. Standard calculations show that (u_n) is bounded in X_ε (see [19, Assertion 2.2]). Hence $\langle I'_\varepsilon(u_n), u_n \rangle \to 0$ and we have that

$$\|u_n\|_\varepsilon^p = \int g(\varepsilon x, u_n) u_n + o_n(1), \tag{3.2}$$

where $o_n(1)$ denotes a quantity approaching zero as $n \to \infty$. Up to a subsequence, we may suppose that

$$\begin{aligned} & u_n \rightharpoonup u && \text{weakly in } X_\varepsilon, \\ & u_n \to u && \text{strongly in } L^s_{loc}(\mathbb{R}^N) \text{ for any } p \le s < p^*, \\ & u_n(x) \to u(x) && \text{for a.e. } x \in \mathbb{R}^N. \end{aligned} \tag{3.3}$$

As proved in [19, Theorem 2.4], u is a critical point of I_ε, and therefore

$$\|u\|_\varepsilon^p = \int g(\varepsilon x, u) u. \tag{3.4}$$

Claim 1. $\displaystyle \lim_{n\to\infty} \int g(\varepsilon x, u_n) u_n = \int g(\varepsilon x, u) u.$

This claim, (3.2) and (3.4) imply that $\|u_n\|_\varepsilon^p \to \|u\|_\varepsilon^p$, from which follows that $u_n \to u$ in X_ε.

In order to prove Claim 1 we first note that, arguing as in [2, Lemma 3.3], we can show that, for any $\zeta > 0$ given, there exists $R > 0$ such that

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N \setminus B_R(0)} (|\nabla u_n|^p + V(\varepsilon x)|u_n|^p) < \zeta.$$

This inequality, (g_2), (f_1), (f_2) and the Sobolev embeddings imply that, for n large enough, there holds

$$\begin{aligned} \int_{\mathbb{R}^N \setminus B_R(0)} g(\varepsilon x, u_n) u_n &\le C_1 \int_{\mathbb{R}^N \setminus B_R(0)} \left(|u_n|^p + |u_n|^q + |u_n|^{p^*}\right) \\ &\le C_2(\zeta + \zeta^{q/p} + \zeta^{p^*/p}) \end{aligned} \tag{3.5}$$

where C_1, C_2 are positive constants. On the other hand, taking R large enough, we can suppose that

$$\int_{\mathbb{R}^N \setminus B_R(0)} g(\varepsilon x, u) u < \zeta.$$

Hence we can use this inequality and (3.5) to conclude that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N\setminus B_R(0)} g(\varepsilon x, u_n)u_n = \int_{\mathbb{R}^N\setminus B_R(0)} g(\varepsilon x, u)u. \tag{3.6}$$

Now we note that, in view of the definition of g, there holds

$$g(\varepsilon x, u_n)u_n \le f(u_n)u_n + a^{p^*} + \frac{V_0}{k}|u_n|^p \quad \text{for any } x \in \mathbb{R}^N \setminus \Lambda_\varepsilon.$$

Since the set $B_R(0) \cap (\mathbb{R}^N \setminus \Lambda_\varepsilon)$ is bounded, we can use the above estimate, (f_1), (f_2), (3.3) and Lebesgue's theorem to conclude that

$$\lim_{n\to\infty} \int_{B_R(0)\cap(\mathbb{R}^N\setminus\Lambda_\varepsilon)} g(\varepsilon x, u_n)u_n = \int_{B_R(0)\cap(\mathbb{R}^N\setminus\Lambda_\varepsilon)} g(\varepsilon x, u)u. \tag{3.7}$$

Claim 2. $u_n \to u$ in $L^{p^*}(\Lambda_\varepsilon)$.

By using the above claim, (g_2), (f_1), (f_2), (3.3) and Lebesgue's theorem again, we conclude that

$$\lim_{n\to\infty} \int_{B_R(0)\cap\Lambda_\varepsilon} g(\varepsilon x, u_n)u_n = \int_{B_R(0)\cap\Lambda_\varepsilon} g(\varepsilon x, u)u.$$

Claim 1 is now a direct consequence of the above expression, (3.6) and (3.7).

It remains to prove the Claim 2. We may suppose that

$$|\nabla u_n|^p \rightharpoonup \mu \quad \text{and} \quad |u_n|^{p^*} \rightharpoonup \nu \qquad \text{(weak}^*\text{-sense of measures)}.$$

Using the concentration compactness principle (cf. [17, Lemma 1.2]) we obtain an at most countable index set Γ, sequences $(x_i) \subset \mathbb{R}^N$, $(\mu_i), (\nu_i) \subset (0, \infty)$, such that

$$\mu \ge |\nabla u|^p + \sum_{i\in\Gamma} \mu_i \delta_{x_i}, \quad \nu = |u|^{p^*} + \sum_{i\in\Gamma} \nu_i \delta_{x_i} \quad \text{and} \quad S\nu_i^{p/p^*} \le \mu_i, \tag{3.8}$$

for all $i \in \Gamma$, where δ_{x_i} is the Dirac mass at $x_i \in \mathbb{R}^N$.

It suffices to show that $\{x_i\}_{i\in\Gamma} \cap \Lambda_\varepsilon = \varnothing$. Suppose, by contradiction, that $x_i \in \Lambda_\varepsilon$ for some $i \in \Gamma$. Define, for $\varrho > 0$, the function $\psi_\varrho(x) := \psi((x - x_i)/\varrho)$ where $\psi \in C_0^\infty(\mathbb{R}^N, [0, 1])$ is such that $\psi \equiv 1$ on $B_1(0)$, $\psi \equiv 0$ on $\mathbb{R}^N \setminus B_2(0)$ and $|\nabla\psi|_\infty \le 2$. We suppose that ϱ is chosen in such way that the support of ψ_ϱ is contained in Λ_ε.

Since $(\psi_\varrho u_n)$ is bounded, $\langle I_\varepsilon'(u_n), \psi_\varrho u_n\rangle \to 0$, and therefore

$$\begin{aligned} \int \psi_\varrho |\nabla u_n|^p \;\; &\le \;\; -\int |\nabla u_n|^{p-2} u_n (\nabla u_n \cdot \nabla \psi_\varrho) \\ &\quad + \int f(u_n)\psi_\varrho u_n + \int \psi_\varrho |u_n|^{p^*} + o_n(1). \end{aligned}$$

Since f has subcritical growth and ψ_ϱ has compact support, we can let $n \to \infty$ and $\varrho \to 0$ to conclude that $\nu_i \ge \mu_i$. It follows from the last statement in (3.8) that

$$\nu_i \ge S^{N/p},$$

and therefore we can use (g_3) and (f_4) to compute

$$\begin{aligned}
c &= I_\varepsilon(u_n) - \frac{1}{p}\langle I'_\varepsilon(u_n), u_n\rangle + o_n(1) \\
&= \int_{\mathbb{R}^N\setminus\Lambda_\varepsilon}\left(\frac{1}{p}g(\varepsilon x, u_n)u_n - G(\varepsilon x, u_n)\right) + \\
&\quad + \int_{\Lambda_\varepsilon}\left(\frac{1}{p}f(u_n)u_n - F(u_n)\right) + \frac{1}{N}\int_{\Lambda_\varepsilon}|u_n|^{p^*} + o_n(1) \\
&\geq \frac{1}{N}\int_{\Lambda_\varepsilon}|u_n|^{p^*} + o_n(1) \geq \frac{1}{N}\int_{\Lambda_\varepsilon}\psi_\varrho|u_n|^{p^*} + o_n(1).
\end{aligned}$$

Hence, taking the limit and using (3.8) we get

$$c \geq \frac{1}{N}\sum_{\{i\in\Gamma: x_i\in\Lambda_\varepsilon\}}\psi_\varrho(x_i)\nu_i = \frac{1}{N}\sum_{\{i\in\Gamma: x_i\in\Lambda_\varepsilon\}}\nu_i \geq \frac{1}{N}S^{N/p},$$

which does not make sense. This conclude the proof of Claim 2 and therefore the lemma is proved. □

We are now ready to present the proof of Proposition 3.1.

Proof of Proposition 3.1. Let $(u_n) \subset \mathcal{N}_\varepsilon$ be such that $I_\varepsilon(u_n) \to c < \frac{1}{N}S^{N/p}$ and $\|I'_\varepsilon(u_n)\|_* \to 0$. Then there exists $(\lambda_n) \subset \mathbb{R}$ such that

$$I'_\varepsilon(u_n) = \lambda_n J'_\varepsilon(u_n) + o_n(1), \tag{3.9}$$

where $J_\varepsilon : X_\varepsilon \to \mathbb{R}$ is given by

$$J_\varepsilon(u) := \int|\nabla u|^p + \int V(\varepsilon x)|u|^p - \int g(\varepsilon x, u)u.$$

Since $(u_n) \subset \mathcal{N}_\varepsilon$, we can use (2.1), (η_3) and (f_4) to get

$$\begin{aligned}
\langle J'_\varepsilon(u_n), u_n\rangle &= \int\left\{(p-1)g(\varepsilon x, u_n)u_n - g'(\varepsilon x, u_n)u_n^2\right\} \\
&= \int_{\Lambda_\varepsilon\cup\{u_n<t_a\}}\left\{(p-1)f(u_n)u_n - f'(u_n)u_n^2\right\} \\
&\quad - \int_{\Lambda_\varepsilon\cup\{u_n<t_a\}}(p^*-p)|u_n|^{p^*} \\
&\quad + \int_{(\mathbb{R}^N\setminus\Lambda_\varepsilon)\cap\{t_a\leq u_n\leq T_a\}}\left\{(p-1)\eta(u_n)u_n - \eta'(u_n)u_n^2\right\} \\
&\leq -(p^*-p)\int_{\Lambda_\varepsilon}|u_n|^{p^*},
\end{aligned}$$

where $g'(x,s)$ means the derivative with respect to the second variable and the numbers t_a and T_a were fixed at the beginning of Section 2.

By the above expression, we may suppose that $\langle J_\varepsilon'(u_n), u_n\rangle \to l \leq 0$. We claim that $l < 0$. If this is the case, it follows from

$$0 = \langle I_\varepsilon'(u_n), u_n\rangle = \lambda_n \langle J_\varepsilon'(u_n), u_n\rangle + o_n(1)$$

that $\lambda_n \to 0$. Hence, use can use (3.9) to conclude that $I_\varepsilon'(u_n) \to 0$ in the dual space of X_ε. We may now invoke Lemma 3.2 to obtain a convergent subsequence of (u_n).

It remains to prove that $l < 0$. Suppose, by contradiction, that $l = 0$. Then $|\langle J_\varepsilon'(u_n), u_n\rangle| \geq (p^* - p)\int_{\Lambda_\varepsilon} |u_n|^{p^*}$ and therefore $u_n \to 0$ in $L^{p^*}(\Lambda_\varepsilon)$. By interpolation, $u_n \to 0$ in $L^p(\Lambda_\varepsilon)$ and $L^q(\Lambda_\varepsilon)$. It follows from (2.3) that $\int_{\Lambda_\varepsilon} g(\varepsilon x, u_n)u_n = o_n(1)$. This and (g_3)(ii) provide

$$\|u_n\|_\varepsilon^p = \int_{\mathbb{R}^N \setminus \Lambda_\varepsilon} g(\varepsilon x, u_n)u_n + \int_{\Lambda_\varepsilon} g(\varepsilon x, u_n)u_n \leq \frac{1}{k}\int_{\mathbb{R}^N \setminus \Lambda_\varepsilon} |u_n|^p + o_n(1),$$

and therefore

$$\left(1 - \frac{1}{k}\right)\|u_n\|_\varepsilon^p = o_n(1),$$

which contradicts (2.4) and proves the lemma. □

4 Multiplicity of solutions for $(\widetilde{P}_\varepsilon)$

In this section we present a multiplicity result for the penalized problem. More specifically, we shall prove the next result.

Theorem 4.1. *For any $\delta > 0$ such that $M_\delta \subset \Lambda$, there exists $\varepsilon_\delta > 0$ such that, for any $\varepsilon \in (0, \varepsilon_\delta)$, the problem $(\widetilde{P}_\varepsilon)$ has at least $\operatorname{cat}_{M_\delta}(M)$ solutions.*

The proof will be done by applying an abstract result of Ljusternik-Schnirelmann type for the functional I_ε constrained to $\mathcal{N}_\varepsilon$. In order to do this, we need perform suitable estimates in the minimax level b_ε. As we will see, it is important to compare it with the minimax level of the limit problem

$$\begin{cases} -\Delta_p u + V_0 u^{p-1} = f(u) + u^{p^*-1} \ \text{ in } \mathbb{R}^N, \\ u \in C^{1,\alpha}_{loc}(\mathbb{R}^N) \cap W^{1,p}(\mathbb{R}^N), \quad u(x) > 0 \ \text{ for all } x \in \mathbb{R}^N, \end{cases} \tag{4.1}$$

whose solutions are the positive critical points of the C^1-functional $I_0 : W^{1,p}(\mathbb{R}^N) \to \mathbb{R}$ given by

$$I_0(u) := \frac{1}{p}\|u\|_X^p - \int F(u) - \frac{1}{p^*}\int |u|^{p^*},$$

where

$$\|u\|_X := \left\{\int (|\nabla u|^p + V_0|u|^p)\right\}^{1/p} \quad \text{for all } u \in X := W^{1,p}(\mathbb{R}^N).$$

Let $\mathcal{M}_0 := \{u \in X \setminus \{0\} : \langle I_0(u), u\rangle = 0\}$ be the Nehari manifold of I_0 and consider the minimization problem

$$c_0 := \inf_{u\in\mathcal{M}_0} I_0(u) = \inf_{u\in X\setminus\{0\}} \sup_{t\geq 0} I_0(tu). \tag{4.2}$$

It can be proved (see [26, Chapter 4]) that c_0 is positive and that, for any $u \in X \setminus \{0\}$, there exists a unique $t_u > 0$ such that $t_u u \in \mathcal{M}_0$. The maximum of the function $t \mapsto I_0(tu)$ for $t \geq 0$ is achieved at $t = t_u$. Moreover, as proved in [19, Lemma 3.4], we have that

$$\lim_{\varepsilon\to 0^+} b_\varepsilon = c_0 < \frac{1}{N} S^{N/p}. \tag{4.3}$$

The proof of the following result is similar to that presented in [1, Theorem 3.1] and it will be omitted.

Lemma 4.2. *Let $(w_n) \subset \mathcal{M}_0$ be such that $I_0(w_n) \to c_0$ and $w_n \rightharpoonup w$ weakly in X. Then there exists a sequence $(y_n) \subset \mathbb{R}^N$ such that $v_n := w_n(\cdot + y_n) \to v \in \mathcal{M}_0$ with $I_0(v) = c_0$. Moreover, if the weak limit w is nonzero, then (y_n) can be taken identically zero and therefore $w_n \to w$ in X.*

We consider $\delta > 0$ such that $M_\delta \subset \Lambda$ and choose $\psi \in C_0^\infty(\mathbb{R}^+, [0, 1])$ satisfying $\psi(s) = 1$ if $0 \leq s \leq \delta/2$ and $\psi(s) = 0$ if $s \geq \delta$. For each $y \in M$, we define the function $\Psi_{\varepsilon,y} : \mathbb{R}^N \to \mathbb{R}$ by setting

$$\Psi_{\varepsilon,y}(x) := \psi(|\varepsilon x - y|)\omega\left(\frac{\varepsilon x - y}{\varepsilon}\right)$$

where ω is a solution of (4.1) such that $I_0(\omega) = c_0$. Note that the existence of ω is assured by the above lemma.

Let $\Phi_\varepsilon : M \to \mathcal{N}_\varepsilon$ be given by

$$\Phi_\varepsilon(y) := t_\varepsilon \Psi_{\varepsilon,y},$$

where $t_\varepsilon > 0$ is the unique number such that $t_\varepsilon \Psi_{\varepsilon,y} \in \mathcal{N}_\varepsilon$. Since for any $x \in B_{\delta/2\varepsilon}(y/\varepsilon)$ we have $\Psi_{\varepsilon,y}(x) = \omega((\varepsilon x - y)/\varepsilon)$ and $y/\varepsilon \in \Lambda_\varepsilon$, Lemma 2.1 shows that Φ_ε is well defined.

Lemma 4.3. *Uniformly for $y \in M$, we have*

$$\lim_{\varepsilon\to 0^+} I_\varepsilon(\Phi_\varepsilon(y)) = c_0.$$

Proof. Suppose that the result is false. Then there exist $\gamma > 0$, $(y_n) \subset M$ and $\varepsilon_n \to 0^+$ such that

$$|I_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - c_0| \geq \gamma > 0. \tag{4.4}$$

By using the change of variables $z := (\varepsilon_n x - y_n)/\varepsilon_n$ we can write

$$\begin{aligned} I_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) &= \frac{t_n^p}{p}\int (|\nabla(\psi(|\varepsilon_n z|)\omega(z))|^p + V(\varepsilon_n z + y_n)|\psi(|\varepsilon_n z|)\omega(z)|^p)\,\mathrm{d}z \\ &\quad - \int F(t_n\psi(|\varepsilon_n z|)\omega(z))\mathrm{d}z - \frac{t_n^{p^*}}{p^*}\int |\psi(|\varepsilon_n z|)\omega(z)|^{p^*}\mathrm{d}z. \end{aligned}$$

Arguing as in [19, Assertion 3.1] we can check that, up to a subsequence, $t_n \to 1$. Thus, letting $n \to \infty$ in the above equality and using Lebesgue's theorem we conclude that

$$\lim_{n\to\infty} I_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) = I_0(\omega) = c_0,$$

which contradicts (4.4) and proves the lemma. $\square$

For any $\delta > 0$, let $\rho = \rho(\delta) > 0$ be such that $M_\delta \subset B_\rho(0)$. Let $\Upsilon : \mathbb{R}^N \to \mathbb{R}^N$ be defined as $\Upsilon(x) := x$ for $|x| < \rho$ and $\Upsilon(x) := \rho x/|x|$ for $|x| \geq \rho$. Finally, let us consider the barycenter map $\beta_\varepsilon : \mathcal{N}_\varepsilon \to \mathbb{R}^N$ given by

$$\beta_\varepsilon(u) := \frac{\displaystyle\int \Upsilon(\varepsilon x)|u(x)|^p \,\mathrm{d}x}{\displaystyle\int |u(x)|^p \,\mathrm{d}x}.$$

Since $M \subset B_\rho(0)$, we can use the definition of Υ and Lebesgue's theorem to conclude that

$$\lim_{\varepsilon\to 0^+} \beta_\varepsilon(\Phi_\varepsilon(y)) = y \quad \text{uniformly for } y \in M. \tag{4.5}$$

As in [8], we introduce a subset Σ_ε of $\mathcal{N}_\varepsilon$ by taking a function $h : [0,\infty) \to [0,\infty)$ such that $h(\varepsilon) \to 0$ as $\varepsilon \to 0^+$, and setting

$$\Sigma_\varepsilon := \{u \in \mathcal{N}_\varepsilon : I_\varepsilon(u) \leq c_0 + h(\varepsilon)\}.$$

Given $y \in M$, we can use Lemma 4.3 to conclude that $h(\varepsilon) = |I_\varepsilon(\Phi_{\varepsilon,y}) - c_0|$ is such that $h(\varepsilon) \to 0$ as $\varepsilon \to 0^+$. Thus, $\Phi_{\varepsilon,y} \in \Sigma_\varepsilon$ and therefore $\Sigma_\varepsilon \neq \varnothing$ for any $\varepsilon > 0$.

Lemma 4.4. *For any $\delta > 0$ we have that*

$$\lim_{\varepsilon\to 0^+} \sup_{u\in\Sigma_\varepsilon} \operatorname{dist}(\beta_\varepsilon(u), M_\delta) = 0.$$

Proof. If $(\varepsilon_n) \subset \mathbb{R}^+$ satisfies $\varepsilon_n \to 0$, then there exists $(u_n) \subset \Sigma_{\varepsilon_n}$ such that

$$\operatorname{dist}(\beta_{\varepsilon_n}(u_n), M_\delta) = \sup_{u\in\Sigma_{\varepsilon_n}} \operatorname{dist}(\beta_{\varepsilon_n}(u), M_\delta) + o_n(1).$$

Thus, it suffices to find a sequence $(y_n) \subset M_\delta$ such that

$$|\beta_{\varepsilon_n}(u_n) - y_n| = o_n(1). \tag{4.6}$$

In order to obtain such sequence, we recall that $(u_n) \subset \Sigma_{\varepsilon_n} \subset \mathcal{N}_{\varepsilon_n}$, and therefore we can use (4.2) and the definition of Σ_{ε_n} to get

$$c_0 \leq \max_{t\geq 0} I_0(tu_n) \leq \max_{t\geq 0} I_{\varepsilon_n}(tu_n) = I_{\varepsilon_n}(u_n) \leq c_0 + h(\varepsilon_n),$$

from which follows that $I_{\varepsilon_n}(u_n) \to c_0$. Arguing as in [19, Lemma 3.2] we can check that

$$\liminf_{n\to\infty} \int_{B_R(\tilde{y}_n)} |u_n|^p \geq \gamma > 0, \tag{4.7}$$

where $(\tilde{y}_n) \subset \mathbb{R}^N$ and $R, \gamma > 0$ are positive constants. At this point we notice that, although v_ε in [19, Lemma 3.2] be a solution of the modified problem, the author in that paper have used only the fact that v_ε belongs to the Nehari manifold. By this same reason, we can argue as in [19, Lemma 3.3] to conclude that $(\varepsilon_n \tilde{y}_n)$ is bounded in $\mathbb{R}^N$.

Claim. Up to a subsequence, $\varepsilon_n \tilde{y}_n \to y \in M$.

Assuming the claim, we can prove the lemma as follows. Since $y \in M$, we have that $y_n := \varepsilon_n \tilde{y}_n \in M_\delta$ for n sufficiently large. Hence,

$$\begin{aligned}
\beta_{\varepsilon_n}(u_n) &= \frac{\displaystyle\int \Upsilon(\varepsilon_n x)|u_n|^p \, \mathrm{d}x}{\displaystyle\int |u_n|^p \, \mathrm{d}x} = \frac{\displaystyle\int \Upsilon(\varepsilon_n z + y_n)|u_n(z+\tilde{y}_n)|^p \, \mathrm{d}z}{\displaystyle\int |u_n(z+\tilde{y}_n)|^p \, \mathrm{d}z} \\
&= y_n + \frac{\displaystyle\int \left(\Upsilon(\varepsilon_n z + y_n) - y_n\right)|u_n(z+\tilde{y}_n)|^p \, \mathrm{d}z}{\displaystyle\int |u_n(z+\tilde{y}_n)|^p \, \mathrm{d}z}.
\end{aligned}$$

It follows from $\varepsilon_n z + y_n \to y \in M$ that $\beta_{\varepsilon_n}(u_n) = y_n + o_n(1)$, and therefore the sequence (y_n) satisfies (4.6).

It remains to check the claim. By the boundedness of $(\varepsilon_n \tilde{y}_n)$ we may suppose that $\varepsilon_n \tilde{y}_n \to y$. We need only to check that $y \in M$. We start by setting $v_n(x) := u_n(x+\tilde{y}_n)$. The expression (4.7) and the boundedness of (u_n) imply that, along a subsequence, $v_n \rightharpoonup v \neq 0$ weakly in X. By taking $(t_n) \subset \mathbb{R}^+$ such that $w_n := t_n v_n \in \mathcal{M}_0$, we get

$$\begin{aligned}
c_0 \leq I_0(w_n) &= \frac{t_n^p}{p}\int (|\nabla v_n|^p + V_0|v_n|^p) - \int F(t_n v_n) \\
&\leq \frac{t_n^p}{p}\int (|\nabla u_n|^p + V(\varepsilon_n x)|u_n|^p) - \int G(\varepsilon_n x, t_n u_n) \\
&= I_{\varepsilon_n}(t_n u_n) \leq I_{\varepsilon_n}(u_n) \leq c_0 + h(\varepsilon_n).
\end{aligned}$$

Hence, $I_0(w_n) \to c_0$ and therefore $w_n \not\to 0$ in X. Since (w_n) and (v_n) are bounded in X and $v_n \not\to 0$ in X, the sequence (t_n) is bounded. Thus, we may suppose that $t_n \to t_0 \geq 0$. If $t_0 = 0$ then $\|w_n\|_X \to 0$, which does not make sense. Hence $t_0 > 0$, and therefore the sequence (w_n) satisfies $I_0(w_n) \to c_0$ and $w_n \rightharpoonup w := t_0 v \neq 0$ weakly in X. It follows from Lemma 4.2 that $w_n \to w$ in X.

We now suppose, by contradiction, that $y \notin M$. Then $V(y) > V_0$ and we can use the convergente of w_n and Fatou's lemma to get

$$\begin{aligned} c_0 &= I_0(w) < \frac{1}{p}\int(|\nabla w|^p + V(y)|w|^p) - \int F(w) - \frac{1}{p^*}\int |w|^{p^*} \\ &\leq \liminf_{n\to\infty}\left\{\frac{1}{p}\int(|\nabla w_n|^p + V(\varepsilon_n z + y_n)|w_n|^p)\ \mathrm{d}z - \int G(\varepsilon_n z + y_n, w_n)\mathrm{d}z\right\} \\ &= \liminf_{n\to\infty} I_{\varepsilon_n}(t_n u_n) \leq \liminf_{n\to\infty} I_{\varepsilon_n}(u_n) = c_0, \end{aligned} \tag{4.8}$$

which does not make sense. The lemma is proved. □

We are now ready to present the proof of Theorem 4.1.

Proof of Theorem 4.1. Given $\delta > 0$ we can use (4.5), Lemmas 4.3 and 4.4, and argue as in [8, Section 6] to obtain $\varepsilon_\delta > 0$ such that, for any $\varepsilon \in (0, \varepsilon_\delta)$, the diagram

$$M \xrightarrow{\Phi_\varepsilon} \Sigma_\varepsilon \xrightarrow{\beta_\varepsilon} M_\delta$$

is well defined and $\beta_\varepsilon \circ \Phi_\varepsilon$ is homotopically equivalent to the embedding $\iota : M \to M_\delta$. Since $c_0 < (1/N)S^{N/p}$, we can use the definition of Σ_ε and Proposition 3.1 to guarantee that I_ε satisfies the Palais-Smale condition in Σ_ε (taking ε_δ smaller if necessary). Ljusternik-Schnirelmann theory for C^1 functionals (see [14, Corollary 4.17]) provides at least $\mathrm{cat}_{\Sigma_\varepsilon}(\Sigma_\varepsilon)$ critical points u_i of I_ε restricted to $\mathcal{N}_\varepsilon$. Arguing along the same lines of the proof of Proposition 3.1 we can check that u_i is a critical point of the unconstrained functional I_ε.

By using the same ideas contained in the proof of [6, Lemma 4.3] we can check that $\mathrm{cat}_{\Sigma_\varepsilon}(\Sigma_\varepsilon) \geq \mathrm{cat}_{M_\delta}(M)$. In order to show that $u := u_i$ is positive, we set $u^- := \max\{-u, 0\}$ and compute

$$0 = \langle I'_\varepsilon(u), u^-\rangle = \|u^-\|^p_\varepsilon - \int g(\varepsilon x, u)u^- = \|u^-\|^p_\varepsilon.$$

Thus $u \geq 0$ in $\mathbb{R}^N$ and we can adapt arguments from [16, Theorem 1.11] to conclude that $u \in L^\infty(\mathbb{R}^N) \cap C^{1,\alpha}_{loc}(\mathbb{R}^N)$ for some $0 < \alpha < 1$. It follows from Harnack's inequality [24] that u is positive in $\mathbb{R}^N$. The theorem is proved. □

5 Proof of Theorem 1.1

In order to prove Theorem 1.1 we need only to verify that, for $\varepsilon > 0$ small enough, the solutions given by Theorem 4.1 satisfy the estimate in (2.2). As in [10], the key step is the following result.

Proposition 5.1. *There exists $\varepsilon^* > 0$ such that, if $\varepsilon \in (0, \varepsilon^*)$ and $u_\varepsilon \in \Sigma_\varepsilon$ is a solution of $(\widetilde{P}_\varepsilon)$, then*

$$\max_{\partial\Lambda_\varepsilon} u_\varepsilon < t_a.$$

Before proving this proposition, let us see how Theorem 1.1 follows from it.

Proof of Theorem 1.1. Given $\delta > 0$ such that $M_\delta \subset \Lambda$, we can invoke Theorem 4.1 to obtain, for any $\varepsilon \in (0, \varepsilon_\delta)$ fixed, $\mathrm{cat}_{M_\delta}(M)$ solution of $(\widetilde{P}_\varepsilon)$. Taking ε_δ smaller if necessary, we can use Proposition 5.1 to guarantee that, if u_ε is one of these solutions, then

$$u_\varepsilon(x) < t_a \quad \text{for all } \in \partial\Lambda_\varepsilon. \tag{5.1}$$

The proof now can be done as in [10]. We sketch it for completeness. Let v_ε be defined as $v_\varepsilon(x) := \max\{u_\varepsilon - t_a, 0\}$ if $x \in \mathbb{R}^N \setminus \Lambda_\varepsilon$, $v_\varepsilon(x) := 0$ otherwise. In view of (5.1), we can take v_ε as a test function for I_ε to get

$$\int_{\mathbb{R}^N \setminus \Lambda_\varepsilon} |\nabla v_\varepsilon|^p + c(x) v_\varepsilon^2 + t_a c(x) v_\varepsilon = 0, \tag{5.2}$$

where

$$c(x) := V(\varepsilon x)|u_\varepsilon(x)|^{p-2} - \frac{g(\varepsilon x, u_\varepsilon(x))}{u_\varepsilon(x)}.$$

Condition (g_3)(ii) implies that $c(x) \geq 0$ in $\mathbb{R}^N \setminus \Lambda_\varepsilon$, and therefore it follows from (5.2) that $v_\varepsilon \equiv 0$. Thus, (2.2) holds and u_ε is a solution of (P_ε). The theorem is proved. □

It remains to prove Proposition 5.1.

Proof of Proposition 5.1. Suppose, by contradiction, that the result is false. Then there exist $\varepsilon_n \to 0^+$, $u_{\varepsilon_n} \in \Sigma_{\varepsilon_n}$ solution of $(\widetilde{P}_{\varepsilon_n})$ such that $u_{\varepsilon_n}(x_n) \geq t_a$, for some point $x_n \in \partial\Lambda_{\varepsilon_n}$. Setting $v_n := u_{\varepsilon_n}(\cdot + x_n)$, we claim that $v_n \to v$ in $C^0_{loc}(\mathbb{R}^N)$. Indeed, the same calculations performed in [19, Proposition 3.6] provide $C > 0$ such that

$$\|v_n\|_{L^\infty(\mathbb{R}^N)} \leq C.$$

If $\Omega \subset \mathbb{R}^N$ is a bounded domain and $\xi > 0$, the above inequality and (2.3) imply that

$$|V(\varepsilon x + \varepsilon_n x_n) v_n^{p-1} - g(\varepsilon x + \varepsilon_n x_n, v_n)| \leq (C_\Omega + \xi) C^{p-1} + C_\xi C^{q-1} + C^{p^*-1}. \tag{5.3}$$

Since v_n satisfies

$$-\Delta_p v_n + V(\varepsilon x + \varepsilon_n x_n) v_n^{p-1} = g(\varepsilon x + \varepsilon_n x_n, v_n),$$

we can use (5.3) and a result of Di Benedetto [11, Theorem 2] to conclude that, for any compact set $K \subset \Omega$, there exists a constant $C_{K,\Omega}$ depending only of C, C_ξ, N, p and $\mathrm{dist}(K, \partial\Omega)$ such that

$$\|v_n\|_{C^{0,\alpha}(\Omega)} \leq C_{K,\Omega},$$

for some $0 < \alpha < 1$. It follows from the Schauder embedding theorem that v_n possesses a convergent subsequence in $C^0_{loc}(\mathbb{R}^N)$.

We now observe that, as in the proof of Lemma 4.4, we have that $I_{\varepsilon_n}(u_n) \to c_0$. Along a subsequence, $v_n \rightharpoonup v$ weakly in X. Since $v_n(0) = u_{\varepsilon_n}(x_n) \geq t_a > 0$, the convergence in $C^0_{loc}(\mathbb{R}^N)$ implies that $v \neq 0$. If $t_n > 0$ is such that $w_n = t_n v_n \in \mathcal{M}_0$ we can argue as in the proof of Lemma 4.4 to conclude that $I_0(w_n) \to c_0$. It follows from Lemma 4.2 that $w_n \to w$ in X and $I_0(w) = c_0$.

Since $\partial\Lambda$ is compact, we may suppose that $\varepsilon_n x_n \to \bar{x} \in \partial\Lambda$. In view of (V_2), we have that $V(\bar{x}) > V_0$. We can now repeat the calculations made in (4.8) and obtain a contradiction. This concludes the proof os the proposition. □

We end the paper by making some comments concerning the concentration behavior of the solutions obtained in Theorem 1.1. If $1 < p \leq 2$ and $\varepsilon > 0$ is sufficiently small, then the same arguments employed in [19] show that the solution u_ε possesses at most one local (hence global) maximum point z_ε in $\mathbb{R}^N$, which is inside Λ, such that

$$\lim_{\varepsilon \to 0^+} V(z_\varepsilon) = V_0 \tag{5.4}$$

and there are positive constants C, α such that

$$u_\varepsilon(z) \leq C \exp\left(-\alpha \left|\frac{z - z_\varepsilon}{\varepsilon}\right|\right) \text{ for all } z \in \mathbb{R}^N.$$

In the proof presented in [19], the restriction $1 < p \leq 2$ is necessary in order to guarantee that the ground-state solution ω of the autonomous problem (4.1) is radially symmetric about some point in $\mathbb{R}^N$ and the corresponding function $\omega(r)$ obeys $\omega'(r) < 0$ for all $r > 0$.

The complementary case $2 < p < N$ was not considered in [19]. However, by adapting the arguments contained in [12] we can obtain a parcial concentration result also in this case. Indeed, by using [12, Lemmas 4.4 and 4.5] and arguing as in the last part of proof of Theorem 1.1 in [12] we can prove that, if $2 < p < N$ and $\varepsilon > 0$ is sufficiently small, then the solutions u_ε have maximum points z_ε contained in a fixed ball $B_R(0) \subset \mathbb{R}^N$ and, moreover, the maximum points also satisfy (5.4).

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